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Notes on the theory of varifolds

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EXPOSÉ n° IV

NOTES ON THE THEORY OF VARIFOLDS

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I. INTRODUCTION.

The purpose of these notes is to state the basic theorems of the theory of varifolds which appear in [1] and to sketch their proofs in such a manner us to make this material immediately accessible to geometers and others who know a modest amount of classical analysis. In order to do this it has been necessary to state some theorems in somewhat less generality than one finds in [1] and to leave out some details and some parts of important ideas. Nonetheless, I feel a great deal has been included. It is worth pointing out that the short list of references includes some expository works. This work rests on the contributions of others ; see [1] for details.

I wish to thank Jean-Pierre Bourguignon and H. Blaine Lawson for inviting me to participate in the Seminar on Minimal Surfaces. My visit to the Ecole Polytechnique and its environs was extremely pleasant.

II. BASIC DEFINITIONS.

Throughout these notes, k and n are integers with $0 \leqslant k \leqslant n$ and Ω is an open subset of $\mathbb{R}^n$. We let

$$V_k(\Omega)$$

be the set of Radon measures on $\Omega \times \mathbb{G}_k(\mathbb{R}^n)$ where $\mathbb{G}_k(\mathbb{R}^n)$ is the Grassmann manifold of k-dimensional linear subspaces of $\mathbb{R}^n$; we call these measures k-<u>dimensional varifolds in</u> Ω. We endow $V_k(\Omega)$ with the weak topology ; thus, a sequence $V_\nu \to V$ in $V_k(\Omega)$ as $\nu \to \infty$ if $\int f\, dV_\nu \to \int f\, dV$ as $\nu \to \infty$ whenever f is a continuous compactly supported real valued function on $\Omega \times \mathbb{G}_k(\mathbb{R}^n)$. For each $V \in V_k(\Omega)$, we let

$$\|V\|(A) = V(A \times \mathbb{G}_k(\mathbb{R}^n)) \quad \underline{\text{for}}\ A \subset \Omega .$$

we let

$$d(V,a,r) = \frac{\|V\| \mathbb{B}(a,r)}{\alpha(k) r^k}$$

whenever $a \in \Omega$, $0 < r < \infty$ where we have set $\mathbb{B}(a,r) = \{x \in \mathbb{R}^n : |x-a| < r\} \subset \Omega$; here, we let $\alpha(k)$ is the k-dimensional area of $\{W \in \mathbb{R}^k : |W| < 1\}$. "d" here stands for "density".

We let

$$d(V,a) = \lim_{r \downarrow 0} d(V,a,r)$$

provided the limit exists. We let

$$\mathbf{M}_k(\Omega)$$

be the set of continuously differentiable embedded submanifolds of $\mathbb{R}^n$ which are subsets of Ω and which have locally finite k-dimensional area in Ω. For each $M \in \mathbf{M}_k(\Omega)$ we define the Radon measure $\|M\|$ on Ω by setting

$$\|M\|(A) = \mathcal{H}^k(A \cap M) \text{ \underline{for} } A \subset \Omega ;$$

here $\mathcal{H}^k$ is k-dimensional Hausdorff measure on $\mathbb{R}^n$. We have a map

$$v : \mathbf{M}_k(\Omega) \to V_k(\Omega)$$

whose value at $M \in \mathbf{M}_k(\Omega)$ is given by

$$v(M)(B) = \|M\|(\{x \in M : (x, T_x M) \in B\})$$

for $B \subset \Omega \times \mathbb{G}_k(\mathbb{R}^n)$; here $T_x M \in \mathbb{G}_k(\mathbb{R}^n)$ is the tangent space to M at x for each $x \in M$. Thus a k-dimensional varifold in Ω can be thought of as a generalized k-dimensional manifold in Ω. In fact, it is easy to see that the closure in $V_k(\Omega)$ of the set of finite positive linear combinations of elements of the range of v equals $V_k(\Omega)$, although this does not seem to be terribly important.

<u>In what follows we frequently identify</u> $S \in \mathbb{G}_k(\mathbb{R}^n)$ <u>with orthogonal projection of</u> $\mathbb{R}^n$ <u>onto</u> S. Suppose $F : \Omega \to \tilde{\Omega}$ carries Ω diffeomorphically onto the open subset $\tilde{\Omega}$ of $\mathbb{R}^n$. We define

$$F_\# : \Omega \to \tilde{\Omega}$$

at $V \in V_k(\Omega)$ by requiring that

$$F_\# V(B) = \int_{\{(x,S) : (F(x), DF(x)(S)) \in B\}} |\Lambda_k DF(x) \circ S| \, dV(x,S)$$

for $B \subset \tilde{\Omega} \times \mathbb{G}_k(\mathbb{R}^n)$. This definition is motivated by the elementary observation that

$$F_{\#}v(M) = v(F(M)) \text{ \underline{whenever} } M \in \mathbf{M}_k(\Omega).$$

One can relax the requirement that F be a diffeomorphism but we will not find this necessary.

III. THE FIRST VARIATION OF GENERALIZED AREA.

Let $\mathcal{X}(\Omega)$ be the vector space of smooth vectorfields on Ω with compact support ; that is, $X \in \mathcal{X}(\Omega)$ if X is a smooth compactly supported $\mathbb{R}^n$-valued function on Ω . We define the linear map

$$\delta V : \mathcal{X}(\Omega) \to \mathbb{R}$$

at $X \in \mathcal{X}(\Omega)$ by letting

$$\delta V(X) = \int DX(x).S \ ;$$

the inner product here is the natural inner product on $\mathrm{End}(\mathbb{R}^n)$. This definition is motivated as follows. Suppose $\varepsilon > 0$ and $F : (-\varepsilon,\varepsilon) \times \Omega \to \Omega$ is such that $F_o(x) = x$ and $\dot{F}_t(x) = X(F_t(x))$ for $(t,x) \in (-\varepsilon,\varepsilon) \times \Omega$. Suppose $(x,S) \in \Omega \times \mathbb{G}_k(\mathbb{R}^n)$ and $(u_1,\ldots,u_k)$ is an orthonormal basis for S . Since

$$|\Lambda_k\, DF_t(x)\circ S| = |DF_t(x)(u_1)\Lambda\ldots\Lambda DF_t(x)(u_k)| \quad \underline{\text{for}}\ |t| < \varepsilon \ ,$$

we have

$$\begin{aligned}\frac{d}{dt}|\Lambda_k\, DF_t(x)\circ S|\Big|_{t=0} &= \frac{d}{dt} DF_t(x)(u_1)\Lambda\ldots\Lambda DF_t(x)(u_k)\Big|_{t=o} . u_1\Lambda\ldots\Lambda u_k \\ &= \sum_{i=1}^{k} \frac{d}{dt} DF_t(x)(u_i)\Big|_{t=o} . u_i \\ &= \sum_{i=1}^{k} DX(x)(u_i).u_i \\ &= DX(x).S \ .\end{aligned}$$

It follows that, if $\|V\|(\Omega) < \infty$, then

$$\begin{aligned}\frac{d}{dt}\|F_{t\#}V\|(\Omega)\Big|_{t=0} &= \frac{d}{dt}\int|\Lambda_k\, DF_t(x)\circ S|\ dV(x,S)\Big|_{t=o} \\ &= \delta V(X) \ .\end{aligned}$$

We say V is <u>stationary</u> if $\delta V = 0$. For each open subset G of Ω we let

$$\|\delta V\|(G) = \sup\{|\delta V(X)| : X \in \mathcal{X}(\Omega), |X| \leq 1 \text{ and spt } X \subset G\} ;$$

thus $\|\delta V\|$ is the total variation of δV ; if $\|\delta V\|(G)$ is finite whenever G has compact closure in Ω, then $\|\delta V\|$ extends to a Radon measure on Ω which we denote by the same symbol in which case we say δV is a measure. In case δV is a measure, it follows from the Radon-Nikodym Theorem that there is a $\|\delta V\|$-measurable $\mathbb{S}^{n-1}$-valued function η such that

(1) $$\delta V(X) = \int X(x) \cdot \eta(x) \, d\|\delta V\| x \quad \text{for } X \in \mathcal{X}(\Omega).$$

IV. EXAMPLES.

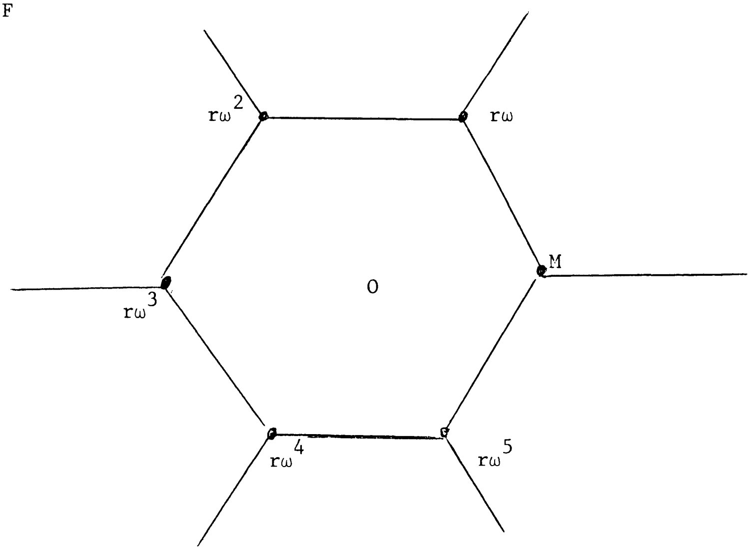
(a) $\Omega = \mathbb{R}^n$. Suppose M is a k-dimensional twice continuously differentiable submanifold of $\mathbb{R}^n$ with boundary B. By advanced calculus,

(2) $$\delta v(M)(X) = -k \int X(x) \cdot H(x) d\|M\| x + \int X(b) \cdot \nu(b) d\|B\| b$$

whenever $X \in \mathcal{X}(M)$, where H is the mean curvature normal field along M and ν is the unit exterior normal to B relative to M.

(b) Suppose $a \in \mathbb{R}^n$, F is a finite subset of $\mathbb{S}^{n-1}$ and $f : F \to \{w : 0 < w < \infty\}$. Let $V = \sum_{u \in F} f(u) \, v(\{a+tu : 0 < t < \infty\}) \in V_1(\mathbb{R}^n)$. It follows from (2) that

$\delta V(X) = - \sum_{u \in F} f(u) \, X(a) \cdot u$ for $X \in \mathcal{X}(\mathbb{R}^n)$.



(c) See [5] for a rather subtle example of a stationary 1-dimensional varifold in $\mathbb{R}^2$.

(d) $\Omega = \mathbb{R}^n$. Suppose $k \leqslant \ell \leqslant n$ and $T \in \mathbb{G}_\ell(\mathbb{R}^n)$. Since $T \in \mathbf{M}_\ell(\mathbb{R}^n)$, $\|T\|$ is defined and we have

(3) $$\int DX(x)(v) \, d\|T\| x = 0$$

whenever $X \in \mathcal{X}(\mathbb{R}^n)$ and $v \in T$. Let ν be a Radon measure on $\mathbb{G}_k(\mathbb{R}^n)$ such that $\operatorname{spt} \nu \subset \{S \in \mathbb{G}_k(\mathbb{R}^n) : S \subset T\}$ and let $V = \|T\| \times \nu \in \mathbf{V}_k(\mathbb{R}^n)$. Then, by (3),

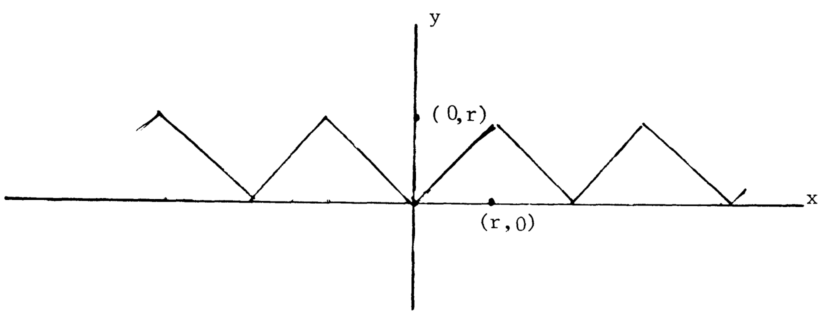
$$\begin{aligned} \delta V(X) &= \int DX(x) . S \, dV(x,S) \\ &= \int (\int DX(x) . S \, d\|T\| x) \, d\nu S \\ &= 0 \end{aligned}$$

whenever $X \in \mathcal{X}(\mathbb{R}^n)$,so V is stationary.

(e) $\Omega = \mathbb{R}^2$ and $k = 1$. Suppose $0 < r < \infty$, let $S_r = \{(x,x) : 0 < x < r\} \in \mathbf{M}_1(\mathbb{R}^2)$ and let $T_r = \{(x, 2r-x) : r < x < 2r\} \in \mathbf{M}_1(\mathbb{R}^2)$. Let

$$V_r = \sum_{j=-\infty}^{\infty} v((2jr,0)+S_r) + v((2jr,0)+T_r) \in \mathbf{V}_1(\mathbb{R}^2) \, ;$$

Below is a picture of $\operatorname{spt} \|V_r\|$.



Let $L = \{(x,0) : x \in \mathbb{R}\}$ and let $L^\pm = \{(x, \pm x) : x \in \mathbb{R}\}$. Thus $\{L, L^+, L^-\} \subset \mathbb{G}_1(\mathbb{R}^2) \cap \mathbf{M}_1(\mathbb{R}^2)$. Let $V = \frac{\sqrt{2}}{2} \|L\| \times (\delta_{L^+} + \delta_{L^-}) \in \mathbf{V}_1(\mathbb{R}^2)$, where $\delta_{L^\pm}$ are the point masses at $L^\pm$,respectively. It is easy to see that $\lim_{r \downarrow 0} V_r = V$. Note that δV is <u>not</u> a measure.

V. THE BASIC THEOREM.

The following theorem while easy to prove is absolutely fundamental. Keep in mind the varifold V in H(f).

THEOREM 4. *Suppose* $V \in V_k(\Omega)$ *and* δV *is a measure. Let* $f : \Omega \to \mathbb{R}$ *be a smooth function such that*

$$\operatorname{spt}\|V\| \subset \{x \in \Omega : f(x) = 0\} .$$

Then $S \subset \ker Df(x)$ *for* V *almost all* (x,S).

Proof. Let φ be a test function on Ω and let $X(x) = \varphi(x) f(x) \operatorname{grad} f(x)$ for $x \in \Omega$. Since δV is a measure and since X vanishes on $\operatorname{spt}\|V\|$, we have $\delta V(X) = 0$. On the other hand, one calculates

$$DX(x).S = f(x) \operatorname{grad}\varphi(x).S[\operatorname{grad} f(x)] + \varphi(x)|S[\operatorname{grad} f(x)]|^2 + \varphi(x) f(x) D(\operatorname{grad} f)(x).S$$

for $(x,S') \in \Omega \times \mathbb{G}_k(\mathbb{R}^n)$ which implies

$$\int \varphi(x)|S[\operatorname{grad} f(x)]|^2 \, dV(x,S) = 0 . \qquad \blacksquare$$

VI. THE MONOTONICITY FORMULA.

This formula ((7) below) may be thought of as the analogue for minimal surfaces of the mean value property for harmonic functions. Its effect on varifold theory and minimal surface theory is profound.

Suppose $0 < R < \infty$, $\Omega = \{x \in \mathbb{R}^n : |x| < R\}$, $V \in V_k(\Omega)$ and δV is a measure. Set $J = (-\infty, R)$. We define bounded Borel functions A,B,C,D on J at $r \in J$ as follows :

$$A(r) = \|V\| \{x : |x| \leqslant r\} ,$$
$$B(r) = \int_{\{(x,S):|x|\leqslant r\}} |x|^{-2}|S(x)|^2 \, dV(x,S),$$
$$C(r) = \int_{\{(x,S):|x|\leqslant r\}} |x|^{-2}|S^{\perp}(x)|^2 \, dV(x,S) ,$$
$$D(r) = \int_{\{x:|x|\leqslant r\}} x.\eta(x) \, d\|\delta V\| \, x ,$$

where in the definition of $D(r)$ we have used the notation of 3(1).

PROPOSITION 5. *We have*

(6) $\|V\| \{0\} = 0$ *and*

(7) $D(r) + r B'(r) = k A(r)$ *for* $r \in J$

in the sense of distributions.

Proof. Suppose $0 < \rho < R$. We choose a test function ψ on Ω such that $\psi \geqslant 0$, $\psi = 1$ near 0, $\psi(x) = 0$ if $|x| \geqslant \rho$ and $|D\psi(x)| \leqslant 2/\rho$ for $x \in \Omega$. Letting $W(x) = \psi(x)x$ for $x \in \Omega$ we obtain

$$\begin{aligned} k\|V\|\{0\} &\leqslant k \int \psi(x) \, dV(x,S) \\ &= \delta V(W) - \int D\psi(x)(S(x)) \, dV(x,S) \\ &\leqslant \delta V(W) + 2\|V\|\{x \in \Omega : \psi(x) \neq 0\}. \end{aligned}$$

Since ρ may be made arbitrarily small, (1) holds. It is a consequence of (1) by a simple argument using cut off functions that

$$(8) \qquad \int X(x).\eta(x)\; d\|\delta V\| x = \int DX(x).S\; dV(x,S)$$

whenever X is a continuous compactly supported vector field on Ω which is smooth away from 0 and which satisfies

$$\sup\{|DX(x)| + |X(x)/|x|| : 0<|x|<R\} < \infty$$

For each test function φ on J, we let $X_\varphi(x) = \varphi(|x|)\,x$ for $|x|<R$ and infer from (8) that

$$\delta V(X_\varphi) = \int \varphi'(|x|)\,|x|^{-1}\,|S(x)|^2 + k\varphi(|x|)\; dV(x,S)$$

so that, with $i(r) = r$ for $r \in J$, we have

$$D'(\varphi) = B'(i\varphi') + k\,A'(\varphi)$$

in the sense of distributions. Thus

$$D' + (i\,B')' = k\,A'$$

in the sense of distributions which implies (7) since A,B,D vanish on $\{r : r<0\}$. ■

What follows is the monotonicity formula.

THEOREM 9. Suppose $V \in V_k(\Omega)$ is stationary. Then

$$(9) \qquad \alpha(k)\,[d(V,a,s) - d(V,a,r)] = \int_{\{(x,S):r<|x-a|\le s\}} |x-a|^{-k-2}\,|S^{\perp}(x-a)|^2\; dV(x,S)$$

whenever $0<r<s$, $a \in \Omega$ and $\mathbb{B}(a,s) \subset \Omega$.

Proof. Let $a = 0$ and let Ω, A,B,C,D be as in the previous paragraph. We have from (7) that, in the sense of distributions,

$$\begin{aligned}\frac{d}{dr}\, r^{-k} A(r) &= -\,k\,r^{-k-1}A(r) + r^{-k}A'(r)\\ &= -\,r^{-k}B'(r) + r^{-k}A'(r)\\ &= r^{-k}C'(r)\end{aligned}$$

for $0<r<R$ since $A = B+C$. Antidifferentiate and note that A and C are continuous on the right. ■

Remark. Thus $d(V,a)$ exists whenever V is stationary and $a \in \Omega$.

Using (7) again we obtain the following extremely useful inequality.

THEOREM 10. Suppose $V \in V_k(\Omega)$, δV is a measure, $a \in \operatorname{spt}\|V\|$, $0 < r < s$ and $\mathbb{B}(a,s) \subset \Omega$. Then

$$(10) \qquad d(V,a,s) \exp \int_r^s \frac{\|\delta V\| \mathbb{B}(a,t)}{\|V\| \mathbb{B}(a,t)} dt \geqslant d(V,a,r) .$$

Proof. Again, let $a = 0$ and let Ω, A, B, C, D be as in the previous paragraph. Since $A-B = C$ is nondecreasing and (7) holds, we have

$$D(t) + t A'(t) \geqslant D(t) + t B'(t) = k A(t),$$

so that

$$\frac{A'(t)}{A(t)} - \frac{k}{t} + \frac{D(t)}{t A(t)} \geqslant 0$$

for $0 < t < R$, all in the sense of distributions. We antidifferentiate and use the continuity on the right of A and D to obtain (10). ∎

VII. THE ISOPERIMETRIC INEQUALITY.

In its simplest form this inequality is as follows. Suppose $V \in V_k(\mathbb{R}^n)$, $\|V\| (\mathbb{R}^n) < \infty$ and

PROPOSITION 11.

(11.1) $\qquad d(V,x) \geqslant 1$ for $\|V\|$ almost all $x \in \mathbb{R}^n$.

Then

(11.2) $\qquad \|V\| (\mathbb{R}^n)^{(k-1)/k} \leqslant C \|\delta V\| (\mathbb{R}^n)$

where C is a constant depending only on k and n . The proof of this uses (10) and the Besicovitch Covering Lemma which we now state. See [6 ,2.8.14] for the proof.

THE BESICOVITCH COVERING LEMMA. Suppose B is a family of closed balls in $\mathbb{R}^n$ with bounded union and A is the set of center points of the balls in B . There are subfamilies $B_1,\ldots,B_{B(n)}$ of B such that

(12.1) B_i is disjointed for $i \in \{1,\ldots,B(n)\}$ and

$$(12.2) \qquad A \subset \bigcup_{i=1}^{B(n)} \cup B_i \ ;$$

here $B(n)$ depends only on n .

The present work or, for that matter, the subject of geometric measure theory would not exist without this Lemma.

COROLLARY 13. Suppose μ and ν are two Radon measures on $\mathbb{R}^n$ with bounded supports. Suppose there are a subset A of $\mathbb{R}^n$ and a function $\rho : A \to \{r : 0 < r < \infty\}$ such that $\mu(\mathbb{R}^n \sim A) = 0$ and $\mu\, \mathbb{B}(a, \rho(a)) \leq \nu\, \mathbb{B}(a, \rho(a))$ for each $a \in A$. Then

$$\mu(\mathbb{R}^n) \leq B(n)\, \nu(\mathbb{R}^n) .$$

Proof. A simple exercise ■

We now prove the isoperimetric inequality. We may assume $\|\delta V\|(\mathbb{R}^n) < \infty$. Suppose $1 < \lambda < \infty$ and let $S = (\lambda \|V\|(\mathbb{R}^n)/\alpha(k))^{1/k}$. From (10) we obtain

$$\exp \int_0^s \frac{\|\delta V\|\, \mathbb{B}(a,t)}{\|V\|\, \mathbb{B}(a,t)}\, dt \geq \frac{1}{d(V,a,s)} \geq \lambda \quad \text{for } a \in A \text{ where}$$

$A = \{a \in \mathbb{R}^n : d(V,a) \geq 1\}$. Thus, for each $a \in A$ there is $\rho(a)$ with $0 < \rho(a) < s$ such that

$$\|V\|\, \mathbb{B}(a, \rho(a)) \leq (s/\log\lambda)\, \|\delta V\|\, \mathbb{B}(a, \rho(a)).$$

Now apply the Corollary to the Besicovitch Covering Lemma. ■

Let M,B,H be as in IV(a) and suppose $\|M\|(\mathbb{R}^n) < \infty$. It follows from (7) that $\|M\|(\mathbb{R}^n)^{(k-1)/k} \leq C(\int |H|\, d\|M\| + \|B\|(\mathbb{R}^n))$. Furthermore, it is shown in [1] and [2] using (7) and a clever but purely elementary analytic argument due to Federer that

$$\left(\int |f|^{(k-1)/k} d\|M\|\right)^{k/(k-1)} \leq C \int |\mathrm{grad}_M f| + |H|\,|f|\, d\|M\|$$

whenever f is a continuously differentiable real valued function on M whose support is a compact subset of M.

VIII. THE RECTIFIABILITY THEOREM.

THEOREM 14. (The rectifiability theorem) Suppose $V \in \mathbf{V}_k(\mathbb{R}^n)$, $\|V\|(\mathbb{R}^n) < \infty$, δV is a measure and $d(V,x) > 0$ for $\|V\|$ almost all $x \in \mathbb{R}^n$.

then, for any $\varepsilon > 0$, there are a positive integer J, elements $M_1, \ldots, M_J \in \mathbf{M}_k(\mathbb{R}^n)$ and positive real numbers $c_1, \ldots c_J$ such that the total variation of the (signed) measure

$$V - \sum_{j=1}^{J} c_j\, v(M_j)$$

on $\mathbb{R}^n \times \mathbb{G}_k(\mathbb{R}^n)$ does not exceed ε.

This theorem says roughly that any varifold satisfying its hypotheses can be strongly approximated by a finite positive real linear combination of continuously differentiable submanifolds. Before continuing, the reader should convince himself that neither the varifold V in IV(e) if $\ell > k$ nor the varifold V in IV(f) satisfies either the hypothesis or the conclusion of the rectifiability theorem. We now sketch its proof.

We shall now discuss homothetic expansions of a varifold about a fixed center point. Whenever $0 < r < \infty$, we let $\mu_r(x) = r\,x$ and, whenever $a \in \mathbb{R}^n$, we let $\tau_a(x) = a + x$ for $x \in \mathbb{R}^n$. For $V \in \mathbf{V}_k(\mathbb{R}^n)$, $a \in \mathbb{R}^n$ and $0 < r < \infty$, we let

$$V_{a,r} = r^{-k}(\mu_r \circ \tau_{-a})_{\#} V \in \mathbf{V}_k(\mathbb{R}^n) .$$

We shall show that under appropriate conditions the varifold $V_{a,r}$ is cone-like for large r. More specifically, let us suppose V satisfies the hypotheses of the rectifiability theorem. We shall indicate how one obtains a $\|V\|$ measurable $\mathbf{G}_k(\mathbb{R}^n)$-valued function T with the property that

(15) $$\lim_{r \downarrow 0} V_{a,r} = d(V,a)\, v(T) \quad \underline{\text{for}}\ \|V\|\ \underline{\text{almost all}}\ a .$$

Using (15) one can show by a fairly straightforward argument using the Bescicovitch Covering Lemma that the conclusion of the rectifiability theorem holds where the M_j, $j \in \{1,\dots,J\}$ are Lipschitzian. Now, it is a well known but by no means trivial theorem in geometric measure theory that a Lipschitzian function off a set a small measure equals a continuously differentiable function. Thus, one completes the proof of the rectifiability theorem. It should be noted that the distinction between Lipschitzian and continuously differentiable is not very important in the present context because the graphs of Lipschitzian functions have nice tangential properties by Rademacher's theorem. We now proceed to give the reader some idea of how one obtains the function T in (15).

We begin with the following theorem

THEOREM 16. <u>Suppose</u> $V \in \mathbf{V}_k(\mathbb{R}^n)$, $a \in \mathbb{R}^n$ <u>and</u>

(16) $$\limsup_{r \downarrow 0} \|\delta V\|\, \mathbf{B}(a,r) / \|V\|\, \mathbf{B}(a,r) < \infty .$$

<u>Then</u>

i) $d(V,a)$ <u>exists</u> ;

ii) <u>any sequence</u> $r_1, r_2, r_3, \dots$ <u>of positive real numbers with limit</u> ∞ <u>has a subsequence</u> $r_{\lambda(1)}, r_{\lambda(2)}, r_{\lambda(3)}, \dots$ <u>such that, for some</u> $C \in \mathbf{V}_k(\mathbb{R}^n)$,

$$\lim_{\nu\to\infty} V_{a,r_{\lambda(\nu)}} = C \;;$$

iii) if C is as in ii) then

(a) $\delta V = 0$.

(b) $d(C,0,r) = d(V,a)$ for $0 < r < \infty$ and

(c) $x \in S$ for C almost almost all (x,S) .

Proof. Thus is a straightforward application of (9) , our definitions and elementary compactness theorems for measures. □

We now study C as above.

THEOREM 17. Suppose C satisfies (iii) (a), (c). Then

$$\|C\|\{r x : x \in A\} = r^k \|C\|\{x : x \in A\} \tag{17}$$

whenever $A \subset \mathbb{R}^n$ and $0 < r < \infty$.

Proof. For any smooth non negative function f on $\mathbb{S}^{n-1}$ we infer from iii)(a),(c) that C_f is stationary where

$$C_f(B) = \int_B f(|x|^{-1}x)\, dC(x,S) \quad \text{for} \quad B \subset \mathbb{R}^n \times \mathbb{G}_k(\mathbb{R}^n) \;. \qquad \blacksquare$$

Now apply (9) to C_f .

Here now is a condition that implies C is planar.

THEOREM 18. Suppose C satisfies (iii) (a), (c) and

$$d(C,x) \geq d(C,0) \text{ for } \|C\| \text{ almost all } x \;. \tag{18.1}$$

Then, for some $T \in \mathbb{G}_k(\mathbb{R}^n)$, we have

$$C = d(C,0)v(T) \;. \tag{18.2}$$

Proof. We use (17) and (9) repeatidly. Suppose $r > 0$, $\lambda > 0$, $a \in \mathbb{R}^n$ and $s > \lambda(|a|+r)$. Then,

$$\begin{aligned} d(C,a,r) &= d(C,\lambda a,\lambda r) \\ &\leq d(C,\lambda a,\; s-\lambda|a|) \\ &\leq \left(\frac{s-\lambda|a|}{s}\right)^{-k} d(C,0,s) \\ &= \left(\frac{s-\lambda|a|}{s}\right)^{-k} d(C,0). \end{aligned}$$

It follows from (18.1) that, for almost all $a \in \mathbb{R}^n$,

(19) $$d(C,a,r) \leq d(C,a) \quad \text{for } 0 < r < \infty .$$

Now, (19) in conjunction with (9) implies

$$x - a \in S \quad \text{for } \|C\| \times C \text{ almost all } (a,x,S).$$

We choose (x,S) such that

$$x - a \in S \text{ for } \|C\| \text{ almost all } a$$

and use iii) of Theorem 16 to complete the proof.

Now, suppose $V \in V_k(\mathbb{R}^n)$,

(20) $$\|\delta V\| \leq M \|V\| \text{ for some } M < \infty \quad \text{and}$$

(21) $$d(V,x) \geq 1 \text{ for } \|V\| \text{ almost all } x \in \mathbb{R}^n .$$

As a consequence of (10) we have

(22) $$e^{Ms} d(V,a,s) \geq e^{Mr} d(V,a,r)$$

whenever $a \in \mathbb{R}^n$ and $0 < r < s < \infty$.

Using the theory of symmetrical derivation, which is based on the Besicovitch Covering Lemma, we see that $\|V\|(\mathbb{R}^n \sim A) = 0$ where A is the set of points $a \in \mathbb{R}^n$ such that, for each $\varepsilon > 0$,

$$\lim_{r \downarrow 0} \frac{\|V\|\{x \in \mathbb{B}(a,r) : |d(V,x) - d(V,a)| \geq \varepsilon\}}{\|V\| \mathbb{B}(a,r)} = 0 .$$

Suppose $a \in A$. Let $r_1, r_2, r_3, \ldots$, λ and C be as in 16). Let $x \in \operatorname{spt}\|C\|$ and suppose $0 < r < s < \infty$. Using (16), we may choose a sequence $a_1, a_2, a_3, \ldots$ in $\mathbb{R}^n$ with limit a such that $\lim_{\nu\to\infty} d(V,a_\nu) = d(V,a)$ and $\lim_{\nu\to\infty} r_{\lambda(\nu)}^{-1}(a_\nu - a) = x$. Since $r < s$ and (22) holds we find

$$\begin{aligned} d(C,x,s) &\geq \limsup_{\nu\to\infty} d(V,a_\nu,r_\nu^{-1} r) \\ &\geq \lim_{\nu\to\infty} e^{Mr_\nu^{-1} r} d(V,a_\nu) \\ &= d(V,a) \\ &= d(C,0) \end{aligned}$$

so that (18.2) holds for some $T \in \mathbb{G}_k(\mathbb{R}^n)$. We would like to know that the limit $C = d(V,0)v(T)$ does not depend on the sequence $r_1, r_2, r_3, \ldots$ or the subsequence $r_{\lambda(1)}, r_{\lambda(2)}, r_{\lambda(3)}, \ldots$ Although it may not be obvious, there is no hope of obtaining a rectifiability theorem unless we can do this ; see [7] and the Topological Disk Theorem of that paper. So, let B be the set of those $a \in A$ such that $\lim_{r \downarrow 0} V_{a,r} = d(V,a)\, v(T)$ for some $T \in \mathbb{G}_k(\mathbb{R}^n)$. Consider V as a $\|V\|$ measurable function on $\mathbb{R}^n$ with values in probability measures on $\mathbb{G}_k(\mathbb{R}^n)$. If $a \in A$ and this function is weakly symmetrically differentiable at a with respect to $\|V\|$, one infers by a straightforward measure-theoretic argument that $a \in B$. But the theory of symmetrical derivation says that $\|V\|(A \sim B) = 0$. Thus, one shows that if V satisfies (20) and (21) above then it satisfies the conclusion of the rectifiability theorem . In order to relax (20) and (21) to get the rectifiability theorem, one employs a tricky but purely measure-theoretic argument which uses the Besicovitch Covering Lemma.

IX. THE REGULARITY THEOREM.

This may be stated as follows.

THEOREM 23. For each ε with $0<\varepsilon<1$ there are $\delta > 0$ and $C > 0$ with the following property :

Suppose

(i) $a \in \mathbb{R}^n$, $0 < r < \infty$ and V is a k-dimensional varifold which is stationary in $\{ x \in \mathbb{R}^n : |x-a| < r\}$;

(ii) $a \in \operatorname{spt}\|V\|$ and $d(V,x) \geqslant 1$ for $\|V\|$ almost all x ;

(iii) $d(V,a,r) \leqslant 1+\delta$.

Then,

(a) $M = \operatorname{spt}\|V\| \cap \{x \in \mathbb{R}^n : |x-a| < (1-\varepsilon)r\} \in \mathbf{M}_k(\mathbb{R}^n)$;

(b) $|T_x M - T_y M| \leqslant C(r|x-y|)^{1-\varepsilon}$

whenever $x, y \in M$.

More is proved in [1] but the above statement goes to the heart of the matter. Consideration of catenoids and complex varieties show that the statement becomes false if hypothesis (ii) is omitted. We now proceed to give a nearly complete proof.

We give four preparatory Lemmas. Let $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$.

LEMMA 24. For all $\varepsilon > 0$, there is $\delta > 0$ with the following property :

Suppose

(i) V is stationary in Ω ;

(ii) $0 \in \operatorname{spt}\|V\|$ and $d(V,x) \geqslant 1$ for $\|V\|$ almost all x ;

(iii) $d(V,0,1) \leqslant 1+\delta$.

Then there is $T \in \mathbb{G}_k(\mathbb{R}^n)$ such that

(24) $$\big|\, \|V\|\, \mathbb{B}(a,r) - \|T\|\, \mathbb{B}(a,r) \,\big| \leqslant \varepsilon$$

whenever $\varepsilon \leqslant r$, $a \in \mathbb{R}^n$ and $\mathbb{B}(a,r) \subset \mathbb{B}(0,1-\varepsilon)$.

Proof. Suppose the Lemma were false. There would be $\varepsilon > 0$ and sequences δ_ν ; V_ν ; a_ν ; r_ν, $\nu = 1,2,3,\ldots$, such that $\delta_\nu \downarrow 0$ as $\nu \to \infty$ and such that, for each $\nu=1,2,3,\ldots$,

$$V_\nu \text{ is stationary in } \Omega \text{ ;}$$

$0 \in \operatorname{spt}\|V_\nu\|$ and $d(V_\nu,x) \geqslant 1$ for $\|V_\nu\|$ almost all x ;

$$d(V_\nu,0,1) \leqslant 1+\delta_\nu \text{ ;}$$

$\varepsilon \leqslant r_\nu$, $a_\nu \in \mathbb{R}^n$ and $\mathbb{B}(a_\nu,r_\nu) \subset \mathbb{B}(0,1-\varepsilon)$ but such that, for all $T \in \mathbb{G}_k(\mathbb{R}^n)$,

(25) $$\big|\, \|V\|\ \mathbb{B}(a_\nu,r_\nu) - \|T\|\, \mathbb{B}(a_\nu,r_\nu) \,\big| > \varepsilon \text{ .}$$

Passing to a subsequence if necessary we could obtain V,a,r such that

$$\lim_{\nu\to\infty} V_\nu = V \text{ , } \lim_{\nu\to\infty} a_\nu = a \text{ , } \lim_{\nu\to\infty} r_\nu = r$$

and such that

$$V \text{ is stationary in } \Omega \text{ ;}$$

$\varepsilon \leqslant r$, $a \in \mathbb{R}^n$ and $\mathbb{B}(a,r) \subset \mathbb{B}(0,1-\varepsilon)$.

Keeping in mind (10), we infer that

$$0 \in \operatorname{spt}\|V\| \text{ and } d(V,x) \geqslant 1 \text{ for } \|V\| \text{ almost all } x \text{ .}$$

Since it is clear that $d(V,0,1) \leqslant 1$, it follows from (18.2) that $V = \mathbf{v}(T \cap \Omega)$ for some $T \in \mathbb{G}_k(\mathbb{R}^n)$. This implies that for any $b \in \mathbb{R}^n$ with $|b| < 1$ and any s with $0 < s < 1-|b|$ we have $\lim_{\nu\to\infty} \|V\|\, \mathbb{B}(b,s) = \|T\|\, \mathbb{B}(b,s)$. This quickly leads to a contradiction of (25). ∎

Remark. Consider again Reifenbergs topological Disk Theorem in [7].

LEMMA 26. For each $\varepsilon > 0$, there is $\delta > 0$ such that if (i), (ii), (iii) of Lemma 25 hold, $T \in \mathbb{G}_k(\mathbb{R}^n)$ and

(26.1) $$\int |S-T|^2 \, dV(x,S) \leqslant \delta$$

then

(26.2) $\operatorname{dist}(x,T) \leq \varepsilon$ whenever $x \in \operatorname{spt}\|V\| \cap \mathbb{B}(0,1-\varepsilon)$.

Proof. Argue as in the proof of Lemma 24 and use . ■

LEMMA 27. (The Lipschitz Approximation.) For each $\varepsilon > 0$ there are $\delta > 0$ and $C > 0$ such that if (i), (ii), (iii) hold, if $T \in \mathbb{G}_k(\mathbb{R}^n)$ and if (26.1) holds then there is a function $f : T \to T^{\perp}$ with Lipschitz constant not exceeding 1 such that

$$(27)\quad \|V\|(\mathbb{B}(0,1-\varepsilon) \sim \operatorname{range} F) + \|T\|(\{w \in T : F(w) \in \mathbb{B}(0,1-\varepsilon) \sim \operatorname{spt}\|V\|\}) \leq C \int |S-T|^2 \, dV(x,S)$$

where $F : T \to \mathbb{R}^n$ is such that $F(w) = w + f(w)$ for $w \in T$.

Proof. L et $\varepsilon > 0$. Suppose $\eta > 0$. Call a point $a \in \operatorname{spt}\|V\|$ good if

$$\int_{\mathbb{B}(a,r)\times \mathbb{G}_k(\mathbb{R}^n)} |S-T|^2 \, dV(x,S) \leq \xi \, \|V\| \, \mathbb{B}(a,r)$$

for every r such that $\mathbb{B}(a,r) \leq \mathbb{B}(0,1)$, where ξ is as δ in Lemma 26 with ε there equal η ; otherwise call it bad. Suppose a is good and $a \in \mathbb{B}(0,1-\varepsilon)$; by making δ sufficiently small and making use of Lemma 24 and (10) we may apply Lemma 26 to the varifolds $V_{a,r}$, $|a|+r < 1$, to infer that

$$(28)\quad \operatorname{dist}(x,T) \leq \eta \quad \text{whenever } x \in \operatorname{spt}\|V_{a,r}\| \cap \mathbb{B}(0,1-\eta).$$

By making δ small it also follows from Lemma 26 that

$$(29)\quad \operatorname{dist}(x,T) \leq \eta \quad \text{whenever } x \in \operatorname{spt}\|V\| \cap \mathbb{B}(0,1-\eta).$$

Combining (28) and (29) and choosing η appropriately we see that the set of good points in $\mathbb{B}(0,1-\varepsilon)$ lies on the range of F where F is as in the statement of the Lemma. To estimate the $\|V\|$-measure of the set of bad points B we use the Corollary to the Besicovitch Covering Lemma to infer that

$$(30)\qquad \|V\|(B) \leq B(n)\,\xi^{-1} \int |S-T|^2 \, dV(x,T).$$

Now, suppose $c \in W = \{w \in T : F(w) \in \mathbb{B}(0,1-\varepsilon) \sim \operatorname{spt}\|V\|\}$.
Let $r = \operatorname{dist}(c, \operatorname{spt}\|V\|)$. Using Lemmas 24 and 26, we see that by making δ and η sufficiently small we can ensure that $\mathbb{B}(F(c),2r) \subset B \cap \mathbb{B}(0,1)$. By (10) ,

$$\|T\|\,\mathbb{B}(c,2r) = \alpha(k)(2r)^k \leqslant 2^k\|V\|(B \cap \mathbb{B}(F(c),2r)).$$

From the Besicovitch Covering Lemma we infer that

$$\|T\|(W) \leqslant 2^k\,\mathbb{B}(n)\,\|V\|\,(B) \quad . \qquad \blacksquare$$

<u>Remark</u>. The above argument is some what different from the one used to prove a similar statement in [1] . See also [4] for a different approach to this problem.

LEMMA 31. <u>Suppose</u> $V \in V_k(\Omega)$ <u>is stationary in</u> Ω , φ <u>is a test function on</u> Ω , $T \in \mathbb{G}_k(\mathbb{R}^n)$ <u>and</u> A <u>is a</u> <u>k-dimensional affine subspace of</u> $\mathbb{R}^n$ <u>parallel to</u> T . <u>Then</u>

$$\int \varphi(x)^2 |S-T|^2\, dV(x,S) \leqslant 2 \int |D\varphi(x)|^2\ \mathrm{dist}(x,A)^2\, d\|V\|x \ .$$

<u>Proof</u>. We may assume $A = T$. Let $X(x) = \varphi(x)^2\, T^{\perp}(x)$ for $x \in \Omega$. Since

$$DX(x).S = 2\varphi(x)\mathrm{grad}\varphi(x).S[T^{\perp}(x)] + \varphi(x)^r T^{\perp}.S$$

and since $|S(T^{\perp}(x))| \leqslant (T^{\perp}.S)^{1/2}|T^{\perp}(x)$ for $(x,S) \in \Omega \times \mathbb{G}_k(\mathbb{R}^n)$
we obtain

$$\int \varphi(x)^2 T^{\perp}.S\ dV(x,S) = -\,2 \int \varphi(x)\mathrm{grad}\varphi(x).S[T^{\perp}(x)]\ dV(x,S)$$

$$\leqslant 2\left(\int \varphi(x)^2 T^{\perp}.S\ dV(x,S)\right)^{1/2} \left(\int |\mathrm{grad}\varphi(x)|^2 |T^{\perp}(x)|^2\ d\|V\|x\right)^{1/2} \quad .$$

Since $2\,T^{\perp}.S = |S-T|^2$, we are done. $\blacksquare$

Now,we are ready to prove the Basic Regularity Lemma from the iteration of which the Regularity Theorem follows by a long but straightforward elementary geometric argument which we omit. We let $A_k(\mathbb{R}^n)$ be the set of k-dimensional affine subspaces of $\mathbb{R}^n$. Whenever $A \in A_k(\mathbb{R}^n)$, $a \in \mathbb{R}^n$, $0 < r < \infty$ and V is a varifold in an open subset of $\mathbb{R}^n$ containing $\{x \in \mathbb{R}^n : |x-a| < r\}$ we let

$$\mu(V,A,a,r) = \left(r^{-k-2} \int_{\{x:|x-a|<r\}} \mathrm{dist}(x,A)^2\, d\|V\|x\right)^{1/2} \quad .$$

THE BASIC REGULARITY LEMMA 32. <u>There exist</u> $\delta, \theta, \lambda, C$ <u>such that</u> $0 < \theta < \lambda < 1$ <u>and with the following property</u> :

<u>Suppose</u>

(i) $a \in \mathbb{R}^n$, $0 < r < \infty$ <u>and</u> V <u>is a stationary varifold in</u> $\{x : |x-a| < r\}$;

(ii) $d(V,x) \geqslant 1$ <u>for</u> $\|V\|$ <u>almost all</u> x ;

(iii) $d(V,a,r) \leqslant 1 + \delta$;

(iv) $$A \in A_k(\mathbb{R}^n) \text{ \underline{and} } \mu(V,A,a,r) \leq \delta \ .$$

Then, there is $\tilde{A} \in A_k(\mathbb{R}^n)$ such that

(32.1) the angle between $\tilde{A}$ and A does not exceed $C\mu(V,A,a,r)$;

(32.2) $$\mu(V,\tilde{A},a,\theta r) \leq \lambda\ \mu(V,A,a,r) \ .$$

Proof. We assume $n = k+1$; The generalization to higher codimensions is easy. C will denote any constant depending only on n . We may assume $a = 0$, $r = 2$ and $A = T \in \mathbb{G}_k(\mathbb{R}^n)$. Set

$B = \{x \in \mathbb{R}^n : |T(x)| < 1 \text{ and } |T^{\perp}(x)| < 1\}$ and let $D = \{w \in T : |w| < 1\}$.

Suppose $\delta > 0$ and V satisfies (16)-(19). Set $\mu = \mu(V,T,0,2)$. Using Lemmas 27 and 31 we infer that if δ is sufficiently small, then

(33) $$\int_B |S-T|^2 \, dV(x,S) \leq C\ \mu^2$$

and there is a function $f : D \to T^{\perp}$ with Lipschitz constant not exceeding 1 such that if $F(w) = w + f(w)$ for $w \in D$ then

(34) $$\|V\|(B \sim X) + \|T\| \ (D \sim W) \leq C\mu^2$$

where we have set $X = \operatorname{spt}\|V\| \cap \operatorname{range} F$ and $W = T(X)$.

We will now estimate the Laplacian of f in a certain sense.
For each test function $\varphi : D \to T^{\perp}$ we let

$$U(\varphi) = \mu^{-1} \int_D Df.D\varphi \ \ d\|T\|$$

and we let

$$U_1(\varphi) = \mu^{-1} \int_{D\sim W} Df.D\varphi \ d\|T\| \ ;$$

$$U_2(\varphi) = \mu^{-1} \int_W Df.D\varphi - (D(\varphi\circ T).S)J \ \ d\|T\|$$

where we have put $S(w) = \operatorname{range} DF(w)$ and $J(w) = |\Lambda_k DF(w)|$ for $\|T\|$ almost all $w \in W$;

$$U_3(\varphi) = \mu^{-1} \int_W (D(\varphi\circ T).S) \ (1-d(V,F(.)))J \ d\|T\| \ ;$$

$$U_4(\varphi) = \mu^{-1}(\int_W D(\varphi\circ T).S \ d(V,F(.)) \ J \ d\|T\| \ - \delta V(\varphi\circ T)) \ ,$$

where $\delta V(\varphi \circ T)$ makes sense because, if δ is sufficiently small, the closure of $T^{\perp}(B \cap \operatorname{spt}\|V\|)$ is a compact subset of $\{y \in T^{\perp} : |y| < 1\}$. Since V is stationary, we have

(35) $$U = \sum_{j=1}^{4} U_j \quad .$$

<u>We shall now estimate</u> $|U_j(\varphi)|$, $j \in \{1,2,3,4\}$. By (34)

$$|U_1(\varphi)| \leq C\mu^{-1} \sup\{|D\varphi(w)| : w \in D\}\ \mu^2 \ .$$

In estimating $|U_2(\varphi)|$ what we will be doing is to compare a first variation of the area integral of f with the first variation of the Dirichlet integral of f . One calculates that

(36) $$D(\varphi \circ T)(W).\ S(W) = D\varphi(W).Df(W)\ J(W)^{-2}$$

so that

$$\begin{aligned} &Df(W).D\varphi(W) - D(\varphi \circ T)(W).S(W)\ J(W) \\ &\quad = Df(W).D\varphi(W)\varphi(1-1/J(W)) \\ &\quad = Df(W).D\varphi(W)\,|Df(W)|^2\ J(W)^{-1}(1+J(W))^{-1} \end{aligned}$$

for $\|T\|$ almost all $W \in D$. It follows that

$$|U_2(\varphi)| \leq \mu^{-1} \sup\{|D\varphi(W)| : W \in D\} \int_D |Df|^2\ d\|T\| \ .$$

Inasmuch as $|S(W)-T|^2 = 2J(W)^{-2}|Df(W)|^2$ for $\|T\|$ almost all $W \in D$ we see that

(37) $$\begin{aligned} \int_D |Df|^2\ d\|T\| &\leq \frac{\sqrt{2}}{2} \int_W |S(w)-T|^2\ J(W)\ d\|T\|\,W + \|T\|\ (D \sim W) \\ &\leq \frac{\sqrt{2}}{2} \int_B |S-T|^2\ dV(x,S) + \|T\|\ (D \sim W) \\ &\leq C\ \mu^2 \end{aligned}$$

by (33) and (34). Using (36) and (37) we estimate

$$\begin{aligned} |U_3(\varphi)| &\leq \mu^{-1} \int_D (d(V,F(w)-1)\,|Df|\,|D\varphi|\ d\|T\| \\ &\leq C \sup\{(d(V,F(w))-1)\,|D\varphi(w)| \ : w \in D\} \ ; \end{aligned}$$

Using (34) we estimate

$$|U_4(\varphi)| = \mu^{-1}\left|\int_{\{(x,S):x\in B\sim X\}} D\varphi(x).S \, dV(x,S)\right|$$

$$\leq C\,\mu\, \sup\{|D\varphi(w)| : w\in D\} .$$

We leave to the reader to use (34) to verify that

(38) $$\int_D |f|^2 \, d\|T\| \leq C\,\mu^2 .$$

Now suppose for each $\nu = 1,2,3,\ldots$ that $\delta_\nu > 0$, V_ν satisfies (32 i) - iv)) and $\lim_{\nu\to\infty} \delta_\nu = 0$. S and $\mu_\nu = \mu(V_\nu,T_\nu,0,2)$ and let f_ν, F_ν, X_ν, W_ν be as above for $\nu = 1,2,3,\ldots$. Setting $g_\nu = \mu_\nu^{-1} f_\nu$ we infer from (37) and (38) that

$$\int_D |g_\nu|^2 + |Dg_\nu|^2 \, d\|T\| \leq C \quad \text{for } \nu = 1,2,3,\ldots$$

so that by Rellich's Lemma there is a $\|T\|$ square summable function $h : D \to T^\perp$ such that, after passing to a subsequence,

$$\eta_\nu = \int_D |g_\nu - h|^2 \, d\|T\| \to 0 \quad \text{as } \nu\to\infty .$$

It follows from our estimates above that, for any test function $\varphi : D \to T^\perp$, we have

$$\int_D h.\Delta\varphi \, d\|T\| = -\lim_{\nu\to\infty} \mu_\nu^{-1}\int_D Df_\nu . D\varphi \, d\|T\| = 0$$

since, by Lemma 24 and (10),

(39) $$\xi_\nu = \sup\{d(V_\nu,x)-1 : x\in B\cap \operatorname{spt}\|V_\nu\|\} \to 0 \quad \text{as } \nu\to\infty$$

<u>Thus</u> h <u>is harmonic</u>. Since $\int_D |h|^2 \, d\|T\| \leq C$ we have

(40) $$|h(0)| \leq C , \quad |Dh(0)| \leq C \quad \text{and}$$

$$s^{-k-2}\int_{\{w:|w|<s\}} |h(w)-h(0)-Dh(0)(w)|^2 \, dw \leq C\, s^2 , \quad 0<s<1/2 ,$$

by standard estimates for harmonic functions. Now, let $\alpha_\nu(w) = \mu_\nu(h(0)+Dh(0)(W))$ for $w\in T$ and let $\tilde{A}_\nu = \{w+\alpha_\nu(w) : w\in T\} \in A_k(\mathbb{R}^n)$ for each $\nu = 1,2,3,\ldots$ It is evident that

(41) $$s^{-k-2}\int_{\{w:|w|<s\}} |f_\nu-\alpha_\nu|^2 \leq 2\, s^{-k-2}\eta_\nu\mu_\nu^2 + C\, s^2\, \mu_\nu^2 , \quad 0<s<1/2 .$$

Now

$$S^{-k-2}\int_{X_\nu \cap \mathbb{B}(0,s)} \operatorname{dist}(x,\tilde{A}_\nu)^2 \, d\|V\| x$$

$$\leq S^{-k-2}\int_{\{w:|w|\leq s\}} (1+\xi_\nu)\operatorname{dist}(F_\nu(w),\tilde{A}_\nu)^2 \, J_\nu(w)\, d\|T\| W$$

$$\leq S^{-k-2}/2\int_{\{w:|w|\leq s\}} (1+\xi_\nu)\,|f_\nu-\alpha_\nu|^2 \, d\|T\|$$

and

$$S^{-k-2}\int_{\mathbb{B}(0,S)\sim X_\nu} \operatorname{dist}(x,\tilde{A}_\nu)^2 \, d\|V_\nu\| x \leq S^{-k-2}\zeta_\nu^2 C \mu_\nu^2$$

where $\zeta_\nu = \sup\{\operatorname{dist}(x,\tilde{A}_\nu) : x \in B \cap \operatorname{spt}\|V_\nu\|\}$. Since $\zeta_\nu \to 0$ as $\nu \to \infty$, by Lemma 1 we see that $\limsup_{\nu\to\infty} \mu_\nu^{-2} S^{-k-2} \int_{\mathbb{B}(0,S)} \operatorname{dist}(x,\tilde{A}_\nu)^2 \, d\|V_\nu\| x \leq C S^2$ for $0 < S < 1/2$. This proves the Lemma. ∎

REFERENCES.

[1] W.K. ALLARD, On the first variation of a varifold, Ann. Math. 95 (1972), 417-491.

[2] W.K. ALLARD, On the first variation of area and generalized mean curvature, C.I.M.E. notes, Edizioni Cremonese, Roma, 1973.

[3] W.K. ALLARD, An a priori estimate for the oscillation of the normal to a hypersurface whose first and second variation with respect to a parametric elliptic integrand is controlled, Inventiones Math. 73 (1983), 287-321.

[4] W.A. ALLARD, An integrality theorem and a regularity theorem for surfaces whose first variation with respect to a parametric elliptic integrand is controlled, Proc. Symp. Pure Mathematics, Volume 44, Amer. Math. Soc. Providence (1986), 1-28.

[5] W.K. ALLARD and F.J. ALMGREN Jr., On the structure of one-dimensional varifolds with positive density, Inventiones Math. 34 (1976), 83-97.

[6] H. FEDERER, Geometric measure theory, Springer-Verlag, New-York, 1969.

[7] E.R. REIFENBERG, Solution of the Plateau problem for m-dimensional surfaces of varying topological type, Acta Math. 104 (1960), 1-92.

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