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SOLUTIONS IN GEVREY SPACES OF PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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§1. INTRODUCTION

In this paper we give a sufficient condition for the existence of a solution u in a given Gevrey space $\Gamma^d(R^n)$, d a rational number ≥ 1, $n \geq 2$, to a linear partial differential equation with constant coefficients $P(D)u = f$, when $f \in \Gamma^d(R^n)$. The result which we state here and the method for its proof may be viewed as an extension of results and methods contained in previous papers by E. De Giorgi and the author [7], [8], [5], [6], [3], [4] concerning the case $d = 1$. For this case see also [1], [14], [15].

By $\Gamma^d(R^n)$, $d > 0$ we denote the set of all C^∞ complex valued functions f in R^n such that for every compact set $K \subset R^n$ there exists a constant $c(K)$, which depends on f, such that

$$\sup_{x\in K} |D^\alpha f(x)| \leq c(K)^{|\alpha|+1}\,\Gamma(d|\alpha|+1)\ ,\quad \alpha \in \mathbb{Z}_+^n\ ,$$

where Γ is the Euler gamma function. Here D, or D_x, stands for $(D_1,\ldots,D_n)$, $D_j = -i\,\partial/\partial x_j$, $j = 1,\ldots,n$, $D^\alpha = D_1^{\alpha_1}\ldots D_n^{\alpha_n}$, $|\alpha| = \alpha_1 +\ldots+ \alpha_n$. We shall also use D_t, D_y when the derivatives are to be considered with respect to the variables t or y respectively.

Here is our main result.

1.1 Theorem.[1)] Let $P(D)$ be a linear differential operator with constant coefficients and let $d \geq 1$ be a given rational number. Suppose that there exists a finite number of vectors $N^j \in R^n \setminus \{0\}$, $j = 1,\dots,\ell$, such that

a) for every $j = 1,\dots,\ell$

$\xi \in R^n$, $t \in R$, $|\xi - \langle \xi, N^j \rangle N^j| \geq k$, $P(\xi + it N^j) = 0$ imply either

$t \geq - c_1 |\xi - \langle \xi, N^j \rangle N^j|^{1/\rho}$ or $t \leq - c_2 |\xi|^{1/d}$, for some constants

$k > 1$, $\rho > d$, $c_1 > 0$, $c_2 > 0$;

b) there exist positive constants γ_j, $j = 1,\dots,\ell$ such that

$$R^n \setminus \{0\} = \bigcup_{j=1}^{\ell} \Delta_j$$

where $\Delta_j = \{y \in R^n;\ |y| < \gamma_j \langle y, N^j \rangle\}$, $j = 1,\dots,\ell$.

Then $P(D)\, \Gamma^d(R^n) = \Gamma^d(R^n)$.

1.2 Remarks.

i) The condition in a) on the roots of the polynomial $P(\xi + it N^j)$ is in particular satisfied if the polynomial $Q(\eta', \eta_n)$, $\eta' = (\eta_1, \dots, \eta_{n-1})$, obtained from P by a change of orthogonal coordinates which sends η_n over N^j, is $\binom{n}{p}$-hypoelliptic of exponent $1/d$ in the sense of E.A. Gorin [10], for every $p = 1,\dots,n$.

ii) Conditions a) and b) are obviously satisfied by every d'-hypoelliptic polynomial P, with $1 \leq d' \leq d$ and by every polynomial P, ρ-hyperbolic in the sense of E. Larsson [16] with respect to a vector $N \in R^n \setminus \{0\}$ (and hence with respect to every vector in some open convex cone containing N), when $d < \rho \leq \infty$.

iii) All polynomials considered by T. Shirota [18], which may be called hybrids between hyperbolic and d'-hypoelliptic polynomials[2)] also satisfy conditions

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1) When $d = 1$ see [4], Theorem 4.6.

2) For the case $\alpha = 1$ see J. Fehrman [9].

a) and b) of Theorem 1.1, when $1 \leq d' \leq d$.

iv) The conclusion of Theorem 1.1 is obviously true for all polynomials which may be written as a product of polynomials of the types considered in ii) and iii). In particular this is the case for every homogeneous polynomial in two variables and any $d \geq 1$.

§2. PROOF OF THEOREM 1.1.

The proof of Theorem 1.1 is based on a representation formula for any $f \in \Gamma^d(R^n)$ and on the construction of a particular solution of the equation $P(D)v = G_1$, where G_1 is a kernel, depending on a parameter, connected with the representation of any $f \in \Gamma^d(R^n)$.

Let $d = r/s, r, s$ relatively prime positive natural numbers, and let $\bar{n} \geq 1$ be a natural number such that $\frac{n}{2s} + \frac{\bar{n}}{2r} > 1$. Put

$$Q(\xi,\tau) = \Big(\sum_{j=1}^{n} \xi_j^2\Big)^s + \Big(\sum_{h=1}^{\bar{n}} \tau_h^2\Big)^r , \quad \xi \in R^n , \quad \tau \in R^{\bar{n}}$$

and let E be the distribution on $R^{n+\bar{n}}$ defined by

$$(2.1) \qquad \langle E,\phi\rangle = (2\pi)^{-(n+\bar{n})} \int_0^{+\infty} d\nu \int_{R^{n+\bar{n}}} e^{-\nu Q(\xi,\tau)} \hat{\phi}(-\xi,-\tau)\, d\xi\, d\tau , \quad \phi \in C_o^\infty(R^{n+\bar{n}}) ,$$

where

$$\hat{\phi}(\xi,\tau) = (\mathscr{F}\phi)(\xi,\tau) = \int_{R^{n+\bar{n}}} e^{-i(\langle x,\xi\rangle + \langle t,\tau\rangle)} \phi(x,t)\, dx\, dt$$

is the Fourier transform of ϕ . It can be easily proved that $E \in C^\infty(R^{n+\bar{n}} \setminus \{(0,0)\})$ and that

(2.1') $$E(x,t) = \int_0^{+\infty} E(x,t;\nu)d\nu \ , \quad (x,t) \in R^{n+\bar{n}} \setminus \{(0,0)\} \ ,$$

where

(2.2) $$E(x,t;\nu) = \mathcal{F}^{-1}_{(\xi,\tau)}(\exp(-\nu Q(\xi,\tau))(x,t) \ .$$

It turns out that E is a fundamental solution to the semi-elliptic operator $Q(D_x,D_t)$ and that for every $\nu > 0, E(x,t;\nu)$ is a radial function with respect to $x \in R^n$ and with respect to $t \in R^{\bar{n}}$, i.e. its values depend only on the values of $|x|$ and $|t|$.

The following representation theorem holds.

<u>2.1. Theorem.</u>[3)] Let $f \in \Gamma^d(R^n)$, $d = r/s$, r,s relatively prime positive natural numbers, and let ϕ be a given positive non increasing function on R. Then there exists a function $g \in C^\infty(\dot{R}^n \times R^+)$ and a positive, non increasing C^∞ function ψ on R such that

(2.3) $$f(x) = \int_{R^n} dy \int_0^{+\infty} G(x-y,\sigma) \ g(y,\sigma)d\sigma \ , \quad x \in R^n \ ,$$

$$\text{supp } g \subset \{(y,\sigma) \in R^n \times R^+; \ \psi(|y|^2) \leq \sigma \leq 2\psi(|y|^2)\} \ ,$$

(2.4) $$\int_0^{+\infty} |g(y,\sigma)|d\sigma \leq \phi(|y|^2) \ ,$$

where $G(x,\sigma) = E(x,t)$, for $|t| = \sigma > 0$, and $E(x,t)$ is given by (2.1), (2.1') with $\bar{n} > 2\max(r,s)$.

Let now $\nu_o > 0$ and let

$$G_1(x,\sigma) = \int_o^{\nu_o} E(x,t;\nu)\Big|_{|t|=\sigma} d\nu \ , \quad x \in R^n, \ \sigma > 0 \ ,$$

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3) When $d = 1$ see [7].

and

$$G_2(x,\sigma) = \int_{\nu_\circ}^{+\infty} E(x,t;\nu)\Big|_{|t|=\sigma} d\nu \quad , \quad x \in R^n,\ \sigma \geq 0 \ .$$

It is easily seen that there exists a positive constant c such that

$$(2.5) \qquad |D_x^\alpha G_2(x,\sigma)| \leq c^{|\alpha|+1}\Gamma(|\alpha|/2s + 1) \ , \ x \in R^n,\ \sigma \geq 0,\ \alpha \in \mathbf{Z}_+^n \ .$$

As a consequence, the following propositions hold.

2.2.Proposition. Let the notation be as in Theorem 2.1 and let ϕ satisfy the condition: $\int_{R^n} \phi(|y|^2)dy < \infty$. Then the function

$$f_2(x) = \int_{R^n} dy \int_0^{+\infty} G_2(x-y,\sigma)g(y,\sigma)d\sigma \quad , \quad x \in R^n$$

is in $\Gamma^{1/2s}(R^n)$ and hence in the space $\gamma^d(R^n)$ of all C^∞ functions on R^n such that for every compact set $K \subset R^n$ and every $\varepsilon > 0$

$$\sup_{\alpha \in \mathbf{Z}_+^n} \varepsilon^{-|\alpha|}\Gamma(d|\alpha|+1)^{-1} \sup_{x\in K}|D^\alpha f(x)| < \infty \ .$$

2.3. Proposition. Let $d = r/s \geq 1$ and let $g \in C^\infty(R^n \times R^+) \cap L^1(R^n \times R^+)$ be such that $\mathrm{supp}\, g \subset \{(y,\sigma) \in R^n \times R^+;\ \sigma \geq \chi(y_n)\}$, where χ is a continuous positive function on R . Then the function

$$h(x) = \int_{R^n} dy \int_0^{+\infty} G_1(x-y,\sigma)g(y,\sigma)d\sigma \ , \ x \in R^n \ ,$$

is in $\Gamma^d(R^n)$.

Recalling that $\gamma^d(R^n) \subset \Gamma^d(R^n)$ and that $P(D)\gamma^d(R^n) = \gamma^d(R^n)$ for every $P(D)$ on R^n and every $d \geq 1$[4], from Proposition 2.2 it follows that Theorem 1.1

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[4] See B. Malgrange [17] for $d = 1$, F. Tréves [20] and G. Björck [2] for $d > 1$.

will be proved if we find a solution $u_1 \in \Gamma^d(R^n)$ to the equation $P(D)u = f_1$, when $f_1(x) = f(x) - f_2(x) = \int_{R^n} dy \int_0^{+\infty} G_1(x-y,\sigma) g(y,\sigma) d\sigma$, $x \in R^n$, and g is as in the representation formula (2.3) for f, when ϕ satisfies the condition of Proposition 2.2.

To this end we shall use the following theorems.

<u>2.4. Theorem</u>. Let $P(D) = \sum_{j=0}^{m} a_j(D')D_n^j$, $D' = (D_1,\dots,D_{n-1})$, be a linear differential operator with constant coefficients and let $d = r/s \geq 1$, $\bar{n} > 2r$, $0 < \nu_o < 1$.

Assume that there exist constants $k > 1$, $\rho > d$, $c_1 > 0$, $c_2 > o$ such that $(\xi',\lambda) \in R^{n-1} \times \mathbb{C}$, $|\xi'| > k$, $P(\xi',\lambda) = 0$ imply

$$\text{(2.6) either} \quad \operatorname{Im}\lambda \geq -c_1|\xi'|^{1/\rho} \quad \text{or} \quad \operatorname{Im}\lambda \leq -c_2(|\xi'| + |\operatorname{Re}\lambda|)^{1/d} .$$

Then for every $\sigma > 0$ there exists a solution $H(\cdot,\sigma) \in C^\infty(R^n)$ to the equation $P(D)v = G_1(x,\sigma)$ such that for every $\alpha \in Z_+^n$

$$|D_x^\alpha H(x,\sigma)| \leq c^{|\alpha|+1} e^{c'|x|} \sigma^{-d|\alpha|} \Gamma(d|\alpha|+1) \cdot \int_0^{\nu_o} \nu^{-\frac{m+n+\mu}{2s} - \frac{\bar{n}}{2r}} \cdot$$

$$\cdot \exp\left[-c''\,\nu^{-1/(2r-1)}\left[\sigma^{2r/(2r-1)} - \tilde{c}(1+|x_n|)^{\frac{2s\rho}{2s\rho-1}}\,\nu^{\frac{2s\rho-2r}{(2s\rho-1)(2r-1)}}\right]\right] d\nu$$

for any $x \in R^n$, and

$$|D_x^\alpha H(x,\sigma)| \leq c^{|\alpha|+1} e^{c'|x|}\, \Gamma(d|\alpha|+1)$$

for any $x \in R^n$ with $x_n < \delta$, $\delta < -\frac{\bar{n}}{2r} + 1$, where $c, c', c'', \tilde{c}$ are positive constants independent of x, σ, α and μ is a non negative number such that, as a consequence of (2.6),[5)]

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5) See Remark 4.2.

$$\sum_{j=0}^{m} |a_j(\xi')| \geq c_3|\xi'|^{-\mu} \ , \ |\xi'| > k+1$$

for some positive constant c_3 .

2.5. Theorem. Let $P(D)$ satisfy the assumptions of Theorem 2.4 and let

$$h(x) = \int_{R^n} dy \int_0^{+\infty} G_1(x-y,\sigma)g(y,\sigma)d\sigma \ , \ x \in R^n \ ,$$

where $g \in C^\infty(R^n \times R^+)$ and

i) $\operatorname{supp} g \subset \{(y,\sigma) \in R^n \times R^+;\ y_n \geq c_o,\ \sigma \geq \chi(y_n)\}$, c_o a constant, χ a positive continuous function on R ;

ii) $\int_{R^n} \exp(c'|y|)dy \int_0^{+\infty} |g(y,\sigma)|d\sigma < \infty$, where c' is the same constant as in the estimates of the function H in Theorem 2.4. With this function H put

$$u(x) = \int_{R^n} dy \int_0^{+\infty} H(x-y,\sigma)g(y,\sigma)d\sigma \ , \ x \in R^n \ .$$

Then $u \in \Gamma^d(R^n)$ and $P(D)u = h$.

2.6. Proof of Theorem 1.1. Let $\{\chi_j\}$, $j = 0,\dots,\ell$ be a C^∞ partition of unity subordinate to the covering $\{\Delta_o,\Delta_1,\dots,\Delta_\ell\}$ of R^n , where Δ_o is an open ball centered at the origin of R^n and Δ_j, $j = 1,\dots,\ell$, are the open cones in condition b). We have

$$f_1(x) = \sum_{j=0}^{\ell} \int_{R^n} dy \int_0^{+\infty} G_1(x-y,\sigma)\ \chi_j(y)\ g(y,\sigma)d\sigma = \sum_{j=0}^{\ell} h_j(x) \ ,$$

where g is the function in the representation formula (2.3) for f , when ϕ is chosen so that $\int_{R^n} e^{c'|y|}\phi(|y|^2)dy < \infty$, c' the same constant as in Theorem 2.4. Application of Theorem 2.5, after a rotation of coordinates which sends the vector $(0,\dots,0,1)$ over N^j and a translation when $j = 0$, yields a solution in

$\Gamma^d(R^n)$ to each equation $P(D)u = h_j$, $j = 0,\dots,\ell$, and hence a solution $u_1 \in \Gamma^d(R^n)$ of the equation $P(D)u = f_1$. According to the remark after Proposition 2.3 this completes the proof of the theorem.

The proofs of Theorem 2.1 and Proposition 2.3 are given in §3. Proposition 2.2 is obvious. Theorems 2.4 is proved in §4. Theorem 2.5 is an easy consequence of Theorem 2.4.

§3. PROOF OF THEOREM 2.1 AND PROPOSITION 2.3

For the proof of Theorem 2.1 we need some auxiliary lemmas. The following lemma contains some properties of the distribution (2.1).

3.1. Lemma. The distribution E on $R^{n+\bar{n}}$ defined by (2.1) is a fundamental solution of the differential operator

$$Q(D_x,D_t) = \Big(\sum_{i=1}^{n} D^2_{x_j}\Big)^s + \Big(\sum_{h=1}^{\bar{n}} D^2_{t_h}\Big)^r \ , \quad n/2s + \bar{n}/2r > 1 \ .$$

E is a C^∞ function on $R^{n+\bar{n}} \setminus \{(0,0)\}$ and a radial function of $x \in R^n \setminus \{0\}$, for every $t \in R^{\bar{n}}$, and of $t \in R^{\bar{n}} \setminus \{0\}$, for every $x \in R^n$.

Furthermore the following estimates hold

$$(3.1) \qquad |D_x^\alpha D_t^\beta E(x,t)| \le c^{|\alpha|+|\beta|+1}\Gamma(|\alpha|q/s+|\beta|q/r+1)$$
$$(|x|^{2s/q}+|t|^{2r/q})^{q(1-(n+|\alpha|)/2s-(\bar{n}+|\beta|)/2r)}$$

for any $(\alpha,\beta) \in \mathbb{Z}^+_n \times \mathbb{Z}^+_{\bar{n}}$, $(x,t) \in R^{n+\bar{n}} \setminus \{(0,0)\}$, where c is a positive constant and $q = \max(r,s)$;

$$(3.2) \qquad |D_x^\alpha E(x,t)| \le c^{|\alpha|+1}(t,K)\Gamma(|\alpha|r/s+1) \ , \ x \in K \ , \ \alpha \in \mathbb{Z}^+_n \ , \ t \in R^{\bar{n}} \setminus \{0\} \ ,$$

where $c(t,K)$ is a positive constant dependent on the compact set $K \subset R^n$, continuous in t and growing to infinity as $|t| \to 0$;

(3.3) $$|D_x^\alpha E(x,t)| \leq c^{|\alpha|+1}(x)\ \Gamma(|\alpha|+1)\ ,\ t \in R^{\bar{n}}\ ,\ x \in R^n \setminus \{0\}\ ,$$

where $c(x)$ is a positive constant continuously dependent on x and growing to infinity as $|x| \to 0$. Finally

$$D_x^\alpha D_t^\beta E(x,t) \in L^1_{loc}(R^{n+\bar{n}})$$

when $|\alpha|/2s + |\beta|/2r < 1$.

Proof. From (2.2) it follows that for $\nu > 0$, $E(x,t;\nu) \in C^\infty(R^n \times R^{\bar{n}})$ and that

$$D_x^\alpha D_t^\beta E(x,t;\nu) = \nu^{-(n+|\alpha|)/2s - (\bar{n}+|\beta|)/2r} \mathscr{F}_\xi^{-1}(\xi^\alpha e^{-|\xi|^{2s}})(\nu^{-1/2s}x) \cdot$$
$$\cdot \mathscr{F}_\tau^{-1}(\tau^\beta e^{-|\tau|^{2r}})(\nu^{-1/2r}t)$$

where $\mathscr{F}_\xi^{-1}$ (respectively $\mathscr{F}_\tau^{-1}$) denotes the inverse of the Fourier transformation with respect to x (with respect to t). Since, as it is easy to see,

(3.4) $$\mathrm{Re}\Big(\sum_{j=1}^{n}(\xi_j+i\eta_j)^2\Big)^s \geq |\xi|^{2s}/2 - b_s|\eta|^{2s}\ ,\ (\xi,\eta) \in R^n \times R^n$$

(3.4') $$\mathrm{Re}\Big(\sum_{h=1}^{\bar{n}}(\tau_h+i\theta_h)^2\Big)^r \geq |\tau|^{2r}/2 - b_r|\theta|^{2r}\ ,\ (\tau,\theta) \in R^{\bar{n}} \times R^{\bar{n}}\ ,$$

for certain constants $b_r, b_s \geq 1$, we have for $(x,t) \in R^n \times R^{\bar{n}}$, $(\alpha,\beta) \in \mathbb{Z}_+^n \times \mathbb{Z}_+^{\bar{n}}$

(3.5) $$|D_x^\alpha D_t^\beta E(x,t;\nu)| \leq c^{|\alpha|+|\beta|+1}\Gamma(|\alpha|/2s+1)\Gamma(|\beta|/2r+1)\nu^{-(n+|\alpha|)/2s-(\bar{n}+|\beta|)/2r}\cdot$$
$$\cdot \exp\Big[-c'(\nu^{-1/2s}|x|)^{2s/(2s-1)} - c''(\nu^{-1/2r}|t|)^{2r/(2r-1)}\Big]$$

where c, c', c'' are positive constants independent of x,t,ν,α,β .

The condition that $n/2s + \bar{n}/2r > 1$ implies that

$$<E,\phi> = \int_0^{+\infty} d\nu \int_{R^{n+\bar{n}}} E(x,t;\nu)\phi(x,t)\,dx\,dt \quad , \quad \phi \in C_o^\infty(R^{n+\bar{n}}) \quad .$$

Thus from (3.5) it follows that in $R^{n+\bar{n}} \setminus \{(0,0)\}$, E is the function

$$E(x,t) = \int_0^{+\infty} E(x,t;\nu)\,d\nu \quad .$$

Moreover

$$(3.6) \qquad D_x^\alpha D_t^\beta E(x,t) = \int_0^{+\infty} D_x^\alpha D_t^\beta E(x,t;\nu)\,d\nu \ , \ (x,t) \neq (0,0)$$

and if K is a compact set contained in $R^{n+\bar{n}} \setminus \{(0,0)\}$

$$(3.7) \qquad |D_x^\alpha D_t^\beta E(x,t)| \leq c(K)^{|\alpha|+|\beta|+1}\Gamma(|\alpha|q/s+|\beta|q/r+1) \ , \ (x,t) \in K$$

where $c(K)$ depends only on K and $q = \max(r,s)$.

It is also easy to see that for any $\lambda > 0$ and $(x,t) \neq (0,0)$

$$D_x^\alpha D_t^\beta E(x,t) = \lambda^{2-(n+|\alpha|)/s-(\bar{n}+|\beta|)/r} D_x^\alpha D_t^\beta E(\lambda^{-1/s}x,\lambda^{-1/r}t) \quad .$$

Hence inequality (3.1) follows choosing $\lambda = (|x|^{2s/q} + |t|^{2r/q})^{q/2}$ and applying (3.7) with $K = \left\{(\lambda^{-1/s}x,\lambda^{-1/r}t),(x,t) \in R^{n+\bar{n}} \setminus \{(0,0)\}\right\}$.

The remaining estimates of the lemma are also easily derived from (3.5) and (3.6).

Finally since for every $\phi \in C_o^\infty(R^{n+\bar{n}})$

$$<Q(D_x,D_t)E,\phi> = (2\pi)^{-(n+\bar{n})} \int_0^{+\infty} d\nu \int_{R^{n+\bar{n}}} e^{-\nu Q(\xi,\tau)} Q(\xi,\tau)\,\tilde{\phi}(-\xi,-\tau)\,d\xi d\tau =$$

$$= -(2\pi)^{-(n+\bar{n})} \int_0^{+\infty} \frac{d}{d\nu}\left(\int_{R^{n+\bar{n}}} e^{-\nu Q(\xi,\tau)}\,\tilde{\phi}(-\xi,-\tau)\,d\xi d\tau\right) d\nu = \phi(0,0) \quad ,$$

E is a fundamental solution of $Q(D_x,D_t)$.

<u>3.2. Lemma</u>. Let $\bar{n} > 2r$ and let $w \in C^\infty(R^{n+\bar{n}})$ satisfy the condition

$$\sup_{x\in R^n}\int_{R^{\bar{n}}}\Big[|Q(D_x,D_t)w(x,t)| + \sum_{\substack{|\alpha|<2s\\|\beta|<2r}}(|D_x^\alpha w(x,t)| + |D_t^\beta w(x,t)|)\Big]dt < +\infty$$

Then

$$w(x,t) = \int_{R^{n+\bar{n}}} E(x-y,t-z)Q(D_y,D_z)w(y,z)\,dy\,dz \quad , \quad (x,t) \in R^{n+\bar{n}} \quad .$$

<u>Proof</u>. Choose $\gamma \in C^\infty(R)$ such that $\gamma(u) = 1$ for $u \leq 0$ and $\gamma(u) = 0$ for $u \geq 1$, and for $\rho > 1$ put

$$w_\rho(x,t) = \gamma((|x|^{2s} + |t|^{2r})^{1/2q}-\rho)w(x,t) \quad , \quad (x,t) \in R^n \times R^{\bar{n}} \quad , \quad q = \max(r,s) .$$

Since $w_\rho \in C_o^\infty(R^{n+\bar{n}})$, from Lemma 3.1 it follows that

$$(3.8) \qquad w_\rho(x,t) = \int_{|y|^{2s}+|z|^{2r}\leq(\rho+1)^{2q}} E(x-y,t-z)Q(D_y,D_z)w_\rho(y,z)\,dydz = I_1 + I_2 \ ,$$

where

$$I_1 = \int_{|y|^{2s}+|z|^{2r}\leq\rho^{2q}} E(x-y,t-z)Qw(y,z)\,dy\,dz \quad ,$$

$$I_2 = \int_{\rho^{2q}\leq|y|^{2s}+|z|^{2r}\leq(\rho+1)^{2q}} E(x-y,t-z)\Big[\gamma Qw+wQ\gamma + \sum_{\substack{|\alpha|+|\alpha^*|=2s\\ \alpha,\alpha^*\neq 0}} c(\alpha,\alpha^*)D_y^\alpha w D_y^{\alpha^*}\gamma +$$

$$+ \sum_{\substack{|\beta|+|\beta^*|=2r\\ \beta,\beta^*\neq 0}} c(\beta,\beta^*)D_z^\beta w D_z^{\beta^*}\gamma\Big]dy\,dz \quad .$$

The condition assumed on the function w and the estimate (3.1) imply that if $\rho \geq 2(|x|^{2s/q} + |t|^{2r/q})^{1/2}$ then

$$|I_2| \leq c\rho^{q(2-n/s-\bar{n}/r)} \int_{\rho^{2q}\leq|y|^{2s}+|z|^{2r}\leq(\rho+1)^{2q}} \Big[|Qw| + \sum_{\substack{|\alpha|<2s\\|\beta|<2r}} (|D_y^\alpha w| + |D_z^\beta w|)\Big] dy\,dz \leq$$

$$\leq c'\rho^{q(2-n/s-\bar{n}/r)}(\rho+1)^{nq/s}$$

for some positive constants c,c'. Since $\bar{n} > 2r$ and $w(x,t) = w_\rho(x,t)$ when $\rho > (|x|^{2s} + |t|^{2r})^{1/2q}$, the conclusion of the lemma follows from (3.8) letting $\rho \to +\infty$.

The following lemma is a straightforward application of a result proved by G. Talenti [19][6)] and of the hypoellipticity of the operator $Q(D_x,D_t)$ considered in Lemma 3.1.

3.3. Lemma. Let $f \in \Gamma^d(R^n)$, $d = r/s > 0$, and let $Q(D_x,D_t)$ be the operator considered in Lemma 3.1. Then there exists an open set $A \subset R^n \times R^{\bar{n}}$ containing the set $\{(x,t) \in R^n \times R^{\bar{n}}; t_{\bar{n}} = 0\}$ and a function $u \in C^\infty(A)$ such that

$$Q(D_x,D_t)u = 0 \quad \text{in} \quad A$$

$$u(x,t',0) = f^*(x,t') \quad , \quad (x,t') \in R^n \times R^{\bar{n}-1}$$

$$D^j_{t_{\bar{n}}} U(x,t',0) = 0 \quad , \quad (x,t') \in R^n \times R^{\bar{n}-1},\ j = 1,\dots,2r-1 \ ,$$

where $f^*(x,t') = f(x)$ for $(x,t') \in R^n \times R^{\bar{n}-1}$, $t' = (t_1,\dots,t_{\bar{n}-1})$. Moreover $u(x,t) \in \Gamma^d(R^n)$ for every $t \in R^{\bar{n}}$.

We can now prove Theorem 2.1. Let A and u be as in Lemma 3.3 where we assume $\bar{n} > 2q$, $q = \max(r,s)$. Let $\Delta(\delta), \delta > 0$, be the distance of the set $\{(x,t)\in R^n \times R^{\bar{n}};\ |x|^2 \leq \delta,\ t = 0\}$ from $\complement A$. Put

$$A(\delta) = \{(x,t) \in R^n \times R^{\bar{n}};\ |x|^2 \leq \delta,\ |t| \leq \Delta(|x|^2)/2r\} \quad .$$

and

6) See Theorems III and V.

$$U(\delta) = \max_{A(\delta)} \{|D_x^\alpha u|, |D_t^\beta u|;\ |\alpha| \le 2s-1,\ |\beta| \le 2r-1\} .$$

Let $\psi \in C^\infty(R)$ be a positive non increasing function such that

$$(3.9) \qquad \psi(\delta) + \sum_{h=1}^{2s} |\psi^{(h)}(\delta)| \le \min \{[\phi(\delta)]^2, \Delta(\delta)/4r, [p+\delta+U(\delta)]^{-4}\}, \delta > 0 ,$$

where ϕ is the given positive non increasing function and p is a number ≥ 1 to be fixed later. Put

$$w(x,t) = \begin{cases} u(x,t)\gamma(|t|^2\psi^{-2}(|x|^2)) & \text{for } (x,t) \in \bigcup_{\delta>0} A(\delta) \\ 0 & \text{for } (x,t) \in R^{n+\bar{n}} \setminus \bigcup_{\delta>0} A(\delta) \end{cases} ,$$

where γ is a C^∞ function on R such that $0 \le \gamma(\tau) \le 1$, $\gamma(\tau) = 0$ when $\tau \ge 4$ and $\gamma(\tau) = 1$ when $\tau \le 1$. Since $\operatorname{supp} \gamma(|t|^2\psi^{-2}(|x|^2)) \subset \bigcup_{\delta>0} A(\delta) \subset A$, $w \in C^\infty(R^n \times R^{\bar{n}})$. Moreover for every $(x,t) \in R^n \times R^{\bar{n}}$ and some positive constant c

$$|Q(D_x,D_t)w(x,t)| \le c\, U(|x|^2)\psi^{-2q}(|x|^2) ,$$

$$|D_x^\alpha w(x,t)| \le c\, U(|x|^2)\psi^{-|\alpha|}(|x|^2) , \quad |\alpha| \le 2s - 1 ,$$

$$|D_t^\beta w(x,t)| \le c\, U(|x|^2)\psi^{-|\beta|}(|x|^2) , \quad |\beta| \le 2r - 1 ,$$

whence it follows, with another constant c,

$$(3.10) \qquad \int_{R^{\bar{n}}} |Qw(x,t)|dt = \int_{\psi(|x|^2)}^{2\psi(|x|^2)} \tau^{\bar{n}-1}d\tau \int_{|\omega|=1} |Qw(x,\omega\tau)|d\omega \le c\, U(|x|^2)\psi^{\bar{n}-2q}(|x|^2) ,$$

$$\int_{R^{\bar{n}}} |D_x^\alpha w(x,t)|dt = \int_0^{2\psi(|x|^2)} \tau^{\bar{n}-1} \int_{|\omega|=1} |D_x^\alpha w(x,\omega\tau)|d\omega \le c\, U(|x|^2)\psi^{\bar{n}-|\alpha|}(|x|^2) ,$$

$|\alpha| \leq 2s - 1$,

$$\int_{R^{\bar{n}}} |(D_t^{\beta} w(x,t)| \leq c\, U(|x|^2)\psi^{\bar{n}-|\beta|}(|x|^2) \;,\quad |\beta| \leq 2r - 1 \;.$$

These estimates together with (3.9) and the assumption $\bar{n} > 2q$ prove that w satisfies the condition of Lemma 3.2. Thus for every $x \in R^n$

$$f(x) = u(x,0) = w(x,0) = \int_{R^{n+\bar{n}}} E(x-y,-z)Q(D_y,D_z)w(y,z)\,dy\,dz \;.$$

Since $E(x,t)$ is a radial function of t , Theorem 2.1 is proved by letting

$$g(y,\sigma) = \int_{|z|=\sigma} Q(D_y,D_z)w(y,z)\,dz,\ \sigma > 0 \;.$$

Estimate (2.4) follows immediately from (3.9) and (3.10), with a suitable choice of the number p .

To prove Proposition 2.3 we note that, as a consequence of (2.5), the estimates (3.1), (3.2), (3.3) can also be used for the function $G_1(x,\sigma)$. Moreover if x is in a compact subset $K \subset R^n$ and $R > o$ is such that $K \subset \{x \in R^n;\ |x_n| \leq R\}$, then $D_x^{\alpha}G_1(x-y,\sigma)$ can be estimated by means of (3.3) when $|y_n| > 2R$ and by means of (3.2) when $|y_n| \leq 2R$. In fact in this case $(y,\sigma) \in \operatorname{supp} g$ implies that σ is greater than a positive constant. Since $d \geq 1$, these remarks prove the proposition.

§4. PROOF OF THEOREM 2.4

The proof of Theorem 2.4 is obtained in some steps, each of them stated below as a lemma.

4.1. Lemma. Assume that $P(D) = \sum_{j=0}^{m} a_j(D')D_n^j$ satisfies the conditions of Theorem

2.4 and choose k such that $2^{-1}c_2 k^{1/d} - c_1 k^{1/\rho} > 1$. Let $\Lambda(\xi'), |\xi'| > k$, be a continuous function such that

$$(4.1) \qquad |\Lambda(\xi')+2^{-1}c_2|\xi'|^{1/d}| \leq 2^{-1}c_2|\xi'|^{1/d} - c_1|\xi'|^{1/\rho}-1$$

and for $|\xi'| > k, \nu \in]0,\nu_o]$ consider the function $w_1(\xi',x_n;\nu)$ defined by

$$w_1(\xi',x_n;\nu) = (2\pi)^{-1} \int_{\mathrm{Im}\lambda=\Lambda(\xi')} \exp[ix_n\lambda-\nu(|\xi'|^2+\lambda^2)^s]/P(\xi',\lambda)d\lambda \ , \ x_n \in R \ .$$

Then

i) $P(\xi',D_n)w_1(\xi',x_n;\nu) = (2\pi)^{-1} \int_{-\infty}^{+\infty} \exp(ix_n\xi_n-\nu|\xi|^{2s})d\xi_n$,

ii) there exists $k^* \geq k$ and positive constants c,c',c_o independent of $\xi',x_n,\nu,\ell \in \mathbb{Z}_+$, such that if $|\xi'| > k^*$

$$(4.2) \qquad |D_n^\ell w_1(\xi',x_n;\nu)| \leq c^{m+\ell+1}(\sum_{j=0}^{m}|a_j(\xi')|)^{-1}\exp(-\nu|\xi'|^{2s}/4+c_1|x_n||\xi'|^{1/\rho}) \cdot$$
$$\cdot \ \nu^{-(m+\ell+1)/2s}\Gamma((m+\ell)/2s+1)\exp(c'|x_n|)$$

for every $x_n \in R$,

$$(4.3) \qquad |D_n^\ell w_1(\xi',x_n;\nu)| \leq c^{m+\ell+1}(\sum_{i=0}^{m}|a_j(\xi')|)^{-1}\exp(-\nu|\xi'|^{2s}/4) \cdot$$
$$\cdot \Big\{\nu^{-(m+\ell+1)/2s}\Gamma((m+\ell)/2s+1)\exp(-(x_n^{2s}/c_o\nu)^{1/(2s-1)}/2)\exp(c'|x_n|) +$$
$$+ |x_n|^{-(m+\ell)d}\Gamma((m+\ell)d+1)\exp(-(3/8)^{2s}c_2|x_n||\xi'|^{1/d}/2)\Big\}$$

for every $x_n < 0$, where if $d = 1$ c_2 is chosen so that $0 < c_2 < 6^{-\frac{1}{2}}$.

Proof. First note that from (4.1) and from the assumption on the roots of the equation $P(\xi',\lambda) = 0$ when $|\xi'| > k$, it follows that $P(\xi',\lambda) = 0$, $|\xi'| > k$, implies $|\mathrm{Im}\lambda - \Lambda(\xi')| \geq 1$. The assertion in i) is an immediate consequence of the Cauchy's theorem applied to the entire function of λ : $\exp[ix_n\lambda-\nu(|\xi'|^2+\lambda^2)^s]$ and of the inequality (3.4).

To prove ii) we shall use the following

<u>Proposition</u>. For every $\varepsilon > 0$ and every $\xi' \in R^{n-1}$ there exists a finite number $\leq m$ of closed disjoint discs $C_h(\xi')$ contained in $\mathbb{C}$ and with radius $\leq \varepsilon$ such that all the roots of the equation in λ $P(\xi',\lambda) = 0$ are contained in $\bigcup_h C_h(\xi')$ and have a distance $\geq 4^{-m+1}\varepsilon$ from the boundary $\partial C_h(\xi')$ of each discs. Moreover if $\lambda \in \mathbb{C} \setminus \bigcup_h C_h(\xi')$ then

$$(4.4) \qquad |P(\xi',\lambda)| \geq c(\varepsilon)^{-1}|\lambda|^{-m}\sum_{j=0}^{m}|a_j(\xi')| \ , \ \xi' \in R^{n-1} \ ,$$

where $c(\varepsilon)$ is a positive constant which depends only on m and ε and grows to infinity as $\varepsilon \to 0$.

Let

$$(4.5) \qquad c_o > 3.2^{2s-1}b_s$$

where b_s is the constant in (3.4). For any $x_n \in R$, $\xi' \in R^{n-1}$ and $\nu \in]0,\nu_o]$ there exists $\delta \in R$ such that $|\delta| \leq m\varepsilon$ and the straight line in $\mathbb{C}$ $\Gamma_o(\xi',x_n;\nu) = \left\{\lambda \in \mathbb{C};\ \mathrm{Im}\lambda = (x_n/c_o\nu)^{1/(2s-1)}+\delta\right\}$ does not contain interior points of the discs $C_h(\xi')$. Denote by $H_1(\xi',x_n;\nu)$ the set of the h's such that $C_h(\xi')$ is contained in the strip $\left\{\lambda \in \mathbb{C};\ \Lambda(\xi') < \mathrm{Im}\lambda < (x_n/c_o\nu)^{1/(2s-1)}+\delta\right\}$. Since $\exp[ix_n\lambda-\nu(|\xi'|^2+\lambda^2)^s]/P(\xi',\lambda)$ is a holomorphic function of λ in $\mathbb{C} \setminus \bigcup_h C_h(\xi')$ we have for any $\ell \in \mathbb{Z}_+$

$$(4.6) \qquad D_n^\ell w_1(\xi',x_n;\nu) = (2\pi)^{-1}\int_{\Gamma_o}\lambda^\ell \exp[ix_n\lambda-\nu(|\xi'|^2+\lambda^2)^s]/P(\xi',\lambda)d\lambda +$$

$$+ (2\pi)^{-1}\sum_{h\in H_1}\int_{\partial C_h(\xi')}\lambda^\ell \exp[ix_n\lambda-\nu(|\xi'|^2+\lambda^2)^s]/P(\xi',\lambda)d\lambda = I_o + \sum_{h\in H_1} I_h \ .$$

In view of (3.4), (4.4), (4.5) and choosing $\varepsilon = (4m)^{-1}$ we obtain the estimates

$$(4.7)\qquad |I_o| \le c^{m+\ell+1}\Big(\sum_{j=0}^{m}|a_j(\xi')|\Big)^{-1}\nu^{-(m+\ell+1)/2s}\exp(-\nu|\xi'|^{2s}/2)\Gamma((m+\ell)/2s+1)\;\cdot$$

$$\cdot\;\exp(|x_n|)\exp[-(x_n^{2s}/c_o\nu)^{1/(2s-1)}/2]\quad,\quad x_n \in R,\ |\xi'| > k\ ,$$

and when $x_n > 0$, $|\xi'| > (4c_1c_0^{1/2s})^{\rho/(\rho-1)}$, $h \in H_1$

$$(4.8)\qquad |I_h| \le c^{m+\ell+1}\Big(\sum_{j=0}^{m}|a_j(\xi')|\Big)^{-1}\nu^{-(m+\ell)/2s}\exp(-\nu|\xi'|^{2s}/4)\Gamma((m+\ell)/2s+1)$$

$$\exp(c'|x_n|)\exp(c_1x_n|\xi'|^{1/\rho})\quad .$$

Let now $x_n < 0$. When $-c_2|\xi'|^{1/d} < (x_n/c_o\nu)^{1/(2s-1)}+\delta < -\,c_1|\xi'|^{1/\rho}$ then H_1 is empty. Otherwise for $h \in H_1$ the following estimates hold:
when $(x_n/c_o\nu)^{1/(2s-1)}+\delta \le -\,c_2|\xi'|^{1/d}$

$$(4.9)\qquad |I_h| \le c^{m+\ell+1}\Big(\sum_{j=0}^{m}|a_j(\xi')|\Big)^{-1}\exp(-\nu|\xi'|^{2s}/4)\Gamma((m+\ell)d+1)|x_n|^{-(m+\ell)d}\;\cdot$$

$$\cdot\;\exp(-(3/8)^{2s}c_2|x_n||\xi'|^{1/d}/2)$$

for $|\xi'| > (c_o^{\,1/2s}\ c_2)^{d/(d-1)}$ if $d > 1$ and c_2 chosen so that $0 < c_2 < c_o^{\,-1/2}$ if $d = 1$[7] , and also

$$(4.9')\qquad |I_h| \le c^{m+\ell+1}\Big(\sum_{j=0}^{m}|a_j(\xi')|\Big)^{-1}\nu^{-(m+\ell)/2s}\exp(-\nu|\xi'|^{2s}/2)\Gamma((m+\ell)/2s+1)\;\cdot$$

$$\cdot\;\exp(c'|x_n|)\quad\text{for}\quad|\xi'| > k\ ;$$

when $(x_n/c_o\nu)^{1/(2s-1)}+\delta \ge -c_1|\xi'|^{1/\rho}$

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[7] Note that if $d = 1$, then $r = s = 1$ and $b_1 = 1$.

$$(4.10)\qquad |I_h| \le c^{m+\ell+1}\Big(\sum_{i=0}^{m}|a_j(\xi')|\Big)^{-1}\nu^{-(m+\ell)/2s}\exp(-\nu|\xi'|^{2s}/4)\Gamma((m+\ell)/2s+1)\cdot$$

$$\cdot\exp(c'|x_n|)\cdot\exp(-(x_n^{2s}/c_o\nu)^{1/(2s-1)}) \quad ,$$

for $|\xi'| > (4c_1c_0^{1/2s})^{\rho/(\rho-1)}$.

Letting $k^* = \max\left\{k, (4c_1c_0^{1/2s})^{\rho/(\rho-1)} , (c_o^{\ 1/2s}\, c_2)^{d/(d-1)}\right\}$ if $d > 1$ and $k^* = \max\left\{k, (4c_1c_0^{1/2})^{\rho/(\rho-1)}\right\}$ if $d = 1$, (4.2) and (4.3) follow from (4.6) in view of (4.7), (4.8), (4.9') and (4.7), (4.9), (4.10) respectively.

<u>4.2. Remark</u>. Note that the condition (2.6) on the roots of the equation $P(\xi',\lambda) = 0$ when $|\xi'| > k$ implies that $\sum_{j=0}^{m}|a_j(\xi')| \ne 0$ when $|\xi'| > k$. Thus, by a lemma of L. Hörmander [11] applied to the polynomial $\sum_{i=0}^{m}|a_j(\xi')|^2$, there exist $\mu \ge 0$ and $c_3 > 0$ such that

$$(4.11)\qquad \sum_{j=0}^{m}|a_j(\xi')| \ge c_3|\xi'|^{-\mu} \ , \ |\xi'| > k + 1 \quad .$$

<u>4.3. Proposition.</u>[8] Let Ω be the ball with center at the origin of $\mathbb{C}^{n-1}$ and radius one and let V be the linear space of the polynomial in $\mathbb{C}^{n-1}$ of degree $\le M - m$. Then there exists a C^∞ map $\Phi : (V\setminus\{0\}) \times \mathbb{C}^{n-1} \to \mathbb{C}$, homogeneous of degree zero with respect to $a \in V\setminus\{0\}$ such that $\operatorname{supp}\Phi \subset (V\setminus\{0\}) \times \Omega$,

$$\int_{\mathbb{C}^{n-1}}\Phi(a;z)d\mu(z) = 1 \quad , \quad \Phi(a;e^{i\theta}z) = \Phi(a;z) \ , \ \theta \in R \ , \ z \in \mathbb{C}^{n-1} \ ^{9)}$$

and

$$\sum_{\alpha\in\mathbb{Z}_+^{n-1}}|a^{(\alpha)}(0)| \le c|a(z)| \ , \ a \in V\setminus\{0\} \ , \ z \in \operatorname{supp}\Phi\,(a;\cdot)$$

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8) See [13], pp. 101-102.

9) Note that these properties imply that $\int_{\mathbb{C}^{n-1}}F(z)\,\Phi\,(a;z)d\mu(z) = F(o)$ for every entire function F and every $a \in V\setminus\{0\}$.

where c is a positive constant and $d\mu$ the Lebesque measure in $\mathbb{C}^{n-1}$.

From this proposition and from a lemma by L. Hörmander[10)] it follows immediately

4.4. Proposition. Assume that the polynomial $P(D',D_n) = \sum_{j=0}^{m} a_j(D')D_n^j$ has order $\leq M$ and denote by $a_{m,\zeta}{}'$, $\zeta' \in \mathbb{C}^{n-1}$, the polynomial $z \to a_m(\zeta'+z)$, $z \in \mathbb{C}^{n-1}$. Then if Φ is a function with the properties indicated in Proposition 4.3, there exist two constants $c_4 > 0$, $c_5 > 1$ such that for every $\zeta' \in \mathbb{C}^{n-1}$ $z \in \operatorname{supp}\Phi(a_{m,\zeta'};\cdot)$, and every root $\lambda \in \mathbb{C}$ of the equation $P(\zeta'+z,\lambda) = 0$

$$|a_m(\zeta'+z)| \geq c_4 ,$$

$$|\lambda| \leq c_5(1+|\zeta'|)^{M-m+1}-1 .$$

Arguing as in the proof of Lemma 4.1 and with the aid of Proposition 4.4 we can prove

4.5. Lemma. Let the notation be as in Proposition 4.4 and let

$$w_2(\xi'+z,x_n;\nu) = (2\pi)^{-1} \int_{\operatorname{Im}\lambda=-c_5(1+|\xi'|)^{M-m+1}} \exp[ix_n\lambda - \nu(\sum_{j=1}^{n-1} (\xi_j+z_j)^2+\lambda^2)^s]/P(\xi'+z,\lambda)d\lambda ,$$

$\xi' \in R^{n-1}$, $z \in \operatorname{supp}\Phi(a_{m,\xi'};\cdot)$, $x_n \in R$, $\nu \in]0,\nu_o]$. Then

i) $P(\xi'+z,D_n)w_2 = (2\pi)^{-1}\int_{-\infty}^{+\infty}\exp[ix_n\xi_n-\nu(\sum_{j=1}^{n-1} (\xi_j+z_j)^2+\xi_n^2)^s]d\xi_n$

ii) there exist positive constants $c,c',c_o' > c_o$ independent of $\xi',x_n,\nu,\ell \in \mathbb{Z}_+$ such that

$$|D_n^\ell w_2(\xi'+z,x_n;\nu)| \leq c^{\ell+1}c_4\nu^{-\ell/2s}\exp(-\nu|\xi'|^{2s}/4)\Gamma(\ell/2s+1)\exp(c'|x_n|) \cdot$$

$$\cdot \Big\{\nu^{-1/2s}\exp(-(x_n^{2s}/c_o'\nu)^{1/(2s-1)}/2) +$$

$$+ \exp[c_5|x_n|(1+|\xi'|^{M-m+1} + c_o'\nu c_5^{2s}(1+|\xi'|)^{2s(M-m+1)}]\Big\}$$

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10) See [12] Lemma A.3.

for every $x_n \in R$,

$$|D_n^{\ell} w_2(\xi'+z,x_n;\nu)| \leq c^{\ell+1} c_4 \nu^{-(\ell+1)/2s} \exp(-\nu|\xi'|^{2s}/4)\Gamma(\ell/2s+1)\exp(|x_n|) \cdot$$
$$\cdot \exp(-(x_n^{2s}/c_o'\nu)^{1/(2s-1)}/2)\exp(c_o'\nu c_5^{2s}(1+|\xi'|)^{2s(M-m+1)})$$

when $x_n < 0$.

From Lemma 4.1, the inequality (4.11) and Lemma 4.5 we obtain easily

4.6. Lemma. Assume that $P(D) = \sum_{j=0}^{m} a_j(D')D_n^j$ has order $\leq M$ and satisfies the conditions of Theorem 2.4. Let $\nu_o = \exp(-8^{-2s} c_2 (k^*+1)^{1/d})$ and for $x \in R^n$, $\nu \in]0,\nu_o]$ let

$$v(x;\nu) = (2\pi)^{-(n-1)} \left\{ \int_{|\xi'|>(8^{2s}|\log\nu|/c_2)^d} e^{i<x',\xi'>} w_1(\xi',x_n;\nu)d\xi' + \right.$$
$$\left. + \int_{|\xi'|\leq(8^{2s}|\log\nu|/c_2)^d} d\xi' \int_{\mathbb{C}^{n-1}} e^{i<x',\xi'+z>} w_2(\xi'+z,x_n;\nu)\,\Phi\,(a_{m,\xi'};z)d\mu(z) \right\} ,$$

where the notation are the same as in Lemma 4.1 and Lemma 4.5.

Then v is a C^∞ function of x in R^n and

i) $P(D)v = \mathscr{F}_\xi^{-1}(\exp(-\nu|\xi|^{2s})(x)$

ii) there exist positive constants c, c', c'', c_o' and $h \geq 0$ independent of $x, \nu, \alpha \in \mathbb{Z}_+^n$ such that

$$|D_x^\alpha v(x;\nu)| \leq c^{|\alpha|+1} \exp(c'|x|)\Gamma(|\alpha|/2s+1)\nu^{-(|\alpha|+m+n+\mu)/2s} \cdot$$
$$\cdot \exp[16(c_1|x_n|)^{2s\rho/(2s\rho-1)}\nu^{-1/(2s\rho-1)} + c''|x_n|(1+|\log\nu|^d)^{M-m+1}]$$

for every $x \in R^n$,

$$|D_x^\alpha v(x;\nu)| \le c^{|\alpha|+1}\Big\{\exp(c'|x|)\Gamma(|\alpha|/2s+1)\nu^{-(|\alpha|+m+n+\mu)/2s} \cdot$$
$$\cdot \exp[-(x_n^{2s}/c_o'\nu)^{1/(2s-1)}/2] + |x_n|^{-(|\alpha|+m+n+\mu-1)d}\Gamma(|\alpha|d+1)\nu^{|x_n|}\Big\}$$

for every $x \in R^n$ with $x_n < 0$.

Recalling that $\mathcal{F}_\tau^{-1}(\exp(-\nu|\tau|^{2r}))(t)$ is a radial function of $t \in R^{\bar{n}} \setminus \{0\}$, Theorem 2.4 follows from Lemma 4.6 if we let

$$H(x,\sigma) = \int_0^{\nu_o} v(x;\nu)\mathcal{F}_\tau^{-1}(\exp(-\nu|\tau|^{2r}))(t)\Big|_{|t|=\sigma} d\nu \ , \ x \in R^n, \ \sigma > 0$$

and note that $\mathcal{F}_\tau^{-1}(\exp(-\nu|\tau|^{2r})(t)$ verifies, as in (3.5), the estimate

$$|\mathcal{F}_\tau^{-1}(\exp(-\nu|\tau|^{2r}))(t)| \le c\ \nu^{-\bar{n}/2r}\exp[-c'(\nu^{-1/2r}|t|)^{2r/(2r-1)}]$$

for suitable positive constants c and c'.

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