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ON THE ANALYTICITY OF SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS AND SYSTEMS

O.A. OLEINIK

The problem of the analyticity of solutions of partial differential-equations is one of the oldest in the theory of partial differential equations. This problem is connected with the 19 th HILBERT problem [1]. It is well-known that all solutions of linear elliptic equations and systems with analytic coefficients are analytic functions (see [2], [3] and a survey in [1]). If a system has constant coefficients and all solutions of this system are analytic functions, then the system is elliptic [3]. It is of great interest to find non elliptic classes of differential equations and systems which have only analytical solutions and to describe classes which do not have such a property. Some results in this direction are given in the papers [3]-[8].

We offer here an approach for the study of the problem of the analyticity of solutions of differential equations and systems. We prove here some a priori estimates in a complex domain for solutions of partial differential equations and systems which have only analytical solutions. Using these estimates we can find classes of partial differential equations and systems which have non analytic solutions.

Let Ω be a bounded domain in R^{m+1} , $x = (x_o, \ldots . x_m)$

DEFINITION 1.-

A function $u(x)$ defined in Ω is called a function of the class $A(\Omega)$ if for any subdomain G with $\bar{G} \subset \Omega$ there exists a constant $\delta(G) > 0$ such that the function $u(x)$ can be extended analytically with respect to all variables x as a function $u(x_o + iy_c, \ldots, x_m + iy_m)$ in a domain $Q_\delta^{2m+2} = \{x \subset G, -\delta(G) < y_j < \delta,(G)$
$j = 0,1, \ldots, m\}$ and also u is bounded in Q_δ^{2m+2}.

We denote by $V(u)$ an analytical extension of u in the domain Q_δ^{2m+2}.

DEFINITION 2.-

A function $u(x)$ defined in Ω is called a function of the class $A_j(\Omega)$, if for any subdomain G with $\bar{G} \subset \Omega$ there exists a constant $\delta(G)$ such that the function $u(x)$ can be extended analytically with respect to the variable x_j as a function $u(x_o, \ldots, x_j + iy_j, \ldots . x_m)$ in a domain :

$$Q_\delta^{m+2} = \{x \subset G , - \delta(G) < y_j < \delta(G)\}$$

and also u and $\frac{\partial u}{\partial x_k}, k \neq j$, are bounded in Q_δ^{m+2} ; $j = 0,1, \ldots, m$.

We denote by $V_j(u)$ an analytical extension of u in the domain Q_δ^{m+2}.
If all components of a vector-function $u = (u_1, \ldots, u_N)$ belong to a linear space B , then we say that $w \in B^N$.

DEFINITION 3.-

A linear system of partial differential equations with analytic coefficients

(1) $L(u) = f$ in Ω , $u=(u_1, \ldots, u_N)$, $f = (f_1, \ldots, f_N)$

is called analytical in Ω , if, for any of its weak solution $u \in \mathcal{D}'(\Omega)$ the condition $f \in A(\Omega)$ implies $u \in A(\Omega)$.

DEFINITION 4.-

System (1) is called analytical with respect to the variable x_j in Ω, if for any of its weak solution $u \in \mathcal{D}'(\Omega)$, the condition $f \in A_j(\Omega)$ implies $u \in A_j(\Omega)$.

THEOREM 1.

Let $B^N(\Omega)$ be a BANACH space consisting of weak solution $u \in (\mathcal{D}'(\Omega))^N$ of the system

(2) $$L(u) = 0 \quad \text{in } \Omega$$

with the norm $\|u\|_{B^N}$ and also the convergence of a sequence in the norm $B^N(\Omega)$ implies its convergence in $\mathcal{D}'(\Omega)$. If system (2) is analytical in Ω, then for any subdomain G with $\bar{G} \subset \Omega$ there exists a constant $\delta(G) > 0$ such that for any solution $u \in B^N(\Omega)$ of system (2) the following estimate is valid :

(3) $$\sup_{Q_\delta^{2m+2}} |V(u)| \leqslant C_1 \|u\|_{B^N},$$

where $|V(u)| = \sum_{k=1}^{N} |V(u_k)|$, $C_1 =$ const.

If system (2) is analytical in Ω with respect to x_j, then for any subdomain G with $\bar{G} \subset \Omega$ there exists a constant $\delta(G) > 0$ such that for any solution $u \in B^N(\Omega)$ of system (2) the following estimate is valid :

(4) $$\sup_{Q_\delta^{m+2}} | V_j(u)| \leqslant C_2 \|u\|_{B^N} ;$$

where

$$|V_j(u)| = \sum_{k=1}^{N} |V_j(u_k)|, \quad C_2 = \text{Const.}$$

PROOF. -

We prove first the estimate (3). Let G be a fixed subdomain with $\bar{G} \subset \Omega$. We set :

$$M_{\gamma,R} = \{u \,;\, u \in B^N(\Omega) \,,\, \sup_{Q_\gamma^{2m+2}} |V(u)| \leqslant R \}$$

where $R = 1, 2, \ldots ; \frac{1}{\gamma} = 1, 2, \ldots$

Since system (2) is analytical, it means that any $u \in B^N(\Omega)$ belongs at least to one of the sets $M_{\gamma,R}$. We show that $M_{\gamma,R}$ are closed in $B^N(\Omega)$.

Let $\{u_n\}$ be a sequence, $u_n \in M_{\gamma,R}$ for $n=1,2,\ldots$ and $\|u_n-u_o\|_{B^N(\Omega)} \to 0$ for $n\to\infty$. We shall prove that $u_o \in M_{\gamma,R}$. Since u_n are uniformly bounded in Q_γ^{2m+2}, there exists a subsequence of $\{u_n\}$ which uniformly converges at any compact set $K \subset Q_\gamma^{2m+2}$ towards an analytic vector-function $\tilde{u}$ in Q_γ^{2m+2}.

The uniform convergence at any compact set $K \subset Q_\gamma^{2m+2}$ of the bounded sequence in Q_γ^{2m+2} implies its convergence in $\mathcal{D}'(G)$. It means that $\tilde{u} = u_o$ in G and therefore u_o can be extended analytically in Q_γ^{2m+2} with the estimate $|V(u_o)| \leqslant R$. It implies that $u_o \in M_{\gamma,R}$. Since $B^N(\Omega)$ is a Banach space and

$$B^N(\Omega) = \bigcup_{\gamma,R} M_{\gamma,R} .$$

according to BAIRE's category theorem [9] - (if B is a complete, it is the second category) - there exists positive constants ε, δ, R_o and a vector-function $u^*(x) \in M_{\delta,R_o}$ such that if $u(x) \in B^N(\Omega)$ and

$$(5) \qquad \|u-u^*\|_{B^N} \leqslant \varepsilon \quad ,$$

then $u(x) \in M_{\delta,R_o}$. That means :

$$(6) \qquad \sup_{Q_\delta^{2m+2}} |V(u)| \leqslant R_o \quad .$$

Let u be any element of $B^N(\Omega)$. Then for

$$u^* + \frac{\varepsilon}{\|u\|_{B^N}} u$$

the inequality (5) is satisfied and therefore according to (6) :

$$(7) \qquad \sup_{Q_\delta^{2m+2}} \left|V(u^*) + \frac{\varepsilon}{\|u\|_{B^N}} V(u)\right| \leqslant R_o \quad .$$

It follows from (7) that :

$$\sup_{Q_\delta^{2m+2}} |V(u)| < 2 R_o \cdot \varepsilon^{-1} \|u\|_{B^N} = C_1 \|u\|_{B^N}$$

The proof is complete. The inequality (4) can be proved similarly. In this case, the uniform convergence of a subsequence of u_n at a compact $K \subset Q_\gamma^{m+2}$ follows from the analyticity u_n with respect to $x_j + iy_j$ in Q_γ^{m+2} and from the boundedness of u_n and their derivatives $\frac{\partial u_n}{\partial x_\ell}$, $\ell \neq j$, in case Q_γ^{m+2}. In this case, we set :

$$M_{\gamma,R} = \{u \; ; \; u \in B^N(\Omega) \quad , \sup \quad [|V_j(u)| + \sum_{\ell \neq j} |\frac{\partial}{\partial x_\ell} V_j(u)|] < R \} .$$

<u>Remark</u> :

Theorem 1 is walid for any BANACH space $B(\Omega)$ such that :

$$B \subset \mathcal{D}'(\Omega) \quad \text{and} \quad B \subset A(\Omega) \quad \text{or} \quad B \subset A_j(\Omega) .$$

The following theorem is a consequence of theorem 1.

<u>THEOREM 2</u>

Suppose that in Ω there exists a family of solutions $u_\rho \in \mathcal{D}'(\Omega)$ of system (2) of the form :

(8) $$u_\rho(x) = e^{i\rho x_j} v_\rho(x) \quad , \quad \rho \in R^1 \quad , \quad \rho > 1 \quad ;$$

such that u_ρ for any $\rho \geq 1$ can be analytically extended with respect to x_j in a domain $Q_{\delta_1}^{m+2}$ for some fixed domain G with $\bar{G} \subset \Omega$, where δ_1 does not depend on ρ, but $|V_1(v_\rho)|, |\frac{\partial}{\partial x_\ell} V_j(v_\rho)|$, $\ell \neq j$, are bounded in $Q_{\delta_1}^{m+2}$ by a constant which can depend on ρ.

Suppose that there exists a BANACH space $B^N(\Omega)$ defined in theorem 1, such that

(9) $$\|u_\rho\|_{B^N} \leq \exp (C_3 \rho^\mu)$$

where the constants μ and $C_3 > 0$ do not depend on ρ and $\mu < 1$. Let there exist a sequence of points $x(\rho) \in G_1$ such that for the points

$$Z_{\rho,h} = (x_o(\rho), \ldots\ldots, x_j(\rho) + ih \ldots\ldots x_m(\rho)),$$

$-\delta_1 \leq h \leq 0$, of $Q_{\delta_1}^{m+2}$, the inequality

(10) $$|V_j(v_\rho(Z_{\rho,h}))| > \exp(-C_4\rho^{+\sigma})$$
is fulfilled, where $C_4 > 0$ ans $\sigma<1$ do not depend on ρ and h.
Then system (2) is not analytical with respect to x_j in Ω.

In order to prove theorem 2 , we show that for solutions u_ρ the estimate (4) is not valid for the large ρ because of (9) and (10).

Example :

As an example of the application of theorems 1 and 2 , let us prove that the BOUENDI-GOULAOUIC equation [6]

(11) $$u_{tt} + t^2 u_{yy} + u_{zz} = 0$$

is not analytical with respect to t and y.

It is easy to see that equation (11) has the family of solutions :

$$u_\rho(x) = \exp\left[i\rho y - \rho\frac{t^2}{2} + \rho^{\frac{1}{2}}(z-1)\right]$$

which do not satisfy the inequality (4) with respect to variables y and t for the large ρ , if $\|u\|_{B^1} = \sup_\Omega |u|$. One can prove, using similar methods than those of [3] and [10] that all solutions of (11) are analytical with respect to z.

In what follows, we shall prove a more general theorem for equations of any order, which includes equation (11).

As a second example of the application of theorem 2, we consider a system with constant coefficient

(12) $$L(u) = f \quad \text{or} \quad \sum_{j=1}^{N} a^{kj} u_j = f_k \quad , \quad K=1,\ldots,N .$$

where $a^{kj}(\mathcal{D}) = \sum_{|\alpha|\leq \gamma_{kj}} a^{kj}_\alpha \mathcal{D}^\alpha$ is a differential operator of the order γ_{kj} with constant coefficients ; $\mathcal{D}^\alpha = \mathcal{D}_0^{\alpha_0} \ldots \mathcal{D}_m^{\alpha_m}$, $\mathcal{D}_j = -i\frac{\partial}{\partial x_j}$.

We set $$r = \sup \sum_{\ell=1}^{N} \gamma_{\ell,k_\ell} ,$$

where χ is any substitution of the form :

$$\chi = \begin{pmatrix} 1,\ldots, N \\ k_1,\ldots.k_N \end{pmatrix}$$

We suppose that the order of the polynomial $P(\xi) = \det\|a^{kj}(\xi)\|$ is equal to r . For such system of differential equations, it is proved [11] that there exists two systems of the integers $(s_1,\ldots,s_N)$ and $(t_1,\ldots.t_N)$ such that for any $k,j = 1,\ldots.N$:

(13) $$s_k + t_j \geqslant \gamma_{kj}$$

(14) $$\sum_{k=1}^{N} (s_k + t_k) = r$$

Let $a^{o}_{kj} = \sum_{|\alpha|=s_k+t_j} a^{kj}_{\alpha} \xi^{\alpha}$.

It is evident that : $\det \|a^{o}_{kj}(\xi)\| = P^{o}(\xi)$

where $P^{o}(\xi)$ is the principal part of the polynomial $P(\xi)$.

If $P^{o}(\xi) \neq 0$ for any $\xi \in R^{m+1}$ with $|\xi| \neq 0$, then the system (12) is called elliptic in the sense of DOUGLIS-NIRENBERG [12] .

If $\gamma_{kj} = m_k$ for $j=1,\ldots, N$ and $P^{o}(\xi) \neq 0$ for $\xi \neq 0$, then the system (12) is elliptic in the sense of PETROVSKY [3] .

It is proved that such elliptic systems are analytical in any domain $\Omega \subset R^{m+1}$, [3],[13].

THEOREM 3.-

If the equation $P^{o}(\xi) = 0$ has a real solution $\hat{\xi} = (\hat{\xi}_o,\ldots,\hat{\xi}_m)$ and $\hat{\xi}_j \neq 0$, then system (12) is not analytical with respect to x_j .

<u>Proof.</u> -

The system (12) for $f = 0$ has a family of solutions of the form :

$$u_\rho(x) = \exp\{ ip(\hat{\xi}, x) + i\lambda(\rho)(\eta,x)\}\ Rp$$
$$\lambda(\rho) = \rho^{\frac{p}{q}} \sum_{j=0}^{\infty} c_j \rho^{\frac{-j}{q}} ,$$

c_j = constant, p,q are positive integers, $p < q$, η is a vector of R^{m+1} , (ξ,η) is a scalar product in R^{m+1} .

It is easy to see that the function u_ρ satisfy the conditions of **theorem 2** if $\|u\|_B = \sup_\Omega |u|$ for the BANACH space B .

In the case $\gamma_{kj} = m_k$, a similar theorem was proved by I. PETROVSKY [3].

Let us consider now a system of first order partial differential equations with analytic coefficients of the form :

(15) $$L(u) = \frac{\partial u}{\partial x_0} + \sum_{j=1}^{m} A_j(x) \frac{\partial u}{\partial x_j} + B(x)\, u = 0 \ .$$

where A_j , B are matrixes of the order $N\times N$. Let us denote :

$$x=(x_0,x_1,\ldots\ldots,x_m) = (x_0,x'),\ y=(y_0,y_1,\ldots\ldots,y_m)=(y_0,y')$$

<u>THEOREM 4.</u>-

Let Ω_1 be a neighbourhood of the origin in the space $R^{m+1} = (x_0,x_1,\ldots\ldots,x_m)$. Suppose that an analytical function $\varphi(x,y')$ with respect to $x' + iy'$, is defined in a domain

$$Q_\gamma(\Omega_1) = \{x,y' : (x_0,x') \in \Omega_1,\ |y'| < \gamma \},\ \gamma = ct.>0$$

Suppose that for some $k>0$

$$\frac{\partial^{2k+1}}{\partial y_s^{2k+1}}(\mathrm{Im}\ \varphi(0,0)) \neq 0, \frac{\partial^s}{\partial y s}(\mathrm{Im}\varphi(0,0)) = 0 \quad \text{for} \quad 0<s<2k+1 ,$$

$$\mathrm{Im}\ \varphi(x,0) < \mathrm{Im}\ \varphi(0,0) \qquad \text{in}\ \Omega_1 \ ;$$

the matrix

(16) $$A(x_0,x'+iy',\ \mathrm{grad}_x\varphi) = E\frac{\partial\varphi}{\partial x_0} + \sum_{j=1}^{m} A_j \frac{\partial\varphi}{\partial x_j} ,$$

(E is the unit matrix), satisfies the following condition :

For any analytical with respect to $x'+iy'$ function $v(x,y')$ in $Q_\gamma(\Omega_1)$ the quadratic form :

(17 $$K(v,v) = \left(\frac{A-\bar{A}^*}{2i} v,v\right).x_0$$

is nonnegative in $Q_\gamma(\Omega_1)$; here (ψ,f) is a scalar product in the complex space C^N , matrix $\bar{A}^*$ is the complex-conjugate matrix for A . Then the system (15) is non analytical with respect to x_j in any neighbourhood Ω of the origin such that $\Omega \subset \Omega_2 \subset \Omega_1$ for some Ω_2 .

In order to prove this theorem, we prove the existence of the family of solutions of system (15) of the form

$$u_\rho(x) = e^{-i\rho\varphi}{}_\rho(x_o, x', y')$$

where $\sup_{\Omega_2} |v_\rho| < C_6 \rho^{2(m+1)}$. Then it can be proved that for these solutions, the inequality (4) is not fulfilled, if $\|u\|_{B^N} = \sup_{\Omega} |u|$.

COROLLARIES.

Let us derive some corollaries of theorem 4 .

COROLLARY 1.-

Consider the case of a first order equation :

(18) $$u_{x_o} + \sum_{k=1}^{m} a^k(x) u_{x_k} + c(x)u = 0$$

Suppose that Im $a^j(x)x_o$ does not change the sign in a neighbourhood of the origin of space $R^{2m+1}(x,y')$. Then according to theorem 4, the equation (18) is not analytical with respect to x_j . In this case, we can take :

$$\varphi = x_j + iy_j .$$

For the MIZOHATA equation [4]

(19) $$u_t + it^s u_x = 0$$

it means that, if s is odd, then equation (19) has non analytic solutions with respect to x.

In paper [4], it is proved that if s is even, then all solutions of equation (19) are analytic functions.

COROLLARY. 2.-

The system (15) is not analytical with respect to x_j , if the matrix A_j is diagonalizable by a unitary transformation and for its eigenvalues λ_k , K=1,...., N , Im $\lambda_k . x_o$ preserves the sign in the neighbourhood of the origin of the space (x,y') and this sign is the same for all k=1,.....,N.

It is easy to see that in this case, the condition (17) is fulfilled with the function $\varphi = x_j + iy_j$.

We consider now some applications of the theorem 1 and 2 to higher order equations and, in particular, to second order equations.

We set :

$$t = (x_o,\ldots\ldots,x_k,0,\ldots\ldots 0) \quad ,y = (0,\ldots\ldots,0,x_{k+1},\ldots\ldots x_\ell,0,\ldots\ldots,0),$$

$$z = (0,\ldots\ldots,0,x_{\ell+1},\ldots\ldots,x_m).$$

Then any point $x \in R^{m+1}$ can be represented in the form $x=t+y+z$.

We denote respectively :

$$\xi' = (\xi_1,\ldots\ldots,\xi_k,0,\ldots\ldots,0),\ \xi'' = (0,\ldots\ldots,\xi_{k+1},\ldots,\xi_\ell,\ldots 0\ldots 0),$$

$$\xi'' = (0,\ldots\ldots,0,\xi_{\ell+1},\ldots\ldots\xi_m) \quad , \quad = (\mathcal{D}_t,\mathcal{D}_y,\mathcal{D}_z,), \quad = \mathcal{D} \quad (-i\frac{\partial}{\partial x_o},\ldots\ldots, -i\frac{\partial}{\partial x_m}).$$

THEOREM 5.-

Suppose that the symbols of the differential operators with analytic coefficients $A(t,y,z,\mathcal{D}_t,\mathcal{D}_y)$, $B(t,y,z,\mathcal{D}_t,\mathcal{D}_y)$, $C(t,y,z,\mathcal{D}_z)$ are given for :

$$|y| \leqslant K \quad , \quad |z| \leqslant K \text{ and for all } t \ ; \ K \text{ is a constant.}$$

Suppose that for some $\tau > 0$, any real number $\rho > 1$, and for some $M_1 > M_2$, any ξ' and $\hat{\xi}''$ $(0,\ldots\ldots,\hat{\xi}_{k+1},0,\ldots\ldots,0)$ with $\hat{\xi}_{k+1} = 1$ we have in the domain $\{|y| < K \ , \ |z| < K \ , \ -\infty < t < \infty\}$

(20) $$A(\rho^{-\tau}t,y,z,\rho^{\tau}\xi',\rho\hat{\xi}'') = \rho^{M_1}A(t,y,z,\xi',\hat{\xi}'') = \rho^{M_1}a(t,\xi') .$$

(22) $$B(\rho^{-\tau}t,y,z,\rho^{\tau}\xi',\rho\hat{\xi}'') = \rho^{M_2}B(t,y,z,\xi',\hat{\xi}'') = \rho^{M_2}b(t,\xi') .$$

and , for some $\hat{\xi}''' \neq 0$ and any complex number γ :

$$C(t,y,z,\gamma\hat{\xi}''') = \gamma^h \quad ,$$

where $h \geqslant [M_1 - M_2] + 1$, $[S]$ is the entire part of S.

If there exists a function $v(t) \in \mathcal{L}_2(R^k)$ such that $v(t) \neq 0$ and

(22) for some λ = constant $\in \mathbf{C}^1$ the function $v(t)$ as a distribution of $\mathcal{D}'(R^{k+1})$ satisfies the equation :

$$a(t, \mathcal{D}_t)v + \lambda b(t, \mathcal{D}_t)v = 0 \quad ,$$

then the equation

$$L(u) = Au + C\ Bu = f$$

is not analytical with respect to variable x_{k+1} in any domain :

$$\Omega = \{x : |t| \leqslant K_1 . |y| \leqslant K_1 \ . \ |z| \leqslant K_1\} \ , \ K_1 = ct. < K.$$

PROOF :

We consider the solution of equation (23) in the form :

$$u_\rho(x) = \exp\{i\rho x_{k+1} + i\rho\mu(z, \hat{\xi}'')\lambda^{\frac{1}{h}}\}\ v(\rho^\tau t)$$

where

$$\mu = \frac{M_1 - M_2}{h} < 1 \quad .$$

and use the BANACH space with the norm :

$$\|u\|_{B(\Omega)} = \left(\int_\Omega |u|^2 dx\right)^{1/2} .$$

It is easy to see that for $u_\rho(x)$, the estimate (4) is not valid.

Thus the theorem is proved.

We can indicate indicate conditions when the solution $v(t)$ of equation (22) with the necessary properties exists.

Suppose that the operators $a(t, \mathcal{D}_t)$ and $b(t, \mathcal{D}_t)$ define in the space $S(R^n)$, (see [14]), the norms

$$H_a(u) = [au, u,] \ , \ H_b(u) = [bu, u]$$

where

$$[f, \psi] = \int_{R^{k+1}} f\ \bar{\psi}\ \ dt \quad .$$

We denote by H_a and H_b the closure of the space $S(R^n)$ with respect to the norm $H_a(u)$ and $H_b(u)$ correspondingly . If the operators a and b are formally selfadjoint and if every bounded set in H_a is compact in H_b , then for some λ = constant, there exists a weak solution of the equation (22) such that $v(t) \in H_a$ and $v(t) \neq 0$.

THEOREM 6.-

Suppose that the operators A,B,C, satisfy the conditions of theorem 5 and the spaces H_a and H_b have the above mentionned properties, $H_a \subset \mathscr{L}_2(R^k)$. Then equation (23) is not analytical with respect to x_{k+1} in any neighbourhood of the origin.

Thus we see that the existence of the eigenfunction of the equation (22) implies the existence of a class of equations (23) that are not analytical. Conservely, if equations (23) are analytical, we can make a statement about the spectrum of equation (22).

We do not state here such theorem about spectrums. Thus we have interesting relation between the spectral theory of differential operators and the problem of the analyticity .

The next theorem follows from theorem 6.

THEOREM 7.-

Let $P_1(\mathscr{D}_t)$, $P_2(\mathscr{D}_y)$, $P_3(\mathscr{D}_z)$ be homogeneous elliptic differential operators with constant coefficients of the order 2p .
The equation

(24) $$P_1(\mathscr{D}_t)u + |t|^{2s} P_2(\mathscr{D}_y)u + |t|^{2d}P_3(\mathscr{D}_z)\, u = f$$

where s and d are integers, s, d>0 , is analytical in a neighbourhood of the origin of the space R^{m+1} , if and only if s=d.

PROOF. -

It is proved in [7], using the MORREY-NIRENBERG method [13] , that equation (24) for s=d is analytical. If $s \neq d$, the equation (24) is not analytical with repect to x_{k+1} in the case d< s and with respect to $x_{k+1+\ell}$

in the case $d > s$, according to theorem 6.

In the paper [6] in other way, it is proved that for $p=1$, $s=1$, $d=0$ equation (24) is not analytical.

The equation (22) for the case (24) has the form :

$$P_1(\mathcal{D}_t)v + |t|^{2s}v + \lambda|t|^{2d}v = q .$$

It is of interest to study the equation of the form (24) with lower order terms.

We consider the second order equation with real analytic coefficients in the neighbourhood Ω of the origin in the space $R^{m+1}(y,x_1,\ldots\ldots,x_m) = (y,z)$

$$L(u) = a^{kj}(x)u_{x_k x_j} + a(x)\, u_{yy} + b^k(x)u_{x_k} + b(x)u_y + c(x)u = f(x,y) \qquad (25)$$

where

$$a^{kj}(x)\xi_x\xi_j \geqslant C_0|\xi|^2 \quad , \; a(x) = |x|^{2s}\,\tilde{a}(x) \quad , \; \tilde{a}(x) > 0 \quad ,$$

C_0 = constant > 0, s in an integer ;

We can prove, using the extension of the solution in the complex domain, that equation (25) is analytical.

In the paper [7], this is proved by the method cf [13] under condition :

$$|b(x)| < C_1|x|^{s-1}$$

Some above mentionned results are contained in the papers [15],[16].

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