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ON CARLEMAN ESTIMATES FOR PSEUDO-DIFFERENTIAL OPERATORS

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Integral estimates with weight functions of the form $e^{\tau\varphi(x)}$, τ large, were introduced by CARLEMAN [1] in his proof of unique continuation of solutions for certain elliptic equations with non-analytic coefficients.

In Chapter 8 of his book [5] HÖRMANDER used similar estimates in his study of the unique continuation problem, and moreover obtained regularity and solvability theorems for a wide class of partial differential operators with variable coefficients. Its natural extension to pseudo-differential operators was given in the paper [6].

Apart from the weight function $e^{\tau\varphi(x)}$ HÖRMANDER also used spaces $\mathcal{H}'_{(\psi)}$ of distributions u such that $(1+|D'|^2)^{\psi(x_n)/2} u \in L_2$ (see [5], Section 8.8). Spaces with the operator $(1+|D'|^2)^{\psi(x_n)/2}$ replaced by the operator $(1+|D|^2)^{s(x)/2}$ of variable order $s(x)$ were studied systematically by UNTERBERGER [8] .

Improvements both in local and global sense of the regularity and solvability theorems were obtained in [7] using Fourier integral operators. Here the study was restricted to the case that the principal part p is either real or such that the Hamilton fields of $\operatorname{Re} p$ and $\operatorname{Im} p$

(i) are linearly independent,

(ii) they do not contain the cone axis in their span,

(iii) and they commute.

Here I want to show how these improvements can be obtained for a class of operators containing those in [6], [7] but without using Fourier integral operators. (Note however that we do not consider the construction of parametrices and solutions with prescribed singularities here. For these, Fourier integral operators are much more essential).

Instead, we use Carleman estimates with the weight function $e^{\tau\varphi(x)}$ replaced by a pseudo-differential operator of the form $e^{\tau\varphi(x,D)} \cdot (1+|D|^2)^{s(x,D)/2}$. Here $\varphi(x,\xi)$ and $s(x,D)$ are real-valued homogeneous C^∞ functions of degree 0 on $T^*(X)\setminus 0$, so we have combined the Carleman weight $e^{\tau\varphi}$ (depending on $(x,\xi) \in T^*(X)\setminus 0$) with an operator of variable order $s(x,\xi)$ (also depending on $(x,\xi) \in T^*(X)\setminus 0$). Detailed proofs can be found in [3].

First some definitions.

$s = s(x,\xi)$ will always be a real-valued homogeneous C^∞ function of degree 0 on $T^*(X)\setminus 0$. The space S^s_ρ of symbols of order s and type ρ is the set of C^∞ functions $a(x,\xi)$ on $T^*(X)\setminus 0$ such that

$$(1) \qquad (1+|\xi|_x^2)^{-s(x,\xi)/2} \cdot a(x,\xi) \in S^0_\rho \quad .$$

Here $|\xi|_x$ is a norm in $T_x(X)^*$ defined by some smooth Riemannian metric. L^s_ρ is the space of pseudo-differential operators of order s, that is the set of locally finite sums of pseudo-differential operators in the usual sense, but with symbols which are in fact of class S^s_ρ. The map assigning to $A \in L^s_\rho$ its **principal symbol** is an isomorphism : $L^s_\rho \,/\, L^{s+1-2\rho'}_\rho \longrightarrow S^s_\rho \,/\, S^{s+1-2\rho'}_\rho$ for $\rho' < \rho$. The choice $\rho' < \rho$ is needed because differentiations of the factor $(1+|\xi|^2_x)^{s(x,\xi)/2}$ introduce factors of the form $\log(1+|\xi|^2_x)^{\frac{1}{2}}$. If $A \in L^s_\rho$, $B \in L^{s'}_\rho$ then :

$$A\,B \in L^{s+s'}_{\rho'} \quad \text{and} \quad [A,B] = AB - BA \in L^{s+s'+1-2\rho'}_{\rho'} .$$

If a (resp. b) are the principal symbols of A (resp. B), then ab (resp $\frac{1}{2}\{a,b\} = \frac{1}{2} H_a b$) are the principal symbols of $A\,B$ (resp. $[A,B]$).
Finally, we define $H_{(s)}$ as the space of $u \in \mathcal{D}'$ such that $A\,u \in L^{loc}_2$ for some elliptic $A \in L^s_\rho$. Ellipticity means as usual that the principal symbol of A has an inverse in S^{-s}_ρ, implying that A has a two sided parametrix of class L^{-s}_ρ.

In the following basic estimate, we use a fixed Hermitian inner product on L^{comp}_2 defined by some smooth positive density on X. All operators are assumed to be properly supported.

LEMMA 1.

Let $P \in L^1_1$ have a homogeneous principal symbol $p(x,\xi)$. Assume that

(2) $$\frac{1}{i}\{\bar{p}, p\} \geq \mu p + \bar{\mu}\bar{p}$$

for some homogeneous C^∞ function μ of degree 0 on $T^*(X)\setminus 0$.

Take $M \in L^0_1$ with principal symbol equal to μ. Let $A \in L^s_1$ be self-adjoint and elliptic with a parametrix $B \in L^{-s}_1$ which can also be chosen self-adjoint.

Define :

(3) $$Q = [P^*, [A,P]].B + [A,P].[P^*,B] + M.[P,A].B.$$

Then we have for each compact subset K of X and any $0 < \delta < 1$ the estimate :

(4) $\|APu\|^2 \geq 2\,\mathrm{Re}\,(QAu,Au) - C_0\|Au\|^2 - C_1.\|APu\|.\|Au\| - C_2\|u\|_{(s-\delta)}.\,\|Au\|$

for all $u \in H^{comp}_{(s+1)}(K)$. The constant C_0 only depends on P, M, K, whereas C_1, C_2 also may depend on A, B, and the choice of the norm $\|u\|_{(s-\delta)}$ in $H^{comp}_{(s-\delta)}$.

Sketch of the proof.

A term will be called negligible if it can be estimated by :

$C.\,(\|APu\|.\|Au\| + \|u\|_{(s-\delta)}.\,\|Au\|)$ with C not depending on u.

All equalities below should be read modulo negligible terms. We then have :

$$AP = PA + [A,P] \quad , \quad \text{so}$$

$$\|APu\|^2 = \|PAu\|^2 + \|[A,P]u\|^2 + 2\,\mathrm{Re}\,([A,P]u,\ PAu)$$

$$= \|PAu\|^2 + \|[A,P]u\|^2 + 2\,\mathrm{Re}\,(P^*[A,P]B\ Au,Au)\ .$$

Interchanging P^* and B in $P^*[A,P]B$ step by step we obtain :

$$P^*[A,P]B = [P^*,[A,P]]B + [A,P].[P^*,B] + [[A,P],B].P^* + B[A,P]P^*.$$

The first two terms already appear in the definition of Q , the third one leads to a term $([[A,P],B]P^*\ Au,Au)$ which is negligible. Finally :

$$2\,\mathrm{Re}\,(B[A,P]P^*\ Au,Au) = 2\,\mathrm{Re}\,(P^*Au,[P^*,A]u)$$

$$= \|(P^*A+[P^*,A])u\|^2 - \|P^*Au\|^2 - \|[P^*,A]u\|^2$$

$$\geq -\|P^*Au\|^2 - \|[P^*,A]u\|^2\ .$$

Now : $\|PAu\|^2 - \|P^*Au\|^2 = ([P^*,P]Au,\ Au)$ and $[P^*,P] = MP + P^*M + R_0$

with $R_0 \in L^1$ having a non-negative principal symbol.

The sharp GÅRDING inequality yields

$$(R_0Au,\ Au) \geq -\ C_0\ \|Au\|^2$$

and the term $\|[A,P]u\|^2 - \|[P^*,A]u\|^2$ is negligible.

This completes the proof.

If the principal symbol of A is equal to $e^{\alpha(x,\xi)}$ (with α real) then the real part of the principal symbol of the operator Q is equal to $\Delta\alpha(x,\xi)$, where the "Laplacian" Δ is defined by

(5) $$\Delta = \operatorname{Re}\,(H_{\bar{p}} + \tfrac{1}{i}\,\mu)H_p \,.$$

If $\Delta\alpha$ is sufficiently large then the term $2\operatorname{Re}(QAu,Au)$ in (4) will dominate the term $C_0.\|Au\|^2$, leading to an a priori estimate

(6) $$\|Au\| \leq C.\ \|APu\| + C'.\|u\|_{(s-\delta)} \,.$$

This in turn leads to a regularity theorem of the form

(7) $$u \in H^{comp}_{(s-\delta)}\ ,\quad Pu \in H_{(s)} \Longrightarrow u \in H_{(s)}.$$

In order to localize this result we say that $u \in H_{(s)}$ at (x,ξ) if $u = u_1 + u_2$ with $u_1 \in H_{(s)}$ and $(x,\xi) \notin WF(u_2)$.

Write $\Sigma_{(s)}(P)$ for the complement in $T^*(X)\setminus 0$ of the (x,ξ) such that

(8) $$u \in \mathcal{D}'\ ,\quad Pu \in H_{(s)} \text{ at } (x,\xi) \Longrightarrow u \in H_{(s+m-1)} \text{ at } (x,\xi).$$

Then we have :

THEOREM 2.

Let $P \in L^m_1$ have homogeneous principal symbol $p(x,\xi)$ of constant degree m. Suppose that (2) holds on the open cone Γ in $T^*(X)\setminus 0$ for a homogeneous C^∞ function μ of degree $m-1$ on Γ. Let φ, s be real homogeneous C^∞ functions of degree 0 such that

(9) $$\Delta\varphi > 0 \quad \text{on} \quad \Gamma \cap \Sigma_{(s)}(P)$$

(10) $$\Delta s \geq \nu.p + \bar{\nu}.\bar{p} \quad \text{on a conic neighborhood in } T^*(X)\setminus 0 \text{ of}$$

$\Gamma \cap \Sigma_{(s)}(P)$, for some homogeneous C^∞ function ν of degree $m-2$.

Then $u \in \mathcal{E}'(X)$, $WF(u) \subset \Gamma$, $Pu \in H_{(s)}(X)$ implies $u \in H_{(s+m-1)}(X)$.

Proof.

Multiplication with an elliptic operator of order $1-m$ reduces to the case $m=1$.

Then take :

$\alpha(x,\xi) = \tau.\varphi(x,\xi) + s(x,\xi).\log(1+|\xi|_x^2)^{\frac{1}{2}}$, that is take A with principal part of the form : $e^{\tau\varphi(x,D)}.(1+|D|^2)^{s(x,D)/2}$.

Then
$$\Delta\alpha = \tau.\Delta\varphi + \Delta s.\ \log(1+|\xi|_x^2)^{\frac{1}{2}} + r ,$$

where $r \in S_1^0$ does not depend on τ . Taking τ sufficiently large, we can then apply Lemma 1.

Remark 1. For constant s condition (10) is automatically satisfied and we obtain a generalization of HÖRMANDER [7]. On the other hand, if we replace (10) by the stronger inequality $\Delta s > 0$ on $\Gamma \cap \Sigma_{(s)}(P)$,then the auxiliary function φ is not needed. This procedure has been followed by UNTERBERGER [8].

Remark 2. P is called subelliptic at (x,ξ) if for some $\delta < 1$:

(11) $$u \in \mathcal{D}' ,\ Pu \in H_{(t)} \text{ at } (x,\xi),\quad t \in \mathbb{R} \Longrightarrow u \in H_{(t+m-\delta)} \text{ at } (x,\xi)$$
P is subelliptic with $\delta=0$ at (x,ξ) if $p(x,\xi) \neq 0$ and with $\delta = \frac{1}{2}$ if $\frac{1}{i}\{\bar{p},p\}(x,\xi) > 0$. See HORMANDER [7] , 1.3.

So $\Gamma \cap \Sigma_{(s)}(P)$ is contained in the set of common zeros of p and $\{\bar{p},p\}$, which implies $d(\frac{1}{i}\{\bar{p},p\} - \mu p - \overline{\mu p}) = 0$ at $\Gamma \cap \Sigma_{(s)}(P)$ in view of (2).

So the function μ is uniquely determined there if d Re p and d Im p are linearly independent there.

Remark 3. On the other hand, the results of HÖRMANDER [7] , section 3.4 imply that $\Gamma \cap \Sigma_{(s)}(P)$ contains all bicharacteristic strips in Γ which are not contained in a cone axis. Here the term bicharacteristic strip in Γ is exclusively reserved for a 0,1 or 2-dimensional immersed C^∞ submanifolds B of Γ such that :

(i) $p = 0$ on B

(ii) the tangent spaces of B are spanned by the Hamilton fields $H_{\mathrm{Re}\, p}$, $H_{\mathrm{Im}\, p}$ of the real, (resp. imaginary) part of p.

(iii) B is maximal with these properties.

So $\Sigma_{(s)}(P)$ is equal to the set of zeros of p if $H_{\mathrm{Re}\, p}$, $H_{\mathrm{Im}\, p}$ are linearly independent and $\{\bar{p}, p\} = 0$ when $p = 0$.

Indeed, in this case $H_{\mathrm{Re}\, p}$, $H_{\mathrm{Im}\, p}$ commute on $p = 0$ so, in view of the Frobenius theorem they are tangent to a 2-dimensional foliation of $p = 0$. The leaves of this foliation are bicharacteristic strips, so $p = 0$ is a union of bicharacteristic strips.

If even after projection to the cotangential sphere bundle $S^*(X)$ the bicharacteristic strips are 2-dimensional, then the existence of a function φ such that $\Delta\varphi > 0$ on $p = 0$ means that no bicharacteristic strip is lying over a compact subset of X (see [2], Th.7.1.5). In this way, Theorem 2 leads to a somewhat more precise version of [2], Th.7.2.4.

Theorem 2 can also be used to generalize the results of EGOROV [4] to the case that bicharacteristic strips are present. We say that P is <u>hypo-elliptic at (x,ξ) with loss of derivatives ≤ 1 if</u>

(12) $$u \in \mathcal{D}' \;, \; Pu = f \implies S_u^*(x,\xi) \geq S_f^*(x,\xi) + m-1 \;.$$

Here we have used the regularity function

(13) $$S_u^*(x,\xi) = \sup\{t \in \mathbb{R} \;;\; u \in H_{(t)} \text{ at } (x,\xi)\} \;.$$

THEOREM 3.

Let $P \in L_1^m$, m constant, have a homogeneous principal symbol $p(x,\xi)$ such that :

(i) $\frac{1}{i}\{\bar{p},p\} \geq \mu p + \bar{\mu}\bar{p}$

for some homogeneous C^∞ function μ of degree $m-1$ on $T^*(X)\setminus 0$.

(ii) For each $(x,\xi) \in T^*(X)\setminus 0$ either P is hypo-elliptic at (x,ξ) with loss of derivatives ≤ 1 or (x,ξ) is lying on a bicharacteristic strip for P.

(iii) No bicharacteristic strip for P stays over the compact subset K of X.

Then we have for any constant $\sigma \in \mathbb{R}$:

(14) $$u \in \mathcal{E}'(K),\ Pu = f,\ S_f^* \geq \sigma \implies S_u^* \geq \sigma + m - 1.$$

Note that (14) implies a corresponding semi-global existence theorem for ${}^tP$. The main difficulty in the proof is that there may exist bicharacteristic strips with 1-dimensional projection in $S^*(X)$ but with arbitrarity close bicharacteristic strips which are 2-dimentional after projection to $S^*(X)$. It turns out that even in this case $\min(S_u^*, \sigma + m-1)$ cannot attain its minimum as a function on $T^*(X)\setminus 0$ on a bicharacteristic strip unless it is constant on the whole bicharacteristic strip.

We finally remark that (i),(ii) are satisfied if $d\,\mathrm{Re}\,p$, $d\,\mathrm{Im}\,p$ are linearly independent at $p = 0$,

(15) $$\frac{1}{i}\{\bar{p},p\} \geq 0 \quad \text{at} \quad p = 0,$$

and finally p is <u>real analytic</u>.

Here (15) is the necessary condition of HÖRMANDER [7] for local solvability for ${}^tP$. The condition (ii) follows in this case from the results of EGOROV [4] on subellipticity, but the situation here is more general because we also allow the existence of bicharacteristic strips.

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