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TORI OF PRESCRIBED MEAN CURVATURE AND THE ROTATING DROP

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1. ROTATING DROPS.

Consider a fluid body rotating with constant angular velocity ω and subject to surface tension. We assume, for example, that gravity is either absent or has no net effect, as would occur if an ambient fluid has equal mass density: $\tilde{\rho} = \rho$. In the rotating system of coordinates, the fluid body experiences a virtual force directed away from the axis of rotation and proportional to $\rho\omega^2 r^2$, where r is the distance to the axis. If the fluid is incompressible, then a constant additional pressure results within the body. Finally, surface tension contributes a force $-2H\tau$ proportional to the mean curvature H of the surface. A surface in equilibrium is therefore described by the condition

$$2H = a + br^2$$

where the constant a results from the volume constraint, and

$$b = (\rho\omega^2 - \tilde{\rho}\tilde{\omega}^2) / 2\tau \quad .$$

Here $\tilde{\rho}$ and $\tilde{\omega}$ are the density and angular velocity of the ambient fluid.

Plateau carried out experiments along these lines, with oil in a solution of water and alcohol, such that $\tilde{\rho} = \rho$ ([3], pp. 10-16). He observed a family of surfaces in the form of oblate spheroids, beginning from the sphere at $\omega = 0$. In a separate experiment, he was able to create a toroidal shape by gradually increasing the rate of rotation of a disk to which the oil had adhered, until the oil left the outer rim of the disk. These experiments have been repeated with greater precision by Wang; see his report in these proceedings.

The mathematical treatment of the spheroidal rotating drop is based on elliptic integrals, and has been studied in detail ([1], pp. 177-201). To our knowledge, however, the existence of tori with prescribed mean curvature $2H = a + br^2$ has remained unproven, despite Plateau's challenge to the geometers ([3], p. 19), and the stimulating questions posed by Poincaré ([5], esp. p. 124).

In the present paper, we shall demonstrate the existence of a large family of rotationally symmetric toroidal rotating drops (Theorem 1), characterize all such tori (Theorem 2), and derive estimates on the possible rates of rotation (Corollary 2). We shall address the question of stability in a future paper.

We would like to express our gratitude to Josef Bemelmans for posing, in a most inspiring fashion, the questions which led us to begin this research.

2. THE EXPLICIT SOLUTION.

Suppose a surface of revolution in euclidean $\mathbb{R}^3$ is given in cylindrical coordinates (r,θ,z) by $z = u(s)$, $r = r(s)$, $0 \leq \theta \leq 2\pi$, where s is an arc-length parameter along the generating curve: $(du/ds)^2 + (dr/ds)^2 = 1$. It is convenient to introduce the variable $v = du/ds = \sin \psi$, where ψ is the angle of inclination of the generating curve to the positive r-axis. Then the curvature of the generating curve is

$$\frac{d\psi}{ds} = \frac{dv/ds}{\cos \psi} = \frac{dv}{dr} \quad .$$

The circle of latitude has curvature $1/r$ and normal curvature v/r. The mean curvature H of the surface is therefore given by the useful formula

$$(1) \qquad 2H = \frac{dv}{dr} + \frac{v}{r} \quad .$$

We are interested in surfaces having the prescribed mean curvature $2H = a + br^2$, for two constants $a \in \mathbb{R}$ and $b > 0$. Using equation (1) for a surface of revolution, this becomes $d(rv)/dr = ar + br^3$, or

$$(2) \qquad v(r) = \frac{C}{r} + \frac{a}{2}r + \frac{b}{4}r^3$$

with a constant of integration C. One may solve for u , introducing a second constant D :

$$(3) \qquad u(r) = \int \frac{v\,dr}{\sqrt{1-v^2}} + D \quad .$$

This exhibits the two-parameter family of surfaces of revolution with prescribed mean curvature $a + br^2$, at all points of the generating curve with $dr/ds \neq 0$. At a general point, the parameterization $z = u(s)$, $r = r(s)$ satisfies a system of two second-order ordinary differential equations.

These surfaces fall into qualitatively distinct families according to the sign of the constant C. For $C = 0$, equations (2) and (3) determine a surface which meets the axis of rotation at right angles, and becomes parallel to the axis at the unique value $r = r_+$ with $v(r_+) = 1$ (recall that $b > 0$). Choosing the constant D so that $u(r_+) = 0$, we may describe the complete surface by $z = \pm u(r)$, $0 \leq r \leq r_+$. The surface is therefore a spheroid, the integral (3) is elliptic and its properties have been extensively studied (for example, [1]; see section 5 below).

The most interesting case is $C < 0$: we shall show that appropriate choices of the parameters a and C lead to a one-parameter family of surfaces of the type of the torus. By rescaling, we may assume that $b = 4/3$, and we write $C = -\gamma^4$. Then

$$(4) \qquad v(r) = -\frac{\gamma^4}{r} + \frac{a}{2}r + \frac{1}{3}r^3 \quad , \text{ and}$$

$$(5) \qquad \frac{dv}{dr} = \frac{\gamma^4}{r^2} + \frac{a}{2} + r^2 \quad .$$

The significance of γ is seen from the formula $d^2v/dr^2 = 2(r^4 - \gamma^4)/r^3$: $r = \gamma$ corresponds to a vertex, or point of minimum curvature, of the generating curve.

The range of $v(r)$ for $0 < r < \infty$ is unbounded above and below. In particular, there are values $0 < r_- < r_+$ with $v(r_-) = -1$ and $v(r_+) = +1$; these values need not be uniquely determined, but for the moment we shall assume r_- and r_+ have been chosen (compare section 4 below). The generating curve is vertical at these points: $du/dr(r_\pm) = \pm\infty$.

Consider the expression

$$F(a,\gamma) = u(r_+) - u(r_-) = \int_{r_-}^{r_+} \frac{v\,dr}{\sqrt{1-v^2}} \quad . \tag{6}$$

A qualitative description of the surface depends on the sign of F. Choose the constant D so that $u(r_-) = 0$. If $F > 0$, then a complete periodic surface is described by $z = 2NF \pm u(r)$, as N ranges over the integers. This surface has the general appearance of one of the nodoids of constant mean curvature of Delauney generated by the roulette of a hyperbola ([2]; cf. [4], pp. 110-126). The apparent singularities at $r = r_\pm$ are regular points of the real-analytic generating curve $z = u(s)$, $r = r(s)$, as follows from a well-known continuation argument for the second-order system of ordinary differential equations. If $F < 0$, then the complete surface may be described as a "reverse nodoid", in the sense that the small closed loops of the generating curve lie on the opposite side from the axis of rotation; there is no surface of revolution of constant mean curvature of this form ([2]). Finally, if $F = 0$, then the surface is in the form of a torus.

We shall show that for each $\gamma \geq (3/8)^{1/3}$, there is a value $a = a_o(\gamma)$ so that $F(a_o(\gamma),\gamma) = 0$, and moreover so that the generating curve $z = \pm u(r)$ is a closed convex curve. Convexity corresponds to $dv/dr \geq 0$ for $r_- \leq r \leq r_+$, which is implied by $dv/dr(\gamma) \geq 0$, since $r = \gamma$ describes the unique minimum of dv/dr. Further, the Four-Vertex Theorem states that a closed convex curve has at least four points of local maximum or minimum of curvature. It follows that $F(a,\gamma) = 0$ can only occur if $r_- < \gamma < r_+$. This conclusion is also a consequence of

<u>Lemma 1</u>. *Suppose* $dv/dr \geq 0$ *on* $[r_-,r_+]$. *If* $r_+ \leq \gamma$, *then* $F(a,\gamma) > 0$; *if* $r_- \geq \gamma$,

then $F(a,\gamma) < 0$.

Proof. We rewrite the integral (6) as

$$F(a,\gamma) = \int_{-1}^{1} \frac{dr}{dv}(v) \frac{v\,dv}{\sqrt{1-v^2}} = \int_{0}^{1} \left[\frac{dr}{dv}(v) - \frac{dr}{dv}(-v)\right] \frac{v\,dv}{\sqrt{1-v^2}} .$$

Recall that $d^2v/dr^2 = 2r^{-3}(r^4 - \gamma^4)$. If $r_+ \leq \gamma$, then dv/dr is a decreasing function on $[r_-, r_+]$, hence dr/dv is increasing, and therefore $F(a,\gamma) > 0$. Similarly, if $\gamma \leq r_-$ then $F(a,\gamma) < 0$.

q.e.d.

The Intermediate-Value Theorem may now be applied to $F(a,\gamma)$ as a function of the parameter a alone, to find $a_o(\gamma)$ such that $F(a_o(\gamma),\gamma) = 0$. However, we need to verify the convexity hypothesis $dv/dr \geq 0$ for two values $a_1(\gamma)$ and $a_2(\gamma)$ with $r_+ \leq \gamma$ and $r_- \geq \gamma$, respectively. Choose $a = a_1(\gamma)$ so that $v(\gamma) = -2\gamma^3/3 + a\gamma/2 = +1$, and choose $a_2(\gamma)$ so that $v(\gamma) = -1$. Clearly, $a_1(\gamma) > a_2(\gamma)$. A straight-forward computation shows that if $\gamma^3 \geq 3/8$, and $a \geq a_2(\gamma)$, then $dv/dr(\gamma) = 2\gamma^2 + a/2 \geq 0$, which implies $dv/dr \geq 0$ everywhere. Further, the monotonicity of v for $a_2(\gamma) \leq a \leq a_1(\gamma)$ ensures that r_- and r_+, and therefore F, are continuous functions of (a,γ). We have proved

Theorem 1. *Given* $\gamma \geq (3/8)^{1/3}$, *there exists* $a_o(\gamma)$ *with* $|a_o(\gamma) - 4\gamma^2/3| < 2/\gamma$ *and* $F(a_o(\gamma),\gamma) = 0$. *The corresponding curve* $z = \pm u(r)$, *where* u *is given by* (3) *and* v *is given by* (4), *is convex and generates a torus of revolution of prescribed mean curvature* $2H = a + 4r^2/3$.

The remaining case is $C > 0$. Here, the function v is convex and assumes a minimum value v_o. There is a corresponding solution u only if $v_o < 1$; in this case, there are unique values $0 < r_+(1) < r_+(2)$ with $v(r_+(1)) = v(r_+(2)) = 1$. Suppose $0 \leq v_o < 1$. Then we may choose $u(r_+(1)) = 0$ and observe that $z = 2N\,u(r_+(2)) \pm u(r)$, as N ranges over the integers, describes a complete periodic surface of revolution, having the qualitative properties of an unduloid of constant mean curvature (whose generating curve is the roulette of an ellipse;

cf. [2]). If $-1 < v_o < 0$, then self-intersection of the complete periodic surface may occur. We have not been able to decide whether compact surfaces may arise under these circumstances. It may be observed, however, that any such surface would be an immersed torus which would intersect itself along a single circle in the plane of symmetry. In fact, the unbounded component of the complement of the generating curve in the (r,z)-plane would lie to the left of the generating curve at $r = r_+(1)$ and to the right at $r = r_+(2)$, where $z = \pm u(r)$ is oriented by $\pm r$. Finally, if $v_o \leq -1$ then there exist $0 < r_-(1) \leq r_-(2)$ with $v(r_-(1)) = v(r_-(2)) = -1$. There are now two surfaces, corresponding to the intervals $r_+(1) \leq r \leq r_-(1)$ and $r_-(2) \leq r \leq r_+(2)$. The surfaces are disjoint if $v_o < -1$, and externally tangent if $v_o = -1$. Using arguments analogous to the proof of Lemma 1, and relying on $d^2v/dr^2 > 0$, one may show that one surface has the qualitative form of a reverse nodoid, the other of a nodoid.

In physical terms, one may think of a finite portion, without self-intersection, of any of these noncompact surfaces as a rotating capillary surface supported by rotationally symmetric solid bodies. For example, an unduloid $z = \pm u(r)$ as described above, restricted to an interval $r_+(1) \leq r \leq R$, lies between two horizontal plates $z = \pm u(R)$ and forms a constant angle of contact.

3. VOLUME AND THE CYLINDRICAL LIMIT.

In physical applications as outlined in section 1 above, a closed surface of prescribed mean curvature $2H = a + br^2$ occurs as a rotating body of fluid having prescribed volume. The constant b is determined by angular velocity, while the constant a is a Lagrange parameter. One would like therefore to consider b and V as given, and to choose a and C to determine a torus bounding the volume V. As we have just seen, a torus without self-intersection can occur only when $C = -\gamma^4 < 0$. If the generating curve is $z = \pm u(r)$, $r_- \leq r \leq r_+$, and $u(r) \leq 0$ as in section 2, then the volume enclosed by a torus is

(7) $$V(a,\gamma) = -4\pi\int_{r_-}^{r_+} r\ u\ dr = 2\pi\int_{r_-}^{r_+} r^2\frac{du}{dr}dr = 2\pi\int_{-1}^{1} r^2\frac{dr}{dv}\frac{v\ dv}{\sqrt{1-v^2}} .$$

Lemma 2. *For* (a,γ) *such that* $\gamma \geq 2$ *and* $F(a,\gamma) = 0$ *, there holds* $V(a,\gamma) < 8\pi\gamma^{-3}$.

Proof. For any value of r in the relevant interval $r_- \leq r \leq r_+$, we have $|v(r)| \leq 1$. For large values of γ , we shall show that this inequality and Lemma 1 force the ratio $\eta = r/\gamma$ to be uniformly close to 1. Geometrically, we have a very thin torus of large radius γ , with cross-section approximately a small circle. A series of elementary computations leads from equation (4) to the formula

$$\eta^2 - 1 = \frac{-[3a/(2\gamma^2) - 2]\eta^2 + 3v(r)\eta\ \gamma^{-3}}{\eta^2 + 3} .$$

Now Lemma 1 implies that $|v(\gamma)| \leq 1$, so that $|a - 4\gamma^2/3| \leq 2/\gamma$. Using $|v(r)| \leq 1$ and the Schwartz inequality $2\sqrt{3}\ \eta \leq \eta^2 + 3$, we derive the estimate

(8) $$|\eta^2 - 1| \leq \gamma^{-3}(3 + \sqrt{3}/2) < 4\gamma^{-3} \leq 1/2 .$$

We shall use inequality (8) to find an upper bound on volume. Subtracting $2\pi\gamma^2\ F(a,\gamma) = 0$ from equation (7) leaves

$$V(a,\gamma) = 2\pi\int_{-1}^{1} (r^2 - \gamma^2)\frac{dr}{dv}\frac{v\ dv}{\sqrt{1-v^2}} .$$

Using inequality (8) and $\gamma \geq 2$, we may compute that

$$|\gamma^{-2}dv/dr - 8/3| = |(a/(2\gamma^2) - 2/3) + \eta^{-2}(1-\eta^2)^2| < \gamma^{-3} + 2\gamma^{-6}(3+\sqrt{3}/2)^2 < 5\gamma^{-3} \leq 5/8 .$$

It follows that $\gamma^2 dr/dv < 24/49 < 1/2$. The estimate (8) now implies

$$V(a,\gamma) < 4\pi\gamma^{-3}\int_{-1}^{1}\frac{|v|\ dv}{\sqrt{1-v^2}} = 8\pi\gamma^{-3} , \qquad \text{q.e.d.}$$

It may be observed that the estimate $|\gamma^{-2}dv/dr - 8/3| < 5\gamma^{-3}$ shows that as $\gamma \to \infty$, the cross-section of the torus tends rapidly in the C^2 sense to a circle of radius $3/(8\gamma^2)$. In particular, the exact asymptotic behavior of volume is $V(a,\gamma) \sim 9\pi^2/(32\gamma^3)$.

Corollary 1. *There is a* $V_o > 0$ *so that given any positive* $V_1 \leq V_o$ *there is a torus of prescribed mean curvature* $2H = \text{const.} + 4r^2/3$ *bounding the volume* V_1.

Proof. Let V_o be the smallest value of $V(a,\gamma_o)$ for all a with $F(a,\gamma_o) = 0$. Here γ_o may be specified by $\gamma_o^3 = 3/8$ or by a particular choice, $\gamma_o^3 \geq 3/8$. Applying Theorem 1 and Lemma 2, we may find a sufficiently large value γ_1 so that $V(a,\gamma_1) \leq V_1$ whenever $F(a,\gamma_1) = 0$. Observe that some connected component of the zero locus $\{(a,\gamma) : F(a,\gamma) = 0\}$ contains points of the form (a_o,γ_o) and (a_1,γ_1). In fact, by Lemma 1, F is positive along the curve $a = 4\gamma^2/3 + 2/\gamma$ and negative along the curve $a = 4\gamma^2/3 - 2/\gamma$, for $\gamma \geq \gamma_o$. It follows, for example via the Alexander Duality Theorem, that some connected component of the zero locus contains points (a_o,γ_o) and (a_1,γ_1). But the continuity of V now implies that the set of values $\{V(a,\gamma) : F(a,\gamma) = 0\}$ includes the closed interval between $V(a_1,\gamma_1)$ and $V(a_o,\gamma_o)$, and V_1 in particular.

q.e.d.

Numerical evidence indicates a strong likelihood that for each $\gamma > 0$ there is a unique value a with $F(a,\gamma) = 0$. If this is true, then the constant V_o is simply the maximum volume among all closed tori of prescribed mean curvature $2H = a + 4r^2/3$. After rescaling we find

Corollary 2. *Given* $V_1 > 0$, *there exists a stationary rotating drop of the type of the torus having volume* V_1 *for any value* ω *of angular velocity not exceeding* $(2V_o\tau/(\rho V_1))^{1/2}$.

4. CLASSIFICATION OF TORI.

Now that we have found a particular one-parameter family of tori with prescribed mean curvature $2H = a + 4r^2/3$, we should like to classify all possible rotationally symmetric embedded tori with this prescribed mean curvature. A simple closed curve $z = u(s)$, $r = r(s)$ must have at least two vertical tangents. If a complete solution curve is reflected in the horizontal line $z = \text{const.}$ through any point with a vertical tangent, then the reflected curve remains a solution of the system of ordinary differential equations. The uniqueness of solutions to the initial-value problem therefore implies that the original curve is symmetric under this reflection. In particular, any solution in the form of a simple closed curve has exactly one horizontal line of symmetry, say $z = 0$, and the curve may be written in the form $z = \pm u(r)$, $r_- \le r \le r_+$. By considering $-u(r)$ if necessary, we may assume that $v = du/ds$ is a solution of equation (1), and therefore is given by equation (2) for some constant C. The arguments of section 2 above now show that for an embedded torus, $C < 0$; we write $C = -\gamma^4$.

If $a + 4\gamma^2 \ge 0$, then v is an increasing function and there are unique values $0 < r_- < r_+$ with $v(r_\pm) = \pm 1$; γ must be chosen to satisfy the condition $F(a,\gamma) = 0$, given by (6). Otherwise, v is not monotone. Nonetheless, there is a unique value $r_+ > 0$ with $v(r_+) = +1$. In fact, one may eliminate the terms of indeterminate sign in equations (4) and (5) to obtain

$$\frac{dv}{dr} = \frac{v}{r} + 2\,\frac{\gamma^4}{r^2} + \frac{2}{3}r^2 \quad .$$

It follows that v is strictly increasing on the interval $r \ge r_o$, where r_o is the first positive zero of v, and that r_+ is unique in particular. On the other hand, it may occur that there are three values $0 < r_1 < r_2 \le r_3$ with $v(r_1) = v(r_2) = v(r_3) = -1$. In this case, none of the three intervals $[r_i, r_+]$ corresponds to a closed curve $z = \pm u(r)$, $r_i \le r \le r_+$. For the function $v(r)$ has a unique inflection point at $r = \gamma$, so that $r_1 < \gamma < r_3$, and $dv/dr \ge 0$ for $r_3 \le r \le r_+$. Lemma 1 now implies that the choice $r_- = r_3$ would yield $F(a,\gamma) < 0$, corresponding to a nodoid. Further, as we have just shown, $r_3 < r_o < r_+$ and therefore $v < 0$ on $[r_1, r_3]$. It follows that

$$\int_{r_1}^{r_+} \frac{v\,dr}{\sqrt{1 - v^2}} < \int_{r_3}^{r_+} \frac{v\,dr}{\sqrt{1 - v^2}} < 0 \quad .$$

This rules out the interval $[r_1, r_+]$. Moreover, in case $r_2 < r_3$, the requirement that $|v| \leq 1$ eliminates the choices $r_- = r_1$ or $r_- = r_2$. After considering the consequences of this analysis for the generating curve (3), we have proved

Theorem 2. *Every embedded torus of revolution with prescribed mean curvature* $2H = a + 4r^2/3$ *has a plane of symmetry. The generating curve may be written as* $z = \pm u(r)$, $r_- \leq r \leq r_+$, *where* $u(r)$ *is given by equation* (3) *and* $v(r)$ *is given by equation* (4) *for some* $\gamma > 0$. *The generating curve may also be written as* $r = f_1(z)$, $r = f_2(z)$ *for two even, real-analytic functions* $f_1(z) < f_2(z)$. *The outside portion* $r = f_2(z)$ *is convex and has positive decreasing curvature for* $z > 0$. *The inside portion* $r = f_1(z)$, *if not convex, has two symmetric intervals* (z_1, z_2) *and* $(-z_2, -z_1)$ *of concavity.*

5. THE SPHEROIDAL LIMIT.

As $\gamma \to 0$, the inner radius $r_- < \gamma$ of a torus (if it exists) tends to zero. The curvature of the generating curve at this innermost point is $dv/dr(r_-) = a + 4r_-^2/3 + 1/r_-$, which increases beyond bound: the boundedness of the parameter a, as well as the existence of the torus, follow from Lemma 3 below. This behavior suggests that the tori are converging to a nonconvex, rotationally symmetric surface of the type of the sphere which is internally tangent to itself at the two umbilic points on the axis of rotation. Our numerical investigations confirm precisely this behavior.

The spheroids of prescribed mean curvature $2H = a + 4r^2/3$, and the internally tangent spheroid in particular, were studied in detail by Beer ([1], pp. 177-201); we give an outline here of one pertinent result. For a spheroid, the function $v = du/ds$ has the form $v = ar/2 + r^3/3$. The integral

$$F(a,0) = \int_0^{r_+} \frac{v\, dr}{\sqrt{1 - v^2}}$$

now describes half the length of the segment of the axis included inside the surface. Here r_+ is defined by $v(r_+) = 1$. By substituting the parameter t, defined by $r_+^2 t^2 = r_+^2 - r^2$, the integral may be written explicitly in terms of r_+. In fact,

$$F(a,0) = r_+ \int_0^1 \frac{\omega\, dt}{\sqrt{\omega^2 + \omega + k}}$$

where $\omega = k - t^2$ and $k = 3r_+^{-3}$. It is readily seen that $\frac{d}{dk}(F(a,0)/r_+) > 0$, that $dr_+/da < 0$, and therefore that $\frac{d}{da}(F(a,0)/r_+) > 0$. This is one step in the proof of

Lemma 3. *There is a unique value* $a = a_o(0) = -1.39$ *for which* $F(a,0) = 0$, *and a continuously differentiable curve* $a = a_o(\gamma)$ *for* $0 \le \gamma < \varepsilon$, *along which* $F(a_o(\gamma),\gamma) = 0$. *These are the only zeroes of* F *in a neighborhood of the* a*-axis*.

Proof. The existence of $a_o(0)$ follows from the Intermediate Value Theorem and from tables of elliptic integrals (see [1], pp. 182 ff.).

We shall first show that F is continuously differentiable in any region of the (a,γ)-plane where r_- has been chosen continuously. For convenience, we extend F to $\gamma < 0$ as an even function of γ. Now $F(a,\gamma)$ is defined by the convergent improper integral (6), so that its differentiability properties are greatly clarified by the introduction of a regularizing parameter t as the variable of integration. We may choose

$$(9) \qquad \frac{t^2}{(1-t)^2} = \frac{r^2 - r_-^2}{r_+^2 - r^2} , \quad 0 \le t \le 1 \quad .$$

Recalling equation (4), we may write $1 - v^2 = P(r)/9r^2$, where $P(r)$ is an even polynomial of degree 8 , having roots $r_\pm$, and hence $P(r) = (r^2 - r_-^2)(r_+^2 - r^2)(r^4 + \alpha r^2 + \beta)$. The last factor is irreducible, and its coefficients $\alpha = r_+^2 + r_-^2 - 3a$ and $\beta = -9\gamma^8 (r_- r_+)^{-2}$ depend smoothly on a and γ. In fact, one may show directly from the definition $v(r_-) = -1$ that $r_- = \gamma^4 - a\gamma^8/2 + O(a^2\gamma^{12})$, with formal differentiation, so that β remains differentiable at $\gamma = 0$. After considerable manipulation using the formula (9), the integral (6) becomes

$$F(a,\gamma) = \frac{3}{r_+^2 - r_-^2} \int_0^1 \frac{v(r)\left(\sqrt{r_+^2 - r^2} + \sqrt{r^2 - r_-^2}\right)}{\sqrt{r^4 + \alpha r^2 + \beta}}\, dt \quad .$$

In the integrand, one understands the nonsingular substitution $r^2 = (r_+^2 t^2 + r_-^2(1-t)^2)/(t^2+(1-t)^2)$.

Considering two points (a,γ) and (a_o,γ_o) , we write the difference of values of $(r_+^2 - r_-^2)F$ in the form $A(a,\gamma,a_o,\gamma_o)(\gamma-\gamma_o) + B(a,\gamma,a_o,\gamma_o)(a-a_o)$; we need to show that A and B are continuous at $(a,\gamma) = (a_o,\gamma_o)$. Continuity follows in a straightforward manner from the explicit formulas if $\gamma_o \neq 0$. When $\gamma_o = 0$, one unbounded term of the form γ^3/r appears in the integrand of A , arising from the difference of values of v. Nonetheless, after integration, the absolute value of the corresponding term in A may be shown to have an upper bound $c\,\gamma^3 \log(2r_+/r_-) \leq c\,\gamma^3 \log(3r_+/\gamma^4)$, for some constant c , which tends to zero as γ approaches $\gamma_o = 0$. It now follows that F is continuously differentiable in any region of the (a,γ)-plane where r_- has been chosen continuously, and in particular on a neighborhood of the zero locus of F (see section 4 above).

By continuity, we have $\frac{\partial}{\partial a}(F(a,\gamma)/r_+) > 0$ on a neighborhood of the a-axis. In particular, along each line $\gamma = \text{constant} < \varepsilon$ there is a unique zero of F in an interval about $a_o(0)$. That is, $F(a_o(\gamma),\gamma) = 0$ for an even C^1 function $a_o(\gamma)$, by the Implicit Function Theorem.

q.e.d.

Restating Lemma 3 in geometric terms, we find

Theorem 3. *The internally tangent spheroid of revolution of prescribed mean curvature* $2H = a_o(0) + 4r^2/3$ *is the weak limit (in the sense, e.g., of integral currents) of a unique one-parameter family of tori of revolution having prescribed mean curvature* $2H = a + 4r^2/3$, *with smoothly varying values of* a. *These tori converge uniformly along with higher derivatives away from the axis of rotation.*

Finally, we would like to point out that the extreme cases $\gamma \ll 1$ examined above and $\gamma \gg 1$ treated in section 3 are both unstable configurations, so that only the tori in an intermediate range can be observed experimentally. Our investigations of stability will appear elsewhere.

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