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FLOW OF OIL AND WATER THROUGH POROUS MEDIA

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We shall prove existence and regularity for the flow of two immiscible fluids through a porous medium. It is described by the following system of degenerate elliptic parabolic equations (see [2], [3]).

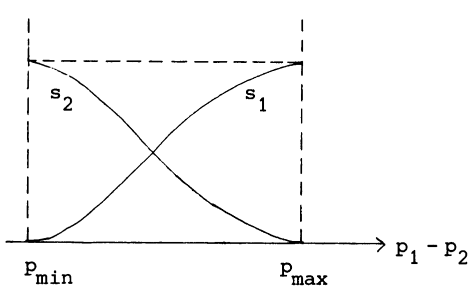
(1) $$\partial_t s_i - \nabla \cdot (k_i(\nabla p_i + e_i)) = 0 \quad \text{in} \quad \Omega_T := \Omega \times]0,T[$$

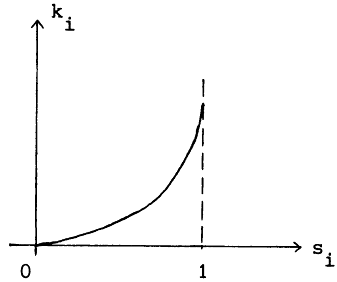
for $i = 1,2$, with side condition

$$s_1 + s_2 = 1 \quad .$$

The porous body Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary. s_i is the fluid content of the i-th fluid depending on $p_1 - p_2$, k_i its conductivity depending on s_i. The hydrostatic pressure is denoted by p_i and e_i is the gravity. s_i and k_i are continuous functions as in the Figure, s_i strictly monotone in $[p_{min}, p_{max}]$, where $-\infty \leq p_{min} < 0 < p_{max} \leq \infty$, and k_i positive in $]0,1]$. Therefore we have the additional side condition

$$p_{min} \leq p_1 - p_2 \leq p_{max} \quad .$$





As initial condition we pose

$$s_i(p_i - p_2)(x,0) = s_i^o(x) \quad \text{for} \quad x \in \Omega \quad ,$$

where s_i are nonnegative measurable functions with $s_1^o + s_2^o = 1$. We assume that $\psi(s_1^o) \in L^1(\Omega)$ where ψ is defined below. The boundary conditions are induced by a partition of $\partial\Omega$ into three measurable sets Γ_1, Γ_2 and Γ_o. We consider Neumann data

$$k_i(\nabla p_i + e_i) \cdot \nu = 0 \quad \text{on} \quad \Gamma_o \times]0,T[$$

and mixed Dirichlet and overflow conditions

$$p_1 = p_1^D$$

$$k_2(\nabla p_2 + e_2) \cdot \nu = 0 \quad \text{if} \quad p_1 - p_2 > p_{min}$$

$$k_2(\nabla p_2 + e_2) \cdot \nu = 0 \quad \text{if} \quad p_1 - p_2 = p_{min}$$

on $\Gamma_1 \times]0,T[$ and similar conditions on $\Gamma_2 \times]0,T[$.

Here

$$p_i^D \in L^\infty(\Omega_T) \cap L^2(0,T;H^{1,2}(\Omega))$$

with

$$p_{min} \leq p_1^D - p_2^D \leq p_{max}$$

and

$$\partial_t p_i^D \in L^1(0,T;L^2(\Omega)) \cap L^r(\Omega_T) \quad \text{for some} \quad r > 1 \quad .$$

Common Dirichlet conditions for p_1 and p_2 are easier to handle.

Multiplying (1) by $p_i - p_i^D$ we see that

$$\sum_{i=1,2} \int_0^T \int_\Omega k_i(s_i(p_i - p_2)) |\nabla p_i|^2 \tag{2}$$

determines the natural topology of the problem. Therefore since k_i degenerates we cannot work in function spaces for p_i. But if we define

$$u_1 := \Phi_1(p_1,p_2) := p_2 + \int_o^{p_1-p_2} \sqrt{\frac{k_1(s_1(\min(\xi,0)))}{k_1(s_1(o))}}\, d\xi \quad ,$$

(3)

$$u_2 := \Phi_2(p_1,p_2) := p_1 - \int_o^{p_1-p_2} \sqrt{\frac{k_2(s_2(\max(\xi,0)))}{k_2(s_2(0))}}\, d\xi \quad ,$$

then (2) is equivalent to the L^2-Norm of $(\nabla u_1, \nabla u_2)$. Also

$$\begin{bmatrix} k_1(s_1)\ \nabla p_1 \\ k_2(s_2)\ \nabla p_2 \end{bmatrix} = K(s_1) \begin{bmatrix} \nabla u_1 \\ \nabla u_2 \end{bmatrix} \quad ,$$

where in the set $\{p_1 \geq p_2\}$ the matrix K is given by

$$K(s_1) = \begin{bmatrix} k_1(s_1) & 0 \\ k_2(s_2) - \sqrt{k_2(s_2(0))k_2(s_2)} & \sqrt{k_2(s_2(0)k_2(s_2)} \end{bmatrix}$$

and similarly in $\{p_1 \leq p_2\}$. Introducing the notation

$$\mathbb{K} := \{(v_1,v_2) \in L^2(0,T;H^{1,2}(\Omega)) \ ; \ v_1 = p_1^D \text{ and } v_1 - v_2 \geq p_{min} \text{ on } \Gamma_1 \times]0,T[\ ,$$

$$v_2 = p_2^D \text{ and } v_1 - v_2 \leq p_{max} \text{ on } \Gamma_2 \times]0,T[\}$$

we can formulate the properties of a weak solution (p_1,p_2) as follows. $p_i : \overline{\Omega} \to \mathbb{R}$ with $p_{min} \leq p_1 - p_2 \leq p_{max}$ and the transformation (u_1,u_2) obtained by (3) (in $\overline{\Omega}$) is of class $L^2(0,T;H^{1,2}(\Omega))$. Furthermore for $(v_1,v_2) \in \mathbb{K}$ with $\partial_t v_i \in L^1(\Omega_T)$ the following inequality holds for almost all t, where $s_i = s_i(p_1 - p_2)$:

$$\int_\Omega (\psi(s_1(t)) - \psi(s_1^o)) - \int_\Omega (s_1(t)(v_1 - v_2)(t) - s_1^o(v_1 - v_2)(0)) + \int_o^t \int_\Omega s_1 \partial_t (v_1 - v_2) +$$

(4)

$$+ \Sigma_i \int_o^t \int_\Omega \left(\Sigma_j\, k_{ij}(s_i) \nabla u_j + k_i(s_i) e_i \right) \cdot \left(\frac{1}{k_i(s_i)}\, \Sigma_j\, k_{ij}(s_i) \nabla u_j - \nabla v_i \right) \leq 0 \quad .$$

Here by convention

$$k_{ij}(0) = 0 \quad \text{and} \quad \frac{k_{ij}}{\sqrt{k_i}}(0) = 0 \quad ,$$

and the convex function ψ is defined by

$$\psi(s_1(z)) := \int_0^z (s_1(z) - s_1(\xi))\, d\xi \quad .$$

Hence formally $\partial_t\psi(s_1(p_1-p_2)) = (p_1-p_2)\partial_t s_1(p_1-p_2)$, therefore the variational inequality (4) formally is equivalent to the above stated initial boundary value problem. We prove

<u>1. Existence Theorem</u>. *Suppose that* $\mathcal{H}^{N-1}(\Gamma_1) > 0$, $p_{min} > -\infty$, *and* $u_{max} := -\Phi_2(0,-p_{max}) < \infty$, *or that* $\mathcal{H}^{N-1}(\Gamma_2) > 0$, $p_{max} < \infty$, *and* $u_{min} := \Phi_1(p_{min},0) > -\infty$. *Then there exists a weak solution.*

<u>Proof</u>. We approximate the conductivity k_i by positive functions

$$k_{\varepsilon i} := \max(\varepsilon^2, k_i) \quad ,$$

and the water content by adding a penalizing term

$$s_{\varepsilon 1}(z) := s_1(z) + \varepsilon z \ , \quad s_{\varepsilon 2}(z) := s_2(z) - \varepsilon z \quad .$$

Furthermore we approximate the time derivative ∂_t by backward difference quotients ∂_t^{-h}. Thus we start with solutions $(p_{h\varepsilon 1}, p_{h\varepsilon 2}) \in \mathbb{K}_h$ of $(p_{h\varepsilon} := p_{h\varepsilon 1} - p_{h\varepsilon 2})$

$$(5) \quad \Sigma_i \int_\Omega \left(\partial_t^{-h} s_{\varepsilon i}(p_{h\varepsilon})(p_{h\varepsilon i} - v_i) + \nabla(p_{h\varepsilon i} - v_i) k_{\varepsilon i}(s_i(p_{h\varepsilon}))(\nabla p_{h\varepsilon i} + e_i) \right) \le 0$$

for all times and for every $(v_1, v_2) \in \mathbb{K}_h$. Here $\mathbb{K}_h$ is defined as $\mathbb{K}$ with p_i^D replaced by

$$p_{hi}^D(t) := \fint_{(j-1)h}^{jh} p_i^D(\tau)\, d\tau \quad \text{for} \quad (j-1)h < t < jh \quad .$$

The initial condition is

$$s_{\varepsilon i}(p_{h\varepsilon})(t) = s_i^o \quad \text{for} \quad -h < t < 0 \quad .$$

The solution $p_{h\varepsilon i}$ of these inductively defined elliptic problems exists since $\mathcal{H}^{N-1}(\Gamma_1 \cup \Gamma_2) > 0$. Setting $v_i = p_{hi}^D$ we obtain for the parabolic part since $s_{\varepsilon i}$ is monotone

$$\int_o^t \int_\Omega \partial_t^{-h} s_{\varepsilon i}(p_{h\varepsilon})(p_{h\varepsilon} - p_h^D) \geq \fint_{t-h}^t \int_\Omega \left(\int_o^{p_{h\varepsilon}} (s_{\varepsilon i}(p_{h\varepsilon}) - s_{\varepsilon 1}(\xi))d\xi - s_{\varepsilon 1}(p_{h\varepsilon})p_h^D \right) +$$

$$+ \int_o^{t-h} \int_\Omega s_{\varepsilon 1}(p_{h\varepsilon}) \partial_t^h p_h^D - C \geq c\ \varepsilon \fint_{t-h}^t \int_\Omega |p_{h\varepsilon}|^2 - \varepsilon \int_o^t \int_\Omega |p_{h\varepsilon}| |\partial_t^h p_h^D| \quad .$$

Together with the elliptic part we obtain the a priori estimate

$$\varepsilon \sup_{0 \leq t \leq T} \int_\Omega |p_{h\varepsilon}|^2 + \Sigma_i \int_o^T \int_\Omega k_{\varepsilon i}(s_i(p_{h\varepsilon})) |\nabla p_{h\varepsilon i}|^2 \leq C \quad .$$

Therefore if $u_{h\varepsilon i}$ are defined as in (3) with respect to $k_{\varepsilon i}$ we can conclude that $\nabla u_{h\varepsilon i}$ are bounded in $L^2(\Omega_T)$. Now in the set $\{p_{h\varepsilon 1} \geq p_{h\varepsilon 2}\}$ by definition of $k_{\varepsilon i}$ (write $u_{h\varepsilon} := u_{h\varepsilon 1} - u_{h\varepsilon 2}$)

$$0 \leq u_{h\varepsilon} \leq u_{max} + C\ \varepsilon\ |p_{h\varepsilon}| \quad .$$

Thus if $u_{max} < \infty$ by the a priori estimate

$$\max(u_{h\varepsilon} - u_{max}, 0) \to 0 \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)) \quad .$$

Similarly if $u_{min} > -\infty$

$$\min(u_{h\varepsilon} - u_{min}, 0) \to 0 \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)) \quad .$$

Together with the boundary condition and the assumptions made this implies that $u_{h\varepsilon i}$ are bounded in $L^2(0,T;H^{1,2}(\Omega))$. Hence for a subsequence $h \to o$, $\varepsilon \to o$

$$u_{h\varepsilon i} \to u_i \quad \text{weakly in} \quad L^2(0,T;H^{1,2}(\Omega))$$

and

$$u_{min} \leq u_1 - u_2 \leq u_{max} \quad .$$

Consequently p_1 and p_2 are well defined by (3).

The next step is to prove compactness of $s_{\varepsilon i}(p_{h\varepsilon})$. We multiply the equation in the time interval $](j-m)h, jh[$ by the time independent function

$$v_i = p_{h\varepsilon i} + \eta^2(u_{h\varepsilon i}(t) - u_{h\varepsilon i}(t-mh)) \quad ,$$

where $\eta \in C_o^\infty(\Omega)$, $j \geq m$, and $(j-1)h < t < jh$. Using the a priori estimate we obtain

$$\int_{mh}^{T}\int_{\Omega} \eta^2 (s_{\varepsilon 1}(p_{h\varepsilon}(t)) - s_{\varepsilon 1}(p_{h\varepsilon}(t-mh))(u_{h\varepsilon}(t) - u_{h\varepsilon}(t-mh)) \leq C\ mh \quad .$$

Since $p_{h\varepsilon}$ is a monotone function of $u_{h\varepsilon}$ and since $\varepsilon|p_{h\varepsilon}| \to 0$ in $L^1(\Omega_T)$ it follows as in [1] that $s_{\varepsilon 1}(p_{h\varepsilon})$ is relative compact in $L^1(\Omega_T)$, hence for a subsequence convergent to $s_1(p_1 - p_2)$ in $L^1(\Omega_T)$ and almost everywhere.

Then also $u_{h\varepsilon 1} - u_{h\varepsilon 2} \to u_1 - u_2$ almost everywhere in Ω_T. Moreover the boundary condition on Γ_i , $i = 1,2$, is of the form

$$u_{h\varepsilon 1} + u_{h\varepsilon 2} = \gamma_\varepsilon(u_{h\varepsilon 1} - u_{h\varepsilon 2}) \quad ,$$

where $\gamma_{\varepsilon i}$ are continuous functions converging uniformly to some γ. This implies that

$$u_1 + u_2 = \gamma(u_1 - u_2) \quad ,$$

that is, (u_1,u_2) is of class $\mathcal{K}$.

Finally we have to show that (u_1,u_2) satisfies the variational inequality. For this write (5) (omitting unessential positive terms on the left) in the form

$$\frac{1}{h}\int_{t-h}^{t}\int_{\Omega}\left(\psi(s_1(p_{h\varepsilon})) - \psi(s_1^o)\right) + \Sigma_i\int_o^t\int_{\Omega}\left(k_{\varepsilon i}(s_i(p_{h\varepsilon}))|\nabla p_{h\varepsilon i}|^2 + k_{\varepsilon i}(s_i(p_{h\varepsilon}))\nabla p_{h\varepsilon i}\cdot e_i\right)$$

$$\leq \frac{1}{h}\int_{t-h}^{t}\int_{\Omega} s_{\varepsilon 1}(p_{h\varepsilon})v_h - \frac{1}{h}\int_o^h\int_{\Omega} s_1^o v_h - \int_o^{t-h}\int_{\Omega} s_{\varepsilon 1}(p_{h\varepsilon})\partial_t^h v_h +$$

$$+ \Sigma_i\int_o^t\int_{\Omega}\left(k_{\varepsilon i}(s_i(p_{h\varepsilon}))\nabla v_{hi}\cdot e_i + k_{\varepsilon i}(s_i(p_{h\varepsilon}))\nabla p_{h\varepsilon i}\cdot\nabla v_{hi}\right) \quad .$$

Here $(v_{h1},v_{h2}) \in \mathcal{K}_h$ is a suitable approximation of a given function (v_1,v_2) with the properties as in (4). Since $s_1(p_{h\varepsilon})$ converges almost everywhere the first integral on the left and all terms on the right except the last one converge to the desired limit. Since

$$k_{\varepsilon i}(s_i(p_{h\varepsilon}))\nabla p_{h\varepsilon i} = \Sigma_j k_{\varepsilon ij}(s_i(p_{h\varepsilon}))\nabla u_{h\varepsilon i}$$

also the last term on both sides converge. The second term on the left is $(\varepsilon \le \varepsilon_o)$

$$\ge \Sigma_i \int_o^t \int_\Omega \frac{1}{k_{\varepsilon_o i}(s_i(p_{h\varepsilon}))} \left| \Sigma_j k_{\varepsilon ij}(s_i(p_{h\varepsilon}))\nabla u_{h\varepsilon i} \right|^2 ,$$

which in the limit $\varepsilon \to 0$, $h \to 0$ is

$$\ge \Sigma_i \int_o^t \int_\Omega \frac{1}{k_{\varepsilon_o i}(s_i(p_1 - p_2))} \left| \Sigma_j k_{ij}(s_i(p_1 - p_2))\nabla u_i \right|^2 .$$

Then let $\varepsilon_o \to 0$.

That weak solutions satisfy the differential equation is stated in the next Lemma.

2. Lemma. *For any weak solution* $\partial_t s_i(p_1 - p_2) \in L^2(0,T;\overset{o}{H}{}^{1,2}(\Omega)^*)$ *with initial values* s_i^o *, that is,*

$$\int_o^T \langle \partial_t s_i(p_1 - p_2), \zeta \rangle + \int_o^T \int_\Omega (s_i(p_1 - p_2) - s_1^o)\partial_t \zeta = 0 \tag{6}$$

for $\zeta \in C_o^\infty(\Omega \times [0,T[)$. *Moreover in the above space*

$$\partial_t s_i(p_1 - p_2) - \nabla \cdot \left(\Sigma_j k_{ij}(s_i(p_1 - p_2))\nabla u_j + k_i(s_i(p_1 - p_2))e_i \right) = 0 . \tag{7}$$

Proof. Formally this follows by setting $v_i = p_i \pm \zeta$ in (4). But since we do not know whether p_i is regular enough to do so, we have to approximate these functions. Choose $u^\rho_{min} \searrow u_{min}$ and $u^\rho_{max} \nearrow u_{max}$ and define

$$u^\rho_{1,2} := \frac{u_1 + u_2}{2} \pm \frac{1}{2} \max\left(u^\rho_{min}, \min(u^\rho_{max}, u_1 - u_2) \right) .$$

Then the corresponding pressures p_i^ρ belong to $L^2(0,T;H^{1,2}(\Omega))$. Similary define $p_i^{D\rho}$. Then

$$w_i := p_i^D + (p_i^\rho - p_i^{D\rho})$$

satisfy $p_{min} \le w_1 - w_2 \le p_{max}$ and the Dirichlet condition on Γ_i. As test function in (4) we use

$$v_{1,2}^{\tau\varepsilon} := \frac{1}{2}\left(w_1^{\tau\varepsilon} + w_2^{\tau\varepsilon}\right) \pm \frac{1}{2}\max\left(p_{min},\ \min(p_{max}, w_1^{\tau\varepsilon} - w_2^{\tau\varepsilon})\right) + \zeta_{1,2} \quad ,$$

where

$$w_i^{\tau\varepsilon}(t) := p_i^D(t) + (p_i^\rho - p_i^{D\rho})(t) +$$

$$+ \max\left(0, 1 - \frac{(j+1)h - \tau - t}{\varepsilon}\right)\left((p_i^\rho - p_i^{D\rho})((j+1)h - \tau) - (p_1^\rho - p_1^{D\rho})(jh - \tau)\right)$$

whenever $jh - \tau \leq t \leq (j+1)h - \tau$, $j = 0, \ldots, j_h$, $t_h = j_h \cdot h$, $t_h - h \leq t_o \leq t_h$ for given $t_o < T$. In this definition $p_i^D(t) := p_i^D(o)$ and $p_i^\rho(t) := p_i^{o\rho}$ for $t < 0$, where $p_i^{o\rho} \in H^{1,2}(\Omega)$ is chosen such that $p_{min} \leq p_1^{o\rho} - p_2^{o\rho} \leq p_{max}$ and

$$\int_\Omega\left(\psi(s_1^o) - \int_o^{p_1^{o\rho} - p_2^{o\rho}} (s_1^o - s_1(\xi))d\xi\right) \to 0 \quad \text{as} \quad \rho \to 0 \quad .$$

Then the ζ_i terms in (4) give the assertion provided we can show that for $\zeta_i = 0$ the right side in (4) does not exceed the left in the limit $\varepsilon \to 0$, $h \to 0$, and $\rho \to 0$.

Let us consider the parabolic terms. For almost all τ almost everywhere in Ω we have (writing $s_1(t)$ for $s_1(x, (p_1 - p_2)(x,t))$, $v^{\tau\varepsilon}$ for $v_1^{\tau\varepsilon} - v_2^{\tau\varepsilon}$ etc.)

$$\int_{jh-\tau}^{(j+1)h-\tau} s_1\, \partial_t v^{\tau\varepsilon} = \int_{jh-\tau}^{(j+1)h-\tau} \chi(\{p_{min} < w^{\tau\varepsilon} < p_{max}\}) \cdot (s_1 - s_1((j+1)h - \tau))\partial_t w^{\tau\varepsilon} +$$

$$+ s_1\Big((j+1)h - \tau\Big)\left(w^{\tau\varepsilon}((j+1)h - \tau) - w^{\tau\varepsilon}(jh - \tau)\right) \geq$$

$$\geq - \int_{jh-\tau}^{(j+1)h-\tau} |s_1 - s_1((j+1)h - \tau)|\ |\partial_t p^D| - \frac{1}{\varepsilon}\int_{(j+1)h-\tau-\varepsilon}^{(j+1)h-\tau} |s_1 - s_1((j+1)h - \tau)| \cdot$$

$$\cdot\ |h\partial_t^h(p^\rho - p^{D\rho})(jh - \tau)| - |s_1((j+1)h - \tau)|\left|\int_{jh-\tau}^{(j+1)h-\tau} |\partial_t(p^D - p^{D\rho})| +\right.$$

$$+ s_1((j+1)h - \tau)\left(p^\rho((j+1)h - \tau) - p^\rho(jh - \tau)\right) \quad .$$

The second term tends to zero as $\varepsilon \to 0$, hence summing over j and integrating over Ω we obtain

$$(8)\quad \lim_{\varepsilon\to 0}\int_o^{t_h-\tau}\int_\Omega s_1\partial_t v^{\tau\varepsilon} \geq \mathfrak{R}_1 + \sum_{j=o}^{j_h-\tau}\int_\Omega s_1\Big((j+1)h-\tau\Big)\Big(p^\rho((j+1)h-\tau) - p^\rho(jh-\tau)\Big) \ .$$

For the second term on the left of (4) we have

$$(9)\quad -\int_\Omega\Big(s_1(t_h-\tau)v^{\tau\varepsilon}(t_h-\tau) - s_1^o\cdot v^{\tau\varepsilon}(o)\Big) \geq \mathfrak{R}_2 - \int_\Omega\Big(s_1(t_h-\tau)p^\rho(t_h-\tau) - s_1^o\cdot p^\rho(o)\Big)$$

Thus the sum of the left sides in (8) and (9) is

$$\geq \mathfrak{R}_3 - \sum_{j=o}^{j_h-1}\int_\Omega\Big(s_1((j+1)h-\tau) - s_1(jh-\tau)\Big)p^\rho(jh-\tau) \geq$$

$$\geq \mathfrak{R}_3 - \int_o^{p^\rho(t_h-\tau)}\Big(s_1(t_h-\tau) - s_1(\xi)\Big)d\xi + \int_\Omega\int_o^{p^{o\rho}}(s_1^o - s_1(\xi))d\xi \geq$$

$$\geq \mathfrak{R}_3 - \int_\Omega\Big(\psi(s_1(t_h-\tau)) - \int_o^{p^{o\rho}}(s_1^o - s_1(\xi))d\xi\Big) \ .$$

Integrating over τ from 0 to h and dividing by h the last integral converges to the first term in (4). The remander $\mathfrak{R}_3$ tends to zero with h and ρ after performing the mean over τ. In the elliptic term we first can go to the limit with ε. After that it is not hard to complete the proof.

3. Remark. In order to show that the weak solution p_1, p_2 satisfies the original problem, we have to show that $s_i(p_1-p_2)$ are continuous in space and time. This would imply that ∇p_i is well defined in the open set $\{k_i(s_i(p_1-p_2))>0\}$.

We need

4. Assumptions. s_i is continuous differentiable with respect to the z variable in $\Omega\times\{p_{min}<z<p_{max}\}$ and

$$(10)\qquad \partial_z s_1 > 0 \ ,$$

$$(11)\qquad \frac{k_1(s_1(z))}{\partial_z s_1(z)} \geq c(\sigma) > 0 \quad \text{for} \quad z \leq p^\rho_{max} \nearrow p_{max} \ ,$$

(12) $$\left| \frac{k_i(s_i(z))}{\partial_z s_i(z)} \right| \le C \quad \text{for} \quad z \le 0 (\ge 0) \quad \text{if} \quad i = 1(2) \quad ,$$

(13) $$|k_1(s_1(z))(\partial_z k_2)(s_2(z)) + k_2(s_2(z))(\partial_z k_1)(s_1(z))| \le C \quad .$$

Let us consider the transformation (see [8], [13])

$$v = s_1 \ , \text{ and } \quad u = p_2 + \int_0^{p_1 - p_2} \frac{k_1(s_1(\xi))}{k_1(s_1(\xi) + k_2(s_2(\xi))} \, d\xi \quad .$$

Then formally v and u locally in Ω are solutions of the system

(14) $$0 = \nabla \cdot (k(v)\nabla u + e(v)) \qquad (\text{define } \vec{v} := -(k(v)\nabla u + e(v))) \quad ,$$

(15) $$\partial_t v = \nabla \cdot (a(v)\nabla v + b(v) + d(v)\vec{v}) \quad ,$$

where

$$k(z) = k_1(z) + k_2(1-z) \quad ,$$

$$e(z) = k_1(z)e_1 + k_2(1-z)e_2 \quad ,$$

$$a(z) = \frac{k_1(z)\ k_2(1-z)}{k(z)} \ \partial_t s_1^{-1}(z) \quad ,$$

$$b(z) = \frac{k_1(z)\ k_2(1-z)}{k(z)} \ (e_1 - e_2) \quad ,$$

$$d(z) = \frac{k_2(1-z)}{k(z)} \quad \text{or} \quad = -\frac{k_1(z)}{k(z)} \quad .$$

The assumptions made imply that these coefficients are bounded and

$$c \le k \le C \ , \quad |\partial_z d| \le C \quad ,$$

$$\phi_0(\omega) := \inf_{z \le 1 - \omega/4} a(z) > 0 \quad \text{for every} \quad \omega > 0 \quad .$$

Then u satisfies an elliptic equation and v a degenerate parabolic equation, coercive near 0.

5. Remark. u and v are solutions of (14) and (15) with ∇v replaced by $\lim_{\rho\to 0} \nabla \min(v, 1-\rho)$ and ∇u replaced by $\lim_{\rho\to 0} \nabla u^\rho$, where $\min(v,1-\rho)$ and u^ρ are in $L^2(0,T;\, H^{1,2}(\Omega))$. u^ρ is defined as u with p_i replaced by p_i^ρ, which is the transformation of u_i^ρ according to (3), and

$$u_i^\rho := \frac{u_1+u_2}{2} \pm \frac{1}{2}\max\left(u^\rho_{min},\ \min(u^\rho_{max},\ u_1-u_2)\right)$$

with $u^\rho_{min} \searrow u_{min}$ and $u^\rho_{max} \nearrow u_{max}$.

Next we show

6. Lemma. *In addition to the assumptions in theorem 1 suppose that if* $H^{N-1}(\Gamma_1) > 0$ *then* $p_{min} > -\infty$ *and*

$$\int_0^{p_{max}} \frac{k_2(s_2(\xi))}{k_1(s_1(\xi)) + k_2(s_2(\xi))}\, d\xi \leq C \quad ,$$

(similar if $H^{N-1}(\Gamma_2) > 0$ *). Then* u *is locally bounded in* Ω_T.

Proof. The assumptions imply that the functions u^ρ defined as above are uniformly bounded on $\Gamma_1 \cup \Gamma_2$ by some C. Then

$$\phi(u^\rho) := \min(u^\rho + C,\ \max(u^\rho - C,\ 0))$$

can be used as test function for the equation (14). This gives that

$$\lim_{\rho\to 0} \|\phi(u^\rho(t))\|_{H^{1,2}(\Omega)}$$

is bounded in t. Then multiplying (14) by $\eta^2 u^\rho$ with $\eta \in C_0^\infty(\Omega)$ we obtain that

$$\lim_{\rho\to 0} \|u^\rho(t)\|_{H^{1,2}_{loc}(\Omega)}$$

is bounded in t. Therefore u^ρ has a weak limit, which is a bounded function satisfying (14).

Now we are able to prove

7. Regularity theorem. *Suppose that the assumptions in* 1. *and* 4. *hold and that* u *is bounded. Then* $s_i(p_1-p_2)$ *are continuous in* Ω_T *, and the modulus of*

continuity can be estimated.

Proof. This follows by an iterative procedure from the two propositions below, and they are proved using the De Giorgi techniques, where the special features here are the degeneracy of the coefficient a in the parabolic equation for v and the coupling to the elliptic equation for u.

8. Notation. Let $(x_o,t_o) \in \Omega_T$. For $R > 0$, $\alpha > 0$, and $0 < \sigma_1$, $\sigma_2 < 1$ we let

$$Q_R^\alpha(\sigma_1,\sigma_2) := B_{(1-\sigma_1)R}(x_o) \times]t_o - (1-\sigma_2)\alpha R^2, t_o[$$

and $Q_R^\alpha = Q_R^\alpha(0,0)$, $Q_R = Q_R^1$. We define

$$\|w\|_{Q_R}^2 := \operatorname*{ess\,sup}_{t_o-R^2 < t < t_o} \int_{B_R(x_o)} |w|^2 + \int_{Q_R} |\nabla w|^2 ,$$

and similar for $Q_R^\alpha(\sigma_1,\sigma_2)$. In the following $0 < R \le R_o$ with $Q_{R_o} \subset\subset \Omega_T$ and μ^+, μ^- are any numbers with

$$\operatorname*{ess\,sup}_{Q_{2R}} v \le \mu^+ \le 1 , \quad \operatorname*{ess\,inf}_{Q_{2R}} v \ge \mu^- \ge 0 ,$$

hence $\operatorname{ess\,osc} v \le \mu^+ - \mu^- \le 1$. Furthermore ω is any positive number satisfying $\mu^+ - \mu^- \le \omega \le 2(\mu^+ - \mu^-)$.

9. Proposition. *There is a small constant such that if*

$$\operatorname{meas}(Q_R \cap \{v > \mu^+ - \tfrac{\omega}{2}\}) \le c_o \phi_1(\omega) \operatorname{meas}(Q_R) ,$$

then

$$\operatorname*{ess\,osc}_{Q_{R/2}} v \le \frac{5}{8}\omega .$$

Here $\phi_1(\omega) := (\omega\, \phi_o(\omega))^{N+2}$.

Proof. Let $v_\omega := \min(v, \mu^+ - \frac{\omega}{4})$ and $\mu^+ - \frac{\omega}{2} \le k \le \mu^+ - \frac{\omega}{4}$ and multiply (15) by $(v_\omega - k)^+ \eta^2$ in the time interval $]t_o - R^2, t[$ with $t < t_o$. Here η is a cut off function with $\eta = 1$ in $Q_R(\sigma_1,\sigma_2)$, $\eta = 0$ on the parabolic boundary of Q_R , and

$$|\nabla\eta| \le C(\sigma_1 R)^{-1} , \quad |\nabla\eta| \le C(\sigma_1 R)^{-2} ,$$

$$0 \le \partial_t \eta \le C(\sigma_2 R^2)^{-1} .$$

We obtain

$$\int_{B_R} \eta(t)^2 \Phi(v(t)) + \int_{t_o-R^2}^t \int_{B_R} a(v)\eta^2 |\nabla(v_\omega - k)^+|^2 = \int_{t_o-R^2}^t \int_{B_R} (\Phi(v)\partial_t \eta^2 -$$

$$- a(v)(v_\omega - k)^+ \nabla v \nabla \eta^2 - (b(v) + d(v)\vec{v})\nabla((v_\omega - k)^+ \eta^2)) ,$$

where

$$\Phi(v) = \frac{1}{2} |(v_\omega - k)^+|^2 + (\mu^+ - \frac{\omega}{4} - k)(v - (\mu^+ - \frac{\omega}{4}))^+ .$$

Since $a(v) \ge \phi_o(\omega)$ in $\{(v_\omega - k)^+ \ne 0\}$ and

$$a(v)(v_\omega - k)^+ \nabla v = (v_\omega - k)^+ \nabla\left(\int_o^v a(\xi)d\xi\right) - \int_o^v a(\xi)d\xi \, \nabla(v_\omega - k)^+$$

we derive using the various properties of the coefficients

$$c\int_{B_R} \eta(t)^2 |(v_\omega(t) - k)^+|^2 + \frac{1}{2}\phi_o(\omega)\int_{t_o-R^2}^t \int_{B_R} \eta^2 |\nabla(v_\omega - k)^+|^2 \le$$

$$\le C\left(\frac{(\sigma_1 R)^{-2}}{\phi_o(\omega)} + (\sigma_2 R^2)^{-1}\right)\int_{Q_R} \chi(\{v > k\}) - \int_{t_o-R^2}^t \int_{B_R} \vec{v} d(v)\nabla((v_\omega - k)^+ \eta^2) .$$

Using the fact that $\vec{v}$ is divergence free the last term equals

$$= - \int_{t_o-R^2}^t \int_{B_R} \vec{v} \int_k^{k+(v_\omega - k)^+} (d(v) - d(\xi))d\xi \, \nabla\eta^2 \le$$

$$\le \delta \int_{t_o-R^2}^t \int_{B_R} |\nabla u|^2 (v_\omega - k)^{+2} \eta^2 + \frac{C}{\delta}\int_{t_o-R^2}^t \int_{B_R} \chi(\{v > k\})|\nabla\eta|^2 .$$

Multiplying (14) with $u((v_\omega - k)^+ \eta)^2$ we see that the integral involving $|\nabla u|^2$ is estimated by

$$C\int_{t_o-R^2}^{t}\int_{B_R}(\eta^2|\nabla(v_\omega-k)^+|^2+\chi(\{v>k\})(|\nabla\eta|^2+\eta^2))\quad .$$

Substituting this estimate we obtain

$$\|(v_\omega-k)^+\|^2_{Q_R(\sigma_1,\sigma_2)}\le\frac{C}{\phi_o(\omega)^2}\left((\sigma_1R)^{-2}+(\sigma_2R^2)^{-1}\right)\cdot\text{meas}(Q_R\cap\{v>k\})\quad .$$

Now we use this over a sequence

$$R_n:=\frac{R}{2}+\frac{R}{2^{n+1}}\quad\text{and}\quad k_n=\mu^+-\frac{\omega}{2}+\frac{\omega}{8}-\frac{\omega}{2^{n+3}}\quad .$$

Using an embedding Lemma [10; II(3.9)] we get

$$\int_{Q_{R_{n+1}}}|(v_\omega-k_n)^+|^2\le C\,\frac{2^{2n}}{\phi_o(\omega)^2R^2}\,\text{meas}(Q_{R_n}\cap\{v_\omega>k_n\})^{1+\frac{2}{N+2}}$$

But the left side controls $(k_{n+1}-k_n)^2\text{meas}(Q_{R_{n+1}}\cap\{v>k_{n+1}\})$, hence

$$y_{n+1}\le\frac{C\,2^{4n}}{\omega^2\phi_o(\omega)^2}\,y_n^{1+\frac{2}{N+2}}\quad ;\quad y_n:=\frac{1}{R^{N+2}}\,\text{meas}(Q_{R_n}\cap\{v>k_n\})\quad .$$

Since by assumption y_o was small enough, $y_n\to 0$ as $n\to\infty$ by [9; 2 Lemma 4.7], which proves the Lemma.

From below we will only assume that

$$\text{(16)}\qquad \text{meas}(Q_R\cap\{v<\mu^-+\tfrac{\omega}{4}\})\le(1-c_o\phi_1(\omega))\text{meas}(Q_R)\quad .$$

But since a is coercive near 0 , we can derive a similar statement to Proposition 9. First we show an uniform estimate in time.

<u>10. Lemma</u>. *Let* $k\le\mu^++\frac{\omega}{4}$ *and* $p\ge 3$. *Then for* $t_o-R^2<t_1<t<t_o$

$$\int_{B_{(1-\sigma_1)R}}\underline{\psi}^2((v(t)-k)^-)\le\int_{B_R}\underline{\psi}^2((v(t_1)-k)^-)+\frac{C\,(p-2)}{\phi_o(\omega)}\left(\frac{1}{\sigma_1^2}+\left(\frac{2^{p_o}R}{\omega}\right)^2\right)R^N\quad ,$$

where

$$\underline{\psi}(z):=\max\left(0,\ \log\frac{\omega/4}{\omega/4-z+\omega/2^p}\right).$$

Proof. Multiply (14) by $-(\underline{\psi}^2)'(v-k)^-)\eta^2$ where η is a cut off function in space with $\eta = 1$ in $B_{(1-\sigma_1)R}$.

We obtain

$$\int_{B_R} \eta^2\underline{\psi}^2((v(t)-k)^-) + \int_{t_1}^{t}\int_{B_R} a(v)\eta^2\underline{\psi}^{2''}((v-k)^-)|\nabla(v-k)^-|^2 =$$

$$= \int_{B_R} \eta^2\underline{\psi}^2((v(t_1)-k)^-) + \int_{t_1}^{t}\int_{B_R} a(v)\underline{\psi}^{2'}((v-k)^-)\nabla v\nabla\eta^2 +$$

$$+ \int_{t_1}^{t}\int_{B_R} (b(v)+d(v)\vec{v})\nabla(\underline{\psi}^{2'}((v-k)^-)\eta^2) \quad .$$

Since $a(v) \geq \phi_o(\omega)$ in $\{\underline{\psi}^{2''}((v-k)^-) \neq 0\}$ and

$$(\underline{\psi}^2)'' = 2(1+\underline{\psi})\underline{\psi}'^2 \quad , \text{ hence } \quad \frac{(\underline{\psi}^2)'^2}{(\underline{\psi}^2)''} \leq 2\underline{\psi} \quad ,$$

we derive that

$$\int_{B_R} \eta^2\underline{\psi}^2((v(t)-k)^-) + c\phi_o(\omega)\int_{t_1}^{t}\int_{B_R} \eta^2\underline{\psi}^{2''}((v-k)^-)|\nabla(v-k)^-|^2 \leq \int_{B_R} \eta^2\underline{\psi}((v(t_1)-k)^-) +$$

$$+ \frac{c}{\phi_o(\omega)}\int_{t_1}^{t}\int_{B_R} ((1+\underline{\psi})\underline{\psi}'^2\eta^2 + \underline{\psi}|\nabla\eta|^2) + \int_{t_1}^{t}\int_{B_R} d(v)\vec{v}\nabla(\underline{\psi}^{2'}((v-k)^-)\eta^2) \quad .$$

Since $\vec{v}$ is divergence free the last term equals

$$= -\int_{t_1}^{t}\int_{B_R} \vec{v}\partial_z d(v)\nabla(v-k)^- \,\underline{\psi}^{2'}((v-k)^-)\eta^2 \leq$$

$$\leq \delta\int_{t_1}^{t}\int_{B_R} \eta^2\underline{\psi}^{2''}((v-k)^-)(|\nabla(v-k)^-|^2+1) + \frac{c}{\delta}\int_{t_1}^{t}\int_{B_R} \eta^2(|\nabla\eta|^2+1)\underline{\psi} \quad .$$

Using

$$(17) \qquad \underline{\psi}((v-k)^-) \leq (\log 2)(p-2) \quad \text{and} \quad \underline{\psi}'((v-k)^-) \leq \frac{1}{\omega/2^p}$$

the assertion follows, where the integral with $|\nabla u|^2$ can be estimated by multiplying (14) with $u\eta^2$.

As a consequence we obtain

11. Lemma. *There is a* $p = p(\omega)$ *such that if* $R \le 2^{-p}\omega$ *and* (16) *hold then*

$$\mathrm{meas}(B_R \cap \{v(t) < \mu^- + 2^{-p}\omega) \le (1-\alpha^2)\mathrm{meas}(B_R)$$

for $t_o - \alpha R^2 < t < t_o$. *Here* $\alpha := \frac{c_o}{2}\phi_1(\omega)$.

Proof. By the previous lemma $(k = \mu^- + \frac{\omega}{4})$

$$(18) \qquad \int_{B_{(1-\sigma_1)R}} \underline{\psi}^2((v(t)-k)^-) \le \int_{B_R} \underline{\psi}^2((v(t_1)-k)^-) + \frac{C(p-2)}{\phi_o(\omega)\sigma_1^2} R^N$$

and by (16) for some $t_1 \in]t_o - R^2, t_o - \alpha R^2[$

$$\mathrm{meas}(B_R \cap \{v(t_1) < k\}) \le \frac{1-2\alpha}{1-\alpha}\mathrm{meas}(B_R)$$

hence using (17)

$$\int_{B_R} \underline{\psi}^2((v(t_1)-k)^-) \le (\log 2)^2(p-2)^2\,\frac{1-2\alpha}{1-\alpha}\mathrm{meas}(B_R) \quad .$$

The left side of (18) is

$$\ge \int_{B_{(1-\sigma_1)R} \cap \{v(t)<\mu^- + 2^{-p}\omega\}} \underline{\psi}^2((v(t)-k)^-) \ge$$

$$\ge \max(0, \log 2^{p_o - 3})^2 \mathrm{meas}(B_{(1-\sigma_1)R} \cap \{v(t) < \mu^- + 2^{-p}\omega\}) \ge$$

$$\ge (\log 2)^2(p_o - 3)^2(\mathrm{meas}(B_R \cap \{v(t) < \mu^- + 2^{-p}\omega\}) - \sigma_1 N\,\mathrm{meas}(B_R)) \quad .$$

Substituting these estimates in (18) we get

$$\frac{\mathrm{meas}(B_R \cap \{v(t) < \mu^- + 2^{-p}\omega\})}{\mathrm{meas}(B_R)} \le \left(\frac{p-2}{p-3}\right)^2 \frac{1-2\alpha}{1-\alpha} + C\,\frac{p-2}{\phi_o(\omega)\sigma_1^2(p-3)^2} + \sigma_1 N \quad .$$

Now choose $\sigma_1 = 3\alpha^2/(2N)$ and p large enough so that

$$\frac{C(p-2)}{\phi_o(\omega)\sigma_1^2(p-3)^2} \le \frac{3}{2}\alpha^2 \quad \text{and} \quad \left(\frac{p-2}{p-3}\right)^2 \le (1-\alpha)(1+2\alpha) \quad .$$

We also need the following estimate

12. Lemma. *There is a constant* c *such that for* $k \leq \mu^- + \frac{\omega}{4}$ *and* $0 < \beta < 1$

$$\|(v-k)^-\|^2_{Q_R^\beta(\sigma_1,\sigma_2)} \leq \frac{c}{\phi_o(\omega)^2}\left((\sigma_1 R)^{-2} + (\sigma_2 \beta R^2)^{-1}\right)\int_{Q_R^\beta} |(v-k)^-|^2 +$$

$$+ \frac{c}{\phi_o(\omega)^2} \operatorname{meas}(Q_R^\beta \cap \{v < k\}) \quad .$$

Proof. This follows similarly to the first part of the proof of Proposition 9 by multiplying (15) with $-(v-k)^-\eta^2$, where η is a suitable cut off function.

Now we are able to show

13. Lemma. *For* $\theta > 0$ *there is a* $q = q(\omega,\theta) > p(\omega)$ *such that if* $R \leq 2^{-p}\omega$ *and* (16) *hold, then*

$$\operatorname{meas}(Q_R^\alpha \cap \{v < \mu^- + 2^{-q}\omega\}) < \theta \operatorname{meas}(Q_R^\alpha) \quad .$$

Proof. Let $q \geq p(\omega)$, $\ell = \mu^- + 2^{-q}\omega$, and $k = \mu^- + 2^{-q-1}\omega$. By Lemma 11 for $t_o - \alpha R^2 < t < t_o$

$$\operatorname{meas}(B_R \cap \{v(t) \geq \ell\}) \geq c\alpha^2 R^N \quad ,$$

therefore using [9; 2 Lemma 3.5]

$$\frac{\omega}{2^{q+1}} \operatorname{meas}(B_R \cap \{v(t) < k\}) \leq \frac{cR}{\alpha^2}\int_{B_R \cap \{k < v(t) < \ell\}} |\nabla((v(t)-\ell)^-)| \quad .$$

Integrating over t yields

$$(19) \qquad \left(\frac{\omega}{2^{q+1}}\right)^2 \operatorname{meas}(Q_R^\alpha \cap \{v < k\})^2 \leq \frac{cR^2}{\alpha^4} \operatorname{meas}(Q_R^\alpha \cap \{k < v < \ell\})\int_{Q_R^\alpha} |\nabla(v-\ell)^-|^2$$

and by lemma 12

$$\int_{Q_R^\alpha} |\nabla(v-\ell)^-|^2 \leq \frac{c}{\phi_o(\omega)^2}\left(\left(\operatorname{ess}_{Q_{2R}} \sup(v-\ell)^-\right)^2 + R^2\right)R^N \leq \frac{c}{\phi_o(\omega)^2}(2^{-q}\omega)^2 R^N \quad .$$

Thus (19) becomes

$$\operatorname{meas}(Q_R^\alpha \cap \{v < k\})^2 \leq \frac{cR^{N+2}}{\alpha^4 \phi_o(\omega)^2} \operatorname{meas}(Q_R^\alpha \cap \{k < v < \ell\}) \quad .$$

Adding this unquality for $q = p(\omega),\ldots,q_o - 1$ we obtain the lemma, if q_o is large enough (depending on ω and θ).

<u>14. Proposition</u>. *There is a* $q = q(\omega)$ *such that if* (16) *holds and* $R \leq 2^{-q}\omega$ *, then*

$$\operatorname{ess}_{Q_{R^*}} \operatorname{osc} v \leq \omega(1 - 2^{-q-1}) \quad ,$$

where $R^* = c_1 R^{7/6}$. *Here* c_1 *is a small constant independent of* R *and* ω.

<u>Proof</u>. Consider the cylinders $Q^\alpha_{R_n}$ and the levels k_n defined by

$$R_n := \frac{R}{2} + \frac{R}{2^{n+1}} \quad \text{and} \quad k_n := \mu^- + \frac{\omega}{2^{q+1}} + \frac{\omega}{2^{q+n+1}} \quad ,$$

where $q = q(\omega,\theta)$, θ to be chosen. By the embedding lemma [10; II (3.9)]

$$\int_{Q^\alpha_{R_{n+1}}} |(v-k_n)^-|^2 \leq C \operatorname{meas}\left(Q^\alpha_{R_{n+1}} \cap \{v<k_n\}\right)^{\frac{2}{N+2}} \|(v-k_n)^-\|^2_{Q^\alpha_{R_{n+1}}} \quad .$$

The left side controls

$$(k_n - k_{n+1})^2 \operatorname{meas}(Q^\alpha_{R_{n+1}} \cap \{v<k_{n+1}\}) \quad ,$$

and by Lemma 12

$$\|(v-k_n)^-\|^2_{Q^\alpha_{R_{n+1}}} \leq \frac{C}{\phi_o(\omega)^2}\left(\left(\frac{2^n}{R} \operatorname{ess\,sup}_{Q^\alpha_{R_n}} (v-k_n)^-\right)^2 + 1\right) \cdot \operatorname{meas}\left(Q^\alpha_{R_n} \cap \{v<k_n\}\right) \quad .$$

Since $(v-k_n)^- \leq 2^{-q}\omega$ on $Q^\alpha_{R_n}$ we get the recursive estimate

$$y_{n+1} \leq \frac{C\alpha^{\frac{2}{N+2}}}{\phi_o(\omega)^2} 2^{4n} y_n^{1+\frac{2}{N+2}}$$

$$y_n := \frac{\operatorname{meas}(Q^\alpha_{R_n} \cap \{v<k_n\})}{\operatorname{meas}(Q^\alpha_{R_n})} \quad .$$

By [10; II Lemma 5.6] we infer that $y_n \to 0$ as $n \to \infty$ if

$$y_o < c\,\frac{\phi_o(\omega)^{N+2}}{\alpha} \quad .$$

But if we choose θ to be the right side of this inequality, this is just the statement in Lemma 13.

15. Remark. In [8] the existence of a classical solution is proved in the case that the equation (15) is strictly parabolic. The paper also contains uniqueness and stability results, but the overflow condition is not included. Some of the arguments are restricted to the two dimensional case.

Recently in [7] the existence of a weak solution was shown for the Dirichlet-Neuman problem. The assumption is that the initial and boundary data stay away from one side of the degeneracy, so that the solution contains only one pure fluid besides the mixture.

In the article presented here the statement of Lemma 6 in connection with the assumption in Theorem 7 is not quite satisfactory, since if $k_1(z) \leq Cz$ condition (11) implies that $p_{min} = -\infty$, but then Lemma 6 does not cover the case $H^{N-1}(\Gamma_i) > 0$.

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