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MULTIPLE VALUED FUNCTIONS IN THE GEOMETRIC CALCULUS OF VARIATIONS

by F.J. ALMGREN and B. SUPER (Princeton University)

1. INTRODUCTION.

This article is intended as an introduction and an invitation to multiple valued functions as a tool of geometric analysis. Such functions typically have a region in $\mathbb{R}^m$ as a domain and take values in spaces of zero dimensional integral currents in $\mathbb{R}^n$. The multiply sheeted "graphs" of such functions f represent oriented m dimensional surfaces S in $\mathbb{R}^{m+n}$, frequently with elaborate topological or singular structure. Perhaps the most conspicuous advantage of multiple valued functions is that one is able to represent complicated surfaces S by functions f having fixed simple domains. This leads, in particular, to applications of functional analytic techniques in ways novel to essentially geometric problems, especially those arising in the geometric calculus of variations.

2. EXAMPLES AND TERMINOLOGY.

(A) We denote by $\mathbb{I}_0(\mathbb{R}^n)$ the space of zero dimensional integral currents in $\mathbb{R}^n$ [F 4.1.24]. Associated to each point $p \in \mathbb{R}^n$ is the current $[\![p]\!]$ which assigns to a test function $\phi : \mathbb{R}^n \to \mathbb{R}$ the value $\phi(p)$ [F 2.5.19]. General zero dimensional integral currents in $\mathbb{R}^n$ are thus representable in the form

$$[\![p_1]\!] + [\![p_2]\!] + \dots + [\![p_P]\!] - [\![q_1]\!] - [\![q_2]\!] - \dots - [\![q_N]\!]$$

corresponding to various choices of $P, N \in \{0,1,2,\dots\}$ and

$p_1, p_2, \ldots, p_P, q_1, \ldots, q_N \in \mathbb{R}^n$; it is not required that the p_i's or q_j's be distinct so that a given point can be taken with higher positive or negative multiplicity.

(B) Consider the function $f : \mathbb{R} \to \mathbb{I}_0(\mathbb{R})$ defined by setting

$$f(x) = [\![x^{1/2}]\!] - [\![-(x^{1/2})]\!] \in \mathbb{I}_0(\mathbb{R}) \quad \text{in case } x > 0$$
$$= 0 \in \mathbb{I}_0(\mathbb{R}) \quad \text{in case } x \leq 0 .$$

We note that f is defined for all real numbers and can check that f is Hölder continuous with exponent $1/2$ and constant 2 in the metric defined on $\mathbb{I}_0(\mathbb{R})$ in (C) below.

Associated with f is its "graph", the one dimensional locally integral current

$$S = -\partial[\mathbb{E}^2 \llcorner \{(x,y) : x > y^2\}] \qquad [\text{F } 4.1.7]$$

which can also be expressed

$$S = ([\![\mathbb{I}_{\mathbb{R}}]\!] \bowtie f)_{\#}(\mathbb{E}^1) \qquad [\text{S1 } 2] \; ;$$

here, in general, if $g : A \to B$, $h : A \to C$ then $g \bowtie h : A \to B \times C$ is the obvious mapping sending $a \in A$ to $(g \bowtie h)(a) = (g(a), h(a))$.
The support of S is, of course, the parabola $x = y^2$.

The slicing technique of H. Federer [F 4.3] enables one to recover f from S . In particular, for *each* $x \in \mathbb{R}$,

$$f(x) = \Pi^2_{\#} \langle S, \Pi^1, x \rangle \; ;$$

here $\Pi^i : \mathbb{R}^2 \to \mathbb{R}$ denotes projection onto the i-th coordinate axis for $i = 1,2$. The symbol $\langle S, \Pi^1, x\rangle$ denotes *the slice of* S *by* Π^1 *at* x .

(C) Corresponding to each fixed pair of nonnegative integers P and N we denote by $\mathcal{Q}(P,N;\mathbb{R}^n)$ the subset of $\mathbb{I}_0(\mathbb{R}^n)$ consisting of zero dimensional currents which can be represented

$$[\![p_1]\!] + \ldots + [\![p_P]\!] - [\![q_1]\!] - \ldots - [\![q_N]\!]$$

corresponding to some $p_1,\ldots,p_P$, $q_1,\ldots,q_N \in \mathbb{R}^n$; it is not required that the p_i's or q_j's be distinct so that both multiplicities and cancellations can occur. The function f of (B) above thus takes values in $\mathbf{Q}(1,1;\mathbb{R})$.

The spaces $\mathbf{Q} = \mathbf{Q}(Q,0;\mathbb{R}^n)$ were introduced in [A3] and further studied in [A2]. The more general spaces $\mathbf{Q}(P,N;\mathbb{R}^n)$ were introduced in [S1].

For $T = [\![p_1]\!] + \ldots + [\![p_P]\!] - [\![q_1]\!] - \ldots - [\![q_P]\!] \in \mathbf{Q}(P,P;\mathbb{R}^n)$ one sets

$$\mathbb{F}(T) = \inf\{\Sigma_{i=1}^P |p_i - q_{\sigma(i)}| : \sigma \text{ is a permutation of } \{1,\ldots,P\}\}$$

and more generally defines the $\mathbb{F}$ metric on $\mathbf{Q}(P,N;\mathbb{R}^n)$ by setting $\mathbb{F}(V,W) = \mathbb{F}(V-W)$ whenever $V, W \in \mathbf{Q}(P,N;\mathbb{R}^n)$, noting, of course, that $V-W \in \mathbf{Q}(P+N,\ P+N;\ \mathbb{R}^n)$.

$\mathbf{Q} = \mathbf{Q}(Q,0;\mathbb{R}^n)$ is readily shown to be in explicit bilipschitz correspondence with a finite polyhedral cone $\mathbf{Q}^*$ (of infinite extent) in a higher dimensional Euclidean vectorspace $\mathbb{R}^{NQ}$ [A3 1.2]. $\mathbf{Q}^*$ is additionally a Lipschitz retract of $\mathbb{R}^{NQ}$ [A3 1.3] , a fact which facilitates such operations as extension, interpolation, and smoothing of $\mathbf{Q}$ valued functions which are indispensable in [A3].

On the other hand B. White has observed that whenever P and N are both positive, then $\mathbf{Q}(P,N;\mathbb{R}^n)$ is in bilipschitz correspondence with no subset of any finite dimensional Euclidean vectorspace. He noted, for example, that in $\mathbf{Q}(1,1;\mathbb{R})$ for each $N \in \{1,2,3,\ldots\}$ distinct members of the family

$$\{[\![2k/2N]\!] - [\![(2k+1)/2N]\!] : k = 0,1,\ldots,N-1\}$$

are each distance exactly $1/N$ from each other. Useful representations of $\mathbf{Q}(P,N;\mathbb{R}^n)$ by polyhedral cones remain possible, however, provided one admits metric degeneracies on various faces of the cones [S1 1.12].

(D) We identify $\mathbb{C} = \mathbb{R}^2$, $\mathbb{C}^2 = \mathbb{R}^4$, and further identify the complex algebraic variety $S = \mathbb{C}^2 \cap \{(z,w) : w^2 = z^3\}$ as a two dimensional locally integral current in $\mathbb{R}^4$ in the usual way [F 5.4.19]. S is representable as the graph of the multiple valued function $f : \mathbb{C} \to \mathbf{Q} = \mathbf{Q}(2,0;\mathbb{C})$ defined by setting $f(0) = 2[\![0]\!]$ and

$$f(z) = \Sigma\{[\![w]\!] : w \in \mathbb{C} \text{ with } w^2 = z^3\} \text{ in case } 0 \neq z \in \mathbb{C} \quad .$$

S is a two dimensional absolutely area minimizing surface in $\mathbb{R}^4$ [F 5.4.19] containing $(0,0)$ as an isolated branch point singularity (of real codimension two in S). As discussed in (5) below, f minimizes Dirichlet's integral for $\mathbb{Q}$ valued functions [A3 2.20].

The multiple valued mapping $([\![\mathbb{I}_{\mathbb{C}}]\!] \bowtie f)_\#$ is naturally defined on general flat chains in $\mathbb{C}$ [A3 1.6]. As an example, corresponding to the oriented circle

$$\Gamma = \partial(\mathbb{E}^2 \llcorner \{z : |z| = 1\}) \in \mathbb{I}_1(\mathbb{C}) \quad ,$$

the current $([\![\mathbb{I}_{\mathbb{C}}]\!] \bowtie f)_\# \Gamma \in \mathbb{I}_1(\mathbb{C}^2)$ is a single simple closed oriented curve lying in S and covering Γ twice.

3. THE INTEGRAL GEOMETRIC ESTIMATES OF D. NANCE.

An oriented m dimensional surface S in $\mathbb{R}^{m+n}$ containing a region perpendicular to $\mathbb{R}^m \times \{0\}$ is not naturally the graph of a multiple valued function $f : \mathbb{R}^m \to \mathbb{I}_0(\mathbb{R}^n)$. Such perpendicularity disappears, of course, with a slight rotation of S or, equivalently, a slight rotation of the orthonormal coordinate system with respect to which perpendicularity is determined. For two dimensional surfaces in $\mathbb{R}^3$, D. Nance gives quantitative estimates for multiple valued representations in the following two theorems.

<u>Multiplicity Theorem</u> [N Theorem 1.10]. *Suppose*

(a) $M \subset \mathbb{R}^3$ *is a compact two dimensional submanifold with boundary, of class* 4.

(b) *for* $v \in \mathbb{S}^2$, $p_v : \mathbb{R}^3 \to \mathbb{R}^2$ *denotes orthogonal projection in direction* v, *and*

$$K(v) = \sup\{\mathcal{H}^0[(p_v|M)^{-1}(y)] : y \in \text{Image}(p_v)\} \quad .$$

Then

$$\int_{v \in \mathbb{S}^2} K(v)\, d\mathcal{H}^2 v \leq C \int_M (\|D\nu\|^2 + \|D^2\nu\|)\, d\mathcal{H}^2 + C \int_{\partial M} (\|D\nu\| + \|D\tau\|)\, d\mathcal{H}^1 \quad ;$$

here C *is a constant independent of* M, ν *is the unit normal to* M, *and* τ *is* *the unit tangent to* ∂M .

Hölder Continuity Theorem [N Theorem 5.14]. *Suppose*

(a) $M \subset \mathbb{R}^3$ *is a compact two dimensional submanifold with boundary, of class* 4, *which is non-positively curved.*

(b) *for* $v \in \mathbb{S}^2$ *and with exponent* 1/3 , HC(v) *is the optimal Hölder constant of the multiple valued function representing* M *in direction* v .

Then

$$\int_{v \in \mathbb{S}^2} HC(v)^{\alpha} d\mathcal{H}^2 v < \infty$$

for each $0 < \alpha < 3/238$ *(and, in fact, can be explicitly estimated in terms of surface and boundary derivatives and boundary geometry).*

In determining HC(v) above, the metric $\mathbb{F}$ is suitably adapted to take into account ∂M .

4. AN OPEN PROBLEM ON CONVEXITY.

Suppose $F : \mathbb{R}^{m+1} \to \mathbb{R}^+$ is a norm and $G : \mathbb{R}^{m+1} \to \mathbb{R}^+$ is continuous and suitably nondecreasing at infinity. Then among all regions $W \subset \mathbb{R}^{m+1}$ with $\mathcal{L}^{m+1}(W) = 1$ there will be one for which the combined surface and volume "energy" integral

$$\int_{\partial W} F \circ \nu \, d\mathcal{H}^m + \int_W G \, d\mathcal{L}^{m+1}$$

will be a minimum [A1 VI] [F 5.1.6]; here ν denotes the unit exterior normal vector to W [F 4.5.5]. Similar results hold with W constrained to be, say, convex or constrained to be within a suitable closed subset Z of $\mathbb{R}^{m+1}$. In case G is constant and no constraints are imposed then (for any continuous positive F , in fact) W can be obtained by Wulff's construction and is hence convex and unique up to translations [T1] [T2]. In case F is the Euclidean norm so that the surface integral above gives m dimensional area then for various interesting (typically convex) G's , W can be shown to be convex by Steiner's symmetrizations

[F 2.10.30] which preserve volume and typically decrease surface area.

In the absence of counterexamples it seems plausible that the convexity of both G and Z should imply the convexity of W. Establishing the truth or falsity of such a straightforward speculation does not seem accessible to present techniques - a rather trantalizing state of affairs. The case $m = 1$ is a bit special. For $m = 2$, $G(x,y,z) = Cz$ for various $0 < C < \infty$, and $Z = \{(x,y,z) : z \geq 0\}$ the convexity question seems of substantial physical interest [ATZ]. As noted above one can minimize in the class of convex W, but it is not clear what special properties the solution will have.

Questions of such convexity frequently can be formulated in terms of multiple valued functions. For example, suppose $Z = \emptyset$, G is of class 2, and F is of class 5 and elliptic in the sense of [A1 IV.1(7)] [B Definition 2] [F 5.1.2] then ∂W will be of class 4. In case $m = 2$, the estimates of D. Nance above are applicable so that, with respect to almost every orthonormal coordinate system, ∂W will be the graph of some Hölder continuous $\mathbf{Q}(P,P;\mathbb{R})$ valued function, with the value of P perhaps depending on the coordinate system. The convexity of W is readily checked to be equivalent to being able to take $P = 1$ in a dense set of coordinate systems. The following proposition illustrates one operation which preserves volume and decreases both surface energy and multiplicity.

<u>Proposition</u>.

Hypotheses.

(a) $A \subset \mathbb{R}^m$ *is bounded and open.*

(b) $P \in \{2,3,4,\ldots\}$ *and* $K \in \{1,2,3,\ldots,P\}$.

(c) $g^1,\ldots,g^P,\ h^1,\ldots,h^P : A \to \mathbb{R}$ *and Lipschitz functions of class* 1 *with*
$g^1 > h^1 > g^2 > h^2 > g^3 > \ldots > g^P > h^P$.

(d) $g^0 = h^K + \Sigma_{i=1}^P (g^i - h^i)$.

(e) $W = A \times \mathbb{R} \cap \{(x,z) : h^i(x) < z < g^i(x) \text{ for some } i = 1,\ldots,P\}$.

(f) $W^0 = A \times \mathbb{R} \cap \{(x,z) : h^K(x) < z < g^0(x)\}$.

(g) $\nu : \partial W \cap A \times \mathbb{R} \to \mathbb{S}^m$ *and* $\nu^0 : \partial W^0 \cap A \times \mathbb{R} \to \mathbb{S}^m$

are the respective unit exterior normal vectors.

Conclusions.

(1) $\mathcal{L}^m(W^0) = \mathcal{L}^m(W)$.

(2) $\int_{\partial W^0 \cap A \times \mathbb{R}} F \circ \nu^0 d\mathcal{H}^m \leq \int_{\partial W \cap A \times \mathbb{R}} F \circ \nu \, d\mathcal{H}^m$ *with strict inequality in case* F *is uniformly convex.*

<u>Proof</u>. Conclusion (1) follows from Fubini's theorem. With regard to conclusion (2) we use the positive homogeneity of F to estimate

$$\int_{\partial W \cap A \times \mathbb{R}} F \circ \nu \, d\mathcal{H}^m = \Sigma_{i=1}^P \int_A F(-\partial g^i/\partial x_1, \ldots, -\partial g^i/\partial x_m, +1) d\mathcal{L}^m$$

$$+ \Sigma_{i=1}^P \int_A F(+\partial h^i/\partial x_1, \ldots, +\partial h^i/\partial x_m, -1) d\mathcal{L}^m \quad ,$$

$$\int_{\partial W^0 \cap A \times \mathbb{R}^n} F \circ \nu^0 d\mathcal{H}^m = \int_A F(-\partial h^K/\partial x_1 - \Sigma_{i=1}^P (\partial g^i/\partial x_1 - \partial h^i/\partial x_1), \ldots,$$

$$- \partial h^K/\partial x_m - \Sigma_{i=1}^P (\partial g^i/\partial x_m - \partial h^i/\partial x_m), + 1) d\mathcal{L}^m$$

$$+ \int_A F(+\partial h^K/\partial x_1, \ldots, +\partial h^K/\partial x_m, -1) d\mathcal{L}^m \quad .$$

One then notes

$$(-\partial h^K/\partial x_1 - \Sigma_{i=1}^P (\partial g^i/\partial x_1 - \partial h^i/\partial x_1), \ldots, - \partial h^K/\partial x_m - \Sigma_{i=1}^P (\partial g^i/\partial x_m - \partial h^i/\partial x_m), + 1)$$

$$= \Sigma_{i=1}^P (-\partial g^i/\partial x_1, \ldots, - \partial g^i/\partial x_m, + 1)$$

$$+ \Sigma_{i=1,\ldots,K-1,K+1,\ldots,P} (+\partial h^i/\partial x_1, \ldots, + \partial h^i/\partial x_m, - 1)$$

and uses the fact that F is a norm to infer conclusion 2.

In the terminology of the proposition $\partial W \cap A \times \mathbb{R}$ is the graph of the function

$$f = [\![g^1]\!] + \ldots + [\![g^P]\!] - [\![h^1]\!] - \ldots - [\![h^P]\!] : A \to \mathbf{Q}(P,P;\mathbb{R})$$

while $\partial W^0 \cap A \times \mathbb{R}$ is the graph of the function

$$f^0 = [\![g^0]\!] - [\![h^K]\!] : A \to \mathbf{Q}(1,1;\mathbb{R})$$

so that W^0 is "more nearly convex" than W.

Although in almost coordinate system a minimizing W can over almost every point in $\mathbb{R}^m$ be represented locally in the manner of hypothesis (e) of the proposition, when $m \geq 2$ there seems no *a priori* way to guarantee that the various h^K's fit together suitably, e.g. suppose W were two disjoint balls with overlapping (but not totally so) projections.

5. THE AREA INTEGRAND, DIRICHLET'S INTEGRAND, AND THE Q-ELLIPTIC INTEGRANDS OF P. MATTILA.

The powerful closure, compactness, and lower semicontinuity theorems of geometric measure theory commonly guarantee rectifiable current or rectifiable set solutions to elliptic variational problems such as the problem of least area. [F 5.1.6] [A1 VI]. Since such rectifiable currents or sets admit approximate tangent planes almost everywhere one is able almost everywhere to study extremal currents or sets in an orthonormal coordinate system (determined by the approximate tangent plane direction, of course) with respect to which most nearby approximate tangent planes are nearly horizontal. The effect of doing this in practice is to enable one to study nearby current structure as though it were nearly the graph of a function minimizing the integral of the nonparametric integrand associated with the particular coordinate system. The nonparametric area integrand $M(p)$ is, of course, the square root of the sum of the squares of the determinants of all $m \times m$ minors of the $(m+n) \times m$ matrix

$$\begin{bmatrix} p_j^i \\ 1\!\!1_{\mathbb{R}^m} \end{bmatrix}$$

which is also expressible.

$$M(p) = 1 + 2^{-1} \Sigma_{i,j} (p_j^i)^2 + \mathbb{O}(|p|^4) \quad .$$

For small values of p , i.e. small slopes, the dominant relevant part of $M(p)$ is one half of Dirichlet's integral

$$\mathrm{Dir}(p) = \Sigma_{i,j} (p_j^i)^2 \quad .$$

Single valued extremals of the associated Dirichlet's integral are, of course, harmonic functions for which numerous *a priori* estimates are available.

Suppose (as a result of coordinate system selection) most tangent planes of an m area minimizing current are nearly horizontal. In case, say, the minimizing current covers a base m disk $\mathbb{U}^m$ once algebraically then one can represent most of the current as the graph of a Lipschitz function $\mathbb{U}^m \to \mathbb{R}^n$ which is weakly nearly harmonic. Current regularity is then inferred ultimately with substantial use of *a priori* estimates for harmonic functions mentioned above.

In case the base disk is covered Q times algebraically similar approximation by a Lipschitz function $\mathbb{U}^m \to \mathbb{Q} = \mathbb{Q}(Q,0;\mathbb{R}^n)$ is possible [A2] [A3 3.28, 3.29] (ultimately one requires a much more delicate curvilinear approximation [A3 4.33]) and one is led to seek information about $\mathbb{Q}$ valued functions minimizing the integral of Q-fold Dirichlet's integrand (defined in a natural and straightforward way). We have, in particular, the following.

<u>Existence and regularity of Dirichlet integral minimizing $\mathbb{Q}$ valued functions</u> [A3 2.2, 2.14]. *For each appropriate function* $g : \partial\mathbb{U}^m \to \mathbb{Q}$ *there exists a (strictly defined but not necessarily unique) function* $f : \mathbb{U}^m \to \mathbb{Q}$ *having boundary values* g *and of least Dirichlet's integral among such functions. Furthermore, each such minimizing* f *is locally Hölder continuous and* $\mathbb{U}^m \times \mathbb{R}^n \cap \{(x,y) : : y \in \mathrm{spt}(f(x))\}$ *is an* m *dimensional real analytic (harmonic) submanifold of* $\mathbb{U}^m \times \mathbb{R}^n$ *except for a closed set of Hausdorff dimension not exceeding* $m-2$.

As examples, for m and n even integers and with the usual complex identifications, the $\mathbb{Q}$ valued function produced by projection mapping slicing of a complex holomorphic chain in $\mathbb{U}^m \times \mathbb{R}^n$ associated with a Q-fold analytic branched covering of $\mathbb{U}^m$ is Dirichlet integral minimizing [A3 2.20]. Our Hausdorff codimension two singularity estimate above is thus the best possible.

For area minimizing surfaces one ultimately concludes the regularity theorem,

Interior regularity theorem for oriented area minimizing surfaces [A3 5.22]. *Suppose* N *is an* $m+\ell$ *dimensional submanifold of* $\mathbb{R}^{m+n}$ *of class* $k+2 \geq 5$ *and that* T *is an* m *dimensional rectifiable current in* $\mathbb{R}^{m+n}$ *which is absolutely area minimizing with respect to* N. *Then there is an open subset* U *of* $\mathbb{R}^{m+n}$ *such that* $\operatorname{spt} T \cap U$ *is an* m *dimensional minimal submanifold of* N *of class* k *and the Hausdorff dimension of* $\operatorname{spt} T \sim (U \cup \operatorname{spt} \partial T)$ *does not exceed* $m-2$.

For parametric integrands other than area, approximation similar to that above leads one to study $\mathbb{Q}$ valued functions minimizing integrals of more general quadratic integrands. The fundamental notion of *Q-ellipticity* has been isolated by P. Mattila who additionally has obtained lower semicontinuity, existence and regularity results in this context [M].

6. B. SOLOMON'S NEW PROOF OF THE CLOSURE THEOREM.

Spaces of Lipschitz $\mathbb{Q}(P,N;\mathbb{R}^n)$ functions were introduced and studied by B. Solomon who, among other things, established a Lipschitz extension theorem [S1 1.00] analogous to Kirszbraun's theorem and showed that Lipschitz functions $\mathbb{U}^m \to \mathbb{Q}(P,N;\mathbb{R}^n)$ have naturally defined integral current "graphs" [S1 2]. The main thrust of Solomon's work in [S1] [S2] was a new proof of the closure theorem [F 4.2.16] [S1 3.00] [S2] independent of H. Federer's structure theorem for sets of finite Hausdorff measure [F 3.3.13]. The closure theorem is, of course, one of the cornerstones of the applications of geometric measure theory to the calculus of variations. Although very considerably more was involved, Lipschitz $\mathbb{Q}$ valued functions were essential in Solomon's proof.

Some alternative multiple valued function approximation and extension techniques are now available which, among other things, work equally well for flat chains modulo ν [A4].

7. B. WHITE'S TANGENT CONE UNIQUENESS PROOF FOR TWO DIMENSIONAL AREA MINIMIZING SURFACES.

Among other applications of multiple valued functions in the calculus of variations we note B. White's use of "harmonic" functions $\mathbb{U}^2 \to \mathbb{Q}$ to generate comparison surfaces for nearly flat two dimensional area minimizing surfaces [W 2.1] as a means of proving an epiperimetric inequality sufficient to give his tangent cone uniqueness.

It is possible that multiple valued function techniques can provide insight into more general questions of tangent cone uniqueness. In particular, non-unique tangent cones to minimal surfaces require unbounded projection multiplicities in an open set of directions.

8. COMPUTATIONAL ALGORITHMS FOR GENERATING MINIMAL SURFACES.

The general existence and regularity (both in the interior and at the boundary) of oriented surfaces S of least area is well in hand with *a priori* derivative estimates of all orders determined solely by boundary data [F 5.1.6, 5.4.15] [HS 0, 12.1] [N 5.15, 5.16]. One is thus able to use Nance's estimates discussed above to infer for most orthonormal coordinate systems *a priori* multiplicity supremum estimates (i.e., which values of P and N to use in $\mathbb{Q}(P,N;\mathbb{R})$) and *a priori* modulus of continuity estimates for functions $\mathbb{R}^2 \to \mathbb{Q}(P,N;\mathbb{R})$ having such S as graph (i.e., the Hölder constant associated with Hölder exponent $1/3$). With such information in hand one is encouraged to attempt to develop machine algorithms to compute and display surfaces of least area in $\mathbb{R}^3$. One such algorithm has been constructed and its implementation [SU] (in the programming language Pascal -- copies are available by writing the second author) has allowed us to see pictures of minimal surfaces on a computer graphics screen and on paper and observe minimal surface phenomena directly from various vantage points. One can see, for example, the effect of varying the boundary on the shape of the minimal surface spanning it.

The input of the present algorithm is a set of boundary curves constrained to lie on the surface of a cylinder. The algorithm initially generates (randomly) a topologically complex surface out of a number of simple nonparametric surfaces or sheets spanning the various boundary curves and stacked in the cylinder with alternating orientations. In particular, regions in which sheets of opposite orientation coincide are cancelled, creating holes, and the edges of these holes are then glued together. Doubling back of boundary curves is similarly generated by orientation cancellation.

The algorithm approximates smooth surfaces by polyhedral ones with triangular faces. Area minimization within the algorithm is accomplished by moving each vertex in the surface by a fixed amount in a direction (not necessarily vertical) which strictly decreases surface area. When no vertex can be moved any longer, the fixed amount by which vertices are moved is decreased and the process is repeated until the desired precision is achieved.

This minimization works effectively only if the triangulation is reasonably uniform, i.e., the triangles are all approximately the same size and nearly equilateral. It turns out that the minimization procedure frequently produces triangulations which are not uniform, even from very uniform initial triangulations. The complete algorithm, however, maintains a uniform triangulation by two methods. When vertices flow too close to each other, causing a clustering or "compression of the material", a sink is created to eliminate some of the material; this is done by merging vertices which come too close to each other into a single vertex. When triangles grow too large causing the surface to "stretch" too much a source is created to generate new material; this is done by subdividing large triangles into smaller ones. We do not know bounds for the amount of source-sink activity which can occur during implementation. Figures 1, 2, 3, 4, 5 illustrate operation of the algorithm.

Figure 1A [resp. 2A] shows initial oriented boundary curves encircling the cylinder. Figure 1B [resp. 2B] shows the same boundary curves after cancellations. Figure 1C [resp. 2C] shows an initial surface made up of stacked sheets of alternating orientations after cancellations. Figure 1D [resp. 2D] shows a final surface with minimized surface area.

Figure 3A shows initial boundary curves after cancellations and figure 3B an initial surface after cancellations. Figure 3C shows an intermediate stage of minimization while figures 3D and 3E are different views of the final minimal surface.

Figures 4ABCDEFG illustrate various stages of area minimization in which a collapse of two handles occurs, while figures 5ABCDE show various views of classical catenoid and coaxial catenoid surfaces generated by the algorithm.

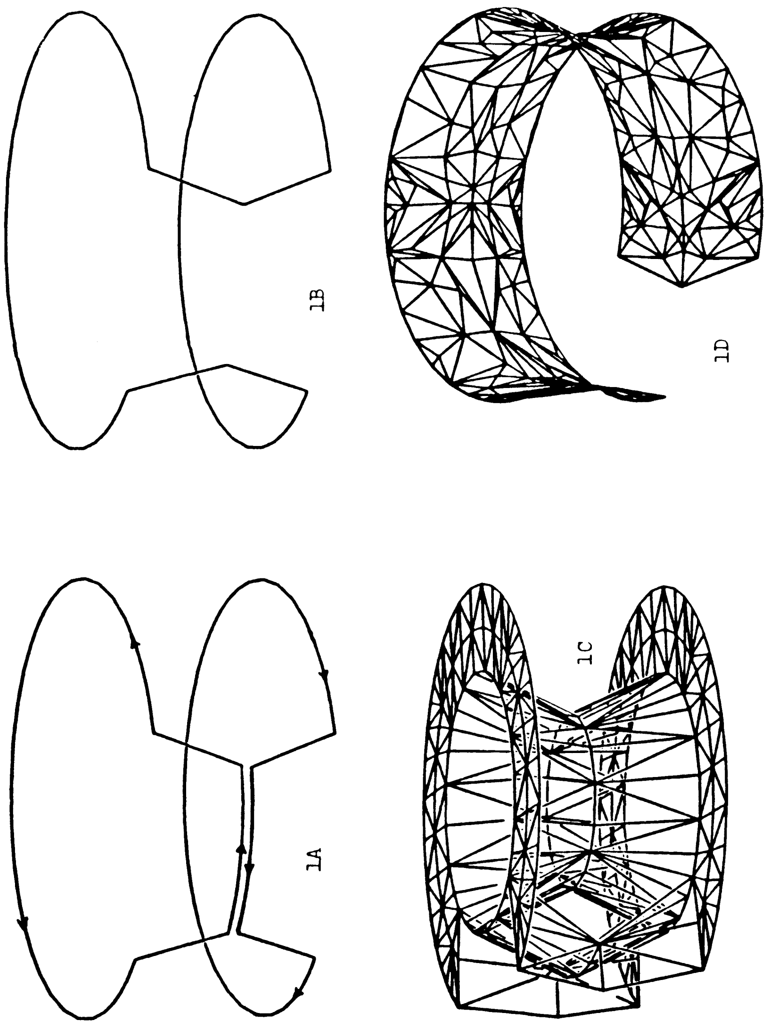

Figure 1

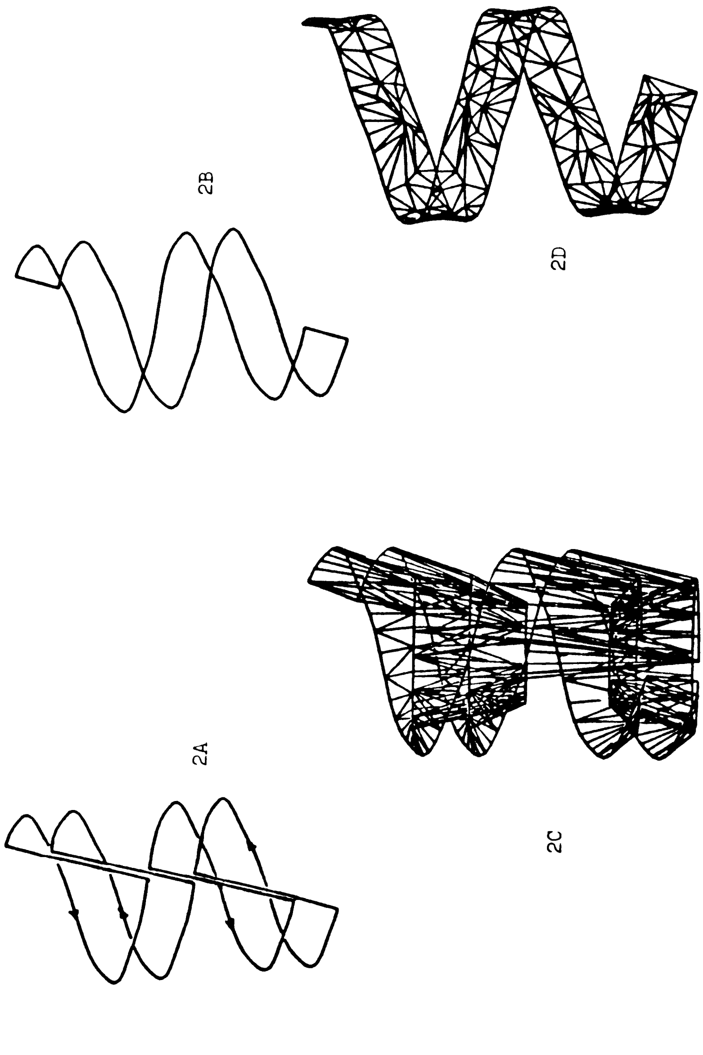

Figure 2

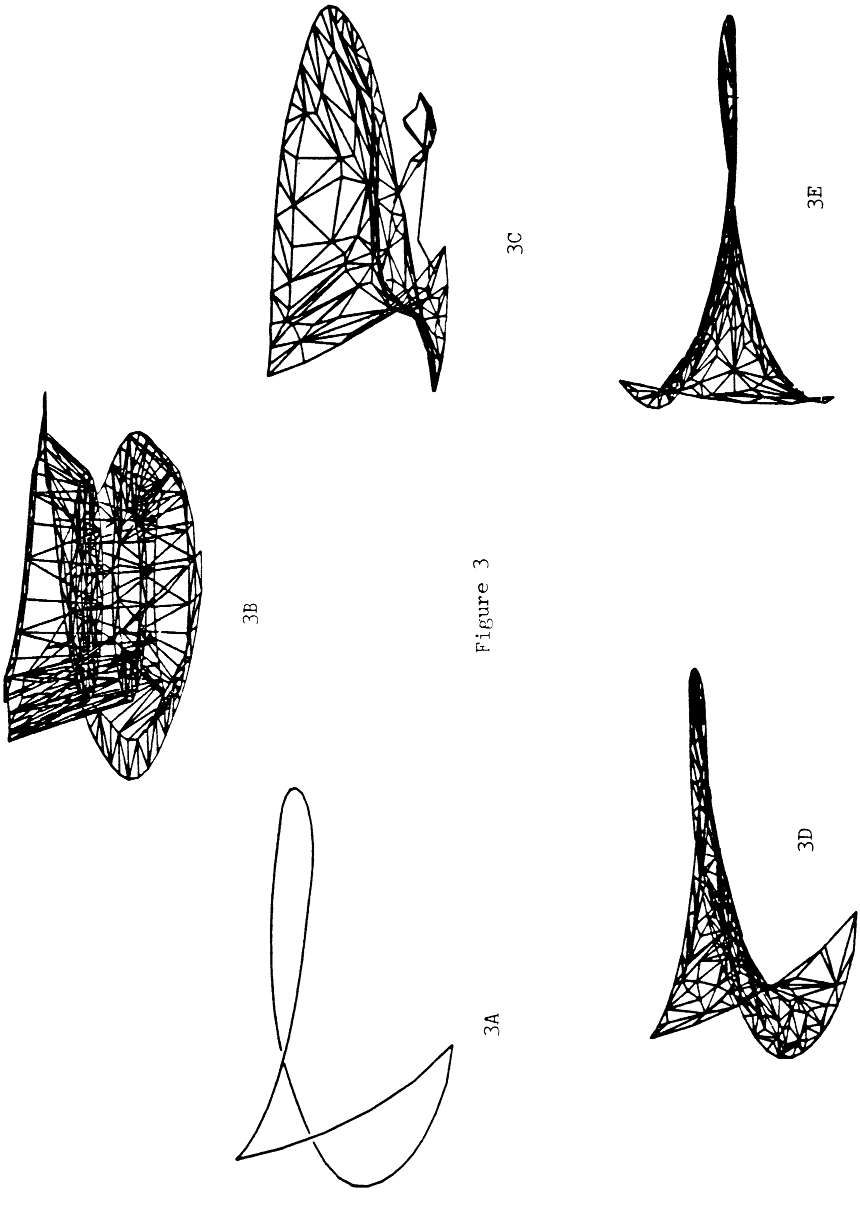

Figure 3

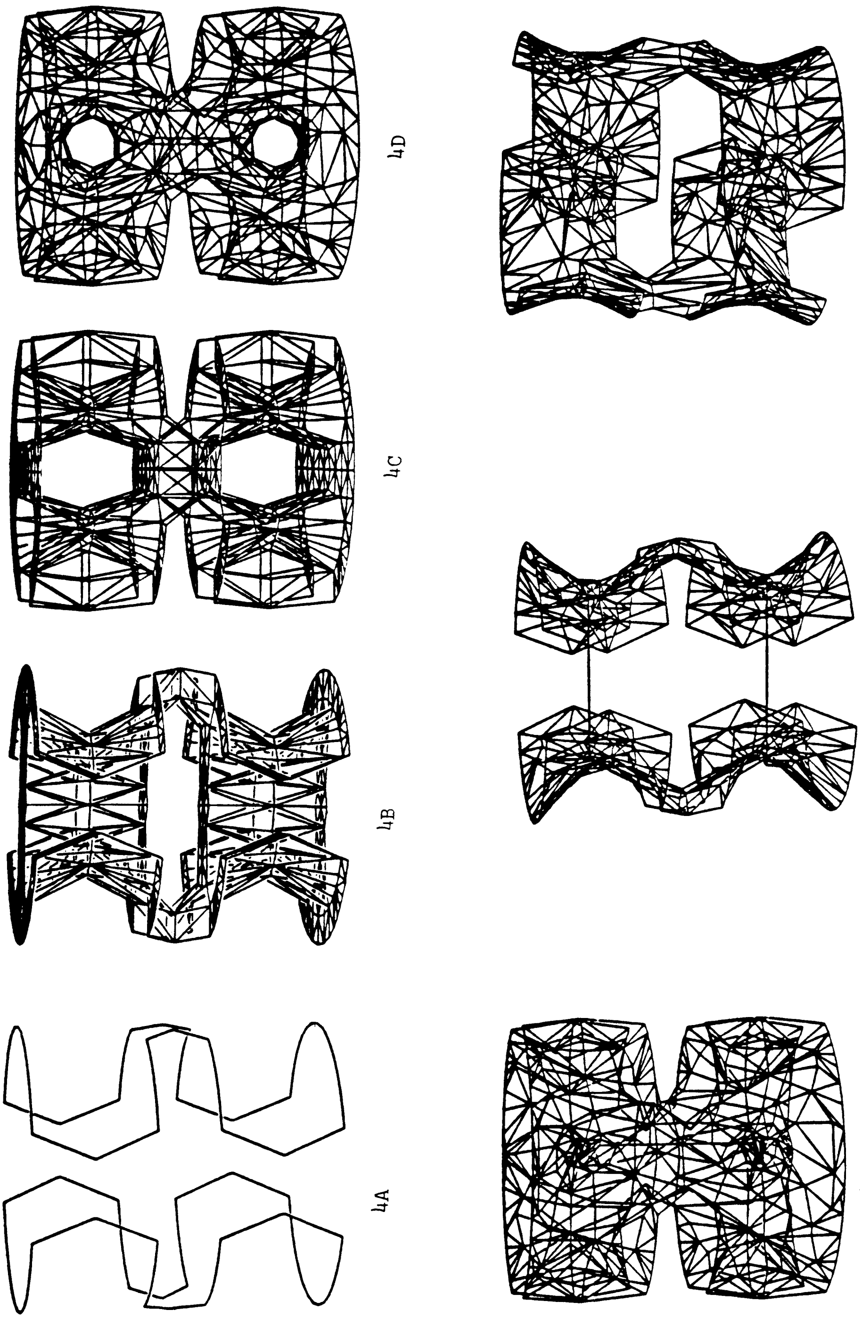

Figure 4

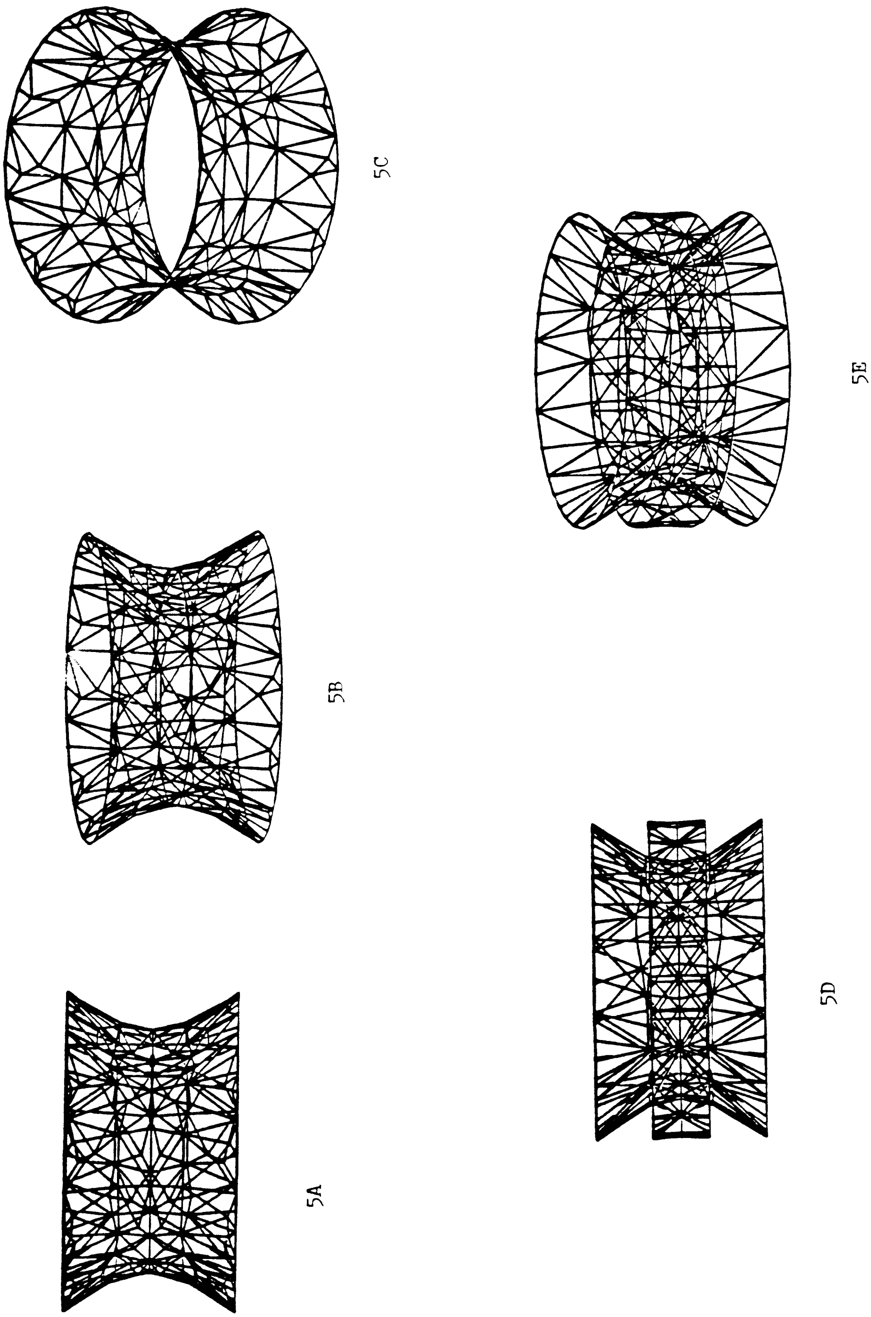

Figure 5

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