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JEAN MOULIN OLLAGNIER

DIDIER PINCHON

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A NOTE ABOUT HEDLUND'S THEOREM

Jean MOULIN OLLAGNIER
Didier PINCHON

In this note, we propose a new proof of Hedlund's theorem for an arbitrary group without using skew-products.

Let X be a compact space and T a minimal homeomorphism of X . Hedlund proved the following theorem (1)

Theorem

Let f be a continuous function on X . If the sums $\sum_{i=0}^{n-1} f \circ T^{-i}$ are uniformly bounded there exists a continuous function F such that

$$f = F \circ T^{-1} - F$$

One can remark that the sums of iterates appearing in this theorem are the values of a cocycle for the powers of transformation T . More generally, let X be a compact space and G a group of homeomorphisms of X . An application V from G to the space $C(X)$ of continuous functions on X ($g \mapsto V_g$) is said to be a cocycle if

$$\forall h, g \in G \quad V_{gh} = V_g + V_h \circ g^{-1}$$

Cocycles appear in the study of quasi-invariant probabilities (2) and in the construction of skew-products.

Theorem

Let X be a compact space and G a group of homeomorphisms of X whose action is minimal i.e. every orbit is dense.

Let V be a cocycle on X for the action of G .

If the functions V_g are uniformly bounded there exists a continuous function F such that

$$\forall g \in G \quad V_g = F \circ g^{-1} - F$$

Proof

1. Let a be a point of X and $O(a)$ its orbit under G (which is dense).

Let us define on $O(a)$ a function f by $f(g^{-1}a) = V_g(a)$

We can do it because $V_g(a)$ does not depend on g but only on the point $g^{-1}a$ of $O(a)$:

If $g^{-1}a = k^{-1}a$, $k = sg$ where s belongs to the group $St(a)$ of all s such that $s(a) = a$.

Then
$$V_{sg}(a) = V_s(a) + V_g(a)$$

But the application from $St(a)$ to \mathbb{R} , $s \mapsto V_s(a)$, is a group homomorphism and from the hypothesis of boundedness is the zero homomorphism.

The function f is bounded.

2. Let F the lower semi-continuous function defined by

$$F(x) = \inf_{y \rightarrow x, y \in O(a)} \lim f(y)$$

F is bounded.

Let us show that the increment of F is V i.e.

$$\forall h \in G \quad F \circ h^{-1} - F = V_h$$

This relation is true for f for all points in $O(a)$:

$$f(h^{-1}b) - f(b) = V_{gh}(a) - V_g(a) = V_h(g^{-1}a) = V_h(b) \quad \text{where } b = g^{-1}a$$

$$\text{Thus} \quad f(h^{-1}b) = f(b) + V_h(b) \quad \forall b \in O(a) \quad (1)$$

$$F(h^{-1}x) = \inf_{y \rightarrow h^{-1}x, y \in O(a)} \lim f(y) = \inf_{b \rightarrow x, b \in O(a)} \lim f(h^{-1}b)$$

with $b = hy$, because h is continuous.

Taking the inf limit in (1)

$$F(h^{-1}x) \geq F(x) + V_h(x)$$

and

$$\forall x \in X \quad \forall h \in G \quad F(h^{-1}x) - F(x) \geq V_h(x)$$

$$\text{Changing } h \text{ in } h^{-1} \text{ and } x \text{ in } h^{-1}x : \quad F(hh^{-1}x) - F(h^{-1}x) \geq V_h(h^{-1}x)$$

$$\text{So} \quad F(h^{-1}x) - F(x) \leq V_h(x) \quad \text{Whence the equality.}$$

3. Let OF be the oscillation function of F

$$OF(c) = \limsup_{x, y \rightarrow c} F(x) - F(y)$$

OF is a bounded upper semi-continuous function.

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$$OF(h^{-1}c) = \sup_{x,y \rightarrow c} \lim F(h^{-1}x) - F(h^{-1}y) \quad \text{for } h^{-1} \text{ is continuous.}$$

But as $F(x) - F(y) = F(h^{-1}x) - F(h^{-1}y) - (V_h(x) - V_h(y))$

Taking the sup limit $OF(c) \leq OF(h^{-1}c)$ because the oscillation of the continuous function V_h is zero.

That shows that the function OF is invariant under G .

By minimality, an upper semi-continuous function invariant is a constant.

4. To complete the proof of the theorem it remains to show that the constant function OF is zero. This result comes from the lemma :
 Let g a lower semi-continuous function on a topological space E , Og its oscillation. If $Og(a)$ is finite then $\inf \lim_{x \rightarrow a} Og(x) = 0$

Proof g being a l.s.c. function

$$Og(x) = \sup_{y \rightarrow x} \lim g(y) - \inf_{y \rightarrow x} \lim g(y) = \sup_{y \rightarrow x} \lim g(y) - g(x)$$

If $Og(a)$ is finite there exists an open neighbourhood V_o of a in which g is bounded. So that for every open neighbourhood $V \subset V_o$, for every ϵ , there exists an x_V in V with

$$g(x_V) \geq \sup_{y \in V} g(y) - \epsilon$$

Then

$$Og(x_V) \leq \sup_{y \in V} g(y) - g(x_V) \leq \epsilon \quad \text{q.e.d.}$$

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Jean MOULIN OLLAGNIER
 Département de Mathématiques
 Université Paris-Nord
 Avenue J.B. Clément
 93430 VILLETANEUSE
 FRANCE

Didier PINCHON
 Laboratoire de Probabilités
 Université Paris 6
 4, place Jussieu
 75230 PARIS CEDEX 05
 FRANCE