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ON INVARIANT MEASURES ON THE SPACE
 OF BILATERAL SEQUENCES

Jan Kwiatkowski

Let $S = \{1, 2, \dots, s\}$, $X = \prod_{-\infty}^{+\infty} S$. Denote by \mathcal{B} the σ -algebra generated by the cylinder sets and by T the shift transformation on X . For $n \geq 1$, we put $X_n = \prod_1^n S$. An element $B = (i_1, i_2, \dots, i_n) \in X_n$ is called a block. We shall identify B with the cylinder $\{x \in X ; x_0 = i_1, x_2 = i_2, \dots, x_{n-1} = i_n\}$. Let us denote $M(X)$ the set of all T -invariant measures on \mathcal{B} . For given $\mu \in M(X)$ we define μ_n on X_n as $\mu_n(B) = \mu(B)$. The measure μ_n may be considered as a point of the space R^{S^n} in this sense that the coordinates of μ_n are indexed by the blocks B , and B -th coordinate of μ_n is equal $\mu_n(B)$.

Denote by K_n the set of all vectors of the form $\langle \mu_n(B) \rangle_{B \in X_n}$, when μ runs over all invariant measures on X . The set K_n may be described by the following conditions :

$$\begin{cases} \sum_{i=1}^s \mu_n(iC) = \sum_{i=1}^s \mu_n(C_i) & , \quad C \in X_{n-1} \quad (n > 1) \\ \sum_{B \in X_n} \mu_n(B) = 1 & , \quad \mu_n(B) \geq 0 \end{cases}$$

In this way K_n is polygon in R^{S^n} . It is easy to see that $\dim K_n = S^{n-1}(S-1)$.

Now, we define a mapping $f_n : K_{n+1} \longrightarrow K_n$ as follows :

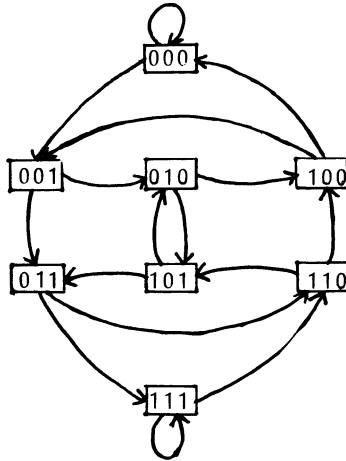
$$(f_n \mu_{n+1})(B) = \sum_{i=1}^S \mu_{n+1}(iB) ,$$

for $B \in X_n$ and $\mu_{n+1} \in K_{n+1}$. We obtain the sequence of the polygons K_n and the sequence of the mapping f_n

$$K_1 \xleftarrow{f_1} K_2 \xleftarrow{f_2} K_3 \xleftarrow{f_3} \dots$$

The set $M(X)$ may be identified with $\varprojlim K_n$.

My first aim is to describe the extremal points of the polygons K_n . In order to do this we use a graph Y_n defined as follows : the vertices of Y_n are the blocks C of the length $n-1$, and two blocks $C_1 = (i_1, i_2, \dots, i_{n-1})$, $C_2 = (j_1, j_2, \dots, j_{n-1})$ are joined by oriented edge iff $(i_2, \dots, i_{n-1}) = (j_1, \dots, j_{n-2})$. If $S = \{0, 1\}$ and $n = 4$ then we have



Y_4

We remark that the edges of Y_n one can identify with the blocks of the length n . Let $\gamma = \{B_1, B_2, \dots, B_{\mathbf{1}}\}$ ($1 \leq \mathbf{1} \leq S^{n-1}$) be a closed path in Y_n such that γ does not contain any loop. Define $\bar{p}\gamma \in K_n$:

$$P_{\gamma}(B) = \begin{cases} \frac{1}{l_{\gamma}} & , \quad B \in \gamma \\ 0 & , \quad B \notin \gamma \end{cases}$$

Theorem 1.

The vector $\mu_n \in K_n$ is an extremal point of K_n iff $\mu_n = \bar{p}_{\gamma}$ for some closed path γ not having any loop.

Now, I would like to describe extremal points of K_n that may be used in a decomposition of Bernoulli measures. We say that two blocks $B, \bar{B} \in X_n$ are ϕ -equivalent iff $\bar{B} = (i_t i_{t+1} \dots i_n i_1 \dots i_{t-1})$ for some $1 \leq t \leq n$, where $B = (i_1 i_2 \dots i_n)$. It is easy to verify that ϕ is an equivalence relation and therefore ϕ induce a partition on disjoint subsets of X_n . Let \mathcal{L}_n be the set of all classes of X_n . If $\gamma \in \mathcal{L}_n$, $\gamma = \{B_1, B_2, \dots, B_{l_{\gamma}}\}$, then l_{γ}/n and γ is a closed path not having any loop.

Take a Bernoulli measure μ given by a probability vector $q = (q_1, q_2, \dots, q_s)$. Then we have

$$\mu_n = \sum_{n_1 + \dots + n_s = n} q_1^{n_1} q_2^{n_2} \dots q_s^{n_s} \sum_{\substack{\gamma \in X(n_1, \dots, n_s) \\ \gamma \in \mathcal{L}_n}} l_{\gamma} \cdot \bar{p}_{\gamma} ,$$

Where n_1, n_2, \dots, n_s is non-negative integers, $X(n_1, \dots, n_s)$ is the set of all blocks B of the length n with the frequency of symbols (n_1, n_2, \dots, n_s) , and l_{γ} is the length of γ .

Note that if $\gamma \in \mathcal{L}_n$ then $\gamma \subset X(n_1, n_2, \dots, n_s)$ since the relation ϕ does not change of the numbers of symbols.

Let L_n be the convex hull spanned by the extremal points \bar{p}_{γ} , $\gamma \in \mathcal{L}_n$. Now, we shall describe all invariant measures μ satisfying conditions $\mu_n \in L_n$ for $n = 1, 2, \dots$

Let $T_S = \{(x_1, x_2, \dots, x_S) ; \sum_{i=1}^S x_i = 1, x_i \geq 0\}$. Let us consider a probability Borel measure $\bar{\mu}$ on T_S . The measure $\bar{\mu}$ allows to define a measure $\mu \in M(X)$ as follows

$$(*) \quad \mu(B) = \int_{T_S} x_1^{n_1} \dots x_S^{n_S} \bar{\mu}(dx), \quad x = (x_1, x_2, \dots, x_S) \in T_S,$$

where B is a block with frequency of symbols (n_1, n_2, \dots, n_S) .

Theorem 2.

Let $\mu \in M(X)$. Then $\mu_n \in L_n$, $n = 1, 2, \dots$, iff there exists a Borel probability measure $\bar{\mu}$ on T_S such that μ has the form (*)

We shall write $\mu = \phi(\bar{\mu})$.

Consider the dynamical system $Z(\bar{\mu}) = (X, \psi(\bar{\mu}), T)$.

Put $H(x_1, x_2, \dots, x_S) = - \sum_{i=1}^S x_i \log x_i$, $(x_1, \dots, x_S) \in T_S$. Then we have $\int_{T_S} H d\bar{\mu} = \langle \phi, \log \phi \rangle = I_S$. Let $\xi = H^{-1}(\epsilon)$, where ϵ is the partition of I_S on points. Therefore the measure $\bar{\mu}$ determines the measure ν on I_S and a conditional measures $\bar{\mu}_a$, $a \in I_S$. Let $m_n(a)$ be the type of $\bar{\mu}_a$ that is $\{m_n(a)\}$ is a sequence of measurable functions on I_S such that $\sum_{n=1}^{\infty} m_n(a) \leq 1$, $m_{n+1}(a) \leq m_n(a)$, $m_n(a) \geq 0$ for $n = 1, 2, \dots$ and for a.e. $a \in I_S$.

We obtain a pair $\theta(\nu, \{m_n(a)\}_{a \in I_S}) = \theta(\bar{\mu})$.

Theorem 3.

Given two Borel probability measures $\bar{\mu}_1, \bar{\mu}_2$ on T_S . The dynamical systems $Z(\bar{\mu}_1), Z(\bar{\mu}_2)$ are isomorphic iff $\theta(\bar{\mu}_1) = \theta(\bar{\mu}_2)$.

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