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SOME RESULTS ON EXPANDING MAPPINGS

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In this note we give some theorems concerning smooth invariant measures for expanding mappings. They complete the results from [3] and [4] .

The following notation and terminology will be used:

M - a compact, connected C^{∞}-manifold,

$\|\cdot\|$ - a C^{∞}-Riemannian metric on M ,

$\mathfrak{B}$ - the family of all Borel subsets of M,

$\mathfrak{M}_r$ - the space of all normalized C^r-measures on M $(r = 0,1,\dots \infty, \omega)$ with C^r-topology $(r = 0,1,\dots \infty)$, 1)

E_r - the space of all expanding C^r-mappings of M $(r = 1,2,\dots \infty, \omega)$ with C^r-topology $(r = 1,2,\dots \infty)$, i.e. all C^r-mappings $\varphi : M \to M$ for which there exist $a > 0$ and $b > 1$ such that $\|D\varphi^n(\sigma)\| \geq ab^n \|\sigma\|$ for $\sigma \in T(M)$ and $n \in N$, 2)

μ - a fixed element of $\mathfrak{M}_{r-1}$ $(r = 1,\dots \infty, \omega)$,

$J\varphi$ - the positive C^{r-1}-function on M such that if $\varphi | A$ is injective, then $\mu(\varphi(A)) = \int_A J\varphi \, d\mu$,where $A \in \mathfrak{B}$ $(\varphi \in E_r)$,

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1),2) If $r = \omega$, then one assumes that M and $\|\cdot\|$ are of class C^{ω} .

U_{φ} - the mapping of $L(\mu)$ into itself such that

$$U_{\varphi}(f)(x) = \sum_{\bar{x}\in \varphi^{-1}(x)} f(\bar{x})(J\varphi(\bar{x}))^{-1} \quad \text{for} \quad x\in M$$

$(\varphi \in E_1)$,

V^n_{φ} - the mapping of $L(\mu)$ into itself such that

$V^n_{\varphi}(f) = n^{-1}U_{\varphi^n}(f \ln J\varphi^n)$ $(n\in N, \ \varphi\in E_1)$,

$C^k(M,\mathbb{R})$ - the space of all real C^k-functions on M $(k = 0,1,\ldots)$,

$\|\cdot\|_k$ - the norm on $C^k(M,R)$,

$C^{k,\alpha}(M,\mathbb{R})$ - the space of all real C^k-functions on M whose k-th derivatives satisfy a Hölder condition with exponent α $(k = 0,1,\ldots, \ \alpha \in \,]0,1]\,)$,

$\|\cdot\|^{\circ}_{k,\alpha}$ - the semi-norm on $C^{k,\alpha}(M,\mathbb{R})$ defined by means of the Hölder constant of k-th derivatives,

$\|\cdot\|_{k,\alpha}$ - the norm $\max(\|\cdot\|_k, \|\cdot\|^{\circ}_{k,\alpha})$ on $C^{k,\alpha}(M,\mathbb{R})$

In the sequel $r = 2,\ldots,$ $\alpha\in \,]0,1]$ and in Theorem 1 $r = 2,\ldots \infty, \omega$. The first part of Theorem 1 and the second one for $r = 2$ is from [3] and [5] .

Theorem 1.

i) For each $\varphi \in E_r$ there exists a unique φ-invariant measure $\mu_{\varphi} \in \mathfrak{M}_{r-1}$

ii) If $r \neq \omega$, then the mapping $E_r \ni \varphi \longrightarrow \mu_{\varphi} \in \mathfrak{M}_{r-1}$ is continuous.

The first author's proof of Theorem 1 $(r \neq \omega)$ was based on the Banach's theorem on contraction, the Rademacher's theorem on differentiability of Lipschitz functions and the following

Proposition. Suppose that h is a real C^k-function $(k = 0,1,\ldots)$ from an open subset A of $\mathbb{R}^m$ such that $D^k h$ is Lipschitz. Let H be a continuous mapping from A to the space of $(k+1)$-linear functionals on $\mathbb{R}^m$ such that $D^{k+1}h = H$ almost everywhere. Then h is of class C^{k+1} and $D^{k+1}h = H$.

We now outline this proof for $r = 2$. One first shows that there exists a function $g_\varphi \in C^{0,1}(M,\mathbb{R})$ such that the measure $\mu_\varphi \in \mathfrak{M}_0$ with the density g_φ is φ-invariant (see b) below). The function g_φ satisfies the equation

1) $U_\varphi(g) = g$.

Therefore Dg_φ (which exists μ-almost everywhere by the Rademacher's theorem) satisfies the equation

2) $A_\varphi(f) = f$

obtained by differentiating both sides of 1) and replacing g and Dg by g_φ and f respectively. Using the Banach's theorem on contraction one proves that 2) has a unique solution in the space of continuous sections of $T^*(M)$ as well as in the space of essentially bounded ones. This and Proposition imply that $g_\varphi \in C^1(M,\mathbb{R})$. It is easy to see that in view of b) the above reasoning also gives ii). This method can also be used for $r > 2$ but is complicated in technical respect.

Let us remark that the above method can be used in proof of the stable manifold theorem for hyperbolic sets.

In the second author's proof one first shows the following

Theorem 2. For each $\varphi \in E_r$ and each α there exist a neighbourhood $\mathcal{V}$ of φ in E_r, numbers $\delta > 0$ and $q \in]0,1[$ such that for $\psi \in \mathcal{V}$ and $n \in N$

$$\| \bar{U}^n_\psi \|_{r-2,\alpha} \leq \delta q^n .$$

In the above theorem for $\varphi \in E_r$ and $n \in N$ $\bar{U}^n_\varphi$ denotes the operator of $L(\mu)$ into itself such that $\bar{U}^n_\varphi(f) = U_{\varphi^n}(f) - g_\varphi \int_M f \, d\mu$, where $g_\varphi \in C^{r-2,\alpha}(M,\mathbb{R})$ is the density of the φ-invariant normalized measure μ_φ with respect to μ. $\bar{U}^n_\varphi$ also acts in $L^p(\mu)$ $(p \geq 1)$, $C^{r-2,\alpha}(M,\mathbb{R})$ and in $C^{r-1}(M,\mathbb{R})$, if in addition $g_\varphi \in C^{r-1}(M,\mathbb{R})$.

The norms $\|\cdot\|_{r-2,1}$ and $\|\cdot\|_{r-1}$ are equivalent in $C^{r-1}(M,\mathbb{R})$ and $C^{r-1}(M,\mathbb{R})$ is closed in $C^{r-2,1}(M,\mathbb{R})$. Therefore Theorem 2 implies Theorem 1 i) $(r \neq \omega)$ and the following

Corollary 1. For each $\varphi \in E_r$ there exist a neighbourhood $\mathcal{V}$ of φ in E_r and numbers $\delta > 0$, $q \in]0,1[$ such that for $\psi \in \mathcal{V}$ and $n \in N$

$$\| \bar{U}^n_\psi \|_{r-1} \leq \delta q^n .$$

It is easy to see that the mapping $E_r \ni \varphi \longrightarrow U_\varphi(1) \in C^{r-1}(M,\mathbb{R})$ is continuous. Therefore Corollary 1 gives Theorem 1 ii).

We now outline the proof of Theorem 2. One first shows

a) For each $\varphi \in E_2$ and each α there exist a neighbourhood $\mathcal{V}$ of φ in E_2, numbers $q \in]0,1[$, $L > 0$ such that for $\psi \in \mathcal{V}$ and $n \in N$

3) $\| U_{\psi^n} \|_0 \leq L$,

4) $\| U_{\psi^n}(f) \|^0_{0,\alpha} \leq L(q^n \|f\|^0_{0,\alpha} + \|f\|_0)$

for $f \in C^{0,\alpha}(M, \mathbb{R})$.

From a) it follows

b) For each $\varphi \in E_2$ there exists a unique φ-invariant measure $\mu_\varphi \in \mathfrak{M}_0$. The mapping $E_2 \ni \varphi \longrightarrow \mu_\varphi \in \mathfrak{M}_0$ is continuous.

For the proof of b) let us remark that in view of a) the set $\{H^n_\varphi(1)\}_{n \in N}$ is relatively compact in $C(M,\mathbb{R})$, where $H^n_\varphi = n^{-1} \sum_{k=0}^{n-1} U_{\varphi^k}$. Therefore there exists a sequence $(H^{k_n}_\varphi(1))$ convergent in $C(M,R)$ to a function g_φ. The measure μ_φ such that $\frac{d\mu_\varphi}{d\mu} = g_\varphi$ has the required properties. The uniqueness of μ_φ follows from [2]. In the proof of the second part of b) one uses relative compactness in $C(M,R)$ of the set $\{g_\psi\}_{\psi \in V}$ and the uniqueness of μ_φ, where V is from a).

We now show

c) For each $\varphi \in E_2$

$$\sup_{\|f\|_{0,\alpha} \leq 1} \| \bar{U}^n_\varphi(f) \|_0 \to 0 .$$

For the proof let us remark that the dynamical system $(M, \mu_\varphi, \varphi)$ is exact [2]. Hence $\bar{U}^n_\varphi(f) \xrightarrow[L]{} 0$ for $f \in L(\mu)$. This and relative compactness in $C(M,R)$ of the set $\{\bar{U}^n_\varphi(f)\}_{n \in N}$ for $f \in C^{0,\alpha}(M,\mathbb{R})$ imply that $\bar{U}^n_\varphi(f) \xrightarrow[C]{} 0$. To complete the proof it is sufficient to use a).

From a) - c) it follows Theorem 2 for $r = 2$. To show this let q_1, q_2, $0 < q_1 < q_2 < 1$ be such that there exists $\bar{n} \in N$ that

$$5) \quad 3L^2 q^{\bar{n}} + Lq_1 \leqslant q_2 .$$

Then, in view of c) one may assume that

$$6) \quad \sup_{\|f\|_{0,\alpha} \leqslant 1} \| \bar{U}^i_{\varphi}(f) \|_0 < q_1$$

for $i = \bar{n}, 2\bar{n}$. Now let us remark that the mapping $E_2 \ni \varphi \longrightarrow U_{\varphi} \in \mathcal{L}(C^{0,\alpha}(M,\mathbb{R}), C(M,\mathbb{R}))$ is continuous. Therefore in view of b) and 6) we may assume that for $\psi \in V$ and $i = \bar{n}$, $2\bar{n}$

$$7) \quad \sup_{\|f\|_{0,\alpha} \leqslant 1} \| \bar{U}^i_{\psi}(f) \|_0 < q_1$$

and in view of a) and b) - that for $\psi \in V$

$$8) \quad \sup_{\|f\|_{0,\alpha} \leqslant 1} \| \bar{U}^{\bar{n}}_{\psi}(f) \|^{o}_{0,\alpha} \leqslant 3L .$$

Now let $\psi \in V$ and $\| f \|_{0,\alpha} \leqslant 1$. Then a), 5), 7) and 8) imply

$$9) \quad \| \bar{U}^{2\bar{n}}_{\psi}(f) \|_{0,\alpha} \leqslant q_2 .$$

Notice that there exist a neighbourhood V_1 of φ in E_2 and $c > 0$ such that **for** $\psi \in V_1$ $\| \bar{U}^1_{\psi} \|_{0,\alpha} \leqslant c$. From this and 9) it follows Theorem 2 for $r = 2$. Further the proof goes by induction on r for all α. In the inductive step the proof is based on

d) For each $\varphi \in E_r$ and each α there exist a neighbourhood $\mathcal{V}$ of φ in E_r, numbers $L > 0$, $q \in]0,1[$ and $\bar{n} \in N$ such that for $\psi \in \mathcal{V}$ and $f \in C^{r-2,\alpha}(M,\mathbb{R})$

$$\| U_{\psi^{\bar{n}}}(f) \|^{o}_{r-2,\alpha} \leqslant q \| f \|^{o}_{r-2,\alpha} + L \| f \|_{r-2} .$$

Now, suppose that Theorem 2 is true for the number $r-1$, where $r > 2$. Then using the induction hypothesis and d) one shows that there exist a neighbourhood $\mathcal{V}_1$ of φ in E_r and number $L_1 > 0$ such that for $\psi \in \mathcal{V}_1$, $n \in N$ and $f \in C^{r-2,\alpha}(M,\mathbb{R})$

$$\| U_{\psi^{n\bar{n}}}(f) \|^{o}_{r-2,\alpha} \leqslant q^n \| f \|^{o}_{r-2,\alpha} + L_1 \| f \|_{r-2} .$$

Further reasoning is similar to that for $r = 2$.

Theorem 2 also implies

Corollary 2. For each $\varphi \in E_2$ there exists a neighbourhood $\mathcal{V}$ of φ in E_2 such that for $f \in C(M,\mathbb{R})$ ($f \in L^p(\mu)$, where $p \in [1,+\infty[$) the sequence $(\bar{U}^n_\psi(f))$ is convergent in $C(M,\mathbb{R})$ $(L^p(\mu))$ to 0 uniformly in $\psi \in \mathcal{V}$.

Corollary 3. For each $\varphi \in E_2$ and each α there exist a neighbourhood $\mathcal{V}$ of φ in E_2 and numbers $\delta > 0$, $q \in]0,1[$ such that for $\psi \in \mathcal{V}$ and $n \in N$

$$\sup_{\|f\|_{o,\alpha} \leqslant 1} \ \sup_{\|g\|_{L(\mu)} \leqslant 1} \left| \int_M fg \circ \psi^n d\mu - \int_M f\, d\mu \int_M g\, d\mu_\psi \right| \leqslant \delta q^n .$$

Corollary 4. For each $\varphi \in E_2$ there exist a neighbourhood $\mathcal{V}$ of φ in E_2 and numbers $\delta > 0$, $q \in]0,1[$ such that for $\psi \in \Upsilon$, $n \in N$ and $A \in \mathcal{B}$

$$|\mu(\psi^{-n}(A)) - \mu_{\psi}(A)| \leqslant \delta q^{n} \mu(A) .$$

We now outline the proof of Theorem 1 for $r = \omega$. For this purpose let us remark that $\pi : \mathbb{R}^m \to M$ is a C^{ω}-universal covering of M, where $m = \dim M$ [9] . Then there exists a C^{ω}-diffeomorphism $\dot{\varphi}$ of $\mathbb{R}^m$ such that $\pi \circ \dot{\varphi} = \varphi \circ \pi$.

We may assume that

$$10) \qquad \| D\dot{\varphi}(\varpi)\|^{\bullet} \geqslant b \|\varpi\|^{\bullet}$$

for $\varpi \in \mathbb{R}^m$, where $b > 1$ and $\|\cdot\|^{\bullet}$ is the lifting of $\|\cdot\|$ to $\mathbb{R}^m$. Let Γ be the group of the deck transformations of the covering π . Then $\gamma \to \dot{\varphi} \circ \gamma \circ \dot{\varphi}^{-1}$ is a homomorphism of Γ onto a subgroup Γ_1 of Γ . Let Γ_2 be a set such that its intersection with each right coset of Γ_1 in Γ is a one-point set. Then $\operatorname{Card} \Gamma_2$ is equal to the multiplicity of φ . If $\dot{C}(\mathbb{R}^m, \mathbb{R})$ is the set of all Γ-invariant $f \in C(\mathbb{R}^m, \mathbb{R})$, then $C(M, \mathbb{R}) \ni f \to \dot{f} = f \circ \pi \in \dot{C}(\mathbb{R}^m, \mathbb{R})$ is a bijection. Therefore

$$11) \qquad \dot{U}_{\varphi}(\dot{f}) = \dot{\widehat{U_{\varphi}(f)}}$$

defines the operator of $\dot{C}(\mathbb{R}^m, \mathbb{R})$ into itself. It turns out that for $n \in N$

$$12) \quad (\dot{U}_{\dot{\varphi}})^{n}(1) = \sum_{(\gamma_1, \dots, \gamma_n) \in \Gamma_2^{n}} \prod_{i=1}^{n} (J\dot{\varphi}^{-1}) \circ \gamma_i \circ \dot{\varphi}^{-1} \circ \gamma_{i-1} \circ \dots \circ \dot{\varphi}^{-1} \circ \gamma_1 \ ,$$

where $J\dot{\varphi}^{-1}$ is defined in the similar way as $J\varphi$. In the definition one replaces μ by the C^{ω}-measure $\dot{\mu}$ on $\mathbb{R}^m$ such that π transforms locally $\dot{\mu}$ on μ. Then Corollary 2 and 11) imply

13) $((\dot{U}_{\varphi})^n(1))$ is uniformly convergent to $\dot{g}_{\varphi}$.

Using 10) one shows that there exist open sets $G_1, G_2 \subset (\mathbb{R}^m)^2 = \mathbb{C}^m$, $G_1 \subset G_2$ such that $\gamma \in \Gamma_2$, $\dot{\varphi}^{-1}$ and $J\dot{\varphi}^{-1}$ restricted to $G_2 \cap \mathbb{R}^m$ [3] have the complex analytic extensions f_γ, g and h to G_2 respectively. Moreover, $\pi(G_1 \cap \mathbb{R}^m) = M$ and for each $i \in N$, if $\gamma_1, \dots \gamma_i \in \Gamma_2$, then for $z \in G_1$

$f_{\gamma_i}(g(f_{\gamma_{i-1}}(\dots g(f_{\gamma_1}(z)))))$ is well-defined and belongs to G_2. Moreover, one shows that (F_n) is uniformly bounded in G_1, where for $z \in G_1$

$$F_n(z) = \sum_{(\gamma_1,\dots\gamma_n)\in\Gamma_2^n} \prod_{i=1}^{n} h(f_{\gamma_i}(g(f_{\gamma_{i-1}}(\dots g(f_{\gamma_1}(z)))))) .$$

This 12), 13) and the Montel theorem for complex analytic functions give the required result.

It turns out that Theorem 1 i) is false for $r = 1$. In fact the following theorem is true.

Theorem 3. The set of all $\varphi \in E_1$ for which there exists a φ-invariant $\nu \in \mathfrak{M}_0$, is of the first category in E_1.

For the proof let $\varphi \in E_1$ and let $\nu \in \mathfrak{M}_0$ be φ-invariant. Then $U_{\varphi^n}(\frac{d\nu}{d\mu}) = \frac{d\nu}{d\mu}$ for $n \in N$. Therefore there exists $k \in N$ such that

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[3] We identify $\mathbb{R}^m$ with $\mathbb{R}^m \times \{0\}$.

14) $k^{-1} \leq U_{\varphi^n}(1) \leq k$ for $n \in N$.

Then, for each $k \in N$ the set A_k of all $\varphi \in E_1$ for which 14) is satisfied, is closed. Moreover, by a perturbation of $\varphi \in A_k$ in a neighbourhood of $\bigcup_{i=1}^{n} \varphi^{-i}(x_0)$, where x_0 is a fixed point of M and n is sufficiently large, one shows that A_k is also a boundary set. This proves the theorem.

It is easy to see that in the above theorem one may replace $\mathfrak{M}_o$ by the set of all normalized measures γ absolutely continuous with respect to μ and such that $\varepsilon^{-1} \leq \frac{d\gamma}{d\mu} \leq \varepsilon$ μ-almost everywhere for a certain number $\varepsilon > 0$.

The following theorem says that in Corollary 2 in the case of the space $L^p(\mu)$ one cannot replace convergence in $L^p(\mu)$ by μ-almost everywhere convergence.

Theorem 4. Let $\varphi \in E_2$ and $p \in [1,+\infty[$. Then the set of all $f \in L^p(\mu)$ for which μ-almost everywhere

$$\sup_{n \geq 1} |U_{\varphi^n}(f)| < +\infty$$

is of the first category in $L^p(\mu)$.

For the proof one may assume that $\mu = \mu_\varphi$. Then, let $G \subset M$ be a non-empty connected open set with sufficiently small diameter and let k be a natural number. Then there exists a component G_k of $\varphi^{-k}(G)$ such that $\mu_\varphi(G_k) \leq N^{-k}\mu_\varphi(G)$, where N is the multiplicity of φ . From this it follows that

$$\sup_{\|f\|_{L^p(\mu)} \leq 1} \mu_\varphi(\{x \in M : \sup_{n \geq 1} |U_{\varphi^n}(f)(x)| \geq N^{\frac{k}{p}}(\mu_\varphi(G))^{-\frac{1}{p}}\}) = 1.$$

This and the Banach's principle [1] give the required result.

We now pass on to some results concerning convergence to invariant smooth measure for expanding mapping in which the rate is linear. For this purpose let for $\varphi \in E_r$ and $n \in N$ $\bar{V}^n_\varphi$ be the operator of $L(\mu)$ into itself such that

$$\bar{V}^n_\varphi(f) = V^n_\varphi(f) - g_\varphi h_\varphi \int_M f \, d\mu \quad ,$$

where h_φ is the entropy of the dynamical system $(M, \mu_\varphi, \varphi)$. $\bar{V}^n_\varphi$ also acts in $L^p(\mu)$ $(p > 1)$, $C^{r-2,\alpha}(M,\mathbb{R})$ and $C^{r-1}(M,\mathbb{R})$. Then the following theorem is true.

Theorem 5. For each $\varphi \in E_r$ and each α there exist a neighbourhood Υ of φ in E_r and number $\delta > 0$ such that for $\psi \in \Upsilon$ and $n \in N$

$$\|\bar{V}^n_\psi\|_{r-2,\alpha} \leq n^{-1}\delta \quad .$$

For the proof let us remark that

15) $\quad h_\psi = \int_M \ln J\psi \, d\mu_\psi \qquad$ for $\quad \psi \in E_r$,

16) the mapping $E_r \ni \psi \longrightarrow h_\psi \in R$ is continuous. Moreover, for $\psi \in E_r$, $f \in C^{r-2,\alpha}(M,R)$ and $n \in N$

$$\bar{V}^n_\psi(f) = n^{-1} \bar{\bar{V}}^n_\psi(f) + h_\psi \bar{U}^n_\psi(f) \quad ,$$

where $\bar{\bar{V}}^n_\psi(f) = U_{\psi^n}(f(\ln J\psi^n - nh_\psi))$. Therefore in view of 16) and Theorem 2, it is sufficient to prove

17) there exist a neighbourhood $\mathcal{V}$ of φ in E_r and number $c > 0$ such that for $\psi \in \mathcal{V}$ and $n \in N$

$$\|\bar{\bar{V}}_\psi^n\|_{r-2,\alpha} \leqslant c .$$

For this purpose let us remark that if in addition $m \in N$, then

$$\bar{\bar{V}}_\psi^{n+m}(f) = \bar{\bar{V}}_\psi^n(\bar{U}_\psi^m(f)) + \int_M f\, d\mu \sum_{i=1}^{n} U_{\psi i}(g_\psi(\ln J\psi - h_\psi)) +$$

$$+ \bar{U}_\psi^n(\bar{\bar{V}}_\psi^m(f)) + g_\psi \sum_{i=0}^{m-1} \left(\int_M f \ln J\psi \circ \psi^i \, d\mu - h_\psi \int_M f\, d\mu \right) .$$

This, 15), 16), Theorem 1 ii), Theorem 2 and Corollary 3 imply the existence of a neighbourhood $\mathcal{V}$ of φ in E_r, numbers $\delta > 0$ and $q \in \,]0,1[$ such that for $\psi \in \mathcal{V}$ and $n, m \in N$

18) $$\|\bar{\bar{V}}_\psi^{n+m}\|_{r-2,\alpha} \leqslant \delta(q^m \|\bar{\bar{V}}_\psi^n\|_{r-2,\alpha} + q^n \|\bar{\bar{V}}_\psi^m\|_{r-2,\alpha} + 1).$$

It is easy to see that 18) gives 17). This completes the proof.

It can be shown that the rate of convergence in Theorem 5 is the best possible. Moreover it turns out that for a fixed $\psi \in E_r$ the rate of convergence in this theorem is the best possible unless μ , μ_ψ and the measure with maximal entropy for ψ are equal.

Corollaries 1-4 and Theorem 4 have their counterparts in the case of the sequence $(\bar{V}_\varphi^n)$ and (V_φ^n) respectively.

We end this note by stating the following theorem whose proof is rather long.

Theorem 6. For each $\varphi \in E_2$ and each α there exist numbers $\delta > 0$ and $q \in]0,1[$ such that for $n \in N$

$$\sup_{\|f\|_{0,\alpha} \leq 1} \left| \sum_{x \in \mathrm{Fix}(\varphi^n)} f(x) \left|\det(D\varphi^n(x))\right|^{-1} - \int_M f \, d\mu_\varphi \right| \leq \delta q^n .$$

The complete proofs will be contained in [6] , [7] and [8].

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