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# Two remarks on structural stability of plane dynamical systems

by
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We shall discuss some questions of structural stability of dynamical systems on the plane. The space of diffeomorphisms or vector fields on open manifold can be endowed with $C^r$ Whitney topology, $r \geqslant 0$. In the case of the plane this topology is defined as topology generated by all sets $B(f, \varepsilon, r) = \{g: |d^i f(x) - d^i g(x)| < \varepsilon(x)$ for $i \in \{0,1,\dots,r\}\}$, f is a diffeomorphism or vector field, r is a non-negative integer, $\varepsilon : R^2 \longrightarrow R$ is a continuous positive function.

Definition 1.

A diffeomorphism f is structurally stable ($C^r$-structurally stable, $r \geqslant 0$) iff there is a neighbourhood ($C^r$ neighbourhood) $B(f, \varepsilon, r)$ of f such that if $g \in B(f, \varepsilon, r)$ then f and g are topologically equivalent i.e. there is a homeomorphism $h: R^2 \longrightarrow R^2$ such that $h \circ f = g \circ h$.

Definition 2.

A vector field X is $C^r$ structurally stable iff there is a $C^r$ neighbourhood $B(X, \varepsilon, r)$ such that if $Y \in B(X, \varepsilon, r)$ then X and Y are topologically equivalent i.e. there is a homeomorphism $h: R^2 \longrightarrow R^2$ which carries X-orbits onto Y-orbits.

There is a question raised several years ago whether $C^n$-structural stability is equivalent to $C^k$-structural stability for $k \neq n$, $k \geqslant 1$, $n \geqslant 1$ (for dynamical systems on closed manifolds). No answer is known up to now for the compact case. In the first part of the paper we shall construct $C^3$-structurally stable diffeomorphism of $R^2$ which is not $C^2$ structurally stable. Our diffeomorphism has a non-hyperbolic fixed point, thus hyperbolicity of fixed points is not necessary for $C^3$ stability on open manifolds. We recall that for $C^r$ stability, $r \geqslant 1$, of diffeomorphisms of closed monifolds hyperbolicity of periodic orbits is a necessary condition, see [5], but, as we know, the analogous theorem for vector fields on closed manifolds has not been proved yet and it is possible that it is not true. In the second part of the paper we shall deal with non-vanishing vector fields on the plane. We shall state a theorem analogous to the well-known theorem of Kupka and Smale and we shall use it to show that there is an open non-empty set of structurally unstable non-vanishing vector fields on $R^2$.

I Let $F: R \to R$ be defined by the formula $F(x) = -x - \frac{1}{14}(x^3-x^5)(1+x^2)^{-2.5}$ and let $r(x) = F(x)+x$. One can easily check that $|r(x)| \leqslant \frac{1}{7}$ and $|r'(x)| \leqslant \frac{1}{2}$ for every real number x. Thus $F'(x) = -1+r'(x) < 0$. This inequality implies that F is an orientation reversing diffeomorphism of R. Therefore F has exactly one fixed point, namely 0. $F'(0) = -1$, i.e. $\{0\}$ is not a hyperbolic fixed point. Because of $r(x)$ and x have the same sign for for x sufficiently close to 0, $\{0\}$ is a repellor. All other non-wandering points of F are periodic points of minimal period 2. It is easy to check that F has only one periodic orbit different from $\{0\}$, namely the orbit consisting of $-1$ and 1. Thus $\Omega(F) = \{-1,0,1\}$.

Let $F_1(x) = F(x) + \frac{3}{224}x(1+x^2)^{-2.5}$ and $F_{-1}(x) = F(x) - \frac{1}{7}x(1+x^2)^{-2.5}$. It is easy to see that $\Omega(F_1) = \left\{-\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right\}$, $\{0\}$ and $\left\{-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right\}$ are hyperbolic attractors, $\left\{-\frac{1}{2}, \frac{1}{2}\right\}$ is a hyperbolic repellor. Analogously $\Omega(F_{-1}) = \{-\sqrt{2}, 0, \sqrt{2}\}$, $\{0\}$ is a hyperbolic repellor and $\{-\sqrt{2}, \sqrt{2}\}$ is a hyperbolic attractor. Thus $F_1$ and $F_{-1}$ are $C^1$ structurally stable.

It has been proved in [6] and also in [3] that if $G$ is sufficiently close to $F$ in $C^3$ topology, then either $G$ is equivalent to $F_1$ or $G$ is topologically equivalent to $F_{-1}$.

Now we can describe the diffeomorphism of $R^2$ which we are looking for. Let $f: R^2 \to R^2$ be a diffeomorphism such that

(i) all lines $L_n = \{(x,y) \in R^2: \; y=n\}$, $n$ is an arbitrary integer are f-invariant and normally hyperbolic, $L_{2n}$ are repellors while $L_{2n+1}$ are attractors,

(ii) $f_{|L_0} = F$, $f_{|L_{2n}} = F_{\operatorname{sgn} n}$ for any integer $n \neq 0$,

(iii) $f_{|L_{2n+1}}$ is an orientation reversing diffeomorphism which has exactly one non-wandering point which is a hyperbolic attractor,

(iv) the stable manifold of the fixed point lying on $L_{2n+1}$ is egual to the open region bounded by the lines $L_{2n}$ and $L_{2n+2}$.

If $g$ is sufficiently close to $f$ in the $C^3$-topology then there are lines $L_n(g)$ which are g-invariant and $C^3$ close to $L_n$. It is clear that $g_{|L_n(g)}$ is $C^3$ close to $f_{|L_n}$ for any integer $n$. Thus $g_{|L_n(g)}$ and $f_{|L_n}$ are topologically equivalent for any $n \neq 0$. If $f_{|L_0}$ and $g_{|L_0(g)}$

are topologically equivalent, then it is possible to construct a conjugacy carrying $L_n$'s onto $L_n(g)$'s. If $f_{|L_o}$ is not topologically equivalent to $g_{|L_o(g)}$ then conjugacy carrying $L_n$, n arbitrary integer, onto $L_{n-2}(g)$ can be constructed. We omit details because it is well-known how such conjugacies can be defined.

f is not $C^2$ structurally stable because in any $C^2$ neighbourhood of it there is a diffeomorphism which has uncountably many of periodic points and therefore it is not topologically equivalent to any Kupka-Smale diffeomorphism.

Similar diffeomorphisms can be constructed on many other open manifolds. The author does not know whether each open connected manifold of dimension not less than 2 admits such diffeomorphism. The author strongly believes that all periodic points of $C^2$ structurally stable diffeomorphism must be hyperbolic. The analogous construction can be made for vector fields, also on some two dimensional manifolds (non-orientable).

II. Let $W^r$ be the set of all $C^r$ vector fields on $R^2$ which have not any non-wandering point. By the Poincaré-Bendixon theorem, $W^r$ consits of all $C^r$ non-vanishing vector fields on $R^2$. Therefore $W^r$ is open in the set of all vector fields on $R^2$ and thus it has the Baire property. Thus it is sensible to investigate structural stability of elements of $W^r$. Recently Z.Nitecki proved that R and $R^2$ are the only manifolds on which the set of all vector fields without non-wandering points is open, so on other connected manifolds we are to deal with another set.

Let $O_X(a)$ denotes the X-orbit of a, $O_X^+(a)$ and $O_X^-(a)$ denote positive and negative semi-orbits of a. We write

$0_X(a) < 0_X(b)$ iff $0_X(a) \neq 0_X(b)$ and for any neighbourhoods $U \ni a$ and $V \ni b$ there is a point $c \in U$ such that $0_X^+(c) \cap V \neq \emptyset$. The pair $\langle 0_1, 0_2 \rangle$ of orbits of vector field X such that either $0_1 < 0_2$ or $0_2 < 0_1$ is called a pair of separatrices, its elements are called separatrices, if $0_1 < 0_2$, then $0_1$ is called the beginning of $\langle 0_1, 0_2 \rangle$ and $0_2$ is called the end of $\langle 0_1, 0_2 \rangle$.

In [2] the following theorem is proved:

Density Theorem.

The set of $X \in W^r$, $r \geq 1$, such that there are no 3 points a,b,c such that $0_X(a) < 0_X(b) < 0_X(c)$ is residual and therefore dense in $W^r$.

In the prove of this theorem the countability of the set of pairs of separatrices is used. This is not true on other manifolds, there exist vector fields on 2-manifolds which have uncountable many pairs of separatrices. Another difficulty in generalizing of this theorem is that $W^r$ is not open on other manifolds (Nitecki's result) and therefore we must be much more carefull when perturbing vector fields.

As an application of this theorem we shall shaw that there is an open non-empty subset of $W^r$, $r \geq 1$, consisting of $C^r$ structurally unstable vector fields.

Let $X \in W^r$ be a vector field on $R^2$ which has the following properties:

(1) X is invariant under the translation $(x,y) \longrightarrow (x,y+1)$ and transversal to the x-axis ,

(2) the orbit of the point $(\frac{k}{2^n}, 0)$, n is a non-negative integer, k is an odd integer, is the beginning of the pair of separatrices the end $e_{n,k}$ of which lies in the half-plane $y \geq n$, the component $G_{n,k}$ of $R^2 \setminus e_{n,k}$ which does not contain $0_X((\frac{k}{2^n}, 0))$ is convex and it is symmetric with

respect to the vertical straight line $L_{n,k}$ which contains the unique lowest point of $e_{n,k}$,

(3) X restricted to an arbitrary $\bar{G}_{n,k}$ is diffeomorphically equivalent to X restricted to $\{(x,y): \quad x \leq 0\}$,

(4) $O_X((\sqrt{2},0))$ is the end of a pair of separatrices the beginning of which is contained in $\{(x,y): \; y \leq -1\}$,

(5) X restricted to $\{(x,y): \quad y \leq 0\}$ is diffeomorphically equivalent to X restricted to $\bar{G}_{0,0}$.

Let B be the union of the beginnings of all pairs of separatrices and let E be the union of the ends of all pairs of separatrices. By the definition of X the set $B \cap \{(x,y): \quad y=0\}$ is dense in x-axis, thus B is dense in $R^2$.

It can be proved, see [2], that pairs of separatrices are stable in the following sense: if $T_1$ and $T_2$ are open transversal sections of vector field $Y \in W^r$, $\varepsilon > 0$, $a \in T_1$, $b \in T_2$, $O_Y(a) < O_Y(b)$ , $C_a \subset O_Y(a)$ is a compact arc, $C_b \subset$ $\subset O_Y(b)$ is a compact arc, then there is a $C^1$ neighbourhood N of Y such that if $Z \in N$, then there are points $a(Z) \in T_1$ and $b(Z) \in T_2$ such that $O_Z(a(Z)) < O_Z(b(Z))$, $\|a(Z)-a\| < \varepsilon$, $\|b(Z)-b\| < \varepsilon$ , $C_a$ is $\varepsilon$-$C^1$-close to $O_Z(a(Z))$, $C_b$ is $\varepsilon$-$C^1$-close to $O_Z(b(z))$. Using this fact and applying the method found by Peixoto and Pugh one easily proves that for any Z sufficiently close to X in the $C^1$ topology $B(Z) \cap \{(x,y): \quad y=0\}$ is dense in x-axis, B(Z) is theunion of all beginnings of pairs of separatrices of vector field Z and that at least one end of a pair of separatrices meets x-axis. This implies that in some neighbourhood of X the vector fields satisfying the assumptions of Density Theorem are dense and the vector fields which do not satisfy these assumptions are also dense in this neighbourhood. Of course no vector field of the first type is equivalent to the vector field of a second type and therefore no vector field in this neighbourhood of X is structurally stable.

The author has almost proved that $X \in W^r$, $r \geqslant 1$, is structurally stable iff $\overline{B(X)} \cap \overline{E(X)} = \emptyset$ (necessity is trivial). Partial result in this direction has been obtained by J.F.Collins in his thesis, who proved under the additional assumption sufficiency of our condition (it was done independently).

Another question is whether $C^1$ structurally stable vector fields (or even) $C^0$ structurally stable) vector fields are $C^0$ dense in $W^r$. The author believes that it is true, note that $C^1$ and $C^0$ structural stability are not equivalent e.g. the vector field $(1+x^2)^{-1.5}((1-x^2)\frac{\partial}{\partial y} - - x(x^2-4)\frac{\partial}{\partial x})$ is $C^1$ structurally stable but not $C^0$ structurally stable, because the trajectory passing through the origin is the beginning of two different pairs of separatrices.

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