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ERGODIC AUTOMORPHISMS OF COMPACT METRIC GROUPS ARE ISOMORPHIC TO BERNOULLI SHIFTS

Nobuo Aoki

I wish to discuss its title. Let X be a compact, metric group and μ be its normalized Haar measure. Then (X,μ) is a Lebesgue space. Let σ be an automorphism of X, then σ is an invertible measure preserving transformation from X onto X. Our problem is concerned with measure theoretic properties of σ.

Today we will outline a proof of the following

Theorem.

An ergodic automorphism of a compact, metric, abelian group is a Bernoulli shift.

In order to outline the Theorem, we prepare the following

Lemma 1. Let X be a compact, metric; abelian group and σ be an ergodic automorphism of X. Then there exists a sequence of subgroup $X \supset X_1 \supset \dots \bigcap_{n=1}^{\infty} X_n = \{e\}$, invariant with respect to σ, such that there are a σ-invariant totally disconnected subgroup $X_D^{(n)}$, a σ-invariant infinite dimensional connected subgroup $X_B^{(n)}$ and a σ-invariant finite dimensional connected subgroup $X_A^{(n)}$, so that $\sigma|_{X_D^{(n)}}$ is isomorphic to a Bernoulli shift, $\sigma|_{X_B^{(n)}}$ is a Bernoulli automorphism (see [28]) and $\sigma|_{X_A^{(n)}}$ is ergodic, and $(X/X_n,\sigma)$ is an algebraic factor of $(X_A^{(n)} \times X_B^{(n)} \times X_D^{(n)}, \sigma\times\sigma\times\sigma)$.

To show the ergodic system $(X_A^{(n)},\sigma)$, $n \geq 1$, have the

Bernoulli properties, let $\overline{T}^r$ be a compact, connected, abelian group with the character group Q^r, being the r-vector of rational numbers and σ be an automorphism of $\overline{T}^r$, then it is sufficient to prove that if $(\overline{T}^r,\sigma)$ is ergodic then $(\overline{T}^r,\sigma)$ is Bernoullian.

We write $Q^r = \{g_1,g_2,\dots\}$. Let Q_n denote a subgroup generated by $\{g_k \circ \sigma^j : j \in Z,\ 1 \leq k \leq n\}$ for $n \geq 1$. Since rank $(Q^r) = r$, there exists an integer $m > 0$ such that rank (Q_n) = rank (Q^r) for $n \geq m$.

We denote by $\overline{T}(Q_n)$ the annihilator of Q_n in $\overline{T}^r$ for $n \geq 1$.

Then we have

<u>Lemma 2</u>. $(\overline{T}^r/\overline{T}(Q_n),\sigma)$ is Bernoullian for $n \geq 1$.

The proof is done by ideas of Katznelson together with the following facts.

We may consider σ as operating on R^r. So we have the decomposition

$$R^r = V_{-p} \oplus \dots \oplus V \oplus \dots \oplus V_g$$

such that $V_j, -p \leq j \leq q$, are σ-invariant, all the eigenvalues of $\sigma_{|V_0}$ have modulus 1, and all the eigenvalues of $\sigma_{|V_j}, j \neq 0$, have the same modulus ρ_j where

$$\rho_{-p} < \dots < 1 = \rho_0 < \dots < \rho_q .$$

<u>Lemma 3</u>. If $\{0\} \neq Q^r \cap \tilde{V}$ where $\tilde{V} = V_{-p} \oplus \dots \oplus V_0$, then R^r is decomposed by subspaces $R^{r_1}(\sigma R^{r_1} = R^{r_1})$ and R^{r-r_1} with the following properties: there exists an automorphism σ' of R^{r-r_1} and a homomorphism λ from R^{r-r_1} into R^{r_1} such that $\sigma(\chi_1,\chi_2) = (\sigma'\chi_1, \lambda(\chi_1) + \sigma\chi_2)$, $(\chi_1,\chi_2) \in R^{r-r_1} \oplus R^{r_1}$, and $\sigma_{|R^{r_1}}$ gives

$$R^{r_1} = V'_{-p} \oplus \dots \oplus V'_0$$

where $V_j' \subset V_j$ for $j = -p, \dots, 0$, and σ' gives

$$R^{r-r_1} = V''_{-p} \oplus \dots \oplus V'' \oplus \dots \oplus V''_q$$

where $V''_j \subset V_j$ for $j = -p, \dots, q$, such that

$$(V''_{-p} \oplus \dots \oplus V'') \cap Q^r = \{0\} .$$

Lemma 4. If σ acts ergodically on Q^r, then so is $\sigma_{|Q^{r_1}}$ and $Q^r \cap V' = \{0\}$.

If $Q^{r-r_1} \cap (V'' \oplus \dots \oplus V''_q) \neq \{0\}$, then we again decompose R^{r-r_1} as in Lemma 3. However, since the dimension of R^r is finite, such the argument ends in finite time.

For $m \geq 0$, we denote by $V^{(m)}$ the set of all r-vectors such that each component of a vector consists of an element of $\{0, \pm 1, \dots, \pm m^{10}\}$. We define

$$V(1,M) = \{ \sum_{m=1}^{M} \sigma^{-m} \tilde{\alpha}_m : \tilde{\alpha}_m \in V^{(M)}, 1 \leq m \leq M\} ,$$

$$V(K,K+M) = \{ \sum_{m=K}^{K+M} \sigma^{m} \tilde{\alpha}_m : \tilde{\alpha}_m \in V^{(m)}, K \leq m \leq K+M\}$$

for $K > 0$ and $M > 0$. Then $V(1,M)$ and $V(K,K+M)$ are subsets of R^r.

Lemma 5. Assume that R^r is decomposed as follows. If $R^r = V_{-p} \oplus \dots \oplus V$, then $V_0 \cap Q^r = \{0\}$, and if $R^r = V_{-p} \oplus \dots \oplus V_0 \oplus \dots \oplus V_q$, then $(V_{-p} \oplus \dots \oplus V) \cap Q^r = \{0\}$ and $(V_0 \oplus \dots \oplus V_q) \cap Q^r = \{0\}$. Then there exists $K_1 > 0$ such that

$$V(1,M) \cap V(K,K+M) = \{0\}$$

for $K \geq K_1$ and an odd integer $M > 0$.

In the proofs of Lemmas 3, 4 and 5, we use no results of number theory.

We now conclude that (X,σ) is Bernoullian. Using theorem and structures of skew product transformations, I can prove the result for the case of non-abelian. But, proofs of these results will appear elsewhere.

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