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INVARIANTS AND SCHUR FUNCTORS IN CHARACTERISTIC $p > 0$

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§1. Introduction

In this note I wish to discuss some very explicit problems concerning representations in characteristic $p > 0$ of cyclic p-groups, in particular the problem of finding the indecomposable representations that arise in the Schur functors applied to these representations. My interest in this problem began when I read a remark of Samuel's -- "We do not know any example of a factorial ring which is not a Cohen-Macaulay ring" [Samuel, page 40, Remark 2]. Subsequently an example was found [Bertin] of a factorial domain of dimension 4 of finite type over a field of characteristic 2. Griffith and I became interested in knowing whether a) there were complete local factorial domains of arbitrarily large dimension with small depth (i.e. not Cohen-Macaulay) and in particular b) was the completion of Bertin's example factorial. The answers to a) and b) were satisfying to some degree [Fossum-Griffith] but did not answer completely all the questions raised. In particular I was led to consider the problem: Given a module V on which $\mathbb{Z}/p\mathbb{Z}$ acts as a group of automorphisms, what is the relation between the socle of V and the socle of the symmetric powers $S_r(V)$ (here we suppose V is a finite dimensional vector space over a field k with characteristic $p > 0$). This problem is stated in its proper context in the next section, where more general problems are posed and some solutions are mentioned.

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§2. Representation Theory

Let k be a field of characteristic $p > 0$. We want to consider $\mathbb{Z}/p^e\mathbb{Z}$ representations of k-modules. So first consider the group ring $k[\mathbb{Z}/p^e\mathbb{Z}]$ and its finitely generated modules. Let τ be a (multiplicative) generator of $\mathbb{Z}/p^e\mathbb{Z}$. Then $k[T] \to k[\mathbb{Z}/p^e\mathbb{Z}]$ given by $T \mapsto \tau$ induces an isomorphism

$$k[T]/(T^{p^e}-1) \xrightarrow{\approx} k[\mathbb{Z}/p^e\mathbb{Z}] .$$

Let $S = T - 1$. Then $T^{p^e} - 1 = (T-1)^{p^e} = S^{p^e}$. The group ring $k[\mathbb{Z}/p^e\mathbb{Z}]$ is a Hopf algebra with co-addition induced by $T \mapsto T \otimes T$. Then $S \mapsto 1 \otimes S + S \otimes 1 + S \otimes S$. The reasons for introducing this structure on the group rings are two-fold:

- The k-tensor product of two modules becomes a module. If V and W are $k[S]/S^n k[S]$-modules, then $V \otimes W$ is again a $k[S]/(S^n)$-module with "diagonal" action

$$S(v \otimes w) := v \otimes Sw + Sv \otimes w + Sv \otimes Sw .$$

- The homomorphisms $k[S]/(S^n) \to k[S]/(S^m)$ for $n \geq m$ are Hopf algebra maps.

So the representations can be placed into one basket and considered as modules of finite length over the power series ring $k[[S]]$.

Let $A := k[[S]]$ which is $\varprojlim k[S]/(S^n)$. Let $A_n := k[[S]]/(S^n)$. If $n = p^e$, then

$$A_n = k[\mathbb{Z}/p^e\mathbb{Z}] .$$

Let $\mathfrak{F}$ denote the category of A-modules of finite length and let $\mathfrak{F}_n$ be the subcategory of $\mathfrak{F}$ consisting of modules annihilated by S^n. We get a filtration

$$\mathfrak{F}_0 \subset \mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \ldots \subset \mathfrak{F} .$$

Note that $\mathfrak{F}_0$ is the category of k vector spaces with trivial S-action. Then $\mathfrak{F}_{p^e}$ is the category of finite dimensional $k[\mathbb{Z}/p^e\mathbb{Z}]$-modules.

Denote by R(p), the abelian group that is universal for maps from $\mathfrak{F}$ to abelian groups that are constant on isomorphism classes and additive on direct sums. The tensor product mentioned above induces a product on R(p). Let $[\ ]: \mathfrak{F} \to R(p)$ be the universal map. Then

$$[V \otimes W] :=: [V][W] .$$

So R(p) becomes a commutative ring with identity [k]. It is possible to describe the structure of this ring, even when $p = 0$.

<u>Proposition 2.1</u>: <u>Let</u> $V_n := k[[S]]/(S^n)$. <u>As an abelian group</u> R(p) <u>is free on</u> $[V_n]$ <u>for</u> $1 \leq n$.

Proof. One needs only to know the indecomposable A-modules of finite length. Let E(M) denote an injective envelope of an A-module M. Then M is indecomposable if and only if E(M) is indecomposable. The proposition follows from the theory of injective modules over a discrete rank one valuation ring. □

Using the rules for decomposition of tensor products of $k[\mathbb{Z}/p^e\mathbb{Z}]$-modules due to [Green] (and later [Srinivasan]), it is possible to describe explicitly the ring structure of R(p). Some notation is useful. Let $X_0, X_1, \ldots$ denote variables over $\mathbb{Z}$. In the following a prime number p is fixed. (It will follow that the formulas will hold for $p = 0$.)

In R(p) let $x_0 = [V_2]$ and for $p^e > 0$ let

$$x_e := [V_{p^e+1}] - [V_{p^e-1}] .$$

In $\mathbb{Z}[X_0, X_1, \ldots]$ define the polynomials

$$W_n := \sum_{\nu=0}^{[\frac{p-1}{2}]} (-1)^\nu \binom{p-1-\nu}{\nu} X_n^{p-1-2\nu} \quad \text{for } n \geq 0 ,$$

$$U_0 := 1 ,$$

$$U_n := \sum_{\nu=0}^{[\frac{p-1}{2}]} (-1)^\nu \{\binom{p-1-\nu}{\nu} X_{n-1} - \binom{p-2-\nu}{\nu}\} X_{n-1}^{p-2-2\nu}$$

and

$$F_n(X_0, X_1, \ldots, X_n) := (X_n - 2U_n)W_n \quad \text{for } n \geq 0 .$$

<u>Proposition 2.2</u>. <u>The map</u>

$$\mathbb{Z}[X_0, X_1, \ldots] \to R(p)$$

<u>induced by</u> $X_i \mapsto x_i$ <u>induces an isomorphism</u>

$$\mathbb{Z}[X_0, X_1, \ldots]/(F_0, F_1, \ldots) \cong R(p) .$$

(If $p = 0$, then $\mathbb{Z}[X_0] \cong R(0)$.)

The proof depends upon the decomposition formulas of [Green] and is available in [Almkvist-Fossum].

The ring R(p) is a limit of subrings $R_e(p)$, which is the representation ring of the group $\mathbb{Z}/p^e\mathbb{Z}$. The surjections

$$\mathbb{Z}/p^e\mathbb{Z} \to \mathbb{Z}/p^f\mathbb{Z} \quad \text{for } e \geq f$$

induce the inclusions

$$R_e(p) \hookrightarrow R_f(p) .$$

As subrings we calculate:

$$\mathbb{Z}[X_0,X_1,\ldots,X_{e-1}]/(F_0,F_1,\ldots,F_{e-1}) \cong R_e(p) .$$

But there are also the inclusions

$$\mathbb{Z}/p^f\mathbb{Z} \to \mathbb{Z}/p^e\mathbb{Z} \quad \text{for } f < e$$

gotten by $\tau \mapsto \tau^{p^{e-f}}$, where τ is a generator (multiplication). At the level of group rings and lifted to A we get the Frobenius (and its iterations) morphism

$$F:A \to A$$

gotten by $F(a) = a^p$. This induces two functors on $\mathfrak{F}$:

- The <u>forgetful functor</u> (corresponding to restriction)

$$r:\mathfrak{F} \to \mathfrak{F}$$

which induces a <u>ring</u> homomorphism

$$r:R(p) \to R(p) .$$

<u>Proposition 2.3</u>. <u>The ring homomorphism</u> $r:R(p) \to R(p)$ <u>is given by</u>

$$r(x_0) = 2$$

<u>and for</u> $n > 0$

$$r(x_n) = x_{n-1} .$$

Proof. That this is a ring homomorphism is clear, since

$$F_n(2,X_0,\ldots,X_{n-1}) = F_{n-1}(X_0,\ldots,X_{n-1})$$

for $n > 0$ and $F_0(2) = 0$. Thus the ideal $(F_0,F_1,\ldots)$ in $\mathbb{Z}[X_0,X_1,\ldots]$ is invariant under the homomorphism induced by $X_n \mapsto X_{n-1}$ for $n > 0$ and $X_0 \mapsto 2$. □

- The <u>Induction functor</u> (corresponding to induction)

$$i:\mathfrak{F} \to \mathfrak{F}$$

given by

$$i(v) := A \otimes_A V ,$$

where A is a right A-module through F. There is a corresponding linear map

$$i : R(p) \to R(p)$$

which has as description on the generators

$$i(x_e) = V_p x_{e+1} ,$$

and in general

$$i(f(x_0, x_1, \ldots, x_n)) = V_p f(x_1, x_2, \ldots, x_{n+1})$$

for a polynomial f. In terms of the indecomposables

$$i(V_n) = V_{pn} \quad \text{for all } n .$$

<u>Proposition 2.4.</u> <u>The composition</u>

$$r \circ i = p ,$$

<u>that is multiplication by</u> p. □

I can now turn to the main problems concerning these representations. Having tensor products, then it is possible to consider Schur functors of these representations. In particular, let V be an object in $\mathfrak{F}$. Then $V^{\otimes n}$ is an object in $\mathfrak{F}$ which becomes a left $\mathcal{K}_n$-module. If S is a left (resp. right) $k\mathcal{K}_n$-module, then the modules

$$V^S := \mathrm{Hom}_{k\mathcal{K}_n}(S, V^{\otimes n})$$

and

$$S(V) := S \otimes_{k\mathcal{K}_n} V^{\otimes n}$$

are again representations of $\mathbb{Z}/p^e\mathbb{Z}$ (for V in $\mathfrak{F}_{p^e}$). Two such functors are

- the exterior power $\wedge^r(V)$
- the symmetric power $S_r(V)$.

<u>Main Problem</u>: <u>Given</u> S <u>as above</u>, <u>determine the indecomposable constituents of</u> V_n^S <u>and</u> $S(V_n)$ <u>for all</u> n.

In particular, the functors $\wedge^r$ and S_r determine pre-λ-ring structures on R(p). Define

$$\lambda_t : R(p) \to R(p)[[t]]$$

and

$$\sigma_t : R(p) \to R(p)[[t]]$$

by

$$\lambda_t([V]) := \sum_{r=0}^{\infty} [\wedge^r(V)]t^r$$

and

$$\sigma_t([V]) := \sum_{r=0}^{\infty} [S_r(V)]t^r \quad \text{for each } V \text{ in } \mathfrak{F}$$

and extend by the rules

$$\lambda_t(x+y) = \lambda_t(x)\lambda_t(y)$$

and

$$\sigma_t(x+y) = \sigma_t(x)\sigma_t(y) .$$

(Note that all these "operations" can also be defined for R(0). At the end of this section we will go through the details for this ring in order to illustrate the differences between positive characteristic and zero characteristic.) The first results are negative in nature.

Proposition 2.5. The series $\lambda_t(x_0)$ and $\sigma_t(x_0)$ are the following

$$\lambda_t(x_0) = 1 + x_0 t + t^2$$

$$\sigma_t(x_0) = \frac{1 - ([V_p] - [V_{p-1}])t^p}{(1-t^p)(1 - x_0 t + t^2)} .$$

Corollary 2.6. It does not follow that

$$\lambda_{-t}(x_0)\sigma_t(x_0) = 1 .$$

Proof of Proposition: Let V be in $\mathfrak{F}_{p^e}$ and set $r = qp^e + r_0$ where $0 \leq r_0 < p^e$. In [Almkvist-Fossum] it is shown that

$$S_r(V) \cong S_{r_0}(V) \oplus nV_{p^e}$$

where

$$n = \frac{1}{p^e} \left\{ \binom{\dim V + r - 1}{r} - \binom{\dim V + r_0 - 1}{r_0} \right\} .$$

So it is enough to know the decompositions of $S_r(V)$ for $0 \leq r \leq p^e$ (when $S^{p^e} V = 0$).

Also let $v_{n+1}(x)$ be the polynomial (with coefficients in $\mathbb{Z}$) with generating function

$$\sum_{n=0}^{\infty} v_n(x)t^n = (1 - xt + t^2)^{-1} .$$

(These are Chebyshev polynomials.) It follows that

$$V_{r+1} \cong S_r(V_2) = v_{n+1}(x_0)$$

for $0 \leq r+1 \leq p$. Hence

$$\sigma_t(x_0) \equiv (1 - xt + t^2)^{-1} \bmod t^p .$$

Further calculation yields the formula for $\sigma_t(x_0)$. The formula for $\lambda_t(x_0)$ is clear since $\wedge^2(x_0) = \wedge^2(V_2) = V_1 = 1$ and $\wedge^r(x_0) = 0$ for $r > 2$. □

A quick calculation shows that R(p) is not a λ-ring for the λ-operations -- for, in case characteristic $p = 2$, then

$$\wedge^2(V_2 \otimes V_2) = \wedge^2(V_2 \oplus V_2) = 2(V_1 \oplus V_2)$$

while the λ-ring structure would yield

$$\begin{aligned}\wedge^2(V_2 \otimes V_2) &= 2\wedge^2(V_2) \otimes (V_2 \otimes V_2 - V_1) \\ &= 2(2V_2 - V_1) .\end{aligned}$$

In characteristic $p = 3$ we calculate

$$S^2(V_2 \otimes V_2) = S^3(V_1 \oplus V_3) = 2V_1 \oplus 6V_3$$

whereas the λ-ring structure would yield

$$S^3(V_2 \otimes V_2) = 4V_1 \oplus 2V_2 \oplus 4V_3 .$$

Hence these pre-λ-rings are <u>not</u> λ-rings

<u>More Particular Problems 2.7</u>.

a. Find the decompositions for $\wedge^r(V_n)$ and $S_r(V_n)$.

b. What are the multiplicative properties of

$$\lambda_t : R(p) \to R(p)$$

$$\sigma_t : R(p) \to R(p) \ ?$$

In particular can one describe $\wedge^r(V_n \otimes V_m)$?

c. What role does

$$\lambda_{-t}(x)\sigma_t(x)$$

play in $R(p)[[t]]$?

It seems as if the divided power functor $S_r(V^\vee)^\vee$ has been ignored in this discussion (where $V^\vee := \mathrm{Hom}_k(V,k)$). In particular ${}^\vee : R(p) \to R(p)$ can be defined. But it is clear that $V_n^\vee \cong V_n$ as A-modules. Hence $D_r(V) := S_r(V^\vee)^\vee$ has the same decomposition as $S_r(V)$. There may be some advantage in working with D_r, but it is not clear that this is the case.

d. The ring A has other formal group laws which induce a product on $\mathfrak{F}$. What can be said about the corresponding ring structure of $R(p)$?

e. Similar questions can be posed for

$$k[[S_1,S_2]] \text{ , etc.}$$

The ring $k[[S_1,S_2]]/(S_1^p,S_2^p)$ has infinitely many indecomposables which makes the problem infinitely more difficult.

This section is concluded with the promised details concerning $R(0)$. The first result is well known.

Lemma 2.8. Let V be a vector space over k. Consider the homomorphisms

$$\wedge^r(V) \otimes S_q(V) \xrightarrow{d_q^r} \wedge^{r-1}(V) \otimes S_{q+1}(V) \text{ ,}$$

gotten by extending the assignment

$$d_q^r((v_1 \wedge v_2 \wedge \dots \wedge v_r) \otimes (w_1 \dots w_q))$$

$$= \sum_{j=1}^{r} (-1)^{r-j}(v_1 \wedge \dots \wedge \hat{v}_j \wedge \dots \wedge v_r) \otimes v_j w_1 \dots w_w \text{ ,}$$

and

$$\wedge^r(V) \otimes S_q(V) \xrightarrow{\sigma_q^r} \wedge^{r+1}(V) \otimes S_{q-1}(V)$$

gotten by extending

$$\sigma_q^r(v_1 \wedge \dots \wedge v_r \otimes w_1 \dots w_q) = \sum_{j=1}^{q} (v_1 \wedge \dots \wedge v_r \wedge w_j) \otimes w_1 \dots \hat{w}_j \dots w_w \text{ .}$$

Then

$$\sigma_{q+1}^{r-1} \circ d_q^r + d_{q-1}^{r+1} \circ \sigma_q^r = (r+q)\mathrm{Id} \text{ .}$$

Proof. Straightforward computation. □

Corollary 2.9. If $n = (r+q)$ is invertible in k, then the complex

$$(K_\cdot(V), d_\cdot)$$

with

$$K_r = \Lambda^r(V) \otimes S_{n-r}(V)$$

and

$$d_r = d^r_{n-r}$$

is a split exact complex (over $GL(V)$). □

In particular, suppose $\operatorname{rk} V = 2$. Then $\Lambda^r V = 0$ for $r > 2$ while $\Lambda^2 V \cong k$. Hence we get a second corollary.

Corollary 2.10. If n is invertible in k and $\operatorname{rk} V = 2$, then there is a $GL_k(V)$ isomorphism

$$V \underset{k}{\otimes} S_{n-1}(V) \cong S_n(V) \oplus S_{n-2}(V) .$$ □

Suppose that char(k) = 0 and consider R(0). It follows that the indecomposable $V_{n+1} \cong S_n(V_2)$. Also (remember $x_0 = V_2$)

$$\lambda_t(x_0) = 1 + x_0 t + t^2$$

and

$$\sigma_t(x_0) = (1 - x_0 t + t^2)^{-1} .$$

Further calculations yield the following formulas:

Proposition 2.11. In R(0)

$$\Lambda^r(V_n) = \frac{V_n \otimes V_{n-1} \otimes \cdots \otimes V_{n-r+1}}{V_r \otimes V_{r-1} \otimes \cdots \otimes V_1}$$

$$S_r(V_{n+1}) = \frac{V_{n+1} \otimes \cdots \otimes V_{n+r}}{V_1 \otimes V_2 \otimes \cdots \otimes V_r} .$$ □

Finally it should be noted that in [Almkvist-Fossum] the decompositions of $\Lambda^r(V)$ and $S_r(V)$ are obtained for all V in $\mathfrak{F}_p$.

§3. Consequences

In this section some of the results related to other topics are mentioned. Again we suppose k is of characteristic $p > 0$. Let V be a vector space over k of finite rank. By $S_{\cdot}(V)$ is meant the symmetric algebra on V and

$$S_{\cdot}(V) = \bigoplus_{r \geq 0} S_r(V) .$$

By $\hat{S}_{\cdot}(V)$ is meant the formal power series on V and

$$\hat{S}_{\cdot}(V) = \prod_{r \geq 0} S_r(V) .$$

Suppose V is an (irreducible) $\mathbb{Z}/p^e\mathbb{Z}$-module. Then the rings $S_{\cdot}(V)$ and $\hat{S}_{\cdot}(V)$ are equipped with a $\mathbb{Z}/p^e\mathbb{Z}$-action, and the invariant rings provide examples of rings satisfying various properties and not satisfying others.

Proposition 3.1 [Ellingsrud-Skjelbred]. If $\dim_k V \geq 4$ and V is irreducible, then

$$\operatorname{depth}(\hat{S}_{\cdot}(V)^{\mathbb{Z}/p^e\mathbb{Z}}) = 3 . \qquad \square$$

Thus these rings are not Cohen-Macauley.

Proposition 3.2. The rings $S_{\cdot}(V)^{\mathbb{Z}/p^e\mathbb{Z}}$ are factorial.

Proof. The divisor class group of the ring of invariants is isomorphic to (a subgroup of)

$$H^1(\mathbb{Z}/p^e\mathbb{Z}, \mu(S_{\cdot}(V))) .$$

But the group of units

$$\mu(S_{\cdot}(V)) \cong \mu(k)$$

on which $\mathbb{Z}/p^e\mathbb{Z}$ acts trivially. Thus

$$H^1(\mathbb{Z}/p^e\mathbb{Z}, \mu(S_{\cdot}(V)) = \operatorname{Hom}(\mathbb{Z}/p^e\mathbb{Z}, \mu(k)) = 0 . \qquad \square$$

This proof is due to [Samuel].

Thus we have examples of factorial domains that are not Cohen-Macaulay. In [Fossum-Griffith] one can find a proof that the completions are also factorial in case $\dim_k V = p^e$.

Problem 3.3. Are the rings of invariants of

$$\hat{S}_{\cdot}(V)$$

factorial?

There is a close relationship between the decomposition problem and the Hilbert-Poincaré Series of the ring of invariants. For

$$\mathrm{rk}_k(S_r(V)^{\mathbb{Z}/p^e\mathbb{Z}}) = \text{number of indecomposables}$$

that appear in $S_r(V)$. The Hilbert-Poincaré Series

$$H_t(S_{\cdot}(V)^{\mathbb{Z}/p^e\mathbb{Z}}) := \sum_{r=0}^{\infty} \mathrm{rk}_k(S_r(V)^{\mathbb{Z}/p^e\mathbb{Z}})t^r .$$

Again the calculations of the decompositions in [Almkvist-Fossum] permit the calculation of certain of these series. Rather than give examples of these calculations in this note, a corollary is stated. Let $f(t) = H_r(S_{\cdot}(V)^{\mathbb{Z}/p\mathbb{Z}})$.

Proposition 3.4. There is a functional equation

$$f(1/t) = (-1)^d t^r f(t)$$

if and only if $\dim_k V$ is even or $\dim_k V = 1$.

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