Astérisque

J. L. BRYLINSKI

Differential operators on the flag varieties

Astérisque, tome 87-88 (1981), p. 43-60
<http://www.numdam.org/item?id=AST_1981__87-88__43_0>

DIFFERENTIAL OPERATORS ON THE FLAG VARIETIES

by J. L. BRYLINSKI

Lecture given at the Conference on "Young tableaux and Schur functors in Algebra and Geometry", held at Torun (Poland) (27 August-3 September 1980).

Let G be a connected semi-simple algebraic group over a field k of characteristic 0. Let X be the flag variety of G, also called the variety of Borel subgroups of G. It is well known that X is a projective variety over k, that G operates on X on the left, in such a way that X = G. x for any x ∈ X, and that the stabilizer of x ∈ X is a Borel subgroup. We let $\mathcal{D}_X$ be the sheaf of algebraic differential operators of finite order on X (a sheaf for the Zariski topology). In this paper, we determine the algebra structure of $\Gamma(X,\mathcal{D}_X)$, the algebra of global differential operators on X.

It is easy to convince oneself that this algebra should be expressed in terms of the Lie algebra $\mathfrak{g}$ of G. Indeed, if $G_1 \to G$ is an isogeny, then G_1 and G have isomorphic flag varieties. Viewing $\mathfrak{g}$ as the Lie algebra of right-invariant vector fields on G, one defines a Lie algebra homomorphism $\mathfrak{g} \xrightarrow{\varphi} \Gamma(X,\mathcal{D}_X)$, whence an algebra homomorphism $U(\mathfrak{g}) \xrightarrow{\varphi} \Gamma(X,\mathcal{D}_X)$, where $U(\mathfrak{g})$ is the enveloping algebra. Let $\underline{Z}$ be the center of $U(\mathfrak{g})$, $J = \underline{Z} \cap U(\mathfrak{g}).\mathfrak{g}$. One first shows that $\varphi(J) = 0$. In other words : every global differential on X, invariant under G, is of order 0. Therefore, defining $I = U(\mathfrak{g}).J$ and $R = U(\mathfrak{g})/I$, one gets an algebra morphism : $\Phi : R \to P(X,\mathcal{D}_X)$.

Theorem : Φ is an isomorphism; Φ is also G-equivariant. Note that G acts on R via the adjoint action.

The method of the proof is to use the action of $\Gamma(X,\mathcal{D}_X)$ on local cohomology groups $H^i_Z(X,\mathcal{O}_X)$ or $H^i_{Z_1/Z_2}(X,\mathcal{O}_X)$, together with the description of these groups as $U(\mathfrak{g})$-modules given by Kempf [11], [12], in case Z, Z_1, Z_2 are Schubert varieties in X. One then applies results of Duflo and of Conze-Berline on Verma modules.

It would be very difficult to compute directly $\Gamma(X,\mathcal{D}_X)$. The method of filtering $\mathcal{D}_X$ in such a way that the quotients are of rank one only lead to a despairing mess. Let me now hazard the following

Conjecture : $H^i(X,\mathcal{D}_X) = 0$ for $i > 0$.

[After this was written, I learnt that Beilinson and Bernstein had a different proof that Φ is an isomorphism. They also showed that $H^i(X,\mathcal{M}) = 0$ for $i > 0$, and any $\mathcal{D}_X$-module $\mathcal{M}$ which is a quasi-coherent $\mathcal{O}_X$-module. This plays an important part in their solution to the Kazhdan-Lusztig conjecture, which they found independently, in the same time as we devised our proof. Also, I learnt from Renée Elkik-Latour that a few years ago, she proved the vanishing of higher cohomology of symmetric powers of the tangent bundle to X, which in particular implies $H^i(X,\mathcal{D}_X(m)) = 0$ for $i > 0$ and therefore $H^i(X,\mathcal{D}_X) = \varinjlim_m H^i(X,\mathcal{D}_X(m)) = 0$.]

One may generalize the theorem as follows. For $\mathcal{L}$ an invertible sheaf on X, we consider the sheaf of algebra $\mathcal{L} \otimes \mathcal{D}_X \otimes \mathcal{L}^{-1}$. This may be called the sheaf of algebra of differential operators on $\mathcal{L}$. Then a morphism analogous to Φ is shown to be an isomorphism.

One should point out that the morphism Φ plays an important part in the proof of the Kazhdan-Lusztig conjecture, found by Kashiwara and myself [ ], [ ]. However, in this proof, we do not need the fact that Φ is an isomorphism. As a conclusion to this lecture, I give a conjectural generalization of the main theorem of [4] where $\mathcal{O}_X$ is replaced by an invertible sheaf, and attempt to describe an action of the Weyl group ($k = \mathbb{C}$) on the K-groups of the following categories :

- the derived category of the cohomology of bounded complexes of $U(\mathfrak{g})$-modules, with a given infinitesimal character, the cohomology spaces of which belong to the category $\tilde{\mathcal{O}}_{triv}$ of [3], [4]
- the derived category of the category of bounded complexes of sheaves, whose cohomology sheaves are constructible.

It would be desirable to make W act on the derived categories themselves, but this does not seem possible, as the example $G = SL(2)$ shows. Perhaps one would hope to make a suitable covering $\tilde{W}$ of W act.

I would like to thank Michel Demazure for several interesting ideas on how to understand Kempf's article [11] and Michel Duflo for a very useful phone conversation (he suggested the use of a theorem of Nicole Conze in order to prove that Φ is surjective). Also, I benefited from a conversation with Fedor Bogomolov and Pierre Deligne.

§ 1. Collection of facts on enveloping algebras and Verma modules

For any Lie algebra $\mathfrak{a}$, we denote by $U(\mathfrak{a})$ its enveloping algebra. Any Lie algebra homomorphism from $\mathfrak{a}$ to an associative algebra B uniquely extends to an algebra homomorphism from $U(\mathfrak{a})$ to B. It follows that one may identify $\mathfrak{a}$-modules and $U(\mathfrak{a})$-modules.

Recall the Poincaré-Birkhoff-Witt theorem (in short P-B-W). Let S be a totally ordered set, $(X_\alpha)_{\alpha \in S}$ a basis for a Lie algebra over a field k. Then the elements $X_{\alpha_1} . X_{\alpha_2} . \dots . X_{\alpha_n}$ where n is any integer $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n$, form a basis of $U(\mathfrak{a})$. This has the following consequence : if $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are sub Lie-algebras of $\mathfrak{a}$ such that $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$, one has : $U(\mathfrak{a}) \cong U(\mathfrak{a}_1) \otimes U(\mathfrak{a}_2)$ (isomorphism of $(U(\mathfrak{a}_1), U(\mathfrak{a}_2))$-bimodules). Similarly, if $\mathfrak{b}$ is a Lie subalgebra of $\mathfrak{a}$, $U(\mathfrak{a})$ is free, as a left or right $U(\mathfrak{b})$-module.

Specialize these considerations to the case of a semi-simple Lie algebra $\mathfrak{g}$ over a field k of characteristic 0. Choose a Borel subalgebra $\mathfrak{b}$, a Cartan subalgebra t. One has the usual decomposition : $\mathfrak{b} = t \oplus n^+$. Also one can choose a nilpotent subalgebra n^- such that $\mathfrak{g} = n^- \oplus \mathfrak{b} = n^- \oplus t \oplus n^+$. From P-B-W, one has a decomposition : $U(\mathfrak{g}) = U(t) \oplus (n^- . U(\mathfrak{g}) + U(\mathfrak{g}) . n^+)$, which gives a projection $p : U(\mathfrak{g}) \to U(t)$.

Now let $\rho \in t^*$ be half the sum of positive roots (= eigenvalues of the adjoint action of t on n^+). Let W be the Weyl group. W operates on U(t) as follows. First $U(t) = S(t)$ is the algebra of regular functions on t^*. So to define the action of W on U(t), it suffices to make W act on the affine space t^*. There is a natural linear action of W on t^*. One just conjugates this action by the translation of vector $+\rho$, so that $-\rho$ is the common fixed point of all elements of W. This "twisted" action is denoted by $(w,\lambda) \to w * \lambda$. With these preparations, one can state the :

Harish-Chandra's theorem : Let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$ then p induces an algebra homomorphism from $Z(\mathfrak{g})$ to $U(t)^W$, the algebra of invariants of W operating on U(t).

Corollary : If $\ell = \dim_k(t)$, $Z(\mathfrak{g})$ is isomorphic to a polynomial algebra in ℓ variables over k.

Let $\rho : U(\mathfrak{g}) \to End(V)$ be a representation of $U(\mathfrak{g})$ in a k-vector space V. Then ρ is said to have infinitesimal character χ (χ a homomorphism $Z(\mathfrak{g}) \to k$) if one has :

$$\rho(z).v = \chi(z).v \quad \text{for all } z \in Z(\mathfrak{g}),\ v \in V.$$

Remark that characters of $Z(\mathfrak{g})$ correspond bijectively to orbits of W operating on t^* (by the action explained above).

Now fix a character λ of t. Extend λ to a character on $\mathfrak{b}$, o on n^+, which is still called λ. Then extend $\lambda : \mathfrak{b} \longrightarrow k$ to a ring homomorphism : $\lambda : U(\mathfrak{b}) \to k$. Then $M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} k_\lambda$ ($U(\mathfrak{b})$ operating on k_λ via λ) is a left $U(\mathfrak{g})$-module, which is called the <u>Verma module with highest weight</u> λ. Since $U(\mathfrak{g}) \cong U(n^-) \otimes U(\mathfrak{b})$, it follows that M_λ is free of rank one as a $U(n^-)$-module, with generator $1 \otimes 1_\lambda$; M_λ has the following properties :

1) for any $u \in M_\lambda$, $\dim_k(U(\mathfrak{b}).u) < \infty$
2) one can write a direct sum decomposition :

$$M_\lambda = \bigoplus_{\substack{\mu \in t^* \\ \mu \leqslant \lambda}} (M_\lambda)^\mu \ , \ \dim(M_\lambda)^\mu < +\infty$$

where $(M_\lambda)^\mu$ is a $U(t)$ submodule on which $U(t)$ operates through the character μ. Here $\mu \leqslant \lambda$ means that $\lambda - \mu = \sum_{\alpha \in R_+} n_\alpha . \alpha$, $\alpha \in \mathbb{N}$.

3) M_λ has inifinitesimal character corresponding to λ. One defines the <u>character</u> of M_λ to be the formal sum

$$ch(M_\lambda) = \sum_\mu \dim (M_\lambda)^\mu \, e^\mu$$

A $U(\mathfrak{g})$-module M is called t-diagonalizable if one can write a decomposition $M = \bigoplus_{\mu \in t^*} M^\mu$ such as in 2). One has the following lemma, which will be used later.

<u>Lemma 1</u> : Let M a $U(\mathfrak{g})$ submodule of M_λ, such that $ch(M) = ch(M_\mu)$. Then M is isomorphic to M_μ.

The homomorphisms from M_μ to M_λ are known. We will only need the following

<u>First theorem of Verma</u> : If $\mathrm{Hom}_{U(\mathfrak{g})}(M_\mu, M_\lambda) \neq 0$, then $\mu \in W * \lambda$. If furthermore λ is <u>antidominant</u>, then $\mu = \lambda$. In that case, M_λ is an irreductible $U(\mathfrak{g})$-module.

I must define what is the condition for λ to be antidominant. For any simple root α, there is a corresponding element $s_\alpha \in W$ of order 2. One has $s_\alpha(\lambda) = \lambda - c_\alpha . \alpha$. Then λ is antidominant if no c_α is equal to 0,1,2,...

<u>Second theorem of Verma</u> : Any homomorphism from M_μ to M_λ is either zero or injective.

Finally there is an interesting category $\mathcal{O}$ of $U(\mathfrak{g})$-modules which contains all Verma modules. A module M is in $\mathcal{O}$ iff

1) for any $u \in M, \dim_k (U(b).u) < \infty$

2) one can write $M = \bigoplus_{u \in t^*} M^{\mu}$, $\dim(M^{\mu}) < \infty$ as above.

3) M is a finitely generated $U(\mathfrak{g})$-module. This category was introduced by Bernstein-Gelfand-Gelfand [1]. For $\lambda \in t^*$, there is a corresponding character χ_λ of $Z(\mathfrak{g})$. Let $\mathcal{O}_\lambda$ be the full subcategory of $\mathcal{O}$ made of modules which have the infinitesimal character χ_λ. For $\lambda = 0$, this category is denoted by $\mathcal{O}_{triv}$.

For any module M in $\mathcal{O}$, M is a union of sub-$U(\mathfrak{b})$-modules of finite dimension.

Finally, let us compute $ch(M_\lambda)$.

<u>Lemma 2</u> : $ch(M_\lambda) = \dfrac{e^\lambda}{\prod_{\alpha \in R_+} (1-e^{-\alpha})}$. Indeed M_λ is isomorphic to $U(n^-) \otimes k_\lambda$ as a U(t)-module. So one has $ch(M_\lambda) = e^\lambda .ch(U(n^-)) = e^\lambda .ch\ (S(n^-))$. Writing $n^- = \bigoplus_{\alpha \in R_+} n_{-\alpha}$, one has :

$$ch(M_\lambda) = e^\lambda . \prod_{\alpha \in R_+} ch(S(n_{-\alpha})) = e^\lambda . \prod_{\alpha \in R_+} (1 + e^{-\alpha} + e^{-2\alpha} + \ldots).$$

Q.E.D.

For the results in this paragraph, one may refer to [7] .

§ 2. <u>Cohomology with support and differential operators</u>

Let X be a topological space, $Z \subset X$ a closed subset, $\mathcal{F}$ a sheaf of abelian groups on X.

<u>Definition 1</u> : i) $\Gamma_Z(X,\mathcal{F}) = \ker \Gamma(X,\mathcal{F}) \longrightarrow \Gamma(X - Z,\mathcal{F})$

ii) $\mathcal{F} \longmapsto H^i_Z(X,\mathcal{F})$ is the i-th right derived function of $\Gamma_Z(X,-)$

iii) one has a long exact sequence

$$\ldots\ldots \longrightarrow H^i_Z(X,\mathcal{F}) \longrightarrow H^i(X,\mathcal{F}) \longrightarrow H^i(X - Z,\mathcal{F}) \xrightarrow{\partial} H^{i+1}_Z(X,\mathcal{F}) \longrightarrow \ldots\ldots$$

iv) if $Z_2 \subset Z_1$ are closed subsets of X, there is a natural map $H^i_{Z_2}(X,\mathcal{F}) \to H^i_{Z_1}(X,\mathcal{F})$ and a morphism of the two exact sequences described in (iii).

v) let U be an open subset of X containing Z. Then the restriction map : $H^i_Z(X,\mathcal{F}) \to H^i_{Z \cap U}(U, \mathcal{F}|_U)$ is an isomorphism..

Now we let X be a smooth algebraic variety over a field k. Let $\mathcal{O}_X$ be the structural sheaf. Let $\mathcal{D}_X$ be the sheaf of differential operators of finite order on X. One has : $\mathcal{D}_X = \bigcup_{m \in \mathbb{N}} \mathcal{D}_X(m)$, where $\mathcal{D}_X(m)$ is the sheaf of differential operators of order $\leqslant m$.

<u>Proposition 1</u> : Let $\mathcal{M}$ be a coherent sheaf of left $\mathcal{D}_X$-modules. Then $\Gamma(X,\mathcal{D}_X)$ operates in a natural way on $H^i_Z(X,\mathcal{M})$. This operation is natural with respect to $\mathcal{M}$, and with respect to Z (see (iv) of Proposition 1). All maps in the exact sequence (iii) are $\Gamma(X,\mathcal{D}_X)$-linear.

To describe, for instance, the action of $\Gamma(X,\mathcal{D}_X)$ on $H^i(X,\mathcal{M})$, one notices that $\mathcal{M}$ is quasi-coherent as a $\mathcal{O}_X$-module (this is because $\mathcal{D}_X$ is a union of coherent $\mathcal{O}_X$-submodules). Then choose an affine open covering $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ of X. Then $H^i(X,\mathcal{M}) \cong H^i(\mathcal{U},\mathcal{M})$, which is the i-th cohomology group of the Czech complex $\check{\mathcal{C}}^\cdot(\mathcal{U},\mathcal{M})$. It suffices to descrive an operation of $\Gamma(X,\mathcal{D}_X)$ on $\Gamma(\bigcap_{\alpha \in B} U_\alpha,\mathcal{M})$ for B a finite subset of A. One has a restriction map :

$\Gamma(X,\mathcal{D}_X) \rightarrow \Gamma(\bigcap_{\alpha \in B} U_\alpha, \mathcal{D}_X)$ and the latter ring operates on $\Gamma(\bigcap_{\alpha \in B} U_\alpha,\mathcal{M})$ because $\mathcal{M}$ is a sheaf of $\mathcal{D}_X$-modules. This operation is obviously compatible with differentials in $\check{\mathcal{C}}^\cdot(\mathcal{U},\mathcal{M})$.

Since $\mathcal{D}_X$ operates on $\mathcal{O}_X$, this Proposition applies to $\mathcal{M} = \mathcal{O}_X$.

Now let us define cohomology with relative support. Let $Z_2 \subset Z_1$ be closed subsets of X (X is again any topological space).

<u>Definition 2</u> : i) $\Gamma_{Z_1|Z_2}(\mathcal{F}) = \operatorname{coker}(\Gamma_{Z_2}(X,\mathcal{F}) \rightarrow \Gamma_{Z_1}(X,\mathcal{F}))$

ii) $\mathcal{F} \mapsto H^i_{Z_1|Z_2}(X,\mathcal{F})$ is the i-th right derived functor of $\Gamma_{Z_1|Z_2}(X,-)$

iii) there exists a long exact sequence

$$\cdots \longrightarrow H^i_{Z_2}(X,\mathcal{F}) \longrightarrow H^i_{Z_1}(X,\mathcal{F}) \longrightarrow H^i_{Z_1|Z_2}(X,\mathcal{F}) \longrightarrow H^{i+1}_{Z_2}(X,\mathcal{F}) \longrightarrow \cdots$$

iv) as in Definition 1, one has functoriality with respect to the pair (Z_1,Z_2).

v) there is an "excision" isomorphism $H^i_{Z_1|Z_2}(X,\mathcal{F}) \xrightarrow{\sim} H^i_{Z_1-Z_2}(X-Z_1,\mathcal{F})$

(first one checks this for i = 0 and $\mathcal{F}$ flasque; the general case follows by considering a flasque resolution).

Now X is again an algebraic variety.

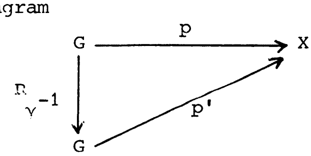
Proposition 2 : If $\mathcal{M}$ is a sheaf of $\mathcal{D}_X$-modules, $\Gamma(X,\mathcal{D}_X)$ operates naturally on $H^i_{Z_1 | Z_2}(X,\mathcal{M})$, etc...

Remark that there is now a very good reference for cohomology with support namely [12], § 7 and 8.

§ 3. Differential operators on the flag variety

Now X is the flag variety of G (see the introduction). One has $\Gamma(X,\mathcal{D}_X) = \bigcup_{m \in \mathbb{N}} \Gamma(X,\mathcal{D}_X(m))$ and each $\Gamma(X,\mathcal{D}_X(m))$ is a finite-dimensional k-vector space, since X is projective and $\mathcal{D}_X^{(m)}$ is a coherent $\mathcal{O}_X$-module.

The Lie algebra $\mathfrak{g}$ of G will be viewed as the Lie-algebra of right invariant vector fields on G. To each $\xi \in \mathfrak{g}$, we associate a vector field $\tilde{\xi}$ on X. To do this, one first chooses a base point x of X, of stabilizer B (if such a point does not exist on k, one just performs a finite extension of k; the construction of $\tilde{\xi}$ will anyhow be independent of the choice of x, so the mapping $\xi \mapsto \tilde{\xi}$ will be defined over k). Now consider the map $p : G \longrightarrow X$, $p(g) = g.x$. Then $\tilde{\xi}$ is such that $dp_g(\xi) = \tilde{\xi}_{p(g)}$. This is well-defined because ξ is right invariant. Now show that $\tilde{\xi}$ does not depend on the choice of x. Consider $x = \gamma.x(\gamma \in G)$. One has a commutative diagram



where $R_{\gamma^{-1}}$ is right translation by γ^{-1}. Then

$$\begin{aligned}\tilde{\xi}_{p(g)} = dp_g(\xi) &= dp'_{g\gamma^{-1}} \circ (d[R_{\gamma^{-1}}]_g(\xi)) \\ &= dp'_{g\gamma^{-1}}(\xi) \\ &= \tilde{\xi}'_{p'(g\gamma^{-1})} \\ &= \tilde{\xi}'_{p(g)}\end{aligned}$$

where right-invariance of ξ has again been used.

One has therefore a Lie algebra homomorphism $\mathfrak{g} \longrightarrow \Gamma(X,\mathcal{D}_X)$ sending ξ to $\tilde{\xi}$. Whence an algebra homomorphism $U(\mathfrak{g}) \xrightarrow{\varphi} \Gamma(X,\mathcal{D}_X)$.

Now let J be the kernel of the character of $\chi_o : Z(\mathfrak{g}) \longrightarrow k$ (one can also describe J as the intersection of $Z(\mathfrak{g})$ with $U(\mathfrak{g}).\mathfrak{g}$.

Proposition 3 : $\varphi(J) = 0$

We will later give a nice proof of this Proposition. Let us briefly outline another, not so nice, proof. One has to show that every element in $\Gamma(X,\mathcal{D}_X)$ which is G-invariant is of order 0 (i.e. a constant). It suffices to show that for $m \geq 1$, $\Gamma(X,\mathcal{D}_X(m)/\mathcal{D}_X(m-1))$ has no non-zero element invariant under G. But the sheaf $\mathcal{D}_X(m)/\mathcal{D}_X(m-1)$ is isomorphic to $S^m(T_X)$, where T_X is the tangent bundle.

Now T_X admits a filtration $\mathcal{F}_o = 0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_{i-1} \subset \mathcal{F}_i \subset \ldots \mathcal{F}_N = TX$ ($N = \dim X$), with $\mathcal{F}_i/\mathcal{F}_{i-1}$, locally free of rank one.

The corresponding characters of T (or of B) are precisely the positive roots.

One deduce for $S^m T_X$ a similar filtration; the associated characters of T are of type $\sum_{\alpha \in R_+} n_\alpha . \alpha$, $n_\alpha \in \mathbb{N}$, $\sum_\alpha n_\alpha \geq 1$. But the theorem of Borel-Weil-Bott (see [13]) implies that $H^o(X,\mathcal{L})^G = 0$ for $\mathcal{L}$ invertible unless $\mathcal{L} \cong \mathcal{O}_X$. Since no element $\sum_{\alpha \in R_+} n_\alpha . \alpha$ as above can be 0, we are done.

Let I be the ideal $U(\mathfrak{g}) . J$ of $U(\mathfrak{g})$. One gets a factorization of φ through $\Phi: U(\mathfrak{g})/I \longrightarrow \Gamma(X,\mathcal{D}_X)$.

Theorem 1 : Φ is an isomorphism.

Corollary : $\Gamma(X,\mathcal{D}_X)$ is generated, as an algebra, by the Lie algebra $\mathfrak{g}$, which is the space of vector fields on X.

Remark : Φ is G-equivariant, G acting on $U(\mathfrak{g})/I$ via adjoint action and on $\Gamma(X,\mathcal{D}_X)$ via its action on X. In particular, as a G-module, $\Gamma(X,\mathcal{D}_X)$ is isomorphic to the space of regular functions on the nilpotent variety of $\mathfrak{g}$ (put together Proposition 2.4.10 and Théorème 8.1.3 of [7]). This was pointed out to me by Procesi. This remark receives a fine explanation in the work of Beilinson and Bernstein.

To prove this theorem, we will make $\Gamma(X,\mathcal{D}_X)$ operate on cohomology groups of $\mathcal{O}_X$ with support in well chosen closed subsets of X. To define these subsets, let B be the Borel subgroup of G with Lie algebra $\mathfrak{b}$. Recall that the orbits of B in X are narurally indexed by W. Indeed, let x be the unique point of X such that B.x = x. Then the Bruhat decomposition $G = \coprod_{w \in W} B w B$ gives $X = G.x = \coprod_{w \in W} (BwBx) = \coprod_{w \in W} Bwx = \coprod_{w \in W} X_w$. Each X_w is a locally closed subset of X

The following facts are known :

- the dimension of Z_w is the length $\ell(w)$ of w (w is a product of $\ell(w)$ elements s_α, α a simple root, but not of k such elements, for $k < \ell(w)$).

- $\underline{Z_w}$ is an affine space
- Z_w is Cohen-Macaulay

We will be interested in the $\Gamma(X,\mathcal{D}_X)$-modules $H^k_{\bar{X}_w/\partial(X_w)}(X,\mathcal{O}_X)$.

Proposition 4 : (i) $H^k_{\bar{X}_w/\partial(X_w)}(X,\mathcal{O}_X) = 0$ for $k \neq N - \ell(w)$

(ii) $U(\mathfrak{g})/I$ acts on $N_w = H^{N-\ell(w)}_{\bar{X}_w/\partial(X_w)}(X,\mathcal{O}_X)$ via Φ . N_w is the union of finite dimensional sub $U(\mathfrak{b})$-modules, on which the action of $\mathfrak{b}$ is the differential of an algebraic action of the algebraic group B.

All this is proved by Kempf [12] . One needs only to remark that, putting $Z_i = \bigcup_{\ell(w) \leqslant i} Z_w$, one has a filtration $Z_0 \subset Z_1 \subset \ldots \subset Z_N = X$, and the excision isomorphism of Proposition 2, (v) implies :

$$H^k_{Z_i/Z_{i-1}}(X,\mathcal{O}_X) \cong \bigoplus_{\ell(w)=i} \cdot H^k_{\bar{Z}_w/\partial(Z_w)}(X,\mathcal{O}_X) .$$

Furthermore, Kempf proves (§ 11 and § 12) that N_w is t-diagonalizable, and computes the character $ch(N_w)$. He shows that $ch(N_w) = \dfrac{e^{-w(\rho)-\rho}}{\prod_{\alpha \in R_+}(1-e^{-\alpha})}$.

Using lemma 2, we get :

Proposition 5 : $ch(N_w) = ch(M_w)$, where M_w is the Verma module $M_{-w(\rho)-\rho}$.

Now we want to identify the $U(\mathfrak{g})$-modules N_w. We first begin with $w = w_0$, the element of longest length in W. Then X_{w_0} is open in X, and we have :

$$N_{w_0} = H^0_{\bar{X}_{w_0}/\partial(X_{w_0})}(X,\mathcal{O}_X) \cong H^0(X_{w_0}, \mathcal{O}_{X_{w_0}})$$

Now, inside $Hom_k(N_{w_0},k)$ let $N^*_{w_0}$ be the space of elements ℓ such that $\dim_k(U(t)\,\ell) < +\infty$. So if $N_{w_0} = \bigoplus_\lambda M^\lambda$, then $N^*_{w_0} = \bigoplus_\lambda (M^\lambda)^*$. Then define a "twisted" action of $U(\mathfrak{g})$ on $N^*_{w_0}$, twisting the natural action by an automorphism τ of $\mathfrak{g}$, which induces -1 on t and sends X_α to $X_{-\alpha}$ (for a given choice of X_α ($\alpha \in R$) in an "épinglage" of $\mathfrak{g}$). Then one has the following result, which was announced by Kempf [12] , but without details.

Proposition 6 : $(N_{w_0})^*$ is isomorphic to M_{w_0} . Indeed, one has $X_{w_0} = N_+ . x_0 \cong N_+$ (N_+ is the unipotent radical of B). Before $N^*_{w_0}$ had a twisted $U(\mathfrak{g})$-module structure,

$(N_{w_o})^*$ was therefore a free $U(n^+)$-module generated by the element of $\mathrm{Hom}_k(N_{w_o},k)$ which sends F to $F(w_o)$. After the $U(\mathfrak{g})$-module structure is twisted, $N^*_{w_o}$ is a free $U(n^-)$-module of rank one. The generator is easily seen to be invariant under T. One deduces that $(N_{w_o})^*$ is the Verma module with highest weight $0 = -w_o(\rho) - \rho$.

One can reformulate this as $N_{w_o} = (M_{w_o})^*$. We want to prove that $N_w = (M_w)^*$ for all $w \in W$. To do this, we will find an injection of N_{w_o} in N_w. For any i, there is a boundary operator :

$$H^{N-i}_{Z_i/Z_{i-1}}(X,\mathcal{O}_X) \longrightarrow H^{N-i+1}_{Z_{i-1}/Z_{i-2}}(X,\mathcal{O}_X)$$

this gives, for each pair of elements $w, w' \in W$ with $\ell(w) = i$, $\ell(w') = i-1$, an operator :

$$\partial_{w,w'} : H^{N-\ell(w)}_{\bar{X}_w/\partial(X_w)}(X,\mathcal{O}_X) \longrightarrow H^{N-\ell(w)+1}_{\bar{X}_{w'}/\partial(X_{w'})}(X,\mathcal{O}_X)$$

<u>Lemma 3</u> : $\partial_{w,w'}$ is surjective whenever $w = s_\alpha . w'$, with α a simple root.

Let U be the open set of X, obtaining by deleting all X_y included in $\bar{X}_w$ and different from $X_{w'}$. One has the following diagram where the first line is exact

$$\begin{array}{ccccccc} H^{N-\ell(w)}_{X_w \cup X_{w'}}(U,\mathcal{O}_X) \longrightarrow & H^{N-\ell(w)}_{X_w}(U-B_{w'},\mathcal{O}_X) & \xrightarrow{\partial} & H^{N-\ell(w)+1}_{X_{w'}}(U,\mathcal{O}_X) & \longrightarrow H^{N-\ell(w)+1}_{X_w \cup X_{w'}}(U,\mathcal{O}_X) \\ & \cong & & \cong & \\ & H^{N-\ell(w)}_{\bar{X}_w/\partial(X_w)}(X,\mathcal{O}_X) & \xrightarrow{\partial_{w,w'}} & H^{N-\ell(w)+1}_{\bar{X}_{w'}/\partial(X_{w'})}(X,\mathcal{O}_X) & \end{array}$$

the top line is exact by Definition 2 (iii); the vertical maps are excision isomorphisms. It suffices therefore to show $H^{N-\ell(w)+1}_{X_w \cup X_{w'}}(U,\mathcal{O}_U) = 0$. Notice $X_w \cup X_{w'} = Bw'x \cup BS_\alpha w'x = (Bw'B \cup BS_\alpha w'B)x = P_{S_\alpha}.w'x$, where P_{S_α} is the parabolic subgroup of rank 1, containing B, associated with the simple root α. Then P_{S_α} is generated by B and by a subgroup L_α, isomorphic to SL(2). The geometric quotient of $X_w \cup X_{w'}$ by the action of SL(2) exists, and it is isomorphic to $X_{w'}$. It is not difficult to find a neighborhood V of $X_w \cup X_{w'}$ in U such that $V \cong A_{N-\ell(w)} \times (X_w \cup X_{w'})$. It suffices to take for V the set $\tilde{U}_w.(X_w \cup X_{w'})$ with $\tilde{U}_w = N^- \cap (wN^- w^{-1})$.

Using a Künneth formula for cohomology with support, one gets :

$$H^{N-\ell(w)+1}_{X_w \cup X_{w'}}(U, \mathcal{O}_U) \cong H^{N-\ell(w)}_{\{0\}}(A_{N-\ell(w)} \quad \mathcal{O}_{A_{N-\ell(w)}}) \otimes H^1(X_w \cup X_{w'}, \mathcal{O}_{X_w \cup X_{w'}})$$

I claim that $H^1(X_w \cup X_{w'}, \mathcal{O}_{X_w \cup X_{w'}}) = 0$. Indeed, there is a smooth and proper morphism $p : X_w \cup X_{w'} \longrightarrow X_{w'}$ such that each fibre is a projective line. In the Leray spectral sequence

$$E_2^{p,q} = H^p(R^q p_* \quad (\mathcal{O}_{X_w \cup X_{w'}})) \Longrightarrow H^{p+q}(X_w \cup X_{w'}, \mathcal{O}_{X_w \cup X_{w'}})$$

all terms $E_2^{p,q}$ are zero for $p > 0$, since $X_{w'}$ is affine. Also I claim that $R^1 p_* (\mathcal{O}_{X_w \cup X_{w'}}) = 0$ Indeed its fibre at a point y of $X_{w'}$ is

$$H^1(p^{-1}(y), \mathcal{O}_{p^{-1}(y)}) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0 .$$

Therefore $H^1(X_w \cup X_{w'}, \mathcal{O}_{X_w \cup X_{w'}}) = 0$ and the lemma is proved .

<u>Lemma 4</u> : For any $w \in W$, there is a surjective $\Gamma(X, \mathcal{D}_X)$-linear-morphism $N_{w_o} \longrightarrow N_w$.

For, let $w_o w^{-1} = s_{\alpha_1} \dots s_{\alpha_{N-\ell(w)}}$ be a reduced decomposition (the α_i being simple roots). Using lemma 3, one has surjections

$$N_{w_o} \twoheadrightarrow N_{s_{\alpha_2} \dots s_{\alpha_{N-\ell(w)}} w} \twoheadrightarrow \dots \twoheadrightarrow N_{s_{\alpha_{N-\ell(w)}} w} \twoheadrightarrow N_w$$

<u>Proposition 7</u> : N_w is isomorphic to M_w^* . Indeed the surjection $N_{w_o} \twoheadrightarrow N_w$ of lemma 4 dualizes to an injection $N_w^* \hookrightarrow N_{w_o}^*$. One knows $N_{w_o} \cong M_{w_o}$ by Proposition 6, and $ch(N_w) = ch(M_w)$ by Proposition 5. One concludes using lemma 1. Notice Proposition 7 is given by Kempf [11], but only with sibylline indications of proofs.

Note that N_1 is isomorphic to M_1^* and that M_1 is a Verma module with highest weight -2ρ , which is antidominant. So M_1 is irreductible as an $U(\mathfrak{g})$-module (first theorem of Verma), and $M_1^* \cong M_1$.

Proposition 8 : $\Gamma(X,\mathcal{D}_X)$ operates faithfully on each space $H^{N-\ell(w)}_{X_w/\partial(X_w)}(X,\mathcal{O}_X)$.

Analogously to lemma 4, there is a surjective $\Gamma(X,\mathcal{D}_X)$-linear morphism $H^{N-\ell(w)}_{\bar{X}_w/\partial(X_w)}(X,\mathcal{O}_X)\longrightarrow H^N_{\{x\}}(X,\mathcal{O}_X)$, so one is reduced to the case w = 1. Notice that x has a neighbourhood U isomorphic to $\mathbb{A}^N$ (e.g. its orbit under N^-). The operation of $\Gamma(X,\mathcal{D}_X)$ factors through $\Gamma(U,\mathcal{D}_U)$ and the restriction map $\Gamma(X,\mathcal{D}_X)\longrightarrow\Gamma(U,\mathcal{D}_U)$ is injective. So it suffices to prove that $\Gamma(U,\mathcal{D}_U)$ operates faithfully there. But this is trivial since $\Gamma(U,\mathcal{D}_U)$ has no proper two-sided ideal [14], page 3.

At this point, I can give the nice proof of Proposition 3 which was promised earlier. For let $z \in J$, then z operates trivially on $N_1 \cong M_1$, because M_1 is in the category $\mathcal{O}_{triv}$. So $\varphi(z) = 0$ by Proposition 8.

Proposition 9 : ker (φ) = I (or Φ is injective).

Indeed, if $\varphi(z) = 0$, then z annihilates the $U(\mathfrak{g})$-module M_1. But this implies $z \in I$ by a theorem of Duflo [8] (see also [7]).

Proposition 10 : Φ is surjective.

By Proposition 8, it suffices to show that given $\xi \in \Gamma(X,\mathcal{D}_X)$, there exists $z \in U(\mathfrak{g})$, I such that $\Phi(Z)$ induces the same action on N_1 as ξ. But ξ belongs to a finite dimensional G-invariant subspace $\Gamma(X, \mathcal{D}_X(m))$. It follows easily that ξ gives a $\mathfrak{g}$-finite endomorphism of N_1. Since N_1 is an irreductible $U(\mathfrak{g})$-module, the conclusion follows from a theorem of Nicole Conze [5] , corollaire 6.9.

So the theorem is proved.

§ 4. A generalization

We assume k is algebraically closed.

Let $\mathcal{L}$ be an invertible sheaf on X (= $\mathcal{O}_X$-module, locally free of rank one). Then instead of $\mathcal{D}_X$, one may consider the sheaf of algebras $\mathcal{D}_X(\mathcal{L}) = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{L}^{-1}$. Of course, locally on X, $\mathcal{D}_X(\mathcal{L})$ is isomorphic to $\mathcal{D}_X$. Also notice that $\mathcal{L}$ is in a natural way a left $\mathcal{D}_X(\mathcal{L})$-module. Indeed, a section $f \otimes D \otimes g$ of $\mathcal{D}_X(\mathcal{L})$ on an open set operates on a section h of $\mathcal{L}$ as follows :

$$(f \otimes D \otimes g).h = D(\langle g,h\rangle).f$$

where <g,h> is the section of $\mathcal{O}_X$ obtained using $\mathcal{L}^{-1} = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L},\mathcal{O}_X)$. So $\mathcal{D}_X(\mathcal{L})$ may rightly be called the sheaf of algebras of <u>differential operators on the sheaf</u> $\mathcal{L}$. This construction was shown to me by Kashiwara.

Analogous results as Propositions 1 and 2 hold for $\mathcal{D}_X(\mathcal{L})$-coherent modules. Now recall that the invertible sheaf $\mathcal{L}$ corresponds to a character of T as follows. Given a character λ of T, one extends it to a character $\lambda : B \to \mathbb{G}_m$ such that $\lambda(N_+) = 1$. Let $\mathcal{L}(\lambda)$ be the coherent sheaf such that for any open set U of X, denoting $p : G \to X$ the projection defined in § 3, one has :

$$\Gamma(U,\mathcal{L}(\lambda)) = \{\text{regular functions } f \text{ on } p^{-1}(U) \text{, such that } f(g.b) = \lambda(b)^{-1}.f(g) \text{ for any } b \in B\}$$

Then $\mathcal{L}(\lambda)$ is an invertible sheaf on X, and there exists exactly one character λ such that $\mathcal{L}(\lambda)$ is isomorphic to $\mathcal{L}$. In other words, the Picard group of X is isomorphic to the character group X(T) (see [6] for details).

We identify X(T) with a subgroup of t* (associating to each character of T its differential, which is a linear form on t). Given $\lambda \in X(T)$, one has a corresponding maximal ideal J_λ of $Z(\mathfrak{g})$ (see § 1) and we let $I_\lambda = U(\mathfrak{g}).J_\lambda$. In the same way as Theorem 1, we can prove

<u>Theorem 2</u> : There is a natural algebra isomorphism :

$$\Phi_\lambda : U(\mathfrak{g})/I_\lambda \xrightarrow{\approx} \Gamma(X,\mathcal{D}_X(\mathcal{L}(\lambda)))$$

In the proof, one must take care that Proposition 6, Lemmas 3 and 4, and Proposition 7 are no longer valid.

Instead, one uses the fact that $ch(N_w) = ch(M_{w*(-\lambda)})$. There exists $w \in W$ such that $w*(-\lambda)$ is antidominant. One deduces that N_w is isomorphic to $M_{w*(-\lambda)}$ and that $\Gamma(X,\mathcal{D}_X)$ operates faithfully on N_w. Then the argument goes through.

Let me remark that the $U(\mathfrak{g})$-modules N_w^* are elements of the category $\mathcal{O}_\lambda$ defined in § 1. They have the same character as Verma modules, but in general are <u>not</u> Verma modules. If λ is dominant (i.e. $s_\alpha(\lambda) = \lambda - n_\alpha.\alpha$ with $n_\alpha \in \mathbb{N}$), it is stated in [11] (and can be proved by the methods in § 3) that N_w^* is a Verma module. In general, the structure of these modules depends only (say for a <u>regular</u> weight λ) on the Weyl chamber to which λ belongs (this is seen easily, using "translation functors"). However, it is a great mistery what happens when reaching or crossing a wall. This seems to be a very deep problem, to which we will deviously return in the next paragraph.

Note that one can define holonomic $\mathcal{D}_X(\mathcal{L})$-modules with regular singularities (R-S) just as in the case $\mathcal{L} = \mathcal{O}_X$, because this definition is of local nature. However, the category of holonomic $\mathcal{D}_X$-modules with R.S is equivalent to that of holonomic $\mathcal{D}_X(\mathcal{L})$-modules with R.S., by the functor :

$$\underline{M} \longmapsto \mathcal{L} \otimes_{\mathcal{O}_X} \underline{M}$$

this is trivial to verify. So for any $\lambda, \mu \in X(T)$, one gets an equivalence between the categories relative to $\mathcal{D}_X(\mathcal{L}(\lambda))$ and to $\mathcal{D}_X(\mathcal{L}(\mu))$, which we may call a geometric translation functor. Applications of this will be hinted at in the next paragraph.

§ 5. Open questions

Now the base field k is $\mathbb{C}$. I first state the main theorem of [4] . Let $\mathcal{M}$ be the category of holonomic $\mathcal{D}_X$-modules with R.S. whose charactersitic varieties are contained in $\bigcup_{w \in W} T^*_{X_w} X$, where $T^*_{X_w} X \subset T^* X$ is the conormal bundle of X_w in X.

Let $\tilde{\mathcal{O}}_{triv}$ be the full subcategory of the category of $U(\mathfrak{g})$-modules, whose objects M admit the "trivial" infinitesimal character and admit a filtration $0 = M_0 \subset M_1 \ldots \subset M_n = M$ such that M_i/M_{i-1} is an object of $\mathcal{O}_{triv}$.

Then $\mathcal{M}$ and $\tilde{\mathcal{O}}_{triv}$ are equivalent, via the following quasi-inverse functors :

$$F : \mathcal{M} \longrightarrow \tilde{\mathcal{O}}_{triv}$$

$$F(\underline{M}) = \Gamma(X, \underline{M})$$

$$G : \tilde{\mathcal{O}}_{triv} \longrightarrow \mathcal{M}$$

$$G(M) = \mathcal{D}_X \otimes_{U(\mathfrak{g})} M$$

Now, it seems reasonable to expect the following generalization for arbitrary $\lambda \in X(T)$ satisfying the regularity condition $\langle \lambda - \rho, \alpha \rangle \neq 0$ for any root α .First define a category $\tilde{\mathcal{O}}_\lambda$ similarly to $\tilde{\mathcal{O}}_{triv}$, but using the character of $Z(\mathfrak{g})$ associated to λ. Then let $\mathcal{D}(\mathfrak{g})_\lambda$ be the derived category of the category of bounded complexes of $U(\mathfrak{g})/I_\lambda$-modules, with cohomology in $\tilde{\mathcal{O}}_\lambda$. Now let $D(\lambda\text{-h.r})$ be the derived category of the category of bounded complexes of sheaves of $\mathcal{D}_X(\mathcal{L}(\lambda))$-modules, the cohomology of which are holonomic with R.S. Then the following functors F_λ and G_λ should be quasi-inverse triangulated equivalences:

$$F_\lambda \;:\; D(\lambda - h.r) \longrightarrow D(\mathfrak{g})_\lambda$$
$$\underline{M} \longmapsto \mathbb{R}\Gamma(X, \underline{M}^\cdot)$$
$$G_\lambda \;:\; D(\mathfrak{g})_\lambda \longrightarrow D(\lambda - h.r)$$
$$M^\cdot \longmapsto \mathcal{D}_X(\mathcal{L}(\lambda)) \overset{\mathbb{L}}{\underset{U(\mathfrak{g})/I_\lambda}{\otimes}} M.$$

Note that for λ dominant, one may define F_λ and G_λ without using derived categories (and get an equivalence of categories). If λ is not dominant, this is not possible, because of the non-vanishing of higher cohomology groups of holonomic $\mathcal{D}_X(\mathcal{L}(\lambda))$-modules with R.S. (for instance $H^i(X, \mathcal{L}(\lambda))$ will often be non-zero, for suitable i).

Now, for any $\lambda, \mu \in X(T)$, we have (see § 4) an equivalence of categories between holonomic $\mathcal{D}_X(\mathcal{L}(\lambda))$-modules with R.s. and holonomic $\mathcal{D}_X(\mathcal{L}(\mu))$-modules with R.S.

$$T_{\lambda,\mu} \;:\; \underline{M} \longmapsto \mathcal{L}(\mu - \lambda) \otimes_{\mathcal{O}_X} M$$

this also gives an equivalence of $D(\lambda - h.r)$ and $D(\mu - h.r)$

$$T_{\lambda,\mu} \;:\; D(\lambda - h.r) \longrightarrow D(\mu - h.r)$$
$$\underline{M}^\cdot \longmapsto \mathcal{L}(\mu - \lambda) \otimes_{\mathcal{O}_X} \underline{M}^\cdot$$

(notice that $\mathcal{L}(\mu - \lambda)$ is flat as an $\mathcal{O}_X$-module). Therefore one has the following diagram, which defines $\tau_{\lambda,\mu}$

$$\begin{array}{ccc}
D(\mathfrak{g})_\lambda & \xrightarrow{\;F_\lambda\;} & D(\mu - h.r) \\
\Big\downarrow{\scriptstyle \tau_{\lambda,\mu}} & & \Big\downarrow{\scriptstyle T_{\lambda,\mu}} \\
D(\mathfrak{g})_\mu & \xleftarrow{\;G_\mu\;} & D(\mu - h.r)
\end{array}$$

Remark again that if λ and μ are dominant, then $\tau_{\lambda,\mu}$ in fact will come from an equivalence of the categrories $\tilde{\mathcal{O}}_\lambda$ and $\tilde{\mathcal{O}}_\mu$. This is probably also true whenever λ and μ belong to the same Weyl chamber. We call again $\tau_{\lambda,\mu}$ the <u>geometric translation functor</u>. It should be interesting to compare it with the translation functor, which is used for instance by Bernstein-Gelfand-Gelfand [2] and in Jantzen's Habilitationschrift [9] .

Now, for $w \in W$, λ and $w * \lambda$ give the same character of $Z(\mathfrak{g})$. Therefore the categories $\tilde{\mathcal{O}}_{\lambda}$ and $\tilde{\mathcal{O}}_{w * \lambda}$ are the same, and the derived categories $D(\mathfrak{g})_{\lambda}$ and $D(\mathfrak{g})_{w * \lambda}$ are the same. So $T_{\lambda, w * \lambda}$ can be interpreted as an automorphism of the category $D(\mathfrak{g})_{\lambda}$, which we denote by $\tilde{w}$. There is an obvious question : does this define an action of W on $D(\mathfrak{g})_{\lambda}$? The answer is <u>no</u>, as explained below in the example G = SL(2) However, it is interesting to compute how $\tilde{w}$ operates in the K_o-group of $D(\mathfrak{g})_{\lambda}$, which is an abelian group generated by the classes $[M_{y * \lambda}]$ of Verma modules.

One has simply $[\tilde{w}(M_{y * \lambda})] = [M_{(yw) * \lambda}]$ so at least one has a representation of W in $K_o(D(\mathfrak{g})_{\lambda})$, which coincides with the one introduced by Bernstein-Gelfand-Gelfand [2].

I come now to the case G = SL(2). Let $\tilde{s}$ be the non trivial element on W. We want to see how s acts on $D(\mathfrak{g})_{\lambda}$ (say for λ dominant), and to check whether $\tilde{s}^2$ is the identity. To simplify things, we identify weights with integers, so that $\rho = 1$ and $\lambda = n, n \geq 0$; one has $s * \lambda = -2-n$. We take the Verma module $M_n \in D(\mathfrak{g})_n$. Then $F_n(M_n)$ is the holonomic $\mathcal{D}_X(\mathcal{L}(n))$-module $\mathcal{H}^o_{X/\{x\}}(\mathcal{L}(n))$. Applying $T_{n,-2-n}$, we get the holonomic $\mathcal{D}_X(\mathcal{L}(-2-n))$-module $\mathcal{H}^o_{X/\{x\}}(\mathcal{L}(-2-n))$. And applying G_{-2-n}, we get the Verma module M_{-2-n}, which is irreductible. Therefore $\tilde{s}(M_n) = M_{-2-n}$. Now start from the object M_{-2-n} of $D(\mathfrak{g})_n$. Then $F_n(M_{-2-n})$ is the $\mathcal{D}_X(\mathcal{L}(n))$-module $\mathcal{H}^1_{\{x\}}(\mathcal{L}(n))$; applying $T_{n,-2-n}$, we act $\mathcal{H}^1_{\{x\}}(\mathcal{L}(-2-n))$. Applying now G_{-2-n}, we get the "twisted dual" M^*_n of M_n. So we have $\tilde{s}^2(M_n) = M^*_n$ and M_n and M^*_n are different objects of $D(\mathfrak{g})_n$ since $\mathrm{Hom}(M_n, L_n) \neq 0$ and $\mathrm{Hom}(M^*_n, L_n) = 0$

Notice however that

$$\begin{aligned} &\mathrm{Ext}^i_{D(\mathfrak{g})_n}(\tilde{s}(M_n), \tilde{s}(M_{-2-n})) \\ &\qquad \| \\ &\mathrm{Ext}^i_{D(\mathfrak{g})_n}(M_{-2-n}, M^*_n) \\ &\qquad \| \\ &\mathrm{Ext}^i_{D(\mathfrak{g})_n}(M_n, M^*_{-2-n}) \\ &\qquad \| \\ &\mathrm{Ext}^i_{D(\mathfrak{g})_n}(M_n, M_{-2-n}) \end{aligned}$$

as we know already.

Still, it might possible that a more clever choice of an identification of $D(\mathfrak{g})_{\lambda}$ to $D(\mathfrak{g})_{s * \lambda}$ would turn the action of W into a group action.

One can do the above constructions in reverse order, define an action $\tilde{w}$ on $D(\lambda\text{-hr})$ by the following diagram

$$\begin{array}{ccccc} & & D(\lambda-h-r) & \xrightarrow{\;G_\lambda\;} & D(\mathcal{O})_\lambda \\ & \swarrow{\scriptstyle \tilde{w}} & & & \big\downarrow \mathrm{Id} \\ D(\lambda-h.r) & \xleftarrow{\;T_{s*\lambda,\lambda}\;} & D(s*\lambda-h.r) & \xleftarrow{\;F_{s*\lambda}\;} & D(\mathcal{O})_{w*\lambda} \end{array}$$

(the vertical map is the natural identification of $D(\mathcal{O})_\lambda$ with $D(\mathcal{O})_{w*\lambda}$). At least for $\lambda = 0$ (the general case is not much different), one has a triangulated equkvalence from $D(\lambda-h-r)$ to the derived category of bounded complexes of sheaves on X (usual topology) with constructible cohomology sheaves, given by

$H_\lambda : D(\lambda-h.r) \longrightarrow D(X) \qquad H_\lambda(\underline{M}) = \mathbb{R}\mathrm{Hom}_{\mathcal{D}_X(\lambda)}(\underline{M}^\cdot, \mathcal{L}(\lambda))$

So $H_\lambda \circ w \circ H_\lambda^{-1}$ gives an automorphism of $D(X)_c$. It is a pleasant exercice to compute this for $G = SL(2)$, in which case $X \cong \mathbb{P}^1$ It is an unclear question whether this automorphism comes from an automorphism of the derived category $D_{\mathbb{Z}}(X)_c$ of complexes of sheaves of abelian groups on X, with cohomology constructible sheaves with fibres of finite type as $\mathbb{Z}$-modules. Of course, one should look for some topological interpretation.

In any case it would be most interesting to bring the group structure of W to bear upon the topology of X or the structure of the categories $\tilde{\mathcal{O}}_\lambda$. After all, Kazhdan-Lusztig polynomials were first defined merely using the Coxeter group W [10], page . One could expect connections with work of Slodowy Springer, Kazhdan and Lusztig on representations of W.

* * *

I would like to thank warmly Michèle Lavallette for her completing of a beautiful typing of a rough manuscript under difficult circumstances.

REFERENCES

[1] Bernstein, I. N., Gelfand, I. M., Gelfand, S. I. : Differential operators on the base affine space and a study of $\mathfrak{g}$-modules in "Lie Groups and their representations", Budapest 1971.

[2] Bernstein, I.N., Gelfand, S.I. : Tensor Products of Finite and infinite dimensional representations of semi-simple Lie algebras, Compositio Math., t.41, fasc.2, (1980), p. 245-285.

[3] Brylinski, J. L., Kashiwara, M. : Démonstration de la conjecture de Kazhdan-Lusztig, C. R. A. S., série A, t. 392, séance du 22/9/1980.

[4] Brylinski, J. L., Kashiwara, M. : Kazhdan-Lusztig conjecture and holonomic systems, manuscript to be submitted to the Inventiones Math.

[5] Conze, N. : Algèbres d'opérateurs différentiels et quotients d'algèbres enveloppantes, Bulletin de la S.M.F., t. 102, fasc. 4, (1974), p.379-415.

[6] Demazure, M. : Désingularisation des variétés de Schubert généralisées, Annales Scientifiques de l'E.N.S., t.7, fasc. 1, 1974.

[7] Dixmier, J. : Algèbres enveloppantes, Cahiers Scientifiques, Gauthier-Villars, 1974.

[8] Duflo, M. : Construction of primitive ideals in induced representations, in "Lie Groups and their representations", Budapest 1971.

[9] Jantzen, : Habilitationsschrift, Bonn 1977.

[10] Kazhdan, D., Lusztig, G. : Representations of Coxeter groups and Hecke algebras, Inv. Math. 53, p. 165-184 (1979).

[11] Kempf, G. : The geometry of homogeneous spaces versus induced representations.

[12] Kempf, G. : The Grothendieck-Cousin complex of an induced representation; Advances in Maths. 29, p.310-396 (1978).

[13] Bott, R. : Homogeneous vector bundles, Annals of Maths 66 (1957) p.203-248.

[14] Björk, J. E. : Rings of differential operators, North-Holland (1979).

****

Brylinski Jean-Luc, Centre de Mathématiques de l'Ecole Polytechnique -
- 91128 Palaiseau Cedex - France. (Equipe de Recherche Associée au CNRS n°169).