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STANISLAW LOJASIEWICZ (JR)

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THE SUFFICIENCY OF MAXIMUM PRINCIPLE

by

S. ŁOJASIEWICZ (Jr)*

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SUMMARY.- We consider a nonlinear control system described by differential inclusion. We give sufficient conditions for characterization of the boundary trajectories of the system by Pontryaguin-Hamilton equations and for the convexity of attainable sets over certain time interval (we give also the best and calculable estimations for the lenght of this interval). We apply it to obtain the sufficiency of maximum principle for a time-optimal problem. Our result about characterization of boundary trajectories generalizes that of [2]. Our assumptions are close to these of [3], but different methods used in the proofs permit us to obtain optimal results. In [1] one can find another approach to the problem with the assumptions which seem to be more restrictive than ours.

NOTATIONS AND DEFINITIONS.- Let u and v be two vectors in k-dimensional euclidian space $\mathcal{E}^k$. We denote : uv (without a dot) the scalar product of u and v, $|v| = (vv)^{\frac{1}{2}}$ the corresponding norm, $[u, v]$ ($]u, v[$) the closed (respectively open) interval with the end points u and v, $B(u ; R)$ the closed ball with center u and radius R. We call an orientor any nonempty compact convex subset of $\mathcal{E}^k$. An orientor is called strictly convex if its boundary does not contain any nondegenerate interval and is called R-convex if it is an intersection of closed balls of radius R. Whene we will not specify R we will speak about uniformly convex of hyperconvex orientors (see [6]). Let A be a strictly convex orientor in $\mathcal{E}^k$ (with $k \geq 2$) and $p \in \mathcal{E}^k : p \neq 0$. We denote $v(A, p)$ the unique point in ∂A such that $pv(A, p) = \max\{pa : a \in A\}$. Let D be an open subset of $\mathcal{E} \times \mathcal{E}^k$.

A multifunction F from D into $\mathcal{R}^k$ is called an orientor field if its values are orientors. Let I be an open interval in the real line $\mathcal{R} = \mathcal{R}^1$. A function x from I into $\mathcal{R}^k$ with graph in D, absolutely continuous on compact subintervals of I is called a trajectory of F if $\dot{x}(t) \in F(t, x(t))$ for almost every $t \in I$. Relation $\dot{x} \in F(t,x)$ is called differential inclusion, sometimes contingens, orientor, or generalised differential equation. Let $t_o \in \mathcal{R}$ and $A \subset \mathcal{R}^k$ be such that $\{t_o\} \times A \subset D$. For $t \geq t_o$ we define an accessible (attainable) set as $E(F, t_o \times A, t) = \{x(t) : x$ is a trajectory of F and $x(t_o) \in A\}$. A trajectory x is called boundary trajectory over an interval J if $x(t) \in \partial E(F, t_o \times A, t)$ for $t \in J$. The standard properties of these notions are collected for instance in [5].

<u>GENERIC CASE</u>.- Let F be an orientor field defined on $\mathcal{R} \times \mathcal{R}^k$ (with $k \geq 2$) by $F(t,x) = f(t,x) + B(0;R)$ where f is a function from $\mathcal{R} \times \mathcal{R}^k$ into $\mathcal{R}^k$. Fix initial data : $t_o \in \mathcal{R}$ and $A_o \subset \mathcal{R}^k$ an R_o-convex orientor.

<u>Hypothesis 1</u>.- The function $f(.,x)$ is measurable for $x \in \mathcal{R}^k$, $f(.,0)$ is locally integrable and there exist $L, M \geq 0$ such that

(1) $$|f(t,x) - f(t,y)| \leq M|x-y|$$

(2) $$|f(t, rx+(1-r)y) - rf(t,x) - (1-r)f(t,y)| \leq Lr(1-r)(x-y)^2$$

hold for $x, y \in \mathcal{R}^k$, $r \in [0,1]$ and $t \in \mathcal{R}$.

<u>Remark 1</u>.- Inequality (2) holds iff for each t the function $f(t,.)$ is of class C^1 and its derivative satisfies Lipschitz condition with the constant $2L$.

Now, we write the Pontryaguin-Hamilton equations (maximum principle) for control system $\dot{x} = f(t,x) + u$, $|u| \leq R$ (corresponding to the orientor field F).

(3) $$\begin{aligned} \dot{x}(t) &= f(t, x(t)) + Rp(t)|p(t)|^{-1} \\ \dot{p}(t) &= -p(t)\, d_x f(t, x(t)) \end{aligned} \quad \text{with } x(t_o) = y(A_o, p(t_o)) \text{ and } |p(t_o)| = 1 .$$

Here d_x denotes the jacobian matrix of f in respect to the second variable. Let $X(.)$ be the solution of Ricatti equation :

(4) $$\dot{X}(t) = R + 3MX(t) + 2LX(t)^2 \text{ with } X(t_o) = R_o \text{ and put } T = \int_{R_o}^{\infty} (R + 3MX + 2LX^2)^{-1} dX .$$

Theorem 1.- Assume hypothesis 1. A boundary trajectory of F over interval $[t_o, t_o+T]$ satisfies (3) with some p and conversely, if (x,p) is a solution of (3) then x is a boundary trajectory of F over $[t_o, t_o+T]$. Moreover $E(F, t_o \times A_o, t)$ are $X(t)$-convex orientors and $x(t) = v(E(F, t_o \times A_o, t), p(t))$.

GENERAL CASE.- Let A and B be two strictly convex orientors in $\mathcal{R}^k$. Put $s(A,B) = \max\{|v(A,p)-v(B,p)| : |p|=1\}$; then s is a metric (introduced in [4]) stronger than Hausdorff metric. For R-convex orientors the both metrics are equivalent. Let F be an orientor field defined on $\mathcal{R} \times \mathcal{R}^k$ (with $k \geq 2$).

Hypothesis 2.- The multifunction $F(.,x)$ is measurable for $x \in \mathcal{R}^k$, $\mathrm{dist}(\{0\}, F(.,0))$ is locally integrable and there exist $L, M, R \geq 0$ such that $F(t,x)$ are R-convex,

$$s(F(t,x), F(t,y)) \leq M|x-y| \tag{5}$$

$$s(F(t, rx+(1-r)y), rF(t,x)+(1-r)F(t,y)) \leq Lr(1-r)(x-y)^2 \tag{6}$$

hold for $x, y \in \mathcal{R}^k$, $r \in [0,1]$ and $t \in \mathcal{R}$.

Now, we can write Pontryaguin-Hamilton equation in the following manner:

$$\begin{aligned} &\dot{x}(t) = v(F(t,x(t)), p(t) \\ &\dot{p}(t) = -p(t)d_x v(F(t,x(t)), p(t)) \end{aligned} \quad \text{with } x(t_o) = v(A_o, p(t_o)) \text{ and } |p(t_o)| = 1 . \tag{7}$$

Theorem 2.- Replace hypothesis 1 by 2 and equations (3) by (7). Then the conclusions of theorem 1 are valid.

Remark 2.- Basing on the same ideas one can reformulate and prove theorem 2 for F defined on some open region in $\mathcal{R} \times \mathcal{R}^k$ and for L, M, R depending on t.

Remark 3.- It is not difficult to see that the coefficients 1, 3 and 2 in the equation (4) cannot be replaced be smaller ones. Similarly, if we replace the Lipschitz condition for derivative of f (see remark 1) by Holder condition or, if we require that $F(t,x)$ are strictly, but not uniformly convex, then one can easily construct an orientor field F (which will satisfy so modified hypothesis 2) and a trajectory x (of F) which will satisfy (7) with a suitable p, but $x(t) \in \text{int}\, E(F, t_o \times A_o, t)$ for $t > t_o$.

APPLICATION : TIME-OPTIMAL PROBLEM.

Let A_o and A_1 be two nonempty disjoint closed subsets of $\mathcal{R}^k$. We wont to find (for a given initial moment t_o) a trajectory x (of a given orientor field) such that $x(t_o) \in A_o$, $x(t_1) \in A_1$ and $t_1 - t_o > 0$ is minimal. Such a trajectory is called optimal. From theorem 2 we get immediately :

Corollary.- Assume that F satisfies hypothesis 2. Let (x, p) be a solution of (7) with $x(t_1) \in A_1$ and $t_1 - t_o \leq T$. Suppose that $\sup\{-p(t)a : a \in A_1\} < -p(t)x(t)$ for $t \in]t_o, t_1[$ (i.e. the strong transversality condition - see [1]). Then x is optimal.

Remark 4.- The corollary can be applied in a simple case in which the methods of [1] does not apply (concavity assumptions are not verified). For instance, take $k=2$, $t_o = 0$, $A_o = \{(0,0)\}$, $A_1 = \{(0, \frac{1}{2})\}$, $F(t, (x,y)) = (0, x^2) + B(0;1)$. Define $x(t) = (0, t)$, $p(t) = (0, 1)$; then x is optimal by corollary.

Remark 5.- Consider control system in $\mathcal{R}^2$ given by (as example 2 in [1]) :

$$(8) \qquad \dot{x} = y,\ \dot{y} = (3 - x\exp(-x^2))/2 + u(3 + x\exp(-x^2))/2 \quad \text{where } u = \pm 1 .$$

The problem is to minimise the time of transfer from $A_o = \{(x,y) : y = 0\}$ to $A_1 = \{(x,y) : y = 1\}$ using trajectories of (8). For a given $x_o \in \mathcal{R}$ let z be a trajectory of (8) defined by $y(t) = 3t$, $x(t) = (3(2)\, t^2 + x_o$. Define an autonomous orientor field F by : $F(x,y) = (y, (3 - x\exp(-x^2))/2) + B(0 ; (3 + x\exp(-x^2))/2)$. By corollary used with F and z as above, $t = 0$, $t_1 = 1/3$, $p(t) = (0,1)$ we get the optimality of z for the problem of transfer $\{(0, x_o)\}$ to A_1 and hence, for the original problem. The optimality of z can be obtained using results of [1] too.

PROOF ; Proof of theorem 1.- First, we are going to specify some particular properties of hyperconvex sets which we will use in our proofs without explicity refer to its. Let A be a strictly convex orientor, then $v(A, .)$ is continuous. If A is R-convex, then $A \subset B(v(A,p) - Rp|p|^{-1} ; R)$. A is R-convex iff $v(A,.)$

satisfies on $\partial B(0,1)$ the Lipschitz condition with the constant R. For $|u-v| \leq 2R$ we denote $S(u,v;R)$ the intersection of all closed balls of radius R containing points u and v. A nonempty, compact set A is R-convex iff $|u-v| \leq 2R$ and $S(u,v;R) \subset A$ for all u and v in A. Put $2a = v-u$, $2b = v+u$ and denote $\Omega(u,v) = \{z \in \mathcal{R}^k : a(z-u) > 0$ and $a(z-v) < 0\}$. Each $z \in \Omega(u,v)$ has the unique orthogonal decomposition : $z = b + xa + y$ where $x \in \mathcal{R}$, $|x| < 1$, $y \in \mathcal{R}^k$, $ya = 0$. We denote $\gamma(u,v;z) = |y|((1-x^2)a^2)^{-1}$ and for $c \geq 0$: $P(u,v;c) = \{z \in \Omega(u,v) : \gamma(u,v;z) \leq c\} \cup \{u,v\}$. If $y \neq 0$, $\gamma(u,v;z)$ has the following geometric interpretation. Let Δ be the parabola passing through the points u, v, z and having as its axis of symmetry the line (lying in the plane (u,v,z)) perpendicular to a and passing through the point b. By a suitable isometric change of coordinates we can uniquely describe Δ by an equation of the form $Y = \alpha \mathbf{X}^2$ with $\alpha > 0$; then $\alpha = \gamma(u,v;z)$. For $|u-v| \leq 2R$ we have : $P(u,v;\frac{1}{2}R) \subset S(u,v;R)$. A nonempty, compact set A is R-convex iff $P(u,v;\frac{1}{2}R) \subset A$ for all u and v in A. Let be $\varepsilon > 0$, and suppose moreover that A is connected. Then A is R-convex iff $P(u,v;\frac{1}{2}R) \subset A$ for all $u \in \partial A$ and $v \in \partial A$ satisfying $|u-v| < \varepsilon$. Now, let u and v be two trajectories of F such that $u(t) \neq v(t)$ for t in some interval $]\alpha,\beta[$. Define the open domain : $D = \{(t,z) : t \in]\alpha,\beta[$ and $z \in \Omega(u(t),v(t))\}$. Put $w(t,z) = (1-x^2)a^2y + 2xy^2a$ for $(t,z) \in D$. We define an orientor field G on D by $G(t,z) = \{p \in F(t,z) :$ $(p-q)\,w(t,z) \leq 0$ for $q \in F(t,z)\}$. Geometrically, $w(t,z)$ is a vector perpendicular to parabola Δ at z.

<u>Lemma 1.</u> - <u>Through each point of</u> D <u>passes a full trajectory of</u> G ; <u>denote it by</u> $z(.)$ <u>and put</u> $\gamma(t) = \gamma(u(t),v(t);z(t))$. <u>Then for a.e.</u> t <u>such that</u> $\gamma(t) \neq 0$ <u>we have</u> :

(9) $$\dot\gamma(t) \leq L + (3M+2L|a(t)|)\gamma(t) + (2R+4M|a(t)|)\gamma(t)^2 .$$

<u>Proof</u> : The existence of $z(.)$ is a classical fact if we remark that G is upper semicontinuous in space variable z. We fix a point t, such that $\dot u(t), \dot v(t), \dot z(t)$ exist and $\gamma(t) \neq 0$. In what follows we suppress dependence on the variable t in order to make our notation simpler. If we calculate the logarithmic derivative of γ we get (using $ya = 0$) :

(10) $$\dot\gamma/\gamma = (\dot y + \dot x a)c + 2xy\dot a((1-x^2)a^2)^{-1} - 2a\dot a(a^2)^{-1} \text{ where } c = y(y^2)^{-1} + 2xa((1-x^2)a^2)^{-1} .$$

We have : $\dot{z} = f(z) - Rc(|c|)^{-1}$. Let n and m be in $B(0;1)$ such that $\dot{v} = f(v) + Rn$ and $\dot{u} = f(u) + Rm$. We can write (10) in the following form :

$$(11) \quad \dot{\gamma}/\gamma = c[f(z) - \tfrac{1}{2}(f(b+a)+f(b-a)) - \tfrac{x}{2}(f(b+a)-f(b+a))] + [xy((1-x^2)a^2)^{-1} - a(a^2)^{-1}] .$$
$$. (f(b+a)-f(b-a)) + R[-|c| - \tfrac{c}{2}(n+m+x(n-m)) + (xy((1-x^2)a^2)^{-1} - a(a^2)^{-1})(n-m)] .$$

From (2) we get : $|f(b+xa) - \tfrac{1}{2}(f(b+a)+f(b-a)) - \tfrac{x}{2}(f(b+a)-f(b-a))| \leq L(1-x^2)a^2$. Using this and (1) we can easily estimate, in (11), the part which does not contain R by $3M+2L|a|+4M|a|\gamma+L/\gamma$. To finish our proof we have to estimate in (11) the part which contains R , by $2R\gamma$. Obviously, it is sufficient to prove that :

$$(12) \qquad |a/a^2 - xy((1-x^2)a^2)^{-1} - \tfrac{c}{2}(1-x)| + |xy((1-x^2)a^2)^{-1} - \tfrac{c}{2}(1+x)| \leq 2\gamma + |c| .$$

We denote $e = 2y^2(a^2)^{-1}$. Multiplying (12) by $2(1-x^2)|y|$ we get an equivalent inequality :

$$(13) \quad [(1+x)^2(1-x^2)^2 + 2(1+x)(1+x^3)e + x^2e^2]^{\frac{1}{2}} + [(1-x)^2(1-x^2)^2 + 2(1-x)(1-x^3)e + x^2e^2]^{\frac{1}{2}} \leq$$
$$\leq 2e + 2[(1-x^2)^2 - 2x^2e]^{\frac{1}{2}} .$$

Let us denote : $s = 1-x^2$ and $g(x) = (1+x)^2(1-x^2)^2 + 2(1+x)(1+x^3)e + x^2e^2$. We square both sides of (13) and after arranging and dividing by 2 we get an equivalent inequality :

$$(14) \qquad [g(x)g(-x)]^{\frac{1}{2}} \leq e^2 + s(e-s)^2 + 4e(s^2+2(1-s)e)^{\frac{1}{2}} .$$

We square both sides of (14) and in the formula we replace $(s^2+2(1-s)e)^{\frac{1}{2}}$ by s . After rearranging and dividing by $4e^2$ we finally get $2e(s^2+s-3) \leq s^3 + se^2$ which is equivalent to (11) and trivially satisfied ; so the proof of lemma 1 is complete.

Lemma 2.- Put $E(t) = E(F, t_o \times A_o, t)$; then $E(t)$ are hyperconvex orientors for t close to t_o .

Proof : It is well known that $E(t)$ are nonempty, compact, connected and depend continuously on t . Let be $\varepsilon > 0$ and $a_o \in E(t_o+\varepsilon)$, $b_o \in E(t_o+\varepsilon)$, $a_o \neq b$. There exist ϕ and ψ , trajectories of F such that $\phi(t_o)$, $\psi(t_o) \in A_o$ and

$\emptyset(t_o+\varepsilon)=a_o$, $\psi(t_o+\varepsilon)=b_o$. We reverse time, i.e. we put $\bar{t}=t_o+\varepsilon+t$, $\overline{x(t)}=x(t)$, $\bar{F}(\bar{t},x)=-F(t,x)$. Then $\dot{x}(t)\in F(t,x(t))$ is equivalent to $\dot{\overline{x(t)}}\in\bar{F}(\bar{t},\overline{x(t)})$. We put : $u(\bar{t})=\emptyset(t)$, $v(\bar{t})=\psi(t)$, $\beta=\max\{\tau\in]0,\varepsilon]:u(\bar{t})\neq v(\bar{t})$ for $\bar{t}\in[0,\tau[\}$ and choose $\alpha<0$ large enough so $u(\bar{t})\neq v(\bar{t})$ for $\bar{t}\in]\alpha,0]$. From lemma 1 with $\bar{F},u,v,\alpha,\beta$, we get the existence of $\varepsilon_o>0$ such that $\gamma(0)\leq 1/_4 R_o$ implies $\gamma(\bar{t})\leq 1/_2 R_o$ for $\bar{t}\in[0,\varepsilon_o]$ and $\varepsilon\in[0,\varepsilon_o]$ (the choice of ε_o depends only on the constants L, M, R and R_o). Hence through each point of $\{0\}\times P(u(0),v(0);1/_4 R_o)$ there passes z, a trajectory of $\bar{F}$, such that $z(\varepsilon)\in A_o$ (we set z to be the z from lemma 1 and if necessary, we extend z using u or v). Hence $P(a_o,b_o,1/4R_o)\subset E(t_o+\varepsilon)$. Therefore, $E(t_o+\varepsilon)$ is $2R_o$-convex which completes proof. Let $]t_o,t_o+r[$ be the maximal open interval such that $E(t)$ are hyperconvex on it.

<u>Lemma 3.- If</u> $R>0$, <u>then</u> $E(t)$ <u>has interior points and the boundary of</u> $E(t)$ <u>is</u> C^1-<u>smooth for</u> $t\in]t_o,t_o+r]$.

<u>Proof</u> : Let G be an orientor field defined by $G(t,z)=g(t,z)+B(0;R)$ with g satisfying the hypothesis 1. Let x and y be the solutions of $\dot{z}=g(t,z)$ with $x(0)=x_o$ and $y(0)=y_o$, then we easily obtain (using standard differential inequality methods) the following formula :

(15) $$E(G,(0,x_o),t)\subset B(x(t);RM^{-1}(\exp(MT)-1))$$

(16) $$B(y(t);RM^{-1}(1-\exp(-Mt))\subset E(G,(0,y_o),t)$$

(17) $$|x(t)-y(t)+y_o-x_o|\leq|x_o-y_o|(\exp(Mt)-1)\ .$$

To prove our lemma it is sufficient to prove that $E(t)$ has unique outward normal at each boundary point. Suppose otherwise ; then there exists $t_1\in]t_o,t_o+r]$, $x_o\in\partial E(t_1)$, $m\in\partial B(0;1)$, $n\in\partial B(0;1)$, $n\neq m$ such that $x_o=v(E(t_1),n)=v(E(t_1),m)$. Put $C=B(x_o;1)\setminus\{x\in\mathscr{E}^k: n(x-x_o)\leq 0$ and $m(x-x_o)\leq 0\}$. Take θ satisfying $(1+mn)\,|m+n|^{-1}<\theta<1$, then $B(x_o;Rt(1+Mt))\subset\bigcup_{y_o\in C}B(y_o;\theta Rt)$ for $t>0$ sufficiently small. Hence using (17) we get for small $t>0$.
$B(x(t);Rt(1+Mt))\subset\bigcup_{y_o\in C}B(y(t),Rt(1-Mt))$ so from (15) and (16) we obtain for small $t>0$:

(18) $$E(G, (0, x_o), t) \subset E(G, \{0\} \times C, t).$$

Let z be a trajectory of F such that $z(t_o) \in A_o$ and $z(t_1) = x_o$. From (18) with $G(t, x) = F(t_1 - t, x)$ we get that there exists $\overline{z}$ a trajectory of G satisfying $\overline{z}(0) \in C$ and $\overline{z}(t_1 - t_2) = z(t_2)$ for some t_2 close to t_1. Hence $\overline{z}(0) \in E(t_1)$ which contradicts $C \cap E(t_1) = \emptyset$ and so the proof is complete.

For $t \in]t_o, t_o + r]$ and $z \in \partial E(t)$ we denote $n(E(t), z)$ the unique (by lemma 3) outward normal to $E(t)$ at the point z. Let z be a boundary trajectory of F over $[t_o, t_o + r]$. By Pontryaguin's maximum principle z satisfies (3) with some p and moreover, $z(t) \in \partial E(t)$ for $t \in [t_o, t_o + r]$.

<u>Lemma 4</u>.- <u>If</u> $R > 0$, <u>then</u> $n(E(t), z(t)) = p(t)\ |p(t)|^{-1}$ <u>for</u> $t \in]t_o, t_o + r]$.

<u>Proof</u> : It is sufficient to prove our equality for $t \in]t_o, t_o + r[$. Suppose otherwise, then there exists a boundary trajectorie z and t_1 such that $n = n(E(t_1), z(t_1)) \neq p(t_1)\ |p(t_1)|^{-1} = m$. Put $G(t, x) = F(t_1 + t, x)$, $x_o = z(t_1)$. Then using (17) and (3) we get for $t \in [0, t_o + r - t_1]$

(19) $$|y_o - y(t) - (x_o + Rtm - (t_1 + t)z)| \leq |y_o - x_o|(\exp(Mt) - 1) + RM^{-1}(\exp(Mt) - 1 - Mt) + \tfrac{1}{2}RMt^2.$$

Let θ be as in the proof of lemma 3. For $\varepsilon > 0$ we put :

$$D_\varepsilon = \{z \in \mathcal{S}^k : |(z - x_o) - n(z - x_o)n| \leq \theta\, |z - x_o| \text{ and } n(x_o - z) \in [0, \varepsilon]\}$$

then for t sufficiently small we get

(20) $$x_o + Rtm \subset \bigcup_{y_o \in D_\varepsilon} B(y_o ; \theta Rt).$$

From lemma 3 we get $D_\varepsilon \subset E(t_1)$ for small ε. Hence using (19) and (20) we obtain $z(t_1 + t) \in \bigcup_{y_o \in E(t_1)} B(y(t) ; Rt(1 - Mt))$ for small t. Therefore from (16) we get $z(t_1 + t) \in \operatorname{int} E(t_1 + t)$ for $t > 0$ which contradicts the fact that z is a boundary trajectory, so the proof of lemma 4 is complete.

<u>Lemma 5</u>.- <u>Suppose that</u> (z, p) <u>is a solution of</u> (3). <u>Then</u> z <u>is a boundary trajectory of</u> F <u>over</u> $[t_o, t_o + r]$.

<u>Proof</u> : If $R = 0$ lemma is trivial, so suppose $R > 0$. Take $t \in]t_o, t_o + r]$.

We define h, a map from $\partial B(0;1)$ into $E(t)$ as follows : $\partial B(0;1) \ni p_o \to h(p_o) = z(t) \in E(t)$ where (z,p) is a solution of (3) with $p(t_o) = p_o$. The function h is well defined because (3) has uniqueness property as Lipschitzian equation. Moreover, from lemma 4 we get that h is injective on $h^{-1}(\partial E(t))$. Obviously h is continuous and $\partial E(t) \subset h(\partial B(0;1))$ by maximum principle. Hence h^{-1} is an homeomorphism from $\partial E(t)$ onto $h^{-1}(\partial E(t))$, but $\partial E(t)$ is homeomorphic to $\partial B(0;1)$, so h^{-1} induces a continuous injection from $\partial B(0;1)$ into $\partial B(0;1)$ such an injection must be onto, hence $h^{-1}(\partial E(t)) = \partial E(0;1)$ and h is a homeomorphism from $\partial B(0;1)$ onto $\partial E(t)$ which completes proof of lemma 5.

<u>Lemma 6.-</u> <u>We have</u> $T \leq r$ <u>and</u> $E(t)$ <u>are</u> $X(t)$-<u>convex orientors for</u> $t \in [t_o, t_o+T[$.

<u>Proof</u> : Take $s \in]t_o, t_o+T[$ with $s \leq t_o + r$. To each $x \in \partial E(s)$ we associate a map $g_x :]t_o, t_o+T[\to \mathcal{E}^k$ defined by $g_x(t) = z(t)$ where (z,p) is a solution of (3) with $z(s) = x$ and with $p(s)|p(s)|^{-1} = n(E(s),x)$ if $R>0$. Then for each $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that $|x-y| \leq \delta(\varepsilon)$ implies $|g_x(t)-g_y(t)| \leq \varepsilon$ for $t \in]t_o, s]$. Moreover, $x \to g_x(t)$ is injective for $t \in]t_o, s]$, hence for $t \in]t_o, t_1[$ with some $t_1 \in]s, t_o+T[$. We have just used lemmas 3, 4 and the regularity of (3).

Now, let Y be a solution of $\dot{Y} = L+(3M+2L\varepsilon)Y+(2R+4M\varepsilon)Y^2$ with $Y(0) \geq 0$. Take $\lambda>0$. We can choose $\varepsilon>0$ such small that $Y(0)(2X(s)+\lambda) \leq 1$ implies $2R_oY(s-t_o) \leq 1$. Let a_o and b_o belong to $\partial E(s)$ and $a_o \neq b_o$, $|a_o - b_o| \leq \delta(\varepsilon)$. We reverse time : $\bar{t} = s-t$, $F(\bar{t},x) = -F(t,x)$ and we put $\alpha = s-t_1$, $\beta = s-t_o$, $u(\bar{t}) = g_{a_o}(t)$, $v(\bar{t}) = g_{b_o}(t)$. From lemma 1 used with E, u, v, α, β we obtain that $\gamma(0)(2X(s)+\lambda) \leq 1$ implies $2R_o\gamma(s-t_o) \leq 1$ (because $\gamma(0) \leq Y(0)$ implies $\gamma(t) \leq Y(t)$ for $t \geq 0$). Hence (by similar argument as in the proof of lemma 2) we get that $E(s)$ is $(X(s)+\lambda)$-convex and therefore $E(s)$ is $X(s)$-convex (because λ can be arbitrary small). We have also $T \leq r$. Otherwise, we can take $s = t_o + r$ and applying lemma 2 with $s \times E(s)$ as initial data we get the hyperconvexity of $E(t)$ for t close to $t_o + r$ which cannot occur because r is maximal. So the proof of lemma 6 is complete.

Theorem 1 results immediately from maximum principle and lemmas 4, 5 and 6.

<u>Proof of Theorem 2</u> : Each boundary trajectory of F over $[t_o, t_o+T]$ satisfies (7) with some p by maximum principle. Conversely, to each solution (x, p) of (7) we associate an orientor field $F_p \supset F$ defined by $F_p(t,x) = f_p(t,x) + B(0;R)$ where $f_p(t,x) = v(F(t,x), p(t)|p(t)|^{-1}) - Rp(t)|p(t)|^{-1}$. Denote $E_p(t) = E(F_p, t_o \times A_o, t)$ and $E(t) = E(E, t_o \times A_o, t)$. Obviously $E(t) \subset E_p(t)$ and (x,p) satisfies (3) with $f = f_p$; therefore by theorem 1 x is boundary trajectory of F_p (hence of F) and $x(t) = v(E_p(t), p(t)) = v(E(t), p(t))$. So, we have only to prove that $E(t)$ are $X(t)$-convex orientors. Take $t_1 \in]t_o, t_o+T[$. We clame : if $\operatorname{int} E(t_1) \neq \emptyset$ then $\operatorname{int} E(t_1) = \operatorname{int} \bigcap_q E_q(t_1)$ where (x,q) are all possible solutions of (7). Hence $E(t_1) = \bigcap_q E_q(t_1)$ which completes the proof for this case, because $E_q(t_1)$ are $X(t_1)$-convex orientors. Otherwise there exists $x_1 \in \partial E(t_1) \cap \operatorname{int} \bigcap_q E_q(t_1)$ and consequently there exists x a boundary trajectory of F (over $[t_o, t_1]$) with $x(t_1) = x_1$. Therefore x satisfies (7) with some p and hence (x,p) satisfies (3) with $f = f_p$. By theorem 1, $x_1 \in \partial E_p(t_1)$ which contradicts $x_1 \in \operatorname{int} \bigcap_q E_q(t_1)$. In the case : $\operatorname{int} E(t_1) = \emptyset$ we have $\operatorname{int} E(t) = \emptyset$ for $t \in [t_o, t_1]$, also A_o is reduced to a single point, say x_o and each trajectory x of F with $x(t_o) = x_o$ is boundary over $[t_o, t_1]$. Moreover, we are going to prove that $F(t, x(t))$ is reduced to a single point for a.e. $t \in [t_o, t_1]$. Consequently $E(t_1)$ will be reduced to a single point which will complete the proof of theorem 2, because a single point is 0-convex and hence $X(t_1)$-convex orientor. Suppose otherwise. For $r > 0$ we put $C_r = \{(t,x) \in \mathscr{E} \times \mathscr{E}^k : t \in]t_o, t_1[\text{ and } |x - x(t)| < r\}$. Fix $n \in \partial B(0;1)$ and put $w(t,x) = \frac{1}{2}(v(F(t,x), n) + v(F(t,x), -n))$. We can choose r and $\varepsilon > 0$ such small that there exists $A \subset]t_o, t_1[$ with positive measure, such that if we define G an orientor field on C_r by : $G(t,x) = w(t,x) + B(0;\varepsilon)$ for $t \in A$ and $G(t,x) = \{w(t,x)\}$ for $t \notin A$, then $G(t,x) \subset F(t,x)$ and $G(t,x) \subset \operatorname{int} F(t,x)$ for $t \in A$. We define y, a trajectory of F by : $y(t) = x(t)$ for $t \leq t_2$ where t_2 is a density point of A and $y(t) = z(t)$ for $t \geq t_2$ where z is a trajectory of G with $z(t_2) = x(t_2)$. Then $\dot{y}(t) \in \operatorname{int} F(t, y(t))$ over a set of positive measure, but this cannot occur because y is a boundary trajectory of F in some neighborhood of t_2, so the proof is complete.

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Stanislaw ŁOJASIEWICZ
Ujejskiego 6 m 3
30-102 KRAKOW (Pologne)