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ON CANONICAL FORM FOR COMPLETELY REFACHABLE DYNAMINCAL SYSTEMS

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INTRODUCTION.

In this paper we investigate the existence of continuous (algebraic) canonical forms for linear, time-invariant, completely reachable dynamical systems on a field K .

Roughly speaking, the situation is the following : a dynamical systems σ is given by a pair (F, G), F and G being respectively n x n and n x m matrices, up to the equivalence induced by a change of basis in state space . A canonical form is the choice of a representative element in the equivalence class of pairs (F, G) which definies σ (see [9]) .

Endowing the set SCR (m, n) of all completely reachable pairs (F, G) with a topological structure (if K coincides with $\mathbb{R}$ or $\mathbb{C}$) or with a geometric one (if K is an algebraically closed field) and interpreting a canonical form as a particular endomorphism c of SCR (m, n) (see 1. 4) , one can demand that c is also continuous or algebraic . Canonical forms of this kind are useful in e. g. identification problems (see [1, 2, 10]), but, as proved in [5] , there are no globally defined continuous (algebraic) canonical forms on SCR (m, n) when m>1 .

Here, we describe (see 1. 6) the equivalence between local continuous (algebraic) canonical forms and local triviality of a particular vector bundle on the

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variety $M_{m,n}$ of completely reachable systems (see [5]). This allows us to determine (see 2.4) a class of open subsets of SCR (m, n), useful in systems theory, on which there exists a local continuous (algebraic) canonical form. Obviously, the property, for a family Σ of systems, to be contained in one of the previous subsets is sufficient to assure that there is one local continuous (algebraic) canonical form defined for all the elements of Σ. Moreover, we prove (see 2.6) that if Σ is finite and K has infinitely many elements the above condition is satisfied.

1. - We consider linear time-invariant completely reachable dynamical systems

$$\dot{x} = Fx(t) + Gu(t) \quad \text{(continuous time) and}$$
$$x(t+1) = Fx(t) + Gu(t) \quad \text{(discreet time)}$$

where F, G are respectively n x n and n x m matrices with entries in a field K. I. e. the state space dimension is n and there are m imputs.

1.1. - A change of basis in state space changes the pair (F, G) as follows

$$(F, G) \longmapsto (T \cdot F \cdot T^{-1}, \; T \cdot G) \quad T \in GL(n, K) \; .$$

Then the systems we are considering are represented by the orbits of the above described action of GL (n, K) on the set SCR (m, n) of all completely reachable pairs (F, G).
This action may be considered from two different points of view. Namely one may assume that K coïncides with $\mathbb{R}$ or $\mathbb{C}$ or that K is an algebraically closed field. So what we have is a continuous action in the first case and an algebraic action in the second (see [3] and [4]).
Our treatment is applicable to both the cases and distinction will be made only when necessary.

1.2. - PROPOSITION. ([5] 3.7 and [7]). *Let* $G_{n,r}$ *be the Grassman variety of n-dimensional subspaces* V^n *of* K^r, $r = (n+1)m$.

The orbit space SCR (m, n)/GL (n, K), denoted $M_{m,n}$, is a quasi projective subvariety of $G_{n,r}$.

1.3. - Since $G_{n,r}$ may be considered as the orbit space of the action, given by rows by columns product, of GL (n, K) on the space $A^{n \times r}_{reg}$ of all maximal rank $n \times r$ matrices, the situation of 1.2 may be described by the following commutative diagram

$$\begin{array}{ccc} SCR\,(m,n) & \xrightarrow{\quad R \quad} & A^{n\times r}_{reg} \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ M_{m,n} & \xrightarrow{\text{inclusion}} & G_{n,r} \end{array}$$

where R is the continuous (algebraic) one-one morphism defined by

$$R(F,G) = (G \;\; FG \;\; F^2G \ldots F^nG)$$

and where p_1, p_2 are the projection onto the orbit spaces .

1.4. - <u>DEFINITION</u> . Let $L \subset A^{n \times r}_{reg}$ be a GL (n, K)-invariant subspace . A <u>continuous</u> (<u>algebraic</u>) <u>canonical form</u> on L is a continuous (algebraic) morphism

$$c : L \longrightarrow L \text{ such that}$$

i) $c(S) = c(S')$ iff $S, S' \in L$ are GL(n, K)-equivalent ;

ii) S and c(S) are GL(n, K)-equivalent for any $S \in L$.

1.5. - Let $L^{\star} = R^{-1}(L) \subset$ SCR (m, n). Since SCR (m, n) is isomorphic to its R-image, a canonical form c on L definies a morphism (see [8] 2.2.)

$$c^{\star} : L^{\star} \longrightarrow L^{\star}$$

which verifies i), ii) of 1.4 modified in the obvious way .
Then $c^{\star}$ is a continuous (algebraic) canonical form for the completely reachable systems which belong to $L^{\star}$.

The question whether a canonical form exists on a given sybset of SCR (m, n) is motivated by e. g. identification of systems theory. To investigate this problem we first need the following .

1\. 6. - PROPOSITION.- Let $\gamma = (E, p, G_{n,r})$ be the n-dimensional (algebraic) canonical vector bundle over $G_{n,r}$ (see [6] 2. 2. 5) and let $L \subset A^{n \times r}_{reg}$. Then there exists a continuous (algebraic) canonical form c on L iff the restricted bundle $\gamma \mid p_2(L)$ is trivial .

Proof. The existence of a continuous (algebraic) canonical form c on L is equivalent to the existence of a continuous (algebraic) morphism

$$\tilde{c} : p_2(L) \longrightarrow L$$

such that $p_2 . \tilde{c}$ = identity .

Now, $\gamma \mid p_2(L)$ is trivial iff there is an isomorphism

$$\alpha : p_2(L) \times K^n \longrightarrow E \mid p_2(L) \quad \text{given by}$$

$$\alpha (x, v) = (x, v . S_x) \quad \text{where } S_x \in A^{n \times r}_{reg} \quad \text{and}$$

$$\tilde{c} : x \longrightarrow S_x$$

is a continuous (algebraic) morphism between $p_2(L)$ and L such that $p_2 . \tilde{c}$ = identity .

1\. 7. - The bundle γ is non trivial and also $\gamma \mid M_{m,n}$ is non trivial if the considered systems have more than one input (i. e. m > 1) (see [5] 6. 2.) .
Our purpose is then to describe a class of proper subset of $G_{n,r}$, useful in systems theory, on which γ is trivial .

2\. - We consider the classical embedding of $G_{n,r}$ into $\mathbb{P}^N$, $N = \binom{r}{n} - 1$, obtained denoting by $x_0, \dots, x_N$ the grassmann cohordinates, lexicographically ordered, of V^n.
Let λ denote a linear homogeous form in the cohordinates $x_0, \dots, x_N$ or, equivalently, a hyperplane of $\mathbb{P}^N$. Both the intersection subvariety $\lambda . G_{n,r}$ and the

corresponding set of V^n in K^r will be denoted by $[\lambda]$.

2.1. DEFINITIONS (see [11] 2) For any linear homogeneous form λ, $[\lambda]$ is called linear complex.

A linear complex $[\lambda]$ is called special if it represent the set of all V^n of K^r which meet a fixed V^{r-n}, axis of the complex, in a proper subspace.

2.2. - PROPOSITION. A linear complex $[\lambda]$ is a special linear complex with axis a fixed V^{r-n} iff the coefficients of λ are the grassmann cohordinates, antilexicographically ordered, of the V^{r-n}.

Proof. Expand by Laplace rule, respect to the first n rows, the determinant of the $r \times r$ matrix $\binom{S}{S_0}$ where $S \in A^{n \times r}_{reg}$ and S_o is an $(r-n) \times r$ matrix whose rows span the fixed V^{r-n}.

Given a linear complex $[\lambda]$ we denote by W_λ the open subvariety of $G_{n,r}$ of the points $x \in G_{n,r}$ such that $x \notin [\lambda]$.

2.3. -REMARK - i) Consider for any $i = 0, \ldots, N$ the linear homogeneous form x_i. We have that $[x_i]$ is a special linear complex and, in particular, the axis of $[x_0]$ is the $V^{r-n} < e_{n+1}, \ldots, e_r >$.

ii) The bundle $\gamma \mid W_{x_i}$ is trivial for any $i = 0, \ldots, N$ (see [6] 3.1.4).

A continuous (algebraic) canonical form on $L_i = p_2^{-1}(W_{x_i})$ is given by

$$L_i \ni S \longmapsto (S_i)^{-1} \cdot S$$

where $\det(S_i)$ is the i-th cohordinate of $p_2(S)$.

2.4. -PROPOSITION Let $[\lambda]$ be a special linear complex, then $\gamma \mid W_\lambda$ is trivial.

Proof. Let $\rho : K^r \longrightarrow K^r$ be a change of basis such that the ρ-image of the $V^{r-n} < e_{n+1}, \ldots, e_r >$, axis of $[x_0]$, is the V^{r-n} axis of $[\lambda]$.

ρ induces a continuous (algebraic) automorphisme ρ^* of γ and, since

$$\rho^*(\gamma \mid W_{x_0}) = \gamma \mid W_\lambda \text{ (see [8] 4.5), by 2.3 ii) } \gamma \mid W_\lambda \text{ is trivial.}$$

2. 5.-COROLLARY. *Let* $L \subset A^{n\times r}_{reg}$. *A sufficient condition for the existence of a continuous (algebraic) canonical form on* L *is that there exists a special linear complex* $[\lambda]$ *with* $p_2(L) \subset W_\lambda$.

2. 6.-PROPOSITION Let the field K have infinitely many elements and let $L = \{S_1, \ldots, S_n\} \times GL(n,K) \subset A^{n\times r}_{reg}$. *Then there exists a special linear complex* $[\lambda]$ *such that* $p_2(L) \subset W_\lambda$.

Proof. Let

$$A_i = \{ S \in A^{(r-n)\times r}_{reg} \text{ such that } \det \begin{pmatrix} S_i \\ S \end{pmatrix} \neq 0, i = 1, \ldots, n\}$$

$\bigcap_{i=1}^{n} A_i$ is non empty (see [8] 4. 4), let X be a point in $\bigcap_{i=1}^{n} A_i$. We have that $\det \begin{pmatrix} T\cdot S_i \\ X \end{pmatrix} \neq 0$ for $T \in GL(n,k)$, $i = 1, \ldots, n$ and so the conclusion follows from 2. 2.

2. 7. - EXAMPLES Let $[\lambda]$ be a special linear complex and let $L_\lambda = p_2^{-1}(W_\lambda)$. By 2. 3 ii) and 2. 4 a continuous (algebraic) canonical forme on L_λ is the following : let ρ be as in 2. 4 and let Y denote the associated $r\times r$ non singular matrix, then

$$L_\lambda \ni S \longrightarrow ((S\cdot Y^{-1})_0)^{-1}\cdot S.$$

In [8] 5 there are examples of families of systems, verifying the condition of 2. 5, on wich the " classical " canonical form of 2. 3 ii) are not defined.

In the same paper a canonical form of the above kind for these families is described.

Here we show that the condition of 2. 5 is not necessary for the existence of continuous canonical form on connected subsets of SCR (m, n) (see also [5] 7. 2.). At this aim assume $K = \mathbb{C}$, $n = 4$, $m = 3$ and denote grassmann cohordinates by $x_{i_1 i_2 i_3 i_4}$.

Let $V_1 = \{ x \in M_{3,4}$ such that

$$\left[\begin{array}{ll} x_{i_1 i_2 i_3 i_4} = 0 & \text{if } \{i_1, i_2, i_3, i_4\} \not\supset \{1,2,3\} \\ x_{1235} + x_{1236} = 0 & \\ x_{1235}\; x_{1239} = (x_{1234})^2 & \end{array}\right. \qquad \}$$

$V_2 = \{ x \in M_{3,4}$ such that

$$\left[\begin{array}{ll} x_{i_1 i_2 i_3 i_4} = 0 & \text{if } \{i_1, i_2, i_3, i_4\} \not\supset \{1,2,3\} \\ x_{1234}\; x_{1236} = (x_{1235})^2 & \\ x_{1235}\; x_{1237} = (x_{1236})^2 & \end{array}\right. \qquad \}$$

Then $p_1^{-1}(V_1) = L_1 = \{ (F_t, C) \times GL(n, K) \subset SCR(3,4)$ with

$$F_t = \begin{bmatrix} 0 & . & . & 0 \\ . & . & . & . \\ 0 & . & . & 0 \\ t & 1 & 1 & t^2 \end{bmatrix} \quad t \in \mathbb{C}, \quad G = \left[\begin{array}{c} I_3 \\ \hline 0\ 0\ 0 \end{array}\right] \qquad \}$$

$p_1^{-1}(V_2) = L_2 = \{ (F'_s, G) \times GL(n, K) \subset SCR(3,4)$ with

$$F'_s = \begin{bmatrix} 0 & . & . & 0 \\ . & . & . & . \\ 0 & . & . & 0 \\ 1 & s & s^2 & s^3 \end{bmatrix} \quad s \in \mathbb{C}, \quad G = \left[\begin{array}{c} I_3 \\ \hline 0\ 0\ 0 \end{array}\right] \qquad \}$$

$L_1 \cup L_2$ is connected since $L_1 \cap L_2 = (F_1, G) \times GL(n,K) = (F'_1, G) \times GL(n,K)$.

The maps

$$c_1 : L_1 \longrightarrow L_1 \quad \text{and} \quad c_2 : L_2 \longrightarrow L_2$$

defined respectively as

$$c_1((F_t, G), T) = (F_t, G) \text{ and } c_2((F'_s, G), T) = (F'_s, G)$$

are both continuous canonical forms on L_1 and on L_2.

Since c_1 and c_2 coïncide on $L_1 \cap L_2$, the map $c : L_1 \cup L_2 \longrightarrow L_1 \cup L_2$ defined as

$$c(A,B) = \begin{cases} c_1(A,B) & \text{if} \quad (A,B) \in L_1 \\ c_2(A,B) & \text{if} \quad (A,B) \in L_2 \end{cases}$$

is a continuous canonical form on $L_1 \cup L_2$.

Moreover if λ is a linear homogeneous form and a_{1234} denotes the coefficient of x_{1234} in λ , then if $a_{1234} = 0, \lambda(p_1(F'_o, G)) = 0$ and, if $a_{1234} \neq 0$, there exists $\bar{t} \in \mathbb{C}$ such that $\lambda(p_1(F_{\bar{t}}, G)) = 0$.

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REFERENCES

[1] M. CLARK. - The consistent selection of parametrization in systems identification. Proc. J.A.C.C. Purdue Univ. Lafayette Indiana (1977).

[2] M.J. DENHAM. - Canonical forms for the identification of multivariable linear systems. IEEE Trans. on Aut. Control. (Dec.1974).

[3] M. HAZEWINKEL. - Moduli and canonical forms for linear dynamical systems. Rep.7609/M Erasmus Univ. Rotterdam (1976).

[4] M. HAZEWINKEL. - Moduli and canonical forms for linear dynamical systems. Rep.7610/M Erasmus Univ. Rotterdam (1976).

[5] M. HAZEWINKEL, R.E. KALMAN. - On invariants, canonical forms and moduli for linear, constant, finite dim. dyn. systems- Math. System Theory, Udine 1975, Springer-Verlag (1976).

[6] D. HUSEMOLLER. - Fibre bundles. Mc Graw Hill (1966).

[7] R.E. KALMAN. - Algebraic geometric description of the class of linear systems of constant dimension. Proc. 8th Annual Princeton Conf., on Inf. Sci. and Systems (1974).

[8] A.M. PERDON, G. CONTE. - Continuous canonical forms for classes of completely reachable Systems. Ricerche di Aut. vol.8 (1977).

[9] V.M. POPOV. - Invariant description of linear time-invariant controllable systems. SIAM J. on Cont., vol.10 (1972).

[10] J. RISSANEN. - Basis of invariants and canonical forms for linear dynamical systems. Automatica, vol.10 (1974).

[11] F. SEVERI. - Sulla varietà che rappresenta gli spazi subordinati di data dimensione, immersi in uno spazio lineare. Ann. di

Matematica Serie III, vol. XXIV (1915).

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