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Astérisque, tome 61 (1979), p. 145-154
<http://www.numdam.org/item?id=AST_1979__61__145_0>

Société Mathématique de France
Astérisque 61 (1979) p. 145-154

SOME RECENT RESULTS IN TRANSCENDENCE THEORY

by

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INTRODUCTION

This article is divided into parts I, II, and III, dealing respectively with elliptic functions, gamma functions, and Siegel E-functions. In each part we describe some recent results and mention a number of open problems.

I - ELLIPTIC FUNCTIONS

M. Anderson [1], [2] has proved several elliptic analogues of A. Baker's inequalities [3] for linear forms in logarithms of algebraic numbers. Let $\wp(z)$ be a Weierstrass elliptic function with algebraic invariants g_2 and g_3. There is a canonical way of choosing a basis ω_1, ω_2 for the period lattice of $\wp(z)$ (see, e.g. [6] p. 421), and we denote by Π the fundamental parallelogram consisting of all points of the form $\theta_1\omega_1 + \theta_2\omega_2$ for real θ_1, θ_2 with $0 \leq \theta_1, \theta_2 < 1$. For $n \geq 1$ let $u_1, \ldots, u_n$ be non-zero points of Π such that $\wp(u_1), \ldots, \wp(u_n)$ are algebraic numbers of heights at most $A \geq 4$ and generate over the rational field $\mathbb{Q}$ and algebraic number field F of degree at most $d \geq 2$. Let $\beta_0, \ldots, \beta_n$ be algebraic numbers of heights at most $B \geq 4$ which generate over F an algebraic number field of degree at most $D \geq 2$ (over $\mathbb{Q}$). Put :

$$\Lambda = \beta_0 + \beta_1 u_1 + \ldots + \beta_n u_n$$

and assume $\Lambda \neq 0$.

All Anderson's results need the additional hypothesis of complex multiplication ; thus throughout this section we shall assume that $\wp(z)$ has complex multiplication over a complex quadratic field K. The main results take the form :

$$\log|\Lambda| > -CD^{\lambda} (\log A)^{\mu} \log B (\log\log B)^{\kappa}$$

where $C > 0$ is effectively computable in terms of g_2, g_3, n and the choice of κ, λ, μ. There are three possibilities for this latter choice ; unconditionally we can take any values satisfying either

$$\kappa > n+1, \ \lambda > 4n^2 + n + 3, \ \mu > n^2 + n$$

or

$$\kappa > n + 2 \text{ , } \lambda > n^2 + 4n + 6, \ \mu > n^2 + n - 1 \quad ;$$

while if $n \geq 2$ and one of $u_1, \ldots, u_n$ is a half-period we can suppose merely that :

$$\kappa > n + 1 \text{ , } \lambda > n^2 + 2n + 3 \text{ , } \ \mu > n^2 - n - 1 \ .$$

Already this presents us with two apparently very difficult problems ; firstly to remove the term $(\log \log B)^{\kappa}$ from these estimates, and secondly to relax the condition on μ to $\mu > n$. Both of these (and much more) have been solved in the case of linear forms in logarithms of algebraic numbers.

Next, suppose that $1, u_1, \ldots, u_n$ are linearly dependent over the field of algebraic numbers. It has been known for some time that then $u_1, \ldots, u_n$ must be linearly dependent over K . Anderson shows in fact that there exist integers $\rho_1, \ldots, \rho_n$ of K , not all zero, with absolute values at most

$$(cn^2 d^4 (\log d)^3 \log A)^{1/2\,(n-1)} \ ,$$

such that

$$\rho_1 u_1 + \ldots + \rho_n u_n = 0 \ .$$

Here $c > 0$ is effectively computable in terms of g_2 and g_3 .

For the rest of this section we shall discuss an interesting feature in the proof of this second result. First we consider the exponential case, due to J.H. Loxton and A.J. Van der Poorten [8] . One approach, not quite theirs, leads to the following problem. Let F be an algebraic number field of degree $d \geq 1$, and for any non-zero α in F define

$$h(\alpha) = \sum_{v} \log \max (1, v(\alpha)) \ ,$$

where the sum is taken over all normalized valuations v of F . Recall that there are d archimedean valuations v such that $v(\alpha)$ is the absolute value of one of the conjugates of α , and every other valuation is associated with a prime ideal $\mathfrak{P}$ of F . In the latter case $v(\alpha) = (N\mathfrak{P})^{-k}$ where k is the exact power of $\mathfrak{P}$ dividing the principal ideal generated by α in F , and $N\mathfrak{P}$ is the norm of $\mathfrak{P}$.

Now, it is not too hard to prove that $h(\alpha) = 0$ if and only if α is a root of unity ; this is essentially Kronecker's Theorem about algebraic integers on the unit circle. The problem referred to above is to find a good positive lower bound for $h(\alpha)$, depending only on d , as α runs over all non-zero elements of F that are not roots of unity.

This is solved as follows. If α is not an algebraic integer then $v(\alpha) > 1$ for some non-archimedean valuation v, and this implies $v(\alpha) \geq 2$, so that we have the lower bound $h(\alpha) \geq \log 2$. On the other hand, if α is an algebraic integer, then, assuming for the moment that α is of exact degree d with conjugates $\alpha_1, \ldots, \alpha_d$, we see that

$$\exp h(\alpha) = \prod_{i=1}^{d} \max(1, |\alpha_i|) .$$

It is a classical problem to find good lower bounds for the right-hand side of this equation, and until recently the best estimate was due to P.E. Blanksby and H.L. Montgomery [4].

They showed that if α is not a root of unity, then

$$\prod_{i=1}^{d} \max(1, |\alpha_i|) > 1 + (52d \log 6d)^{-1} .$$

We deduce that if $d \geq 2$ then

$$h(\alpha) > c(d \log d)^{-1} \qquad (*)$$

for some absolute constant $c > 0$, and it is easy to check that this remains valid even when α has degree less than d. It follows that (*) is valid for any non-zero α in F that is not a root of unity.

In 1977 C.L. Stewart found a new proof of (*), with a slightly smaller value of c, by applying techniques from transcendence theory (see [12]). Recently, E. Dobrowolski used similar methods to obtain a significant improvement on these estimates, in which the order of magnitude $(d \log d)^{-1}$ is replaced by $(\log \log d / \log d)^3$.

To explain the elliptic analogue, we note that h is a natural height function associated with the multiplicative group $\mathbb{C}^\times$ of non-zero complex numbers. This group can be identified with the curve $\mathcal{C}$ of points (x, y) satisfying $xy = 1$, for example. Then h is defined on the subgroup $\mathcal{C}(F) = F^\times$ of points (α, β) on $\mathcal{C}$ with coordinates α, β in F.

Moreover h vanishes exactly at the torsion points of $\mathcal{C}(F)$. Now let $\mathcal{E}$ denote the elliptic curve associated with $\wp(z)$, consisting of points (x, y) satisfying $y^2 = 4x^3 - g_2 x - g_3$ together with the point at infinity ∞.

This curve has an additive group structure, and, provided g_2 and g_3 lie in the algebraic number field F, so has the subset $\mathcal{E}(F)$ of points $P = (\alpha, \beta)$ on $\mathcal{E}$ with coordinates α, β in F (together with $P = \infty$).

There is a natural height function $\hat{h}$ on $\mathcal{E}(F)$ which vanishes exactly at the torsion points of $\mathcal{E}(F)$; this is the Tate height, defined in the following way.

For a finite point $P = (\alpha, \beta)$ of $\mathcal{E}(F)$ let

$$h(P) = \sum_{v} \log \max (1, v(\alpha), v(\beta)) \quad ,$$

where the sum is as before, and put $h(\infty) = 0$. Then for any P in $\mathcal{E}(F)$ the limit

$$\hat{h}(P) = \lim_{m \to \infty} m^{-2} h(mP)$$

can be shown to exist and have the required vanishing properties.

The problem we have to solve for this height function $\hat{h}$ is the same as before ; to find a good positive lower bound for $\hat{h}(P)$, depending only on the degree d of F , as P runs over all non-torsion points of $\mathcal{E}(F)$. The solution seems difficult if the approach of Blanksby-Montgomery is adopted. But by modifying the methods introduced by Stewart, Anderson was able to establish the following result. When $\wp(z)$ has complex multiplication, there is a constant $c > 0$, effectively computable in terms of g_2 and g_3 , such that for any algebraic number field F of degree at most $d \geq 2$ containing g_2 and g_3 we have

$$\hat{h}(P) > c\,(d \log d)^{-3}$$

for any non-torsion point P in $\mathcal{E}(F)$.

It is an interesting problem to extend this result to elliptic curves without complex multiplication. An easy argument shows that always $\hat{h}(P) > \exp(-cd^2)$ for some c independent of d , but it seems hard to prove $\hat{h}(P) > cd^{-\kappa}$ for some absolute constant κ . One might even ask whether $\hat{h}(P)$ is bounded below independently of d . The analogous question for $h(\alpha)$ was asked some time ago by D.H. Lehmer, but it remains unanswered to this day.

II - GAMMA FUNCTIONS

G.V. Chudnovsky's proof [5] of the transcendence of $\Gamma(1/4)$ shows in fact that the numbers π, $\Gamma(1/4)$ are algebraically independent over $\mathbb{Q}$ (see also [14]) . A similar remark applies to $\Gamma(1/3)$. By easy calculations we deduce that $\Gamma(x)$ is transcendental for the following values of x with $0 < x < 1$;

$$1/6\,,\ 1/4,\ 1/3,\ 1/2,\ 2/3,\ 3/4,\ 5/6\ .$$

Naturally, the next problem is to complete the Farey series of order 6 by adding the values 1/5, 2/5, 3/5, 4/5 to the list. This problem seems intimately connected, via the Chowla-Selberg relations, to the perhaps no more difficult question of the algebraic independence of periods of abelian varieties with complex multiplication. Recently P. Deligne found an extensive family of new algebraic relations between these periods, thereby substantially reducing the upper bound for the transcendence degree of the field they generate. Chudnovsky in his talk at the Helsinki Congress conjectured that the reduced upper bound is the correct value of the transcendence degree ; this amounts to saying that there are essentially no more algebraic relations. All that is known so far is that the transcendence degree is always at least 2 . This implies, for example, that at least two of the numbers π, $\Gamma(1/5)$, $\Gamma(2/5)$ are algebraically independent over $\mathbb{Q}$.

On a less exciting level, one can consider linear independence. In [9] I asked if the values $B(m/5, n/5)$ of the classical beta function span over the field of algebraic numbers a vector space of dimension 6 as m and n run over all positive integers. Recently I proved that this is so ; writing $\theta = \Gamma(1/5)$, $\Phi = \Gamma(2/5)$ we deduce that the numbers

$$1,\ \pi,\ \theta^2/\Phi,\ \pi\Phi/\theta^2,\ \theta\Phi^2/\pi\ ,\ \pi^2/\theta\Phi^2$$

are linearly independent over the field of algebraic numbers.

Finally let us mention an interesting quantitative sharpening of Chudnovsky's results. A subfield F of $\mathbb{C}$ that is finitely generated over $\mathbb{Q}$ is said to be of finite transcendence type if for any $\theta_1, \ldots, \theta_n$ in F there exist $C > 0$, $\tau > 0$ with the following property. For any polynomial P in $\mathbb{Z}[x_1, \ldots, x_n]$ with degree at most $d \geq 1$ and coefficients at most $H \geq 2$ in absolute value such that $P(\theta_1, \ldots, \theta_n) \neq 0$, we have (+)

$$\log |P(\theta_1, \ldots, \theta_n)| > -C(d + \log H)^{\tau} \ . \qquad (*)$$

(it suffices to check this for a single transcendence basis $\theta_1, \ldots, \theta_n$ of F over $\mathbb{Q}$) .

(+) It is probably too late to change the nomenclature now, but it would have been nice to define the transcendence type of F as the infimum τ_0 of all numbers τ such that (*) holds. If then (*) happens to hold for $\tau = \tau_0$ we could have said that F has strict transcendence type τ_0 (cf. the order of an entire function).

Classical results show that $\mathbb{Q}(e)$, $\mathbb{Q}(\pi)$, $\mathbb{Q}(e^{\pi})$ are all of finite transcendence type (and much more) ; but until recently no such field of transcendence degree 2 was known. The work of Chudnovsky provides several examples, including $\mathbb{Q}(\pi, \Gamma(1/4))$ and $\mathbb{Q}(\pi, \Gamma(1/3))$.

III - E-FUNCTIONS

Recall that $f(z) = \sum_{k=0}^{\infty} \alpha_k z^k/k!$ is defined to be an E-function if there is an algebraic number field F and a constant $c > 0$ such that, for each $m \geq 1$, there exists a positive integer $d_m \leq c^m$ such that $d_m \alpha_0, \ldots, d_m \alpha_m$ are algebraic integers of F with all their conjugates of absolute values at most c^m . The fundamental theorem of Siegel-Shidlovsky [11] is as follows. For $n \geq 1$ let $f_1(z), \ldots, f_n(z)$ be E-functions, algebraically independent over $\mathbb{C}(z)$, that satisfy a system of linear differential equations

$$f_i'(z) = q_{i0}(z) + \sum_{j=1}^{n} q_{ij}(z) f_j(z) \qquad (1 \leq i \leq n)$$

with rational functions $q_{ij}(z)$ in $\mathbb{C}(z)$. Then for any non-zero algebraic number α distinct from the poles of the $q_{ij}(z)$, the values $f_1(\alpha), \ldots, f_n(\alpha)$ are algebraically independent over $\mathbb{Q}$.

In 1962 , S. Lang obtained a quantitative version of this (see [7]) . For brevity put $\theta_i = f_i(\alpha)$ $(1 \leq i \leq n)$. He proved that for any $d \geq 1$ there exist constants $c > 0$, $C > 0$, depending only on d, α, and the functions $f_1(z), \ldots f_n(z)$, with c independent of d , having the following property. For any non-zero polynomial P in $\mathbb{Z}[x_1, \ldots, x_n]$ of degree at most d and coefficients with absolute values at most $H \geq 2$, we have

$$|P(\theta_1, \ldots, \theta_n)| > CH^{-cd^n} .$$

Since C may depend on d , this does not show that the field $\mathbb{Q}(\theta_1, \ldots, \theta_n)$ has finite transcendence type ; and indeed this has not been proved even for the simple example $\mathbb{Q}(e, e^{\sqrt{2}})$.
Recently, Ju. V. Nesterenko [10] published a proof that we can take

$$C^{-1} = \exp\exp(c' d^{2n} \log d)$$

for some c' depending only on α and $f_1(z), \ldots, f_n(z)$.

This relatively weak estimate reflects the fact that Siegel's method is designed to operate with linear independence of power products rather than directly with algebraic independence itself. Incidentally, it does not seem clear from Nesterenko's proof that c' is in fact effectively computable in all cases, and it would be an interesting exercise to verify this point.

The main part of Nesterenko's paper is concerned with estimates for zeroes of functions that are polynomials in $f_1(z), \ldots, f_n(z)$. The arguments are algebraic in nature rather than analytic, and the paper contains techniques from commutative algebra that should be applicable elsewhere in transcendence theory. Recently, Dale Brownawell and myself have obtained similar zero-estimates for solutions of certain non-linear differential equations. Among other things these lead to a quantitative version of the Schneider-Lang Theorem [7], [13].

Let us first recall this result. Let $f_1(z), \ldots, f_n(z)$ be meromorphic functions of finite growth order ρ. Suppose they satisfy differential equations of the form

$$f_i'(z) = P_i(f_1(z), \ldots, f_n(z)) \qquad (1 \leq i \leq n) \quad ,$$

where $P_1, \ldots, P_n$ are polynomials with coefficients in an algebraic number field F of degree at most $d \geq 1$. Suppose further that at least two of $f_1(z), \ldots, f_n(z)$ are algebraically independent over F. Then if $m > 2\rho d$ and $w_1, \ldots, w_m$ are any distinct points at which $f_1(z), \ldots, f_n(z)$ are analytic, not all the values $f_i(w_j)$ $(1 \leq i \leq n,\ 1 \leq j \leq m)$ can lie in F.

Thus, if β_{ij} $(1 \leq i \leq n,\ 1 \leq j \leq m)$ are elements of F with heights at most $B \geq 1$, the expression

$$\mathcal{U} = \max |f_i(w_j) - \beta_{ij}|$$

never vanishes, and we can ask for a positive lower bound for $\mathcal{U}$ as a function of B. In fact, a further hypothesis is needed before we can give a satisfactory answer.

For, if $\lambda_1, \ldots, \lambda_n$ are complex numbers algebraically independent over $\mathbb{Q}$, the constant functions $f_i(z) = \lambda_i$ $(1 \leq i \leq n)$ satisfy all the conditions of the theorem, and, if also $\lambda_1, \ldots, \lambda_n$ are arbitrarily well approximated by numbers of F, then $\mathcal{U}$ can be an equally arbitrarily small function of B.

However, let us exclude this possibility by assuming that there exists an additional point w_0, at which $f_1(z), \ldots, f_n(z)$ are analytic, such that $f_1(w_0), \ldots, f_n(w_0)$ lie in F. Then we can show that given any $\varepsilon > 0$, there is a constant $c > 0$, depending in a simple way on n and ε, such that if $m > c\rho d$ and $w_1, \ldots, w_m$ are points as above, then

$$\mathcal{U} > C \exp(-B^{\varepsilon})$$

for some $C > 0$ depending only on $f_1(z), \ldots, f_n(z)$, $w_0, \ldots, w_m$, ε, and F.

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