A. Fathi
M. Shub

Some dynamics of pseudo-Anosov diffeomorphisms

Astérisque, tome 66-67 (1979), p. 181-207
<http://www.numdam.org/item?id=AST_1979__66-67__181_0>

EXPOSÉ 10

SOME DYNAMICS OF PSEUDO-ANOSOV DIFFEOMORPHISMS

by A. FATHI and M. SHUB

We prove in this "exposé" that a pseudo-Anosov diffeomorphism realizes the minimum of topological entropy in its isotopy class. In section I, we define topological entropy and give its elementary properties. In section II, we define the growth of an endomorphism of a group and show that the topological entropy of a map is greater than the growth of the endomorphism it induces on the fundamental group. In section III, we define subshifts of finite type and give some of their properties. In section IV, we prove that the topological entropy of a pseudo-Anosov diffeomorphism is the growth rate of the automorphism induced on the fundamental group, it is also $\log \lambda$, where $\lambda > 1$ is the stretching factor of f on the unstable foliation. In section V, we prove the existence of a Markov partition for a pseudo-Anosov diffeomorphism, this fact is used in section IV. In section VI, we show that a pseudo-Anosov map is Bernoulli.

§ I. - TOPOLOGICAL ENTROPY

Topological entropy was defined to be a generalization of measure theoretic entropy [1] . In some sense, entropy is a number (possibly infinite) which describes "how much" dynamics a map has. Here the emphasis, of course, must be on asymptotic behaviour. For example, if $f : X \to X$ is a map and $N_n(f)$ is the cardinality of the fixed point set of f^n , then $\limsup \frac{1}{n} \log N_n(f)$ is one measure of "how much" dynamics f has ; but, if we consider $f \times R_\theta : X \times T^1 \to X \times T^1$ to be $(f \times R_\theta)(x,\alpha) = (f(x), \theta + \alpha)$ where $T^1 = \mathbb{R}/\mathbb{Z}$ and θ is irrational, then $N_n(f \times R_\theta) = 0$, and yet $f \times R_\theta$ should have at least as "much" dynamics as f . Topological entropy is a topological invariant which overcomes this difficulty.

We describe a lot of material frequently without crediting authors.

<u>Definitions</u>. Let $f : X \to X$ be a continuous map of a compact topological space X . Let $\mathcal{A} = \{A_i\}_{i\in I}$ and $\mathcal{B} = \{B_j\}_{j\in J}$ be open covers of X , the open cover $\{A_i \cap B_j\}_{\substack{i\in I \\ j\in J}}$ will be denoted by $\mathcal{A} \vee \mathcal{B}$. If $\mathcal{A}$ is a cover, $N_n(f,\mathcal{A})$ denotes the minimum cardinality of a subcover of $\mathcal{A} \vee f^{-1}\mathcal{A} \vee \dots \vee f^{-n+1}\mathcal{A}$, and $h(f,\mathcal{A}) = \limsup \frac{1}{n} \log N_n(f,\mathcal{A})$. The topological entropy of f is $h(f) = \sup_{\mathcal{A}} h(f,\mathcal{A})$ where the supremum is taken over all open covers of X .

<u>Proposition</u>. Let X and Y be compact spaces. Let $f : X \to X$, $g : Y \to Y$ and $h : X \to Y$ be continuous. Suppose that h is surjective and $hf = gh$:

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow{\ g\ } & Y \end{array} \quad ;$$

then, $h(f) \geq h(g)$.

In particular, if h is a homeomorphism, $h(f) = h(g)$. So <u>topological entropy</u> is a <u>topological invariant</u>.

<u>Proof</u>. Pull back the open covers of Y to open covers of X . □

For metric spaces, compact or not, Bowen has proposed the following definition. Suppose $f : X \to X$ is a continuous map of a metric space X and suppose

$K \subset X$ is compact. Let ε be > 0. We say that a set $E \subset K$ is (n, ε)-separated if, given $x, y \in E$ with $x \neq y$, there is $0 \leq i < n$ such that $d(f^i(x), f^i(y)) \geq \varepsilon$. We let $s_K(n, \varepsilon)$ be the maximal cardinality of an (n, ε)-separated set contained in K. We say that the set E is (n, ε)-spanning for K if, given $y \in K$, there is an $x \in E$ such that $d(f^i(x), f^i(y)) < \varepsilon$ for each i with $0 \leq i < n$. We let $r_K(n, \varepsilon)$ be the minimal cardinality of an (n, ε)-spanning set contained in K. It is easy to see that $r_K(n, \varepsilon) \leq s_K(n, \varepsilon) \leq r_K(n, \frac{\varepsilon}{2})$. We let $\bar{s}_K(\varepsilon) = \limsup \frac{1}{n} \log s_K(n, \varepsilon)$ and $\bar{r}_K(\varepsilon) = \limsup \frac{1}{n} \log r_K(n, \varepsilon)$. Obviously, $\bar{s}_K(\varepsilon)$ and $\bar{r}_K(\varepsilon)$ are decreasing functions of ε, and $\bar{r}_K(\varepsilon) \leq \bar{s}_K(\varepsilon) \leq \bar{r}_K(\frac{\varepsilon}{2})$. Hence, we may define $h_K(f) = \lim_{\varepsilon \to 0} \bar{s}_K(\varepsilon) = \lim_{\varepsilon \to 0} \bar{r}_K(\varepsilon)$. Finally, we put $h_X(f) = \sup \{h_K(f) \mid K \text{ compact} \subset X\}$.

Proposition [2], [4]. If X is a compact metric space and $f : X \to X$ is continuous, then $h_X(f) = h(f)$.

The proof is rather straightforward. By the Lebesgue covering lemma, every open cover has a refinement which consists of ε-balls.

The number $h_X(f)$ depends on the metric on X and makes best sense for uniformly continuous maps.

Suppose that X and Y are metric spaces, we say that $p : X \to Y$ is a metric covering map if it is surjective and satisfies the following condition : there exists $\varepsilon > 0$ such that, for any $0 < \delta < \varepsilon$, any $y \in Y$ and any $x \in p^{-1}(y)$, the map $p : B_\delta(x) \to B_\delta(y)$ is a bijective isometry (here $B_\delta(\cdot)$ is the δ-ball).

The main example we have in mind is the universal covering $p : \widetilde{M} \to M$ of a compact differentiable manifold M.

Proposition. Suppose $p : X \to Y$ is a metric covering and $f : X \to X$, $g : Y \to Y$ are uniformly continuous. If $pf = gp$, then $h_X(f) = h_Y(g)$.

Proof. It should be an easy estimate. The clue is that for $\ell > 0$ and for any sequence a_n, we have $\limsup \frac{1}{n} \log(\ell a_n) = \limsup \frac{1}{n} \log a_n$. If $K \subset X$ and $K' \subset Y$ are compact and $p(K) = K'$, then there is a number $\ell > 0$ such that $[\text{cardinality } p^{-1}(y)] \leq \ell$ for all $y \in K'$. In fact, we may choose ℓ such that, if $\delta > 0$ is small enough, then $p^{-1}(B_\delta(y)) \cap K$ can be covered by at most ℓ 2δ-balls centered at points in $p^{-1}(B_\delta(y)) \cap K$.

By the uniform continuity of f, we can find a δ_0 $(< \varepsilon)$ such that $x, x' \in X$ and $d(x,x') < \delta_0$ implies $d(f(x), f(x')) < \varepsilon$, where $\varepsilon > 0$ is the one given in the definition of a metric covering. If $2\delta < \delta_0$, it is easy to see that if $E' \subset K'$ is an (n,δ)-spanning set for g, then there exists an $(n,2\delta)$-spanning set $E \subset K$ for f, such that $\text{card } E \leq \ell \text{ card } E'$. So, we have $r_K(n,2\delta) \leq \ell r_{K'}(n,\delta)$, hence $\bar{r}_K(f,2\delta) \leq \bar{r}_{K'}(g,\delta)$ and $h_K(f) \leq h_{K'}(g)$.

On the other hand, if $E \subset K$ is (n,η)-spanning (with $0 < \eta < \varepsilon$) then $p(E) \subset K'$ is (n,η)-spanning. So $r_{K'}(n,\eta) \leq r_K(n,\eta)$, hence $h_{K'}(g) \leq h_K(f)$. Consequently $h_K(f) = h_{K'}(g)$. Since we sup over all compact sets and since p is surjective, we obtain $h_X(f) = h_Y(g)$. □

We add one additional fact.

Proposition. If X is compact and $f : X \to X$ is a homeomorphism, then $h(f^n) = |n| h(f)$.

For a proof, see [1] or [2].

§ II. - THE FUNDAMENTAL GROUP

Given a finitely generated group G and a finite set of generators $\mathcal{G} = \{g_1, \ldots, g_r\}$ of G, we define the length of an element g of G by $L_{\mathcal{G}}(g)$ = minimum length of a word in the g_i's and the g_i^{-1}'s representing the element g.

It is easy to see that if $\mathcal{G}' = \{g'_1, \ldots, g'_s\}$ is another set of generators, then :

$$L_{\mathcal{G}}(g) \leq (\max L_{\mathcal{G}}(g'_i)) \, L_{\mathcal{G}'}(g) \; .$$

If $A : G \to G$ is an endomorphism, let :

$$\gamma_A = \sup_{g \in G} \limsup \frac{1}{n} \log L_{\mathcal{G}}(A^n g) = \sup_{g_i \in \mathcal{G}} \limsup \frac{1}{n} \log L_{\mathcal{G}}(A^n g_i) \; .$$

So γ_A <u>is finite</u> and by the inequality given above, γ_A does not depend on the set of generators.

<u>Proposition</u> 1. If $A : G \to G$ is an endomorphism and $g \in G$, define $gAg^{-1} : G \to G$ by $[gAg^{-1}](x) = gA(x)g^{-1}$. We have $\gamma_A = \gamma_{gAg^{-1}}$.

<u>Caution</u> : $(gAg^{-1})^n \neq gA^n g^{-1}$.

First, we need a lemma.

<u>Lemma</u> 1. Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences with a_n and $b_n \geq 0$ and k be > 0. We have :

i) $\limsup \frac{1}{n} \log(a_n+b_n) = \max(\limsup \frac{1}{n} \log a_n, \limsup \frac{1}{n} \log b_n)$

ii) $\limsup \frac{1}{n} \log k a_n = \limsup \frac{1}{n} \log a_n$

iii) $\limsup \frac{1}{n} \log a_n \leq \limsup \frac{1}{n} \log(a_1+\ldots+a_n) \leq \max(0, \limsup \frac{1}{n} \log a_n)$

<u>Proof</u>. Put $a = \limsup \frac{1}{n} \log a_n$ and $b = \limsup \frac{1}{n} \log b_n$.

i) The inequality $\max(a,b) \leq \limsup \frac{1}{n} \log(a_n+b_n)$ is clear.

If $c > \max(a,b)$, then we can find $n_0 \geq 1$ such that $n \geq n_0$ implies $a_n \leq e^{nc}$ and $b_n \leq e^{nc}$. We obtain for $n \geq n_0$:

$$\frac{1}{n}\log(a_n + b_n) \leq \frac{1}{n}\log(2e^{nc}) .$$

Hence $\limsup \frac{1}{n}\log(a_n+b_n) \leq \limsup \frac{1}{n}\log(2e^{nc}) = c$.

ii) is clear.

iii) The inequality $a \leq \limsup \frac{1}{n}\log(a_1+\ldots+a_n)$ is clear.

Suppose $c > \max(0,a)$. We can find then $n_0 \geq 1$ such that $a_n \leq e^{nc}$ for $n \geq n_0$. We have for $n \geq n_0$:

$$a_1 + \ldots + a_n \leq \sum_{i=1}^{n_0-1} a_i + \frac{e^{(n+1-n_0)c} - 1}{e^c - 1} e^{n_0 c} .$$

It follows clearly that $\limsup_n \frac{1}{n}\log(a_1+\ldots+a_n) \leq c$. □

<u>Proof of Proposition</u> 1. If $x \in G$, we have :

$$(gAg^{-1})^n(x) = gA(g)\ldots A^{n-1}(g)A^n(x)A^{n-1}(g^{-1})\ldots A(g^{-1})g^{-1} .$$

Suppose first that $A^{n_0}(g) = e$ for some n_0 , then it is clear that by lemma 1 i) :

$$\limsup \frac{1}{n}\log L_{\mathfrak{G}}[(gAg^{-1})^n(x)] \leq \limsup \frac{1}{n}\log L_{\mathfrak{G}}(A^n(x)) .$$

If $A^n(g) \neq e$ for each $n \geq 1$, we have $L_{\mathfrak{G}}(A^n(g)) \geq 1$, for each $n \geq 1$; hence $\limsup \frac{1}{n}\log L_{\mathfrak{G}}(A^n(g)) \geq 0$. By lemma 1 i) & iii) , we obtain :

$$\limsup \frac{1}{n}\log L_{\mathfrak{G}}[(gAg^{-1})^n(x)]$$
$$\leq \max(\limsup \frac{1}{n}\log L_{\mathfrak{G}}(A^n(g)), \limsup \frac{1}{n}\log L_{\mathfrak{G}}(A^n(x))) .$$

This gives us $\gamma_{gAg^{-1}} \leq \gamma_A$, and by symmetry, we have $\gamma_{gAg^{-1}} = \gamma_A$. □

For a compact connected differentiable manifold, we interpret $\pi_1(M)$ as the group of covering transformations of the universal covering space $\widetilde{M}$ of M. If $f: M \to M$ is continuous, then there is a lifting $\widetilde{f}: \widetilde{M} \to \widetilde{M}$. If $\widetilde{f}_1$ and $\widetilde{f}_2$ are both liftings of f, then $\widetilde{f}_1 = \theta \widetilde{f}_2$ for some covering transformation θ. A given lifting $\widetilde{f}_1$ determines an endomorphism $\widetilde{f}_{1\#}$ of $\pi_1(M)$ by the formula $\widetilde{f}_1 \alpha = \widetilde{f}_{1\#}(\alpha)\widetilde{f}_1$ for any covering transformation α. If $\widetilde{f}_1$ and $\widetilde{f}_2$ are two liftings of f, then $\widetilde{f}_1 = \theta \widetilde{f}_2$ for some covering transformation θ and $\widetilde{f}_1 \alpha = \theta \widetilde{f}_2 \alpha = \theta \widetilde{f}_{2\#}(\alpha)\widetilde{f}_2 = \theta \widetilde{f}_{2\#}(\alpha)\theta^{-1}\widetilde{f}_1$, so $\widetilde{f}_{1\#} = \theta \widetilde{f}_{2\#}\theta^{-1}$ and $\gamma_{\widetilde{f}_{1\#}} = \gamma_{\widetilde{f}_{2\#}}$. Thus, we may define $\gamma_{f\#} = \gamma_{\widetilde{f}\#}$ for any lifting $\widetilde{f}: \widetilde{M} \to \widetilde{M}$ of f. If f has a fixed point $m_0 \in M$, then there is also a map $f_\#$: $\pi_1(M, m_0) \to \pi_1(M, m_0)$. The group $\pi_1(M, m_0)$ is isomorphic to the group of covering transformations of $\widetilde{M}$ and f may be lifted to $\widetilde{f}$ such that $\widetilde{f}_\# : \pi_1(M) \to \pi_1(M)$ is identified with $f_\# : \pi_1(M, m_0) \to \pi_1(M, m_0)$ by this isomorphism. Thus $\gamma_{f\#}$ makes coherent sense in the case that f nas a fixed point as well.

We suppose now that M has a Riemannian metric and we put on $\widetilde{M}$ a Riemannian metric by lifting the metric on M via the covering map $p : \widetilde{M} \to M$. The map p is then a metric covering and the covering transformations are isometries. We have the following lemma due to Milnor [8].

<u>Lemma</u> 2. Fix $x_0 \in \widetilde{M}$. There exist two constants $c_1, c_2 > 0$ such that for each $g \in \pi_1(M)$, we have :

$$c_1 L_{\mathcal{G}}(g) \leq d(x_0, gx_0) \leq c_2 L_{\mathcal{G}}(g) .$$

<u>Proof</u> [8]. Let $\delta = \operatorname{diam} M$, and define $N \subset \widetilde{M}$ by $N = \{x \in \widetilde{M} \mid d(x, x_0) \leq \delta\}$. We have $p(N) = M$. Remark that $\{gN\}_{g \in \pi_1(M)}$ is a locally finite covering of $\widetilde{M}$ by compact sets. Choose as a finite set of generators $\mathcal{G} = \{g \in \pi_1(M) \mid gN \cap N \neq \emptyset\}$ and notice that $g \in \mathcal{G} \iff g^{-1} \in \mathcal{G}$. Suppose $L_{\mathcal{G}}(g) = n$, then we can write $g = g_1 \cdot \ldots \cdot g_n$, with $g_i N \cap N \neq \emptyset$. It is easy to see then that $d(x_0, gx_0) \leq 2\delta n$. Hence, we obtain :

$$d(x_0, gx_0) \leq 2\delta L_{\mathcal{G}}(g) \ .$$

Now, put $\nu = \min\{d(N, gN) \mid N \cap gN = \emptyset\}$, by compactness $\nu > 0$. Let k be the minimal integer such that $d(x_0, gx_0) < k\nu$. Along the minimizing geodesic from x_0 to gx_0 , take $k+1$ points $y_0 = x_0$, $y_1 \dots y_{k-1}$, $y_k = gx_0$, such that $d(y_i, y_{i+1}) < \nu$ for $i = 0, \dots, k-1$. Then, for $1 \leq i \leq k-1$, choose $y'_i \in N$ and $g_i \in G$ such that $y_i = g_i y'_i$, and put $g_0 = e$ and $g_k = g$. We have $d(g_i y'_i, g_{i+1} y'_{i+1}) < \nu$, hence $g_i^{-1} g_{i+1} \in \mathcal{G}$. From $g = (g_0^{-1} g_1) \dots (g_{k-1}^{-1} g_k)$, we obtain $L_{\mathcal{G}}(g) \leq k$.

Since k is minimal, we have :

$$L_{\mathcal{G}}(g) \leq \frac{1}{\nu} d(x_0, gx_0) + 1 \leq \left(\frac{1}{\nu} + \frac{1}{\mu}\right) d(x_0, gx_0)$$

where $\mu = \min\{d(x_0, g\,x_0) \mid g \neq e,\ g \in \pi_1(M)\}$. □

Consider now $f : M \to M$ and let $\tilde{f} : \tilde{M} \to \tilde{M}$ be a lifting of f . Applying the lemma above, we obtain, for each $x_0 \in \tilde{M}$:

$$\gamma_{f_\#} = \max_{g \in \pi_1(M)} \limsup \frac{1}{n} \log d(x_0, \tilde{f}^n_\#(g)\, x_0) \ .$$

We next prove the following lemma :

<u>Lemma</u> 3. Given $x, y \in \tilde{M}$, we have :

$$\limsup \frac{1}{n} \log d(\tilde{f}^n(x), \tilde{f}^n(y)) \leq h(f) \ .$$

<u>Proof</u>. Choose an arc α from x to y . If $y_1, \dots, y_\ell \in \alpha$ is $(n+1, \varepsilon)$-spanning for α and $\tilde{f}$, then $\tilde{f}^n(\alpha) \subset \bigcup_{i=1}^{\ell} B(\tilde{f}^n(y_i), \varepsilon)$. Since $\tilde{f}^n(\alpha)$ is connected, this implies $\operatorname{diam} \tilde{f}^n(\alpha) \leq 2\varepsilon\ell$. Hence :

$$d(\tilde{f}^n(x), \tilde{f}^n(y)) \leq 2\varepsilon\ell \ .$$

By taking ℓ to be minimal, we obtain :

$$d(\tilde{f}^n(x), \tilde{f}^n(y)) \leq 2\varepsilon r_\alpha(n+1, \varepsilon) \ .$$

From this, we get :

$$\limsup_{n\to\infty} \frac{1}{n} \log d(\tilde{f}^n(x), \tilde{f}^n(y)) \le \limsup_{n\to\infty} \frac{1}{n} \log [2\varepsilon r_\alpha(n+1,\varepsilon)] = \bar{r}_\alpha(\varepsilon) \le$$

$$\le h_\alpha(\tilde{f}) \le h(\tilde{f}) = h(f) . \qquad \square$$

We are now ready to prove :

<u>Theorem</u>. If $f : M \to M$ is a continuous map, then :

$$h(f) \ge \gamma_{f_\#} .$$

<u>Proof</u>. Since $\gamma_{f\#} = \max_{g\in\pi_1(M)} [\limsup \frac{1}{n} \log d(x_0, \tilde{f}^n_\#(g)x_0)]$, we have to prove that for each $g \in \pi_1(M)$:

$$\limsup \frac{1}{n} \log d(x_0, \tilde{f}^n_\#(g)x_0) \le h(f) .$$

We have :

$$d(x_0,\tilde{f}^n_\#(g)x_0) \le d(x_0,\tilde{f}^n(x_0)) + d(\tilde{f}^n(x_0),\tilde{f}^n_\#(g)\tilde{f}^n(x_0)) + d(\tilde{f}^n_\#(g)\tilde{f}^n(x_0),\tilde{f}^n_\#(g)x_0) .$$

Since $\tilde{f}^n_\#(g)\tilde{f}^n = \tilde{f}^n g$, and the covering transformations are isometries, we obtain :

$$d(x_0,\tilde{f}^n_\#(g)x_0) \le 2\, d(x_0,\tilde{f}^n(x_0)) + d(\tilde{f}^n(x_0),\tilde{f}^n(g x_0)) .$$

Remark also that :

$$d(x_0,\tilde{f}^n(x_0)) \le d(x_0,\tilde{f}(x_0)) + d(\tilde{f}(x_0),\tilde{f}^2(x_0)) + \dots + d(\tilde{f}^{n-1}(x_0),\tilde{f}^n(x_0)) .$$

By applying lemma 3 and lemma 1 (together with the fact $h(f) \ge 0$) , we obtain :

$$\limsup \frac{1}{n} \log d(x_0,\tilde{f}^n_\#(g)\, x_0) \le h(f) . \qquad \square$$

The proof of the following lemma is straigth forward.

<u>Lemma</u> . If G_1 and G_2 are finitely generated groups, if $A : G_1 \to G_1$, $B : G_2 \to G_2$

and $p : G_1 \to G_2$ are homomorphisms with p surjective and $pA = Bp$:

$$\begin{array}{ccccc} G_1 & \xrightarrow{p} & G_2 & \longrightarrow & 0 \\ \downarrow{\scriptstyle A} & & \downarrow{\scriptstyle B} & & \\ G_1 & \xrightarrow{p} & G_2 & \longrightarrow & 0 \end{array}$$

then , $\gamma_A \geq \gamma_B$.

Applying this lemma to the fundamental group of M mod the commutator subgroup, we have :

$$\begin{array}{ccccc} \pi_1(M) & \longrightarrow & H_1(M) & \longrightarrow & 0 \\ {\scriptstyle f_\#}\downarrow & & \downarrow{\scriptstyle f_{1*}} & & \\ \pi_1(M) & \longrightarrow & H_1(M) & \longrightarrow & 0 \end{array}$$

so we obtain Manning's theorem [5] .

<u>Theorem</u>. If $f : M \to M$ is continuous, then $h(f) \geq \gamma_{f_{1*}} = \max \log \lambda$, where λ ranges over the eigenvalues of f_{1*} .

<u>Remark</u> 1. For $\alpha \in \pi_1(M, m_0)$, we denote by $[\alpha]$ the class of loops freely homotopic to α . If M has a Riemannian metric, let $\ell([\alpha])$ be the minimum length of a (smooth) loop in this class. If $f : M \to M$ is continuous, $f[\alpha]$ is clearly well defined as a free homotopy class of loops. Let $G_f([\alpha]) = \limsup_n \frac{1}{n} \log[\ell(f^n[\alpha])]$ and let $G_f = \sup_\alpha G_f([\alpha])$.

It is not difficult to see that $G_f \leq \gamma_{f\#}$. In fact, we have $\ell(f^n[\alpha]) \leq$ $\leq d(x_0, \tilde{f}_\#^n(\alpha)x_0)$, since the minimizing geodesic from x_0 to $\tilde{f}_\#^n(\alpha)x_0$ has an image in M which represents $f^n[\alpha]$.

<u>Remark</u> 2. It occured to various people that Manning's theorem is a theorem about π_1 .

Among these are Bowen, Gromov and Shub. Manning's proof can be adapted. The proof above is more like Gromov [4] or Bowen [2], but we take responsability for any error. At first, we assumed that f had a periodic point or we worked with G_f. After reading Bowen's proof [3], we eliminated the necessity for a periodic point.

Remark 3. If $x \in M$ and ρ is a path joining x to $f(x)$, we call $\rho_\#$ the homomorphism

$$\begin{array}{ccc} \pi_1(M,f(x)) & \to & \pi_1(M,x) \\ [\gamma] & \longmapsto & [\rho^{-1}\gamma\rho] \end{array}$$

. Since $f_\# : \pi_1(M,x) \to \pi_1(M,f(x))$, the composition $\rho_\# f_\#$ is a homomorphism of $\pi_1(M,x)$ into itself. This homomorphism can be identified with $\tilde{f}_\#$ for a lifting $\tilde{f}$ of f. Thus our result is the same as Bowen's [3].

§ III. - SUBSHIFTS OF FINITE TYPE

Let $A = (a_{ij})$ be a $k \times k$ matrix such that $a_{ij} = 0$ or 1, for $1 \leq i, j \leq k$, that is A is a $0-1$ matrix. Such a matrix A determines a subshift of finite type as follows. Let $S_k = \{1,\dots,k\}$ and let $\Sigma(k) = \prod_{i=-\infty}^{i=\infty} S_k^i$, where $S_k^i = S_k$ for each $i \in \mathbb{Z}$. We put on S_k the discrete topology and on $\Sigma(k)$ the product topology. The subset $\Sigma_A \subset \Sigma(k)$ is the closed subset consisting of those bi-infinite sequences $\underline{b} = (b_n)_{n\in\mathbb{Z}}$ such that $a_{b_i b_{i+1}} = 1$ for all $i \in \mathbb{Z}$.

Pictorially, we image k boxes : [1] [2] .... [k] and a point which at discrete "time n" can be in any one of the boxes. The bi-infinite sequences represent all possible histories of points. If we add the restriction that a point may move from box i to box j, if and only if $a_{ij} = 1$, then the set of all possible histories is precisely Σ_A.

The shift $\sigma_A : \Sigma_A \to \Sigma_A$ is defined by $\sigma_A[(b_n)_{n\in\mathbb{Z}}] = (b'_n)_{n\in\mathbb{Z}}$, where $b'_n = b_{n+1}$ for each $n \in \mathbb{Z}$. Clearly, σ_A is continuous. Let $C_i \subset \Sigma(k)$ be defined by $C_i = \{\underline{x} \in \Sigma(k) \mid x_0 = i\}$. Let $D_i = C_i \cap \Sigma_A$, then $\mathcal{D} = \{D_1,\dots,D_k\}$ is an open cover of Σ_A by pairwise disjoint elements. For any $k \times k$ matrix $B = (b_{ij})$, we define the norm $\|B\|$ of B by $\|B\| = \sum_{i,j=1}^{k} |b_{ij}|$. It is easy to see that $N_n(\sigma_A, \mathcal{D}) = \mathrm{card}\,(\mathcal{D} \vee \ldots \vee \sigma_A^{-n+1}\mathcal{D}) \leq \|A^{n-1}\|$ because the integer $a_{ij}^{(n)}$ is equal to the number of sequences $(i_0, \ldots, i_n)$ with $i_\ell \in \{1, \ldots, k\}$, $i_0 = i$, $i_n = j$ and $a_{i_\ell i_{\ell+1}} = 1$. So $\limsup \frac{1}{n}\log(N_n(\sigma_A, \mathcal{D})) \leq \limsup \frac{1}{n} \log \|A^{n-1}\| = \limsup \log \|A^n\|^{1/n}$. This later number is recognizable as log (spectral radius A) or $\log \lambda$, where λ is the largest modulus of an eigenvalue of A. In fact, we have :

<u>Proposition</u>. For any subshift of finite type $\sigma_A : \Sigma_A \to \Sigma_A$, we have $h(\sigma_A) = \log \lambda$, where λ is the spectral radius of A.

<u>Proof</u>. We begin by noticing that each open cover $\mathcal{U}$ of Σ_A is refined by a cover of the form $\bigvee_{i=-\ell}^{\ell} \sigma_A^{-i}\mathcal{D}$. This implies, with the notations of section I :

$$N_{n+1}(\sigma_A, \mathcal{U}) \leq \mathrm{card}\,(\bigvee_{j=-\ell}^{j=n+\ell} \sigma_A^{-j}\mathcal{D}) = \mathrm{card}(\bigvee_{j=0}^{j=n+2\ell} \sigma_A^{-j}\mathcal{D}) = N_{n+2\ell+1}(\sigma_A, \mathcal{D}) \ .$$

Hence, we obtain : $h(\sigma_A, \mathcal{U}) \leq h(\sigma_A, \mathcal{D})$.

This shows that : $h(\sigma_A) = h(\sigma_A, \mathcal{D})$.

We now compute $h(\sigma_A, \mathcal{D})$. We distinguish two cases.

<u>First case</u>. Each state $i = 1, \ldots, k$ occurs. This means that $D_i \neq \emptyset$ for each $D_i \in \mathcal{D}$. It is not difficult to show by induction that we have in fact

$$N_{n+1}(\sigma_A, \mathcal{D}) = \mathrm{card}\,(\mathcal{D} \vee \ldots \vee \sigma_A^{-n}\mathcal{D}) = \|A^n\| \ .$$

This proves the proposition in this case, as we saw above.

<u>Second case</u>. Some states do not occur. One can see that a state i occurs, if, and only if, for each $n \geq 0$, we have :

$$\sum_{j=1}^{k} a_{ij}^{(n)} > 0 \quad \text{and} \quad \sum_{j=1}^{k} a_{ji}^{(n)} > 0 \qquad \text{where } A^n = (a_{ij}^{(n)}) \ .$$

Notice that if $\sum_{j=1}^{k} a_{ij}^{(n_0)} = 0$, then $\sum_{j=1}^{k} a_{ij}^{(n)} = 0$ for $n \geq n_0$, this is because each $a_{\ell m}$ is ≥ 0.

Now, we partition $\{1,\dots,k\}$ into three subsets X, Y, Z, where :

$$X = \{i \mid \forall n \geq 0 \quad \sum_{j=1}^{k} a_{ij}^{(n)} > 0 , \quad \sum_{j=1}^{k} a_{ji}^{(n)} > 0\}$$

$$Y = \{i \mid \exists n > 0 \quad \sum_{j=1}^{k} a_{ij}^{(n)} = 0\} = \{i \mid \text{for } n \text{ large } \sum_{j=1}^{k} a_{ij}^{(n)} = 0\}$$

$$Z = \{1,\dots,k\} - (X \cup Y) .$$

We have :

$$Z \subset \{i \mid \text{for } n \text{ large } \sum_{j=1}^{k} a_{ji}^{(n)} = 0\} .$$

By performing a permutation of $\{1,\dots,k\}$, we can suppose that we have the following situation :

$$\{\underbrace{1,\dots,t}_{X} , \underbrace{t+1,\dots,s}_{Y} , \underbrace{s+1,\dots,k}_{Z}\}$$

If B is a $k \times k$ matrix, we write :

$$B = \begin{pmatrix} B_{XX} & B_{XY} & B_{XZ} \\ B_{YX} & B_{YY} & B_{YZ} \\ B_{ZX} & B_{ZY} & B_{ZZ} \end{pmatrix}$$

where B_{KL} corresponds to the subblock of B have row indices in K and column indices in L.

It is easy to show that :

$$N_{n+1}(\sigma_A, \mathcal{L}) = \text{card}(\mathcal{L} \vee \dots \vee \sigma_A^{-n} \mathcal{L}) = \|(A^n)_{X,X}\| .$$

On the other hand, by the definition of Y and Z, for n large, A^n has the form :

$$A^n = \begin{pmatrix} (A^n)_{X,X} & (A^n)_{X,Y} & 0 \\ 0 & 0 & 0 \\ (A^n)_{Z,X} & (A^n)_{Z,Y} & 0 \end{pmatrix}$$

This implies that for n large, A^n and $(A^n)_{X,X}$ have the same non zero eigenvalues, in particular :

$$\log(\text{spectral radius } A^n_{X,X}) = n \log\lambda .$$

Remark also that we get, for n large and $k \geq 1$:

$$(A^{kn})_{X,X} = [(A^n)_{X,X}]^k .$$

This gives us, for n large :

$$\limsup_{k\to\infty} \frac{1}{kn+1} \log N_{kn+1}(\sigma_A, \mathfrak{L}) = \limsup_{k\to\infty} \frac{1}{kn+1} \| [(A^n)_{X,X}]^k \| = \log\lambda$$

This implies that :

$$\log\lambda \leq h(\sigma_A, \mathfrak{L}) = \limsup_{n\to\infty} \frac{1}{n} \log N_n(\sigma_A, \mathfrak{L}) .$$

As we showed the reverse inequality, we have :

$$\log\lambda = h(\sigma_A, \mathfrak{L}) = h(\sigma_A) . \qquad \square$$

§ IV. - THE ENTROPY OF PSEUDO-ANOSOV DIFFEOMORPHISMS

Now we suppose that we have a compact, connected 2-manifold M without boundary with genus ≥ 2, and a pseudo-Anosov diffeomorphism $f : M \to M$. Hence there exists a pair $(\mathfrak{F}^u, \mu^u)$ and $(\mathfrak{F}^s, \mu^s)$ of transverse measured foliations with (the same) singularities such that $f(\mathfrak{F}^s, \mu^s) = (\mathfrak{F}^s, \frac{1}{\lambda}\mu^s)$ and $f(\mathfrak{F}^u, \mu^u) = (\mathfrak{F}^u, \lambda\mu^u)$ where $\lambda > 1$. This means, in particular, that f preserves the two foliations $\mathfrak{F}^s$ and $\mathfrak{F}^u$; it contracts the leaves of $\mathfrak{F}^s$ by $\frac{1}{\lambda}$ and it expands the leaves of $\mathfrak{F}^u$ by λ.

Let us recall that for any non trivial simple closed curve α we have $\log\lambda = G_f(\alpha)$ (see exposé 9, prop. 19), hence we get $\log\lambda \leq G_f$. [For the definition of G_f, look at the end of section II.]

Proposition. If $f : M \to M$ is pseudo-Anosov, then $h(f) = \gamma_{f\#}$. So in particular, f

has the minimal entropy of anything in its homotopy class. Moreover $h(f) = \log \lambda$, where λ is the expanding factor of f .

<u>Proof</u>. Since $G_f \geq \log \lambda$, it suffices to show that $h(f) \leq \log \lambda$ for a pseudo-Anosov diffeomorphism f . To do this, we find a subshift of finite type $\sigma_A : \Sigma_A \to \Sigma_A$ and a surjective continuous map $\Sigma_A \to M$ such that :

$$\begin{array}{ccc} \Sigma_A & \xrightarrow{\sigma_A} & \Sigma_A \\ \theta \downarrow & & \downarrow \theta \\ M & \xrightarrow{f} & M \end{array} \qquad \text{commutes,}$$

and log (spectral radius A) $= h(\sigma_A) = \log \lambda$ for this same λ . Thus we will have $\log \lambda \leq G_f \leq \gamma_{f\#} \leq h(f) \leq h(\sigma_A)$ or $\log\lambda \leq h(f) \leq \log \lambda$. □

In the following, we construct A and θ via Markov partitions.

First some definitions.

<u>Definitions</u> (compare exposé 9). A subset R of M is called a $(\mathcal{F}^s, \mathcal{F}^u)$ - rectangle or birectangle, if there exists an immersion $\varphi : [0,1] \times [0,1] \to M$ whose image is R and such that :

- $\varphi \mid]0,1[\times]0,1[$ is an imbedding ;

- $\forall t \in [0,1]$, $\varphi(\{t\} \times [0,1])$ is included in a finite union of leaves and singularities of $\mathcal{F}^s$, and in fact in one leaf if $t \in]0,1[$.

- $\forall t \in [0,1]$, $\varphi([0,1] \times \{t\})$ is included in a finite union of leaves and singularities of $\mathcal{F}^u$, and in fact in one leaf if $t \in]0,1[$.

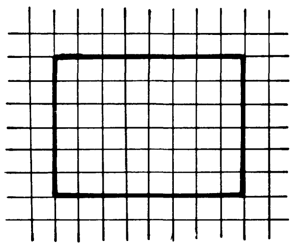
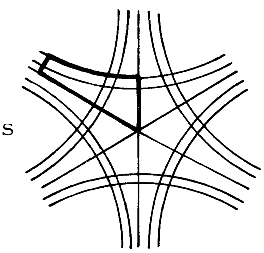
We adopt the following notations :

int $R = \varphi(]0,1[\times]0,1[)$, $\partial^0_{\mathcal{F}^s} R = \varphi(\{0\} \times [0,1])$, $\partial^1_{\mathcal{F}^s} R = \varphi(\{1\} \times [0,1])$, $\partial_{\mathcal{F}^s} R = \partial^0_{\mathcal{F}^s} R \cup \partial^1_{\mathcal{F}^s} R$, and in the same way, we define $\partial^0_{\mathcal{F}^u} R$, $\partial^1_{\mathcal{F}^u} R$, $\partial_{\mathcal{F}^u} R$.

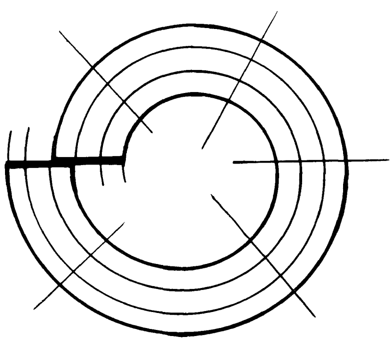
Remark that Int R is disjoint form $\partial_{\mathfrak{F}^s} R \cup \partial_{\mathfrak{F}^u} R$, because $\varphi \mid]0,1[\times]0,1[$ is an embedding.

We call a set of the form $\varphi(\{t\} \times [0,1])$ (resp. $\varphi([0,1] \times \{t\})$) a $\mathfrak{F}^s$-fiber (resp. a $\mathfrak{F}^u$-fiber) of R . We will call a birectangle <u>good</u> if φ is an imbedding.

If R is good birectangle, a point x of R is contained in only one $\mathfrak{F}^s$-fiber which we will denote by $\mathfrak{F}^s(x,R)$. In the same way, we define $\mathfrak{F}^u(x,R)$.

good rectangles



birectangle (not good)

Remarks. 1) If R is a $\mathfrak{F}^u$-rectangle (see exposé 9) and $\partial^0_\tau R$ and $\partial^1_\tau R$ are contained in a union of $\mathfrak{F}^s$-leaves and singularities, it is easy to see that R is in fact a birectangle.

2) We used the word birectangle instead of rectangle, even though rectangle is the standard word in Markov partition, because this word was already used in exp. 9.

3) If R_1 and R_2 are birectangles and $R_1 \cap R_2 \neq \emptyset$, then it is a finite union of birectangles and possibly of some arcs contained in $(\partial_{\mathfrak{F}^s} R_1 \cup \partial_{\mathfrak{F}^u} R_1) \cap (\partial_{\mathfrak{F}^s} R_2 \cup \partial_{\mathfrak{F}^u} R_2)$.

Moreover the birectangles are the closures of the connected components of $\operatorname{Int} R_1 \cap \operatorname{Int} R_2$.

If R is a birectangle, we define the width of R by :

$$\mathcal{W}(R) = \max \{ \mu^u(\mathcal{F}^s\text{-fiber}), \mu^s(\mathcal{F}^u\text{-fiber}) \} .$$

Lemma 1. There exists $\epsilon > 0$ such that, if R is a birectangle with $\mathcal{W}(R) < \epsilon$, then it is a good rectangle.

Sketch of Proof. If a birectangle is contained in a coordinate chart of the foliations, then it is automatically a good birectangle. The existence of ϵ follows from compactness. □

Lemma 2. There exists $\epsilon > 0$ such that if α (resp. β) is an arc contained in a finite union of leaves and singularities of $\mathcal{F}^s$ (resp. $\mathcal{F}^u$) with $\mu^u(\alpha) < \epsilon$ (resp. $\mu^s(\beta) < \epsilon$), then the intersection of α and β is at most one point.

Definition of Markov partition. A Markov partition for the pseudo-Anosov diffeomorphism $f : M \to M$ is a collection of birectangles $\mathcal{R} = \{R_1, \dots, R_k\}$ such that :

1) $\bigcup_{i=1}^{k} R_i = M$;

2) R_i is a good rectangle ;

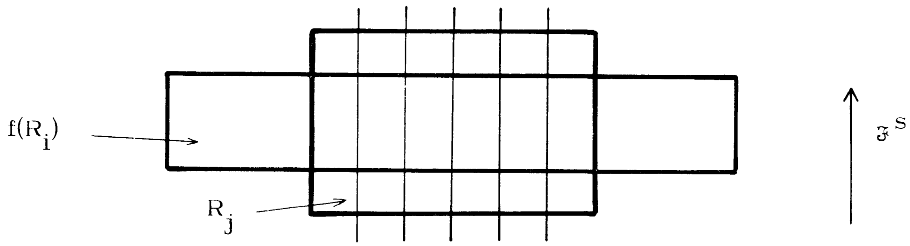
3) $\operatorname{Int} R_i \cap \operatorname{Int} R_j = \emptyset$ for $i \neq j$;

4) If x is in $\operatorname{Int}(R_i)$ and $f(x)$ is in $\operatorname{Int}(R_j)$, then $f(\mathcal{F}^s(x, R_i)) \subset \mathcal{F}^s(f(x), R_j)$, and $f^{-1}(\mathcal{F}^u(f(x), R_j)) \subset \mathcal{F}^u(x, R_i)$;

5) If x is in $\operatorname{Int}(R_i)$ and $f(x)$ is in $\operatorname{Int}(R_j)$, then

$$f(\mathcal{F}^u(x, R_i)) \cap R_j = \mathcal{F}^u(f(x), R_j) \quad \text{and} \quad f^{-1}(\mathcal{F}^s(f(x), R_j)) \cap R_i = \mathcal{F}^s(x, R_i) ;$$

This means that $f(R_i)$ goes across R_j just one time.



We will show in next section how to construct a Markov partition for a pseudo-Anosov diffeomorphism.

Given a Markov partition $\mathcal{R} = \{R_1, \dots, R_k\}$, we construct the subshift of finite type Σ_A and the map $h : \Sigma_A \to M$ as follows. Let A be the $k \times k$ matrix defined by $a_{ij} = 1$ if $f(\text{Int } R_i) \cap \text{Int } R_j \neq \emptyset$, and $a_{ij} = 0$ otherwise. If $\underline{b} \in \Sigma_A$, then $\bigcap_{i \in \mathbb{Z}} f^{-i}(R_{b_i})$ is non empty and consists in fact a single point. This will follow from the following lemma.

<u>Lemma</u> 3. i) Suppose $a_{ij} = 1$, then $f(R_i) \cap R_j$ is a non empty (good) birectangle which is a union of $\mathcal{F}^u$-fibers of R_j .

ii) Suppose moreover that C is a birectangle contained in R_i which is a union of $\mathcal{F}^u$-fibers of R_i , then $f(C) \cap R_j$ is a non empty birectangle which is a union of $\mathcal{F}^u$-fibers of R_j .

iii) Given $\underline{b} \in \Sigma_A$, for each $n \in \mathbb{N}$, $\bigcap_{i=-n}^{n} f^{-i}(R_{b_i})$ is a non empty birectangle. Moreover, we have $\mathcal{W}(\bigcap_{i=-n}^{n} f^{-i}(R_{b_i})) \leq \lambda^{-n} \max\{\mathcal{W}(R_1), \dots, \mathcal{W}(R_k)\}$

<u>Proof</u>. Since $a_{ij} = 1$, we can find $x \in \text{Int }(R_i) \cap f^{-1}(\text{Int } R_j)$. We have $f(\mathcal{F}^s(x, R_i)) \subset$ $\subset \mathcal{F}^s(f(x), R_j) \subset R_j$. Since each $\mathcal{F}^u$-fiber of R_i intersects $\mathcal{F}^s(x, R_i)$, we obtain that the image of each $\mathcal{F}^u$-fiber of R_i intersects R_j . Moreover, by condition 5) $f[R_i - \partial_{\mathcal{F}^u} R_i] \cap R_j$ is an union of $\mathcal{F}^u$-fibers of R_j , hence $f(R_i) \cap R_j = \overline{f(R_i - \partial_{\mathcal{F}^u} R_i) \cap R_j}$ is also a union of $\mathcal{F}^u$-fibers of R_j . This proves i) . The proof of ii) is the same.

To prove iii) , remark first that it follows by induction on n using ii) that each set of the form $f^n R_{b_i} \cap f^{n-1}(R_{b_{i+1}}) \cap \ldots \cap R_{b_{i+n}}$ is a non empty birectangle which is a union of $\mathfrak{F}^u$-fibers of $R_{b_{i+n}}$. In particular, $\bigcap_{i=-n}^{n} f^{-i}(R_{b_i})$ is a non empty birectangle in R_{b_0}. The estimate of the width is clear. □

By the lemma, if $\underline{b} \in \Sigma_A$, the set $\bigcap_{i \in \mathbb{Z}} f^{-i}(R_{b_i})$ is the intersection of a decreasing sequence of non empty compact sets, namely the sets $\bigcap_{i=-n}^{n} f^{-i}(R_{b_i})$ for $n \in \mathbb{N}$. Hence $\bigcap_{i \in \mathbb{Z}} f^{-i}(R_{b_i})$ is non void. It is reduced to one point because $\mathcal{W}(\bigcap_{i=-n}^{n} f^{-i}(R_{b_i}))$ tends to zero as n goes to infinity.

The map $\theta : \Sigma_A \to M$ given by $\theta(\underline{b}) = \bigcap_{i \in \mathbb{Z}} f^{-i}(R_{b_i})$ is well defined, it is easy to see that it is continuous and that the following diagram commutes :

$$\begin{array}{ccc} \Sigma_A & \xrightarrow{\sigma_A} & \Sigma_A \\ \theta \downarrow & & \downarrow \theta \\ M & \xrightarrow{f} & M \end{array}$$

We show now that θ is surjective. First remark that, for each $i = 1, \ldots, k$, the closure of $\operatorname{Int}(R_i)$ is R_i. Hence $V = \bigcup_{i=1}^{k} \operatorname{Int}(R_i)$ is a dense open set. By the Baire category theorem $U = \bigcap_{i \in \mathbb{Z}} f^{-i}(V)$ is dense in M. If $x \in U$, then for each $n \in \mathbb{Z}$, the point $f^n(x)$ is in a unique $\operatorname{Int}(R_{b_n})$ and $\underline{b} = \{b_n\}_{n \in \mathbb{Z}}$ is an element of Σ_A. It is clear that $\theta(\underline{b}) = x$. Thus $\theta(\Sigma_A) \supset U$. As Σ_A is compact and h continuous, we have $\theta(\Sigma_A) = M$.

Up to now, we have obtained that :

$$\log \lambda \leq G_f \leq \gamma_{f\#} \leq h(f) \leq h(\sigma_A) = \log(\text{spectral radius of } A).$$

All that remains is to show that :

$$(\text{spectral radius of } A) = \lambda .$$

To see this, we do the following thing. Put $y_i = \mu^u(\mathfrak{F}^s\text{-fiber of } R_i)$, it is clear that this quantity is independant of the $\mathfrak{F}^s$-fiber of R_i and also $y_i > 0$.

We have trivially the following equality :

$$y_j = \sum_{i=1}^{k} \frac{y_i}{\lambda} a_{ij} ,$$

which gives :

$$\lambda y_j = \sum_{i=1}^{k} y_i a_{ij} \quad [\text{in particular } \lambda \text{ is an eigenvalue of } A] .$$

Hence, we obtain :

$$\lambda y_j \geq \left(\sum_{i=1}^{k} a_{ij} \right) \min_i y_i .$$

This gives us :

$$\lambda \left(\sum_j y_j \right) \geq \| A \| \min_i y_i$$

where $\| \ \|$ is the norm introduced in section III.

In the same way, we obtain for each $n \geq 2$:

$$\lambda^n \left(\sum_j y_j \right) \geq \| A^n \| \min_i y_i .$$

Hence :

$$\lambda \geq \| A^n \|^{1/n} \left(\frac{\min(y_1, \ldots, y_k)}{\sum_i y_j} \right)^{1/n} .$$

Since $\min(y_1, \ldots, y_k) > 0$, $\lim_{n \to \infty} \left(\frac{\min(y_1, \ldots, y_k)}{\sum_j y_j} \right)^{1/n} = 1$.

We thus obtain :

$$\lambda \geq \lim_{n \to \infty} \| A^n \|^{1/n} = \text{spectral radius of } A .$$

Since λ is an eigenvalue of A , we obtain :

$$\lambda = \text{spectral radius of } A .$$

§ V. - CONSTRUCTION OF MARKOV PARTITIONS FOR PSEUDO-ANOSOV DIFFEOMORPHISMS

In this section, we still consider $f : M \to M$ a pseudo-Anosov diffeomorphism and we keep the notations of the last section. We sketch the proof of the following proposition.

Proposition. A pseudo-Anosov diffeomorphism has a Markov partition.

Using the methods given in exposé 9 , § V , it is easy, starting with a family of transversals to $\mathcal{F}^u$ contained in $\mathcal{F}^s$-leaves and singularities, to construct a family $\mathcal{R}$ of $\mathcal{F}^u$-rectangles $R_1, \ldots, R_\ell$, such that :

i) $\bigcup_{i=1}^{\ell} R_i = M$;

ii) $\operatorname{Int}(R_i) \cap \operatorname{Int}(R_j) = \emptyset$ for $i \neq j$;

iii) $f^{-1}\left(\bigcup_{i=1}^{\ell} \partial_{\mathcal{F}^u} R_i\right) \subset \bigcup_{i=1}^{\ell} \partial_{\mathcal{F}^u} R_i$, $f\left(\bigcup_{i=1}^{\ell} \partial_{\mathcal{F}^s} R_i\right) \subset \bigcup_{i=1}^{\ell} \partial_{\mathcal{F}^s} R_i$.

By the remark following the definition of birectangles, the R_i' s are birectangles since the system of transversals is contained in $\mathcal{F}^s$-leaves and singularities.

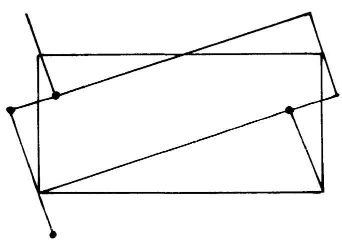
We define for each n a family of birectangles $\mathcal{R}_n$ in the following way : the birectangles of $\mathcal{R}_n$ will be the closures of the connected components of the non empty open sets contained in $\bigvee_{i=-n}^{n} f^i \overset{o}{\mathcal{R}} = \{ \bigcap_{i=-n}^{n} f^i(\operatorname{Int} R_{a_i}) \mid R_{a_i} \in \mathcal{R} \}$.

It is easy to see that $\mathcal{R}_n$ still satisfies the properties i) , ii) and iii) given above. Moreover, if $R \in \mathcal{R}_n$, we have $\mathcal{W}(R) \leq \lambda^{-n} \max \{\mathcal{W}(R_i) \mid R_i \in \mathcal{R}\}$. In particular, by lemma 1 of last section, for n sufficiently large, each birectangle R in $\mathcal{R}_n$ is a <u>good</u> one.

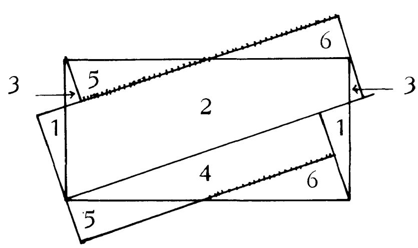
We assert that for n sufficiently large $\mathfrak{R}_n$ is a Markov partition. All that remains is to verify properties 4) and 5) of a Markov partition. It is an easy exercise to show that property 4) is a consequence of property iii) given above (see exposé 9, lemme 10). By lemma 2 of section IV, if n is sufficiently large and $R, R' \in \mathfrak{R}_n$, then if $x \in R$, $f(\mathfrak{F}^u(x,R))$ intersects in at most one point each $\mathfrak{F}^s$-fiber of R'. Property 5) follows easily from the combination of this fact and of property 4). □

Example of Markov partition on T^2. Let $A : T^2 \to T^2$ be the linear map defined by $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Here $T^2 = \mathbb{R}^2/\mathbb{Z}^2$; and A acts on $\mathbb{R}^2$ preserving $\mathbb{Z}^2$, thus A defines a map of T^2. The translates of the eigenspaces of A foliate T^2. The map A on T^2 is Anosov. The foliation of T^2 corresponding to the eigenvalue $\frac{3+\sqrt{5}}{2}$ is expanded, the foliation corresponding to $\frac{3-\sqrt{5}}{2}$ is contracted.

We draw a fundamental domain with eigenspaces approximately drawn in.



The endpoints of the short stable manifold are on the unstable manifolds after equivalences have been made. Filling in to maximal rectangles gives the following picture.

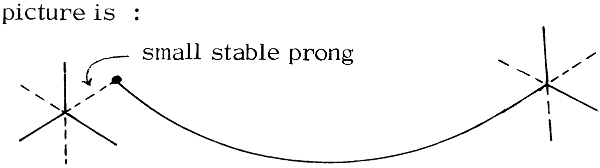


The hatched line is the extension of the unstable manifold. Identified pieces are numbered similarly. One rectangle is given by 1,2,3,6 and the other by 4,5. This partition in two rectangles gives a Markov partition by taking intersections with direct and inverse images.

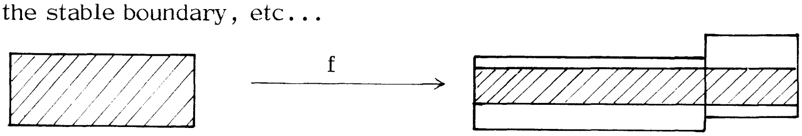
The construction of the Markov partition of a pseudo-Anosov diffeomorphism $f: M \to M$, which preserves orientation and fixes the prongs of $\mathfrak{F}^s$ and $\mathfrak{F}^u$, is the same as in the example above. We sketch here the argument, hoping that it will aid the reader to understand the general case.

Since the unstable prongs are dense, we may pick small stable prongs whose endpoints lie on unstable prongs.

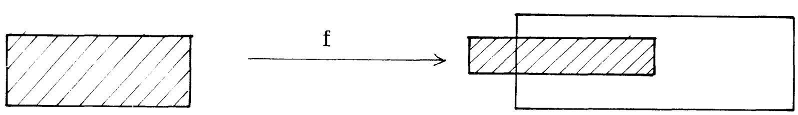
Roughly, the picture is :



We may extend these curves to maximal birectangles leaving the drawn curves as boundaries. By density of the leaves, every leaf crosses a small stable prong, so the rectangles obtained this way cover M^2. The extension process requires that the unstable prongs be extended perhaps but the extension remains connected. Thus we have a partition by birectangles with boundaries the unions of connected segments lying on stable or unstable prongs. Consequently an unstable leaf entering the interior of a birectangle under f can't end in the interior, because the stable boundary has been taken to the stable boundary, etc...



but not



The only thing left is to make the partition sufficiently small. To do this, it is sufficient to take the birectangles obtained by intersections $f^{-n}\mathfrak{R} \vee \dots \vee \mathfrak{R} \vee \dots \vee f^{n}(\mathfrak{R})$ for n sufficiently large.

§ VI. - PSEUDO-ANOSOV DIFFEOMORPHISMS ARE BERNOULLI.

A pseudo-Anosov diffeomorphism $f : M \to M$ has a natural invariant probability measure μ which is given locally by the product of μ^s restricted to plaques of $\mathfrak{F}^u$ with μ^u restricted to plaques of $\mathfrak{F}^s$. The goal of this section is to sketch the proof of the following theorem.

Theorem. The dynamical system (M,f,μ) is isomorphic (in the measure theoretical sense) to a Bernoulli shift.

Recall that a Bernoulli shift is a shift $(\Sigma(\ell),\sigma)$ together a measure ν which is the infinite product of some probability measure on $\{1,\dots,\ell\}$. Obviously, ν is invariant under σ, see [9],[11].

We will have to use the notion and properties of measure theoretic entropy, see [11]. We will also need the following two theorems on subshifts of finite type.

Let A be a $k \times k$ matrix and (Σ_A,σ_A) be the subshift of finite type obtained from it.

Theorem (Parry) [10]. Suppose that A^n has all its entries > 0 for some n. Then, there is a probability measure ν_A invariant under σ_A such that the measure theoretic entropy $h_{\nu_A}(\sigma_A)$ is equal to the topological entropy $h(\sigma_A)$. Moreover, ν_A is the only invariant probability measure having this property, and $(\Sigma_A, \sigma_A, \nu_A)$ is a mixing Markov process.

Theorem (Friedman-Ornstein) [9] . A mixing Markov process is isomorphic to a Bernoulli shift. In particular, the $(\Sigma_A, \sigma_A, \nu_A)$ above is Bernoulli.

Now we begin to prove that (M, f, μ) is Bernoulli. For this, we will use the subshift (Σ_A, σ_A) and the map $\theta : (\Sigma_A, \sigma_A) \to (M, f)$ obtained from the Markov partition $\mathfrak{R} = \{R_1, \ldots, R_k\}$.

Lemma 1. There exists $n \geq 1$ such that A^n has > 0 entries.

Proof. Given R_i , we can find a periodic point $x_i \in \overset{\circ}{R}_i$, call n_i its period. Consider the unstable fiber $\mathfrak{F}^u(x_i, R_i)$; we have, for $\ell \geq 0$, $f^{\ell n_i}(\mathfrak{F}^u(x_i, R_i)) \supset \mathfrak{F}^u(x_i, R_i)$. Moreover the μ^s-length of $f^{\ell n_i}(\mathfrak{F}^u(x_i, R_i))$ goes to infinity, since it is $\lambda^{\ell n_i} \mu^s(\mathfrak{F}^u(x_i, R_i))$. This implies that $f^{\ell n_i}(\mathfrak{F}^u(x_i, R_i)) \cap \overset{\circ}{R}_j \neq \emptyset$, $\forall\, j = 1, \ldots, k$, for ℓ large because the leaves of $\mathfrak{F}^u$ are dense. Now, if $n = \ell \cdot \prod_{i=1}^{k} n_i$ with ℓ large enough, we get $f^n(\overset{\circ}{R}_i) \cap \overset{\circ}{R}_j \neq \emptyset$ for each pair (i,j) . Hence, we obtain that $a_{ij}^{(n)} > 0$ for each (i,j) , where $A^n = (a_{ij}^{(n)})$. □

This lemma shows that $(\Sigma_A, \sigma_A, \nu_A)$ is Bernoulli by the results quoted above. All we have to do now is to prove that (M, f, μ) is isomorphic to $(\Sigma_A, \sigma_A, \nu_A)$.

Lemma 2. The measure theoretic entropy $h_\mu(f)$ is $\log \lambda$.

Proof. Since topological entropy is the supremum of measure theoretical entropies, see [2, 5] , we have $h_\mu(f) \leq \log \lambda$. Consider now the partition $\overset{\circ}{\mathfrak{R}} = \{\text{Int}\, R_i\}$, its

μ-entropy $h_\mu(f,\overset{o}{\mathfrak{R}})$ with respect to f is given by :

$$h_\mu(f,\overset{o}{\mathfrak{R}}) = \lim_n -\frac{1}{n} \Sigma\, a_{ij}^{(n)} \lambda^{-n} y_i x_j \log (\lambda^{-n} y_i x_j)$$

where $y_i = \mu^u(\mathfrak{F}^s$ fiber of $R_i)$ and $x_j = \mu^s(\mathfrak{F}^u$ fiber $R_j)$. As we saw and the end of section IV , $\frac{a_{ij}^{(n)}}{\lambda^n} \leq \frac{\|A^n\|}{\lambda^n}$ is bounded (by $\frac{\Sigma y_j}{\min y_i}$) . This implies :

$$\lim_n -\frac{1}{n} \Sigma\, a_{ij}^{(n)} \lambda^{-n} y_i x_j \log y_i x_j = 0 .$$

We have also :

$$\Sigma\, a_{ij}^{(n)} \lambda^{-n} y_i x_j = \Sigma\, y_j x_j = \Sigma\, \mu(\overset{o}{R}_j) = \mu(M) = 1 .$$

By putting these facts together, we obtain : $h_\mu(f,\overset{o}{\mathfrak{R}}) = \log \lambda$. Hence, $h_\mu(f) = \log \lambda$, because $\log \lambda = h_\mu(f,\overset{o}{\mathfrak{R}}) \leq h_\mu(f) \leq h(f) = \log \lambda$. □

<u>Proof of the theorem</u>. Put $\partial \mathfrak{R} = \bigcup_{i=1}^{k} \partial R_i$, we have $\mu(\partial \mathfrak{R}) = 0$. This implies that the set $Z = M - \bigcup_{i \in \mathbb{Z}} f^i(\partial \mathfrak{R})$ has μ-measure equal to one. We know by section IV that θ induces a (bicontinuous) bijection of $\theta^{-1}(Z)$ onto Z , we can then define a probability measure ν on Σ_A by $\nu(B) = \mu(\theta[\theta^{-1}(Z) \cap B])$ for each borel set $B \subset \Sigma_A$. It is easy to see that ν is σ_A invariant ; moreover, θ gives rise to a measure theoritic isomorphism between $(\Sigma_A, \sigma_A, \nu)$ and (M,f,μ) . In particular, $h_\nu(\sigma_A) = h_\mu(f) = \log \lambda$. Since $\log \lambda$ is also the topological entropy of σ_A , we obtain from Parry's theorem that $\nu = \nu_A$ and that $(\Sigma_A, \sigma_A, \nu)$ is a mixing Markov process. By the Friedman-Ornstein theorem, $(\Sigma_A, \sigma_A, \nu)$ is Bernoulli, hence (M,f,μ) is also Bernoulli. □

REFERENCES

[1] R.L. ADLER, A.G. KONHEIM and M.H. McANDREW, Topological entropy, Trans. Amer. Math. Soc. 114 (1965), p. 309-319.

[2] R. BOWEN, Entropy for group endomorphisms and homogeneous spaces, Trans. Amer. Math. Soc. 153 (1970), p. 401-413.

[3] R. BOWEN, Entropy and the fundamental group, in the Structure of attractors in Dynamical Systems, Lecture notes in Math., vol 668, Springer-Verlag, New-York, 1978 .

[4] E.F. DINABURG, On the relations of various entropy characteristics of dynamical systems, Math. of the USSR, Izvestija, 35 (1971) p. 337 (english translation), p. 324 (russian).

[5] T.N.T. GOODMAN, Relating topological entropy with measure theoritic entropy, Bull. London Math. Soc. 3 (1971), p. 176-180.

[6] M. GROMOV, Three remarks on geodesic dynamics and fundamental group (preprint).

[7] A. MANNING, Topological entropy and the first homology group, in Dynamical Systems, Warwick 1974, Lecture notes in Math. Vol. 468, Springer-Verlag, New-York, 1975 .

[8] J. MILNOR, A note on curvature and the fundamental group, Journal of Diff. Geometry 2 (1968), p. 1 -70.

[9] D. ORNSTEIN, Ergodic theory, randomness and dynamical systems, Yale Mathematical monographs 5, Yale University Press, New Haven, 1974 .

[10] W. PARRY, Intrinsic Markov chains, Trans. Amer. Math. Soc. 112 (1964) p. 55-66.

[11] Y.G. SINAI, Introduction to ergodic theory, Mathematical notes 18, Princeton University Press, Princeton , 1976 .