Systems of symplectic forms on four-manifolds

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Abstract. We study almost-Hermitian 4-manifolds with holonomy algebra, for the canonical Hermitian connection, of dimension at most one. We show how Riemannian 4-manifolds admitting five orthonormal symplectic forms fit in this picture and we classify them. In this set-up we also fully describe almost-Kähler 4-manifolds.

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1. Introduction

The existence of orthogonal harmonic forms on an oriented Riemannian four-manifold (M^4, g) typically encodes relevant properties of the metric. An orthonormal frame of closed 1-forms, for instance, will flatten g.

Also closed, orthonormal 2-forms impose severe constraints on (M^4, g), essentially according to the choices of orientation available. Many cases have been addressed in the literature: couples or triples of this kind, defining the same orientation, were studied in [20,21,27], whereas [10] dealt with pairs of oppositely-oriented symplectic forms.

The aim of this note is to consider smooth Riemannian manifolds (M^4, g) admitting a system of *five* symplectic forms $\{\omega_k, 1 \leq k \leq 5\}$ such that

$$g(\omega_i, \omega_j) = \delta_{ij} \qquad i, j = 1, \ldots, 5. \tag{1.1}$$

As we will see in Section 2, equation (1.1) is equivalent to considering, on a 4-manifold M, a so-called 5-*frame*, that is five non-degenerate 2-forms satisfying

$$\omega_i \wedge \omega_j = \pm \delta_{ij}\, \omega_5 \wedge \omega_5 \qquad i, j = 1, \dots, 4 \tag{1.2}$$

at each point of M. A *closed* 5-*frame* is a 5-frame of symplectic forms. It is known that if $\omega_1, \dots, \omega_6$ is an orthonormal frame of closed 2-forms then the metric g must be flat, a case we will exclude a priori.

It is easy to see that, up to re-ordering, three of the 2-forms are anti-self-dual, and furnish a hyperKähler (hence Ricci-flat) metric g. The remaining 2-forms in the frame are self-dual and give a holomorphic-symplectic form: this in turn yields a complex structure I, for which g is necessarily Hermitian.

Our first result is:

Theorem 1.1. *Let (M, g) be a non-flat Riemannian 4-manifold equipped with a closed* 5-*frame. Then*

(i) *there exists a tri-holomorphic Killing vector field for the hyperKähler structure;*

(ii) *(M, g) is locally isometric to $\mathbb{R}^+ \times \mathrm{Nil}^3$ equipped with metric*

$$dt^2 + \left(\tfrac{2}{3}t\right)^{3/2}(\sigma_1^2 + \sigma_2^2) + \left(\tfrac{2}{3}t\right)^{-2/3}\sigma_3^2,$$

where $\{\sigma_i\}$ is a basis of left-invariant one-forms on the Heisenberg group Nil^3 satisfying $d\sigma_1 = d\sigma_2 = 0, d\sigma_3 = \sigma_1 \wedge \sigma_2$.

Moreover the 5-frame is unique, up a constant rotation in $O(3) \times U(1)$.

We are thus dealing with Hermitian, self-dual, Ricci-flat surfaces, classified in [5]. To prove Theorem 1.1 we need to determine, first, which metrics in [5] admit a holomorphic-symplectic form, and then determine that form explicitly.

The technique used for the proof is indicative of another point of view for looking at closed 5-frames, namely that of holonomy. We prove that the curvature tensor $\widetilde{R}$ of the canonical Hermitian connection of (g, I), *cf.* Section 3, is algebraically defined by the structure's Lee and Kähler forms (Proposition 5.3). This is used to show that the holomorphic Killing field X, coming from the Goldberg-Sachs theorem [4, 25] for Einstein-Hermitian metrics, is actually tri-holomorphic for the hyperKähler structure. This proves part (i), and as a consequence, we know that g is essentially described by the celebrated Gibbons-Hawking Ansatz.

By changing the sign of I on the distribution spanned by X, IX we obtain a Kähler structure (g, J) with "negative" orientation. This allows us to use [5, Theorem 1], and eventually the metric g is given as in (ii). Note this is a cohomogeneity-one Bianchi metric of type II (see [29] for details).

The holonomy approach to close 5-frames leaves much space (Section 5) for investigating almost-Hermitian 4-manifolds (M, g, I) for which the holonomy group of the canonical Hermitian connection is "small": since this has dimension

bounded by four, small will mean of dimension zero or one. Then there are three possibilities for the two-form corresponding to the holonomy generator. It can vanish identically, in which case we prove g must be flat (Theorem 5.1), generalising earlier results [8,9]. It can be proportional to the Kähler form ω_I of (g, I), and this is precisely the set-up of self-dual Ricci-flat 4-manifolds (Proposition 5.3). In the third case the holonomy generator has a component orthogonal to ω_I that defines a Kähler structure reversing the orientation. In this situation almost-Kähler structures are explicitly classified (Theorem 5.5). The corresponding examples are built deforming the product of $\mathbb{R}^2$ with a Riemann surface, in the spirit of [1,15].

2. Two-forms on 4-manifolds

Consider a smooth oriented Riemannian 4-manifold (M, g). The Hodge star operator $\star$ acting on the bundle of two-forms Λ^2 is an involution, with the rank-three subbundles of self-dual and anti-self-dual 2-forms $\Lambda^\pm = \ker(\star \mp \mathrm{Id}_{\Lambda^2})$ as eigenspaces. The resulting decomposition

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^- \tag{2.1}$$

is loosely speaking the "adjoint" version of the fact that the Lie algebra $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ is semisimple.

Now let I in $T^*M \otimes TM$ be an orthogonal almost complex structure, that is

$$I^2 = -\mathrm{Id}_{TM}, \qquad g(I\cdot, I\cdot) = g(\cdot, \cdot).$$

The Kähler form $\omega_I = g(I\cdot, \cdot)$ is non-degenerate at any point of M, and induces, by decree, the positive orientation: $\omega_I^2 = \mathrm{vol}(g)$. The almost complex structure I extends to the exterior algebra by

$$(I\alpha)(X_1, \ldots, X_p) = \alpha(IX_1, \ldots, IX_p),$$

where α is a p-form on M and $X_1, \ldots, X_p$ belong to TM. We shall work with real-valued forms, unless specified otherwise. Notation-wise, $()^\flat : TM \to \Lambda^1 M$ is the isomorphism induced by the metric, with inverse $()^\sharp$.

The bundle of real 2-forms also decomposes under $U(2)$ as

$$\Lambda^2 = \lambda^{1,1} \oplus \lambda^2, \tag{2.2}$$

where $\lambda^{1,1}$ denotes I-invariant two-forms, and λ^2 anti-invariant ones. If $\langle\omega_I\rangle$ is the real line through ω_I we further have $\lambda^{1,1} = \langle\omega_I\rangle \oplus \lambda_0^{1,1}$, where the space of primitive (1, 1)-forms $\lambda_0^{1,1}$ is the orthogonal complement to $\langle\omega_I\rangle$ in $\lambda^{1,1}$. The real rank-two bundle λ^2 has a complex structure $\mathcal{I}\alpha = \alpha(I\cdot, \cdot)$ that makes it a complex line bundle isomorphic to the canonical bundle of (M, I). So in presence of an almost-Hermitian structure, Λ^+ splits as

$$\Lambda^+ = \langle\omega_I\rangle \oplus \lambda^2$$

under $U(1)$, and comparing (2.2) and (2.1) leads to $\Lambda^- = \lambda_0^{1,1}$.

Slightly changing the point of view, we briefly recall how to recover a conformal structure in dimension four from rank-three subbundles of two-forms. Consider a smooth oriented manifold M of real dimension four with volume form $\mathrm{vol}(M)$. The bundle Λ^2 possesses a non-degenerate bilinear form q given by

$$\alpha \wedge \beta = q(\alpha, \beta)\mathrm{vol}(M)$$

whenever α, β belong to Λ^2. Any subbundle E of Λ^2 of rank three and maximal, in the sense that $q_{|E}$ is positive, determines a unique conformal structure on M such that $E = \Lambda^+$ [26]. The proof of this descends from the fact that q has signature (3, 3) and that at any point x of M the set of maximal subspaces of $\Lambda^2 T_x M$ is parametrised by

$$\frac{\mathrm{GL}(4, \mathbb{R})}{\mathrm{CO}(4)} \cong \frac{\mathrm{SL}(4, \mathbb{C})}{\mathrm{SO}(4)} \cong \frac{\mathrm{SO}_0(3, 3)}{\mathrm{SO}(3) \times \mathrm{SO}(3)}.$$

Here $\mathrm{CO}(4) = \mathbb{R}^+ \times \mathrm{SO}(4)$ is the conformal group. In particular:

Lemma 2.1. *A closed 5-frame on a smooth 4-manifold determines*

(i) *a Riemannian metric g such that*

$$\mathrm{span}\{\omega_1, \omega_2, \omega_3\} = \Lambda^-, \ \{\omega_4, \omega_5\} \subset \Lambda^+, \tag{2.3}$$

up to re-ordering;

(ii) *an orthogonal complex structure I defined by $\omega_5 = \omega_4(I\cdot, \cdot)$.*

Proof. (i) The ω_i are linearly independent as the metric q is neutral; for the same reason, in (1.2) there are effectively 3 minus signs and 2 pluses,[1] *e.g.* $\omega_1^2 = \omega_2^2 = \omega_3^2$, and in the chosen conformal class there exists a unique Riemannian metric g such that $\{\omega_4, \omega_5\}$ are orthonormal, so that $-\omega_1^2 = \omega_4^2 = \omega_5^2$.

(ii) The closure of ω_4 and ω_5 implies automatically that I is integrable [27]. $\square$

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[1] Or the other way around, but we will suppose three minuses.

The convention throughout this note will be that of (2.3). Then the fundamental form $\omega_I = g(I\cdot, \cdot) \in \Lambda^+$ completes the 5-frame to an orthonormal basis of Λ^2. When the metric g is flat the forms ω_4, ω_5 must be parallel for the Levi-Civita connection of g by a result of [7], so from now we consider non-flat 5-frames, that is, g will be assumed not flat.

Since the triple $\{\omega_1, \omega_2, \omega_3\}$ defines a hyperKähler structure, the metric g is Ricci-flat and self-dual. By Lemma 2.1 the classification of closed 5-frames amounts to that of Ricci-flat, self-dual Hermitian structures (g, I) equipped with a complex-symplectic structure, that is a closed, constant-length two-form $\omega_4 + i\omega_5$ in $\Lambda_I^{0,2}M = \Lambda^2(M, \mathbb{C}) \cap \ker(\mathcal{I} + 2i)$.

The article [5] contains the complete local-structure theory for Ricci-flat, self-dual Hermitian surfaces; to locate closed 5-frames in that classification we will set up, in the next section, an equivalent curvature description. Before doing so we remind that Gibbons and Hawking [22] have generated, locally, all hyperKähler 4-manifolds admitting a tri-holomorphic Killing vector field using Laplace's equation in Euclidean three-space. We outline below their construction to show how closed 5-frames fit therein.

Take a hyperKähler 4-manifold (M^4, g, J_1, J_2, J_3) with a vector field X such that $L_X J_i = 0, 1 \leq i \leq 3$ and $L_X g = 0$. Choose a local system of co-ordinates (u, x, y, z) on M with $X = \frac{d}{du}$ and where x, y, z, given by $X \lrcorner \omega_{J_1} = dx$, etc., are the momentum maps. In these co-ordinates the metric reads

$$g = U(dx^2 + dy^2 + dz^2) + U^{-1}(du + \Theta)^2,$$

where $U(x, y, z) = \|X\|^{-2}$ is defined on some domain in $\mathbb{R}^3$, and the connection one-form Θ is invariant under X and such that $\Theta(X) = 0$. The fundamental forms of $(g, J_k), 1 \leq i \leq 3$,

$$\omega_{J_1} = U\,dy\,dz + dx(du + \Theta), \quad \omega_{J_2} = U\,dx\,dy + dz(du + \Theta),$$
$$\omega_{J_3} = U\,dz\,dx + dy(du + \Theta)$$

are closed if and only if Θ satisfies the monopole equation $d\Theta = \star_{\mathbb{R}^3} dU$. In particular U is harmonic on some open region of $\mathbb{R}^3$, and conversely such a function completely determines the geometry as explained above.

Moreover the (non-necessarily closed) forms

$$\omega_{I_1} = U\,dy\,dz - dx(du + \Theta), \quad \omega_{I_2} = U\,dx\,dy - dz(du + \Theta),$$
$$\omega_{I_3} = U\,dz\,dx - dy(du + \Theta)$$

are orthonormal and yield a trivialisation of Λ^+.

Example 2.2. Imposing that the forms ω_{I_1} and ω_{I_2} be closed forces $U_x = U_z = 0$, so $U = ay + b$ for real constants a, b; one can explicitly take $\Theta = \frac{a}{2}(z\,dx - x\,dz)$. In this situation ω_{I_1} and ω_{I_2} build, together with $\omega_{J_k}, 1 \leq k \leq 3$, a closed 5-frame, which is not flat for $a \neq 0$ since $d\omega_{I_3} = 2a\,dx\,dy\,dz$ does not vanish.

Roughly speaking, Theorem 1.1 explains why this example is no coincidence.

3. The curvature of the canonical connection

In order to characterise closed 5-frames in terms of curvature we need some facts from almost-Hermitian geometry; these will serve us beyond the 5-frame set-up as well, so the presentation will be general.

Let (M^4, g, I) be almost-Hermitian. The Levi-Civita connection ∇ of g defines the so-called intrinsic torsion of (g, I)

$$\eta = \frac{1}{2}(\nabla I)I \in \Lambda^1 \otimes \lambda^2.$$

Its knowledge is the main tool to capture, both algebraically and not, the geometry of almost-Hermitian manifolds: indeed, the components of η inside the irreducible $U(2)$-modules into which $\Lambda^1 \otimes \lambda^2$ decomposes determine the type and features of the structure under scrutiny (*e.g.*, Kähler, Hermitian, conformally Kähler and so on). When indexing a differential form with a vector we shall mean $\beta_X = X \lrcorner \beta = \beta(X, \cdot, \ldots, \cdot)$, and in particular $\eta_X = \frac{1}{2}(\nabla_X I)I$.

The canonical connection

$$\widetilde{\nabla} = \nabla + \eta$$

of the almost-Hermitian structure (g, I) is a linear connection that preserves Riemannian and almost complex structures, $\widetilde{\nabla}g = 0$ and $\widetilde{\nabla}I = 0$, hence it is both metric and Hermitian. It coincides with the Chern connection (see [19]) if I is integrable. Since the torsion tensor T of $\widetilde{\nabla}$ is given by

$$T_X Y = \eta_X Y - \eta_Y X$$

for any tangent vectors X, Y, we have $\eta = 0$ if and only if (g, I) is Kähler. The canonical Hermitian connection naturally induces an exterior derivative on bundle-valued differential forms. For instance if α belongs to $\Lambda^1(M, \lambda^2)$, we have $d^{\widetilde{\nabla}}\alpha(X, Y) = (\widetilde{\nabla}_X \alpha)Y - (\widetilde{\nabla}_Y \alpha)X$ for any X, Y in TM. On ordinary differential forms one defines $d^{\widetilde{\nabla}} : \Lambda M \to \Lambda M$ in analogy with the usual exterior derivative, that is $d^{\widetilde{\nabla}} = \sum_{i=1}^4 e_i \wedge \widetilde{\nabla}_{e_i}$ where $\{e_i, 1 \leq i \leq 4\}$ is an orthonormal basis of each tangent space. If the action of T on some 1-form α is defined by $(T\alpha)(X, Y) = \alpha(T_X Y)$ for any X, Y in TM, we may compare differentials

$$d^{\widetilde{\nabla}}\alpha = d\alpha - T\alpha.$$

Given a local gauge I_1, that is a locally-defined orthogonal complex structure such that $I_1 I + I I_1 = 0$, we define $I_2 = I_1 I$, and write

$$\nabla I = a \otimes I_2 + c \otimes I_1, \quad \text{or equivalently} \quad 2\eta = -a \otimes I_1 + c \otimes I_2, \qquad (3.1)$$

for local 1-forms a, c on M. The curvature tensor $\widetilde{R} \in \Lambda^2 \otimes \lambda^{1,1}$ of the canonical connection, defined by $\widetilde{R}(X, Y) = -[\widetilde{\nabla}_X, \widetilde{\nabla}_Y] + \widetilde{\nabla}_{[X,Y]}$, X, Y in TM, has in general not all of the symmetries enjoyed by the Riemannian counterpart R. It fails to be

symmetric in pairs, and does not satisfy the first Bianchi identity, due to the terms involving the intrinsic torsion in

$$\widetilde{R}(X, Y) = R(X, Y) - d^{\widetilde{\nabla}}\eta(X, Y) + [\eta_X, \eta_Y] - \eta_{T_X Y} \tag{3.2}$$

for any X, Y in TM, see $e.g.$ [14]. The algebraic summands above can be computed locally from (3.1); in particular

$$[\eta_X, \eta_Y] = \frac{1}{2}\Phi(X, Y)I \tag{3.3}$$

for all X, Y in TM, where $\Phi = a \wedge c$. The first Chern form $\widetilde{\gamma}_1$ of the canonical connection is the 2-form defined by

$$\widetilde{\gamma}_1(X, Y) = \langle \widetilde{R}(X, Y), \omega_I \rangle$$

for any X, Y in TM, where the brackets $\langle \cdot, \cdot \rangle$ denote the inner product on forms. The differential Bianchi identity forces it to be closed, $d\widetilde{\gamma}_1 = 0$, and moreover $\frac{1}{2\pi}\widetilde{\gamma}_1$ is a de Rham representative for $c_1(M, I)$. Splitting the Ricci tensor $\mathrm{Ric} = \mathrm{Ric}' + \mathrm{Ric}''$ into invariant and anti-invariant parts under I, and taking the scalar product with ω_I in (3.2), yields

$$\widetilde{\gamma}_1 = \rho^I + W^+\omega_I + \Phi - \frac{s}{6}\omega_I. \tag{3.4}$$

Here $\rho^I = \langle \mathrm{Ric}' I \cdot, \cdot \rangle \in \lambda^{1,1}$, s is the scalar curvature of g and $W^{\pm} = \frac{1}{2}(W \pm W\star)$ are the positive and negative halves of the Weyl curvature considered as a bundle-valued 2-form

$$W = W^+ + W^-,$$

reflecting (2.1). The 4-manifold is called self-dual or anti-self-dual according to whether $W^- = 0$ or $W^+ = 0$.

One may also compute the first Chern form locally, by expanding the covariant derivative of a local gauge I_1:

$$\widetilde{\nabla}I_1 = -b \otimes I_2, \quad \widetilde{\nabla}I_2 = b \otimes I_1 \tag{3.5}$$

where b is a local 1-form on M, which implies

$$\widetilde{\gamma}_1 = -db. \tag{3.6}$$

Expression (3.2) for $\widetilde{R}$ simplifies considerably if one uses the Weyl tensor. Let Ric_0 denote the trace-free component of the Ricci tensor and $h = \frac{1}{2}(\mathrm{Ric}_0 + \frac{s}{12}g)$ be the reduced Ricci tensor of g. Then $R = W + h \wedge g$, where $(h \wedge g)(X, Y) = hX \wedge Y + X \wedge hY$ for any X, Y in TM. The latter can be written as $(h \wedge g)F = \{F, h\}$ for any F in $\Lambda^2 \cong \mathfrak{so}(TM)$, where $\{\cdot, \cdot\}$ denotes the anti-commutator. Due to the isomorphism $\mathrm{Sym}_0^2 \cong \Lambda^+ \otimes \Lambda^-$ described by the map $S \mapsto S^-$, where $S^-(F) = \{S, F\}^-$, F in Λ^2, we also know that $\{\mathrm{Sym}_0^2, \Lambda^{\pm}\} \subseteq \Lambda^{\mp}$.

The next lemma generalises a statement of [18].

Lemma 3.1. *On an almost-Hermitian manifold* (M^4, g, I) *the curvature of the canonical connection can be decomposed as*

$$\widetilde{R} = W^- + \frac{s}{12} \operatorname{Id}_{\Lambda^-} + \frac{1}{2} \operatorname{Ric}_0^- + \frac{1}{2} \widetilde{\gamma}_1 \otimes \omega_I.$$

Proof. Expanding (3.2), and using (3.3) along the way, we obtain

$$\widetilde{R}(X, Y) = W^-(X, Y) + W^+(X, Y) + hX \wedge Y + X \wedge hY$$
$$- d^{\widetilde{\nabla}} \eta(X, Y) + \frac{1}{2} \Phi(X, Y) \omega_I - \eta_{T_X Y}$$

for any X, Y in TM. Now, since $d^{\widetilde{\nabla}} \eta(X, Y) + \eta_{T_X Y}$ belongs to λ^2 and $\widetilde{R}$ lives in $\Lambda^2 \otimes \lambda^{1,1}$, by taking into account that W^+ only acts on $\langle \omega_I \rangle \oplus \lambda^2$, we can project onto invariant 2-forms and infer

$$\widetilde{R} = W^- + (h \wedge g)_{\lambda^{1,1}} + \frac{1}{2}(W^+ \omega_I + \Phi) \otimes \omega_I.$$

From $\lambda^{1,1} = \Lambda^- \oplus \langle \omega_I \rangle$ we further get $(h \wedge g)_{\lambda^{1,1}} = (h \wedge g)^- + \frac{1}{2} \langle \{h, I\} \cdot, \cdot \rangle \otimes \omega_I$, and the claim follows by definition of h and equation (3.4). □

3.1. Elements of Hermitian geometry

We now specialise the facts above to a Hermitian (M^4, g, I). Equivalently $\eta \in \lambda^{1,1} \otimes \Lambda^1$, which in a local gauge I_1 means that the 1-forms c, a of (3.1) satisfy

$$c = -Ia, \quad \theta = 2I_1 a \tag{3.7}$$

where the Lee form θ is defined by $d\omega_I = \theta \wedge \omega_I$. A simple computation yields

$$\eta_U = \frac{1}{4}(U^\flat \wedge \theta + (IU)^\flat \wedge I\theta)$$

for any U in TM. Consequently $\eta_\zeta = \eta_{I\zeta} = 0$, where $\zeta = \theta^\sharp$. It easily follows that

$$\Phi = \frac{1}{4}(\theta \wedge I\theta + |\theta|^2 \omega_I). \tag{3.8}$$

Let $\kappa = 3\langle W^+ \omega_I, \omega_I \rangle$ be the conformal scalar curvature, which differs from the usual scalar curvature by

$$\kappa - s = -3d^\star \theta - \frac{3}{2}|\theta|^2, \tag{3.9}$$

see [18]. Given a one-form α we denote by $d^\pm \alpha$ the components of $d\alpha$ in $\Lambda^\pm$ respectively, so that $d\alpha = d^- \alpha + d^+ \alpha$. An important property [4] of the positive Weyl tensor of the Hermitian structure is

$$W^+ = \frac{\kappa}{4}\left(\frac{1}{2}\omega_I \otimes \omega_I - \frac{1}{3}\operatorname{Id}_{|\Lambda^+}\right) - \frac{1}{4}\Psi \otimes \omega_I - \frac{1}{4}\omega_I \otimes \Psi$$

where $\Psi = d^+\theta(I\cdot, \cdot)$ belongs to λ_I^2. In particular $W^+\omega_I = \frac{\kappa}{6}\omega_I - \frac{1}{2}d^+\theta(I\cdot, \cdot)$, hence (3.4) updates, with the aid of (3.8) and (3.9), to

$$\widetilde{\gamma}_1 = \rho^I - \frac{1}{2}(d^\star\theta)\omega_I + \frac{1}{4}\theta \wedge I\theta - \frac{1}{2}\Psi. \tag{3.10}$$

It is well known that $d^+\theta = 0$ is equivalent to demanding W^+ to be degenerate, which is a short way of saying that W^+ has a double eigenvalue.

4. Proof of Theorem 1.1

As mentioned earlier, closed 5-frames are equivalently described by Ricci-flat, self-dual Hermitian manifolds (M^4, g, I) that admit a holomorphic-symplectic structure compatible with the complex orientation.

The crucial observation is the following characterisation of such structures by means of the curvature of their canonical Hermitian connection. This approach will be taken up in the next section, in a more general situation.

Proposition 4.1. *A Hermitian manifold (M^4, g, I) admits, around each point, a closed 5-frame if and only if*

$$\widetilde{R} = -\frac{1}{4}d(I\theta) \otimes \omega_I.$$

Proof. Lemma 3.1 guarantees that the metric g is Ricci-flat and self-dual if and only if $\widetilde{R} = \frac{1}{2}\widetilde{\gamma}_1 \otimes \omega_I$. Equivalently, there exists a g-compatible hyperKähler structure $\{\omega_1, \omega_2, \omega_3\}$ spanning Λ^- around each point in M. There remains to show that the existence of an orthonormal pair ω_4, ω_5 of closed forms in λ_I^2 is the same as $\widetilde{\gamma}_1 = -\frac{1}{2}d(I\theta)$.

Suppose I_1 is a local gauge for (g, I) in the notation of (3.5). Then by writing $\nabla I_2 = -a \otimes \omega_I + b \otimes \omega_{I_1}$, the closure of ω_{I_2} is equivalent to $-a \wedge \omega_I + b \wedge \omega_{I_1} = 0$. But equation (3.7) says $a \wedge \omega_I = -\frac{1}{2}I_1\theta \wedge \omega_I = \frac{1}{2}I\theta \wedge \omega_{I_2}$, hence $b = \frac{1}{2}I\theta$.

Now, assume first that $\widetilde{R} = -\frac{1}{4}d(I\theta) \otimes \omega_I$, so that $\widetilde{\gamma}_1 = -\frac{1}{2}d(I\theta)$. A straightforward computation shows that the Hermitian connection $D = \widetilde{\nabla} - \frac{1}{4}I\theta \otimes I$ has zero curvature. Take a local orthonormal frame e_1, $e_2 = Ie_1$, e_3, $e_4 = Ie_3$ such that $De_k = 0, 1 \le k \le 4$. Then $\omega_I = e^{12} + e^{34}$ (e^{12} meaning $e^1 \wedge e^2$), and the other self-dual forms $e^{14} + e^{23}, e^{13} + e^{42}$ can be written as $g(I_1\cdot, \cdot), g(I_2\cdot, \cdot)$ respectively, with $I_2 = I_1I$. Since I_2 is D-parallel we have $\widetilde{\nabla}I_2 = \frac{1}{2}I\theta \otimes I_1$. Equation (3.5) gives $b = \frac{1}{2}I\theta$, hence ω_{I_2} is closed, and so is ω_{I_1} [27]. The anti-self-dual forms $e^{12} - e^{34}, e^{13} - e^{42}, e^{23} - e^{14}$ are D-parallel by construction. But they are ∇-parallel as well, for $D - \nabla$ belongs to $\Lambda^1 \otimes \Lambda^+$, and self-dual and anti-self-dual forms commute. The construction of the 5-frame is thus complete.

Vice versa, assume that $\omega_4 = g(I_1\cdot, \cdot)$, $\omega_5 = g(I_2\cdot, \cdot)$, with $I = I_2I_1$, are closed forms in λ_I^2. The above argument gives $b = \frac{1}{2}I\theta$, hence again $\widetilde{\gamma}_1 = -\frac{1}{2}d(I\theta)$ by (3.6). $\qquad\square$

If in addition M is simply connected the 5-frame is global. At this point we invoke the theory of self-dual Hermitian-Einstein manifolds, as presented in [5]. Well-known facts are collected in the following:

Proposition 4.2 ([4,16,25]). *Let (M^4, g, I) be Hermitian, Ricci-flat and self-dual, but not flat. Then:*

  (i) *ω_I is an eigenform of W^+, i.e., $W^+\omega_I = \frac{\kappa}{6}\omega_I$;*

 (ii) *the conformal scalar curvature κ and Lee form θ satisfy $\kappa\theta + \frac{2}{3}d\kappa = 0$, and $(\kappa^{\frac{2}{3}}g, I)$ is Kähler;*

(iii) *$X = I\,\mathrm{grad}(\kappa^{-\frac{1}{3}})$ is a Hamiltonian Killing vector field;*

(iv) *$d^+X^\flat = -\frac{1}{12}\kappa^{\frac{2}{3}}\omega_I$.*

Conditions (i) and (ii) are equivalent on any compact, not necessarily Einstein, Hermitian complex surface [5,13]. Part (i) holds for compact self-dual Hermitian surfaces [3] as well.

Proof of Theorem 1.1. Since a closed 5-frame induces a self-dual, Ricci-flat Hermitian metric, in order to use the classification of [5] we will show first that the Killing field X above is tri-holomorphic for the local hyperKähler structure. Examples in [30] confirm that this is not true in general.

(i - ii) The conformal scalar curvature κ is nowhere zero, otherwise the metric g would be flat. Ricci-flatness implies $d\theta = 0$ by (ii) in the proposition above. By Proposition 4.1 we have $\widetilde{\gamma}_1 = -\frac{1}{2}d(I\theta)$, hence (3.10) implies

$$-\frac{1}{2}d(I\theta) = \frac{1}{4}\theta \wedge I\theta - \frac{d^\star\theta}{2}\omega_I.$$

Using Proposition 4.2 (ii), and the comparison formula (3.9), we get $d(\kappa^{-\frac{1}{3}}I\theta) = -\kappa^{-\frac{1}{3}}(\frac{\kappa}{3} + \frac{|\theta|^2}{2})\omega_I$. Since $X^\flat = \frac{\kappa^{-\frac{1}{3}}}{2}I\theta$ we obtain $d^-X^\flat = 0$; it follows that the Killing vector field X is tri-holomorphic with respect to the local hyperKähler structure.

At the same time, by comparing with Proposition 4.2 (iv), it follows that $|X|^2 + \frac{1}{12}\kappa^{\frac{1}{3}} = 0$ or equivalently $|\theta|^2 = -\frac{\kappa}{3}$. Then $d\ln|\theta| = -\frac{3}{4}\theta$ belongs to the distribution $\mathcal{D}$ spanned by $\theta^\sharp, I\theta^\sharp$; this means, according to [5, Theorem 1], that the orthogonal almost complex J, obtained by reversing the sign of I along $\mathcal{D}$, is integrable. Its fundamental form

$$\omega_J = \omega_I + 2|\theta|^{-2}\theta \wedge I\theta$$

belongs to Λ^- and is closed; indeed $d\theta = 0$, and since X is tri-holomorphic, $dX^\flat = -\frac{1}{12}\kappa^{\frac{2}{3}}\omega_I$ by Proposition 4.2, so

$$d\omega_J = \theta \wedge \omega_I - 2\theta \wedge d(|\theta|^{-2}I\theta) = \theta \wedge \omega_I + 12\theta \wedge d(\kappa^{-\frac{2}{3}}X^\flat)$$

$$= \theta \wedge \omega_I + 12\kappa^{-\frac{2}{3}}\theta \wedge dX^\flat = 0.$$

Therefore (g, J) is a Kähler structure. Now [5, Theorem 1], case b1) of its proof to be precise, warrants that self-dual Ricci-flat Hermitian 4-manifolds with X tri-holomorphic and (g, J) Kähler reduce to the Gibbons-Hawking Ansatz with $U = ay + b$ for constants a, b. Since g is not flat we can take $a \neq 0$, and without of loss of generality let $a = 1$ by rescaling the metric, and $b = 0$. In this way g is of the form claimed, with $t = \frac{2}{3} y^{3/2}$, $\sigma_1 = dz$, $\sigma_2 = dx$, $\sigma_3 = du + \Theta$ in the notation of Example 2.2.

As for the theorem's last statement, the anti-self-dual part of a closed 5-frame is unique up to an $O(3)$-rotation. Let now ω_4', ω_5' be orthonormal and closed in Λ^+. Up to a sign they determine [27] the same orthogonal complex structure as ω_4, ω_5, since W^+ is degenerate and never zero and so they belong to λ_I^2. Then $\omega_4' + i\omega_5' = f(\omega_4 + i\omega_5)$, with $f : M \to U(1)$ holomorphic with respect to I due to the closure of the forms, and therefore constant. $\qquad\square$

At this juncture a few comments are in order. First, a non-flat closed 5-frame is incompatible with the manifold being compact. In fact, if the induced metric were even only complete, X would become a global tri-holomorphic Killing vector field, in contradiction to [12, Theorem 1 (iii)]. Secondly, Theorem 1.1 can be considered as a local 4-dimensional analogue, for two-forms, of the following result:

Theorem [23]. *Let (M, g) be a compact Riemannian n-manifold with $b_1(M) = n - 1$ and such that every harmonic 1-form has constant length. Then (M, g) is a quotient of a nilpotent Lie group with 1-dimensional centre, equipped with a left-invariant metric.*

5. Small holonomy and further examples

The proof of Theorem 1.1 suggests a wider perspective should be considered, namely that of almost-Hermitian 4-manifolds with small holonomy.

Given (M^4, g, I) almost-Hermitian, consider the holonomy algebra $\widetilde{\mathfrak{hol}} \subseteq \mathfrak{u}(2)$ of the canonical connection at a given point of M, and assume it at most 1-dimensional. Then any generator of $\widetilde{\mathfrak{hol}}$ must be invariant under parallel transport by $\widetilde{\nabla}$, so it must extend to an element F of Λ^2 such that $\widetilde{\nabla} F = 0$. Since the curvature tensor $\widetilde{R}$ takes its values in $\widetilde{\mathfrak{hol}}$ we can write

$$\widetilde{R} = \gamma \otimes F$$

for some two-form γ on M. As $\widetilde{\nabla}$ is Hermitian F must have type $(1, 1)$, hence we can split

$$F = F_0 + \alpha\omega_I,$$

where F_0 is in $\lambda_0^{1,1}$ and α a real number.

Three possible scenarios unfold before us: either the entire curvature $\widetilde{R}$ vanishes, or F_0 is zero, or F_0 is non-zero.

5.1. The flat case

We begin with the simplest situation, in which the almost-Hermitian manifold (M^4, g, I) has $\widetilde{R} = 0$ everywhere.

Theorem 5.1. *Let (M^4, g, I) be almost-Hermitian and such that $\widetilde{R} = 0$. Then*

  (i) *the metric g is flat*;
 (ii) *if $(\sigma_1, \sigma_2, \sigma_3)$ is a ∇-parallel orthonormal basis of self-dual forms,*

$$\omega_I = \sigma_1 \cos\varphi \cos\psi + \sigma_2 \cos\varphi \sin\psi + \sigma_3 \sin\varphi$$

where φ and ψ are locally-defined maps on M subject to $d\psi \wedge d\varphi = 0$.

Proof. (i) By Lemma 3.1 the tensors W^-, Ric and $\widetilde{\gamma}_1$ all vanish. Therefore, $W^+\omega_I = -\Phi$ by (3.4), implying the two-form Φ belongs to Λ^+. But Φ is decomposable in any local gauge, hence it squares to zero. This means that Φ vanishes, too:

$$W^+\omega_I = 0, \quad \Phi = 0.$$

Since g is Einstein and W^+ has zero determinant, [11, Proposition 16.72] forces $W^+ = 0$, and g is indeed a flat metric.
(ii) A local gauge for ω_I is given by

$$\omega_{I_1} = \sigma_1 \sin\varphi \cos\psi + \sigma_2 \sin\varphi \sin\psi - \sigma_3 \cos\varphi$$
$$\omega_{I_2} = -\sigma_1 \sin\psi + \sigma_2 \cos\psi.$$

A straightforward computation yields $a = -d\varphi$, $c = \cos\varphi d\psi$, $b = \sin\varphi d\psi$. From the proof of part (i), $\widetilde{R} = 0$ is equivalent to $\Phi = a \wedge c = 0$ when g is flat. Therefore $\cos\varphi d\varphi \wedge d\psi = 0$, and we conclude by a density argument. $\square$

In addition, the theorem of Frobenius tells that $\psi = \psi(\varphi)$ is a local function of one variable.

Corollary 5.2. *Let (M^4, g, I) be either Hermitian or almost-Kähler, with $\widetilde{R} = 0$. Then (g, I) is a flat Kähler structure.*

Proof. The Hermitian and almost-Kähler conditions are both characterised in a local gauge by $c = \mp Ia$, so the claim follows from $a \wedge c = 0$ and Theorem 5.1 (i). $\square$

In the almost-Kähler case the corollary was proved in [17] assuming compactness, albeit differently and for arbitrary dimensions. Similar results can be found in [8,9], again for compact M.

5.2. The case $F_0 = 0$

This is a very familiar situation as the next observation shows.

Proposition 5.3. *On an almost-Hermitian manifold (M^4, g, I) the following are equivalent:*

(i) *the curvature of the canonical connection is generated by the Kähler form of I:*

$$\widetilde{R} = \frac{1}{2} \widetilde{\gamma}_1 \otimes \omega_I \tag{5.1}$$

(ii) $\mathrm{Ric} = 0$ *and* $W^- = 0$.

Proof. The statement is an immediate consequence of Lemma 3.1. $\qquad\square$

Note that on a self-dual, Ricci-flat manifold any positive orthogonal almost complex structure satisfies (5.1).

5.3. The case $F_0 \neq 0$

Because F_0 is $\widetilde{\nabla}$-parallel, it has constant length. By rescaling γ, if necessary, we may parametrise $F_0 = \omega_J = g(J\cdot, \cdot)$ by means of an orthogonal complex structure J with orientation opposite to that of I.

Proposition 5.4. *Let (M^4, g, I) be almost-Hermitian. The following statements are equivalent*:

(i) *the holonomy algebra of the canonical connection is 1-dimensional, generated by F in $\lambda_I^{1,1}$ with non-vanishing primitive part*;
(ii) $\widetilde{\nabla}$ *is not flat and there is a negatively-oriented, g-compatible Kähler structure J such that $\widetilde{\gamma}_1 = \alpha \rho^J$, where α is a non-zero real constant.*

Either assumption implies

$$\widetilde{R} = \frac{\rho^J}{2} \otimes (\alpha \omega_I + \omega_J).$$

Proof. (i) $\Rightarrow$ (ii) It is clear that $\widetilde{\nabla} J = 0$. Since $\eta = \widetilde{\nabla} - \nabla$ belongs to $\Lambda^1 \otimes \lambda^2 \subseteq \Lambda^1 \otimes \Lambda^+$, it follows that $\nabla J = 0$, for self-dual and anti-self-dual forms commute. Therefore (g, J) is Kähler and compatible with the negative orientation. In particular the Ricci tensor of g is J-invariant and

$$W^- = \begin{pmatrix} \frac{s}{6} & 0 \\ 0 & -\frac{s}{12} \end{pmatrix}$$

with respect to $\Lambda^- = \langle \omega_J \rangle \oplus \lambda_J^2$. Equivalently,

$$W^- + \frac{s}{12} \mathrm{Id}_{\Lambda^-} + \frac{1}{2} \mathrm{Ric}_0^- = \frac{1}{2} \rho^J \otimes \omega_J. \tag{5.2}$$

Lemma 3.1 gives then $\widetilde{R} = \frac{1}{2}\rho^J \otimes \omega_J + \frac{1}{2}\widetilde{\gamma}_1 \otimes \omega_I$. From $\widetilde{R} = \gamma \otimes F$

$$\frac{1}{2}\rho^J \otimes \omega_J + \frac{1}{2}\widetilde{\gamma}_1 \otimes \omega_I = \gamma \otimes \omega_J + \alpha\gamma \otimes \omega_I$$

follows, and proves the claim.

The implication (ii) $\Rightarrow$ (i) is a direct consequence of (5.2) and Lemma 3.1, which also prove the final assertion. $\square$

In the rest of this section we explicitly determine the almost-Kähler structures (M^4, g, I) with $\dim \widetilde{\mathfrak{hol}} \leq 1$. We first describe a slightly larger class of almost-Kähler 4-manifolds.

Let $(\Sigma, g_\Sigma, I_\Sigma)$ be a Riemann surface with area form ω_Σ; we equip $\mathbb{R}^2$ with co-ordinates x, y and let $z = x + iy$. For any $w : \mathbb{R}^2 \times \Sigma \to \{z \in \mathbb{C} : |z| < 1\}$ we consider the almost-Kähler structure (g, I) on $\mathbb{R}^2 \times \Sigma$, where

$$g = \frac{4}{1 - |w|^2}(dz - wd\bar{z}) \odot (d\bar{z} - \overline{w}dz) + g_\Sigma$$
$$g(I\cdot, \cdot) = \frac{i}{2}dz \wedge d\bar{z} + \omega_\Sigma. \tag{5.3}$$

We assume w holomorphic in the Σ-variable, that is $I_\Sigma d_\Sigma w = i d_\Sigma w$ where d_Σ is differentiation on Σ; then (g, J) is Kähler, where

$$g(J\cdot, \cdot) = -\frac{i}{2}dz \wedge d\bar{z} + \omega_\Sigma.$$

These examples generalise the construction in [1] where w is chosen to depend only on Σ; also, they particularise the twisting construction in [15] to the case of the trivial line bundle over a Riemann surface.

Theorem 5.5. *Let (M^4, g, I) be almost-Kähler with* $\dim \widetilde{\mathfrak{hol}} \leq 1$.*Then:*

(i) *I is integrable, or*
(ii) *g is Ricci-flat and self-dual, or*
(iii) *(g, I) is locally given by (5.3), where w does not depend on $\mathbb{R}^2$ and the metric* $(1 - |w|^2)^{\frac{1}{4(\alpha-1)}} g_\Sigma, \alpha \in \mathbb{R}\backslash\{\pm 1\}$ *is flat, or*
(iv) *(g, I) is locally given by (5.3), where g_Σ is flat.*

Proof. By the previous results only the case $\widetilde{\mathfrak{hol}} = \mathbb{R}(\alpha\omega_I + \omega_J)$, where α is a real number and J is an orthogonal, negative Kähler structure, has to be looked at. We shall also assume that I is non-integrable.

The rank-two distributions $\mathcal{D}_\pm = \ker(IJ \mp \mathrm{Id})$ are parallel for the canonical Hermitian connection and allow to decompose $\omega_I = \omega_+ + \omega_-$, $\omega_J = -\omega_+ + \omega_-$.

Since (g, I) is almost-Kähler, $\eta_{IX}IY = -\eta_X Y$ for all X, Y in TM; it follows that the restrictions of η to $\mathcal{D}_\pm$ are symmetric. Since the latter are $\widetilde{\nabla}$-parallel the distributions $\mathcal{D}_+$ and $\mathcal{D}_-$ must be both integrable. In particular the Levi-Civita connections of the induced metrics coincide with the restrictions of $\widetilde{\nabla}$ to $\mathcal{D}_\pm$. Let $s_\pm = 2\langle \widetilde{R}(\omega_\pm), \omega_\pm \rangle$ be the corresponding scalar curvatures; from the general formula $\widetilde{R} = \frac{1}{2}\rho^J \otimes \omega_J + \frac{1}{2}\widetilde{\gamma}_1 \otimes \omega_I$ we get

$$-s_+ = \langle \widetilde{\gamma}_1 - \rho^J, \omega_+ \rangle, \quad s_- = \langle \widetilde{\gamma}_1 + \rho^J, \omega_- \rangle. \tag{5.4}$$

We now parametrise $\rho^J = \frac{s}{4}\omega_J + \mu\omega_I + \varphi_1$, with φ_1 in λ_I^2; in particular $\rho^I = \frac{s}{4}\omega_I + \mu\omega_J$. Let us also write $W^+\omega_I = \frac{\kappa}{6}\omega_I + \varphi_2$, where φ_2 belongs to λ_I^2 and κ is the conformal scalar curvature of (g, I). We fix a local gauge I_1 for I with connection forms a and b and impose $\tilde{\gamma}_1 = \alpha\rho^J$. Since (g, I) is almost-Kähler, using (3.4) with $c = Ia$ and $\Phi = a \wedge Ia$ we get

$$\varphi_2 = \alpha\varphi_1$$
$$a \wedge Ia = \left(\alpha\mu - \frac{s}{12} - \frac{\kappa}{6}\right)\omega_I + \left(\frac{\alpha s}{4} - \mu\right)\omega_J. \tag{5.5}$$

Note that $\kappa = s + 6|a|^2$.

In particular Ia is orthogonal to J_1a, J_2a where J_1 is a local gauge for J and $J_2 = J_1J$. Since it is also orthogonal to a it follows that $Ia = \pm Ja$ on the open set where $a \neq 0$.

a) $Ia = -Ja$. Then a belongs to $\Lambda^1\mathcal{D}_+$, therefore $\mathcal{D}_-$ is the Kähler nullity of (g, I), that is $\eta_{\mathcal{D}_-} = 0$. Since $\mathcal{D}_- = \ker a \cap \ker(Ia)$ is integrable, the structure equations
$$da + b \wedge Ia = R(\omega_{I_2}), \ d(Ia) - b \wedge a = R(\omega_{I_1})$$
of (g, I) imply that $R(\omega_-)$ is orthogonal to λ_I^2. But the component in λ_I^2 of $R(\omega_-) = \frac{1}{2}(R(\omega_I) + R(\omega_J)) = \frac{1}{2}\{h, I\} + \frac{1}{2}(W^+(\omega_I) + \rho^J)$ is precisely $\frac{1}{2}(\varphi_1 + \varphi_2)$ and therefore it must vanish.

a$_1$) When $\alpha \neq -1$, we have $\varphi_1 = \varphi_2 = 0$ by (5.5). This means Ric is I-invariant and ω_I is an eigenform of W^+. These almost-Kähler manifolds form the so-called class $\mathcal{AK}_3$; using their classification in [1] we get that (g, I) is locally given by (5.3), where w depends only on Σ. Consequently Ric $= 0$ on $\mathcal{D}_+$, and as $a \wedge Ia = -|a|^2\omega_+$, the scalar relations in (5.5) are equivalent to $\frac{\alpha-1}{2}s = |a|^2$. But
$$|a|^2 = \frac{|d|w||^2}{2(1 - |w|^2)^2},$$
and since w is holomorphic it is easy to check that $(dI_\Sigma d)\ln(1 - |w|^2) = 8|a|^2\omega_\Sigma$. The flatness of $(1 - |w|^2)^{\frac{1}{4(\alpha-1)}}g_\Sigma$ follows now from the conformal transformation rule of the scalar curvature.

a$_2$) When $\alpha = -1$ the bundle $\mathcal{D}_-$ is flat for the canonical connection. The second equation in (5.5), now equivalent to $2\mu + \frac{s}{2} + |a|^2 = 0$, contains no further information; its left hand side computes in fact s_- by (5.4).
Pick locally-defined unit vectors $e_2 = Ie_1$ such that $\tilde{\nabla}e_k = 0, k = 1, 2$. Because (g, I) is almost-Kähler with Kähler nullity $\mathcal{D}_-$, it follows that the dual forms satisfy $de^1 = de^2 = 0$. Now write locally $M = \Sigma \times \mathbb{R}^2$ for some 2-dimensional manifold Σ where the co-ordinates x, y on $\mathbb{R}^2$ are such that $e_1 = \frac{d}{dx}$, $e_2 = \frac{d}{dy}$. Since the distribution $\mathcal{D}_+$ is also integrable Σ can be chosen to correspond to $\mathcal{D}_+$, i.e., to admit local co-ordinates t, u such that

$\mathcal{D}_+ = \mathrm{span}\{\frac{d}{dt}, \frac{d}{du}\}$ and $\omega_+ = dt \wedge du$. Then $J(\frac{d}{dx}) = \frac{d}{dy}$ whereas on Σ the complex structure J is determined by a family of complex structure compatible with ω_+, possibly depending on x, y. Therefore $J = (1-S)^{-1}I_0(1-S)$ on Σ where I_0 is given by $I_0(du) = dt$, $S = \begin{pmatrix} \mathrm{Re}\, w & \mathrm{Im}\, w \\ \mathrm{Im}\, w & -\mathrm{Re}\, w \end{pmatrix}$ in the basis $\{du, dt\}$, and $w : \mathbb{R}^2 \times \mathbb{R}^2 \to \{z : |z| < 1\}$. Now requiring J to be integrable amounts to

$$J_y = J J_x$$

on Σ. However this linearises as $S_y = I_0 S_x$ and the claim easily follows.

b) $Ia = Ja$. Replace J by $-J$ and apply part (a). Note that α transforms into $-\alpha$. □

Part (ii) in the above theorem is a manifestation of a closed '4-frame', whose local geometry is more complicated. The only known explicit examples are given by the Gibbons-Hawking Ansatz for a translation-invariant harmonic map, see [6] and its generalisations [2].

In the compact case Theorem 5.5 can be enhanced as follows.

Theorem 5.6. *A compact almost-Kähler 4-manifold (M, g, I) with $\dim \widetilde{\mathfrak{hol}} \leq 1$ must be Kähler.*

Proof. If $\widetilde{\mathfrak{hol}} = \{0\}$ this is granted by corollary 5.2. Assume now that $\widetilde{\mathfrak{hol}}$ is generated by $F = F_0 + \alpha \omega_I$. If $F_0 = 0$ the metric g is Ricci-flat by Proposition 5.3 and the integrability of I follows from [28].

There remains to treat the case when $F_0 \neq 0$, when after re-normalisation(see Section 5.3) we may assume that $F_0 = \omega_J$ where (g, J) is Kähler compatible with the negative orientation. If $\alpha \neq \pm 1$ a case-by-case inspection of the proof of Theorem 5.5 shows that $\varphi_1 = \varphi_2 = 0$ in $\{x \in M : \eta_x \neq 0\}$. Now in any open set U where $\eta = 0$ the structure (g, I) is Kähler hence g is a local product; from the definition of φ_1 and φ_2 it is easy to see they vanish in U as well. By a standard density argument we conclude that $\varphi_1 = \varphi_2 = 0$ over M hence the almost-Kähler structure (g, I) belongs to the class $\mathcal{AK}_3$. Because M is compact it follows that (g, I) is Kähler by the classification of Apostolov-Armstrong-Drăghici in [1]. To complete the proof there remains to examine the following cases.

a) $\alpha = -1$. We will first show that $\eta_{\mathcal{D}_-} = 0$ everywhere in M; note that it suffices to prove this at points where $\eta \neq 0$. Working around such points and using the same local choices as in the proof of Theorem 5.5, the second equation in (5.5) reads

$$a \wedge Ia = -\frac{\kappa - s}{6}\omega_+ - \left(2\mu + \frac{2s + \kappa}{6}\right)\omega_- = -|a|^2\omega_+ - \left(2\mu + \frac{2s + \kappa}{6}\right)\omega_-.$$

In particular $\langle a \wedge Ia, \omega_+ \rangle + |a|^2 = 0$ after taking the scalar product with ω_+. If $Ia = Ja$ the form a vanishes on $\mathcal{D}_+$ hence $\langle a \wedge Ia, \omega_+ \rangle = 0$ and further $a = 0$,

contradicting the assumption that $\eta \neq 0$. Therefore around points where $\eta \neq 0$ we have $Ia = -Ja$ hence $\eta_{\mathcal{D}_-} = 0$. The claim on the vanishing of $\eta_{\mathcal{D}_-}$ in M is therefore proved. Since $\mathcal{D}_-$ is parallel with respect to $\widetilde{\nabla}$ it follows that $\mathcal{D}_-$ is totally geodesic.

Now having $\alpha = -1$ means that bundle $\mathcal{D}_-$ is flat with respect to the canonical connection, in particular $s_- = 0$. In the terminology of [24] the foliation induced by the integrable distribution $\mathcal{D}_+$ is transversally totally geodesic with vanishing transverse Ricci curvature. Because the Kähler manifold (M, g, J) is compact, Proposition 2.1 in [24], applied to the foliation induced by $\mathcal{D}_+$, shows that the latter is parallel with respect to the Levi-Civita connection of g. Equivalently, (g, I) is Kähler and the theorem is proved in this case.

b) $\alpha = 1$. Replace J by $-J$ and apply part a) above. $\qquad\qquad\square$

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