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CONTROL OF UNDERWATER VEHICLES IN INVISCID FLUIDS II. FLOWS WITH VORTICITY

Rodrigo Lecaros[1] and Lionel Rosier[2]

Abstract. In a recent paper, the authors investigated the controllability of an underwater vehicle immersed in an infinite volume of an inviscid fluid, assuming that the flow was irrotational. The aim of the present paper is to pursue this study by considering the more general case of a flow *with vorticity*. It is shown here that the local controllability of the position and the velocity of the underwater vehicle (a vector in $\mathbb{R}^{12}$) holds in a flow *with vorticity* whenever it holds in a flow *without vorticity*.

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1. Introduction

An accurate model for the motion of a boat (without rudder) equipped with tunnel thrusters was investigated in [5]. In that paper, using Coron's return method (see [2]), the authors proved that it was in general possible to control both the position and the velocity of the boat (a vector in $\mathbb{R}^6$) by using *two* control inputs. The fluid was assumed to be inviscid, but not necessarily irrotational, and its motion was described by Euler equations for incompressible fluids.

In [11], the authors started the study of the controllability of an underwater vehicle $\mathcal{S}$ (*e.g.* a submarine) immersed in an infinite volume of an inviscid fluid (filling $\mathbb{R}^3 \setminus \mathcal{S}$). Assuming that the fluid was irrotational, they proved by using Coron's return method the controllability of both the position and the velocity of the vehicle (a vector in $\mathbb{R}^{12}$) by using 6, or 4, or merely 3 control inputs for appropriate geometries. The aim of the present paper is to pursue this study by considering the more general case of a flow *with vorticity*. We will show that the local controllability of both the position and the velocity of the underwater vehicle holds in a flow *with vorticity* whenever it holds in a flow *without vorticity*. The method of proof is inspired by the one of [5]: the extension of the exact controllability to a system with a (small) vorticity is achieved by a perturbative approach relying on a topological argument. Next, the small vorticity assumption is removed by using a scaling argument. However, to prove the wellposedness of the complete system we shall use here the contraction mapping theorem instead of the Schauder fixed-point theorem as in [5]. This choice leads to a more straightforward proof.

Keywords and phrases. Underwater vehicle, fluid-structure interaction, Euler equations, vorticity, exterior domain, exact controllability, return method.

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Our fluid-structure interaction system can be described as follows. The underwater vehicle, represented by a rigid body occupying a connected compact set $\mathcal{S}(t) \subset \mathbb{R}^3$, is surrounded by a homogeneous incompressible perfect fluid filling the open set $\Omega(t) := \mathbb{R}^3 \setminus \mathcal{S}(t)$ (as *e.g.* for a submarine immersed in an ocean). We assume that $\Omega(t)$ is C^∞ smooth and connected. Let $\mathcal{S} = \mathcal{S}(0)$ and $\Omega(0) = \mathbb{R}^3 \setminus \mathcal{S}(0)$ denote the initial configuration $(t = 0)$. Then, the dynamics of the fluid-structure system are governed by the following system of PDE's

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0, \quad t \in (0,T),\ x \in \Omega(t), \tag{1.1}$$

$$\operatorname{div} u = 0, \quad t \in (0,T),\ x \in \Omega(t), \tag{1.2}$$

$$u \cdot n = (h' + \zeta \times (x - h)) \cdot n + w(t,x), \quad t \in (0,T),\ x \in \partial\Omega(t), \tag{1.3}$$

$$\lim_{|x|\to+\infty} u(t,x) = 0, \quad t \in (0,T), \tag{1.4}$$

$$m_0 h'' = \int_{\partial\Omega(t)} pn \,\mathrm{d}\sigma, \quad t \in (0,T), \tag{1.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(QJ_0Q^*\zeta) = \int_{\partial\Omega(t)} (x - h) \times pn \,\mathrm{d}\sigma, \quad t \in (0,T), \tag{1.6}$$

$$Q' = S(\zeta)Q, \quad t \in (0,T), \tag{1.7}$$

$$u(0,x) = u_0(x), \quad x \in \Omega(0), \tag{1.8}$$

$$(h(0), Q(0), h'(0), \zeta(0)) = (h_0, Q_0, h_1, \zeta_0) \in \mathbb{R}^3 \times \mathrm{SO}(3) \times \mathbb{R}^3 \times \mathbb{R}^3. \tag{1.9}$$

In the above equations, u (resp. p) is the velocity field (resp. the pressure) of the fluid, h denotes the position of the center of mass of the solid, ζ denotes the angular velocity and $Q \in \mathbb{R}^{3\times3}$ the rotation matrix giving the orientation of the solid. The positive constant m_0 and the matrix J_0, which stand for the mass and the inertia matrix of the rigid body, respectively, are defined as

$$m_0 = \int_{\mathcal{S}} \rho(x)\mathrm{d}x, \quad J_0 = \int_{\mathcal{S}} \rho(x)(|x|^2 Id - xx^*)\mathrm{d}x,$$

where $\rho(\cdot)$ represents the density of the rigid body. The vector n is the outward unit vector to $\partial\Omega(t)$, $x \times y$ is the cross product between the vectors x and y, and $S(y)$ is the skew-adjoint matrix such that $S(y)x = y \times x$, *i.e.*

$$S(y) = \begin{pmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{pmatrix}.$$

The neutral buoyancy condition reads

$$\int_{\mathcal{S}} \rho(x)\mathrm{d}x = \int_{\mathcal{S}} 1\mathrm{d}x. \tag{1.10}$$

When f is a function depending on t, f' (or $\dot{f}$) stands for the derivative of f with respect to t. For $A \in \mathbb{R}^{M\times N}$ $(M, N \in \mathbb{N}^*)$, A^* denotes the transpose of the matrix A, and Id denotes the identity matrix. The term $w(t,x)$, which stands for the flow through the boundary of the rigid body, is taken as control input. Its support will be strictly included in $\partial\Omega(t)$, and actually only a finite dimensional control input will be considered here (see below (1.17) for the precise form of the control term $w(t,x)$).

When no control is applied (*i.e.* $w(t,x) = 0$), then the existence and uniqueness of strong solutions to (1.1)–(1.9) was obtained first in [12] for a ball embedded in $\mathbb{R}^2$, and next in [13] for a rigid body $\mathcal{S}$ of arbitrary form (still in $\mathbb{R}^2$). The case of a ball in $\mathbb{R}^3$ was investigated in [14], and the case of a rigid body of arbitrary form in $\mathbb{R}^3$ was studied in [17]. (See also [16] for the motion of a rigid body in the inviscid limit of Navier–Stokes equations and [6] for the time regularity of the flow.) The detection of a rigid body $\mathcal{S}(t)$ from a partial measurement of the fluid velocity (or of the pressure) has been tackled in [3] when $\Omega(t) = \Omega_0 \setminus \mathcal{S}(t)$ ($\Omega_0 \subset \mathbb{R}^2$ denoting a fixed cavity) and in [4] when $\Omega(t) = \mathbb{R}^2 \setminus \mathcal{S}(t)$.

Note also that since the fluid is flowing through a part of the boundary of the rigid body, additional boundary conditions are needed to ensure the uniqueness of the solution of (1.1)–(1.9) (see [7], [9]). In dimension three, one can specify the tangent components of the vorticity $\omega(t,x) := \mathrm{curl}\, v(t,x)$ on the inflow section; that is, one can set

$$\omega(t,x)\cdot\tau_i = g_0(t,x)\cdot\tau_i \ \text{ for } w(t,x) < 0,\ i = 1,2, \tag{1.11}$$

where $g_0(t,x)$ is a given function and τ_i, $i = 1,2$, are linearly independent vectors tangent to $\partial\Omega(t)$. On the other hand, since ω is divergence-free in Ω, we have that $\int_{\partial\Omega(t)} \omega(t,x)\cdot n\, \mathrm{d}\sigma = 0$.

In order to write the equations of the fluid in a *fixed frame*, we perform a change of coordinates. We set

$$x = Q(t)y + h(t), \tag{1.12}$$
$$v(t,y) = Q^*(t)u(t, Q(t)y + h(t)), \tag{1.13}$$
$$\mathbf{q}(t,y) = p(t, Q(t)y + h(t)), \tag{1.14}$$
$$l(t) = Q^*(t)h'(t), \tag{1.15}$$
$$r(t) = Q^*(t)\zeta(t). \tag{1.16}$$

Then x (resp. y) represents the vector of coordinates of a point in a fixed frame (respectively in a frame linked to the rigid body). Note that, at any given time t, y ranges over the fixed domain

$$\Omega := Q_0^*(\Omega(0) - h_0)$$

when x ranges over $\Omega(t)$. Finally, we assume that the control takes the form

$$w(t,x) = w(t, Q(t)y + h(t)) = \sum_{j=1}^{m} w_j(t)\chi_j(y), \tag{1.17}$$

where $m \in \mathbb{N}^*$ stands for the number of independent inputs, and $w_j(t) \in \mathbb{R}$ is the control input associated with the function $\chi_j \in C^\infty(\partial\Omega)$. To ensure the conservation of the mass of the fluid, we impose the relation

$$\int_{\partial\Omega} \chi_j(y)\mathrm{d}\sigma = 0 \ \text{ for } 1 \le j \le m. \tag{1.18}$$

Then the functions $(v, \mathbf{q}, l, r)$ satisfy the following system

$$\frac{\partial v}{\partial t} + ((v - l - r\times y)\cdot\nabla)v + r\times v + \nabla\mathbf{q} = 0,\ t\in(0,T),\ y\in\Omega, \tag{1.19}$$
$$\mathrm{div}\ v = 0,\ t\in(0,T),\ y\in\Omega, \tag{1.20}$$
$$v\cdot n = (l + r\times y)\cdot n + \sum_{1\le j\le m} w_j(t)\chi_j(y),\ t\in(0,T),\ y\in\partial\Omega, \tag{1.21}$$
$$\lim_{|y|\to+\infty} v(t,y) = 0,\ t\in(0,T), \tag{1.22}$$
$$m_0\dot{l} = \int_{\partial\Omega} \mathbf{q}n\, \mathrm{d}\sigma - m_0 r\times l,\ t\in(0,T), \tag{1.23}$$
$$J_0\dot{r} = \int_{\partial\Omega} \mathbf{q}(y\times n)\, \mathrm{d}\sigma - r\times J_0 r,\ t\in(0,T), \tag{1.24}$$
$$(l(0), r(0)) = (l_0, r_0) := (Q_0^* h_1, Q_0^*\zeta_0),\ v(0,y) = v_0(y) := Q_0^* u_0(Q_0 y + h_0). \tag{1.25}$$

The initial velocity field $v_0 \in C^{2,\alpha}(\overline{\Omega})$ has to satisfy the following compatibility conditions

$$\begin{cases} \operatorname{curl} v_0 = \omega_0 & \text{in } \Omega, \\ \operatorname{div} v_0 = 0 & \text{in } \Omega, \\ v_0 \cdot n = (l_0 + r_0 \times y) \cdot n + \sum\limits_{1\le j\le m} w_j(0)\chi_j(y) & \text{on } \partial\Omega, \\ \lim\limits_{|y|\to+\infty} v_0(y) = 0 \end{cases} \tag{1.26}$$

where $\omega_0 := \operatorname{curl}\, v_0$ is the *initial vorticity*.

Once (l, r) is known, the motion of the underwater vehicle is described by the system

$$Q'(t) = Q(t)S(r(t)), \tag{1.27}$$

$$h'(t) = Q(t)l(t), \tag{1.28}$$

$$\zeta(t) = Q(t)r(t). \tag{1.29}$$

Using quaternions, the rotation matrix Q can be parametrized by

$$\boldsymbol{q} \in B_1(0) := \{\boldsymbol{q} = (q_1, q_2, q_3) \in \mathbb{R}^3;\ \ |\boldsymbol{q}| := \sqrt{q_1^2 + q_2^2 + q_3^3} < 1\}$$

(see *e.g.* [11]); namely, we can write $Q = R(\boldsymbol{q})$ where

$$R(\boldsymbol{q}) := \begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_2q_1 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_3q_1 - q_0q_2) & 2(q_3q_2 + q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{pmatrix}.$$

with $\boldsymbol{q} = (q_1, q_2, q_3) \in B_1(0)$ and $q_0 := \sqrt{1 - |\boldsymbol{q}|^2}$. Let $Q_0 = R(\boldsymbol{q}_0)$ with $\boldsymbol{q}_0 = (q_{1,0}, q_{2,0}, q_{3,0})$.

Then the dynamics of $\boldsymbol{q}$ and h are given by

$$\begin{cases} h'(t) = (1 - |\boldsymbol{q}|^2)l + 2\sqrt{1 - |\boldsymbol{q}|^2}\, \boldsymbol{q} \times l + (l \cdot \boldsymbol{q})\boldsymbol{q} - \boldsymbol{q} \times l \times \boldsymbol{q}, \\ \boldsymbol{q}'(t) = \frac{1}{2}(\sqrt{1 - |\boldsymbol{q}|^2}\, r + \boldsymbol{q} \times r), \\ h(0) = h_0, \quad \boldsymbol{q}(0) = \boldsymbol{q}_0. \end{cases} \tag{1.30}$$

When there is no vorticity ($\omega \equiv 0$), sufficient conditions of local exact controllability for $(h, \boldsymbol{q}, l, r)$ were derived in ([11], Thm. 3.10). That result was applied to the controllability of an ellipsoidal submarine with a small number of controls: $m \in \{3, 4, 6\}$. The reader is referred to [11] for precise statements. The method of proof of ([11], Thm. 3.10), inspired by the one of ([5], Thm. 2.1), combined Coron's return method (see [2]), the flatness approach (for the construction of the reference trajectory) and a variant of Silverman−Meadows criteria. In the following, we shall assume that the conclusion of ([11], Thm. 3.10) (controllability without vorticity) holds; namely,

(H) *For any $T > 0$, there exist a number $\eta > 0$ and a map $W \in C^1(B_{\mathbb{R}^{24}}(0, \eta), C^1([0, T]; \mathbb{R}^m))$ which associates with any $(h_0, \boldsymbol{q}_0, l_0, r_0, h_T, \boldsymbol{q}_T, l_T, r_T) \in B_{\mathbb{R}^{24}}(0, \eta)$ a control $w \in C^1([0, T], \mathbb{R}^m)$ with $w(0) = 0$ steering the state of system* (1.19)–(1.25) *and* (1.30) without vorticity *from $(h_0, \boldsymbol{q}_0, l_0, r_0)$ at $t = 0$ to $(h_T, \boldsymbol{q}_T, l_T, r_T)$ at $t = T$.*

In (H), we used the obvious notation: $B_{\mathbb{R}^N}(0, \eta) := \{x \in \mathbb{R}^N;\ |x| < \eta\}$.

The aim of this work is to extend that property to the more general case of fluids with vorticity. Here, we shall use the contraction mapping theorem (instead of a compactness approach as in [5]) to obtain in a direct way the existence and uniqueness of the solution of (1.19)–(1.25). The main result in this paper is the following

Theorem 1.1. *Assume that the assumption* (H) *is fulfilled, and pick any* $T_0 > 0$*. Then there exists* $\eta > 0$ *such that for any* $(h_0, \boldsymbol{q}_0, l_0, r_0) \in \mathbb{R}^{12}$ *and any* $(h_T, \boldsymbol{q}_T, l_T, r_T) \in \mathbb{R}^{12}$ *with*

$$|(h_0, \boldsymbol{q}_0)| < \eta, \quad |(h_T, \boldsymbol{q}_T)| < \eta,$$

and for any $\omega_0 \in C^{1,\alpha}(\overline{\Omega}) \cap M^p_{1,\delta+2} \cap M^p_{0,\delta+3}$ *(see below for the definition of these spaces) with*

$$\begin{aligned}
&|\omega_0(y^1) - \omega_0(y^2)| \le \frac{K}{[1+\min(|y^1|,|y^2|)]^\kappa}|y^1 - y^2|, \quad \forall (y^1, y^2) \in \Omega^2,\\
&\left|\frac{\partial \omega_0}{\partial y}\right| = O(|y|^{-1}) \quad \text{as } |y| \to +\infty,\\
&\left|\frac{\partial \omega_0}{\partial y}(y^1) - \frac{\partial \omega_0}{\partial y}(y^2)\right| \le \frac{K}{1+\min(|y^1|,|y^2|)}|y^1 - y^2|, \quad \forall (y^1, y^2) \in \Omega^2
\end{aligned}$$

for some constants $p \in (3,4]$, $\delta \in [0, 1-\frac{3}{p})$, $\alpha \in (0, 1-\frac{3}{p}]$, $\kappa > 3 + \delta + \frac{3}{p}$ *and* $K > 0$*, if* v_0 *denotes the solution of* (1.26) *with* $w_j(0) = 0$ *for* $1 \le j \le m$*, then there exist a time* $T \in (0, T_0]$ *and a control input* $w \in C^1([0,T];\mathbb{R}^m)$ *with* $w(0) = 0$ *such that the system* (1.19)–(1.25) *and* (1.30) *admits a solution* $(h, \boldsymbol{q}, l, r, v, \mathbf{q})$ *satisfying*

$$(h, \boldsymbol{q}, l, r)_{|t=T} = (h_T, \boldsymbol{q}_T, l_T, r_T).$$

Remark 1.2. In our previous control result ([11], Thm. 3.10) for a system without vorticity, it was required that the initial/final velocities be small, but this restriction could easily be removed by using the same scaling argument as in the proof of Theorem 1.1.

The paper is organized as follows. In Section 2 we prove the existence and uniqueness of the solution of the control problem (1.19)–(1.25) (the vorticity being extended to $\mathbb{R}^3$) by applying the contraction mapping theorem in Kikuchi's spaces. The proof of Theorem 1.1 is given in Section 3.

2. Wellposedness of the system with vorticity

Let us introduce some notations. For $k \in \mathbb{N}$ and $\alpha \in (0,1)$, let $C^{k,\alpha}(\overline{\Omega})$ denote the classical Hölder space endowed with the norm

$$\|f\|_{C^{k,\alpha}(\overline{\Omega})} = \sum_{\substack{\beta = (\beta_1,\beta_2,\beta_3) \in \mathbb{N}^3\\ \beta_1+\beta_2+\beta_3 \le k}} \Big(\|\partial^\beta f\|_{L^\infty(\Omega)} + |\partial^\beta f|_{0,\alpha} \Big),$$

where

$$|f|_{0,\alpha} = \sup \left\{ \frac{|f(x) - f(y)|}{|x-y|^\alpha} \ ; \ x \in \overline{\Omega},\ y \in \overline{\Omega},\ x \ne y \right\}.$$

We also need some notations borrowed from [10]. Let $\langle y \rangle = (1+|y|^2)^{\frac{1}{2}}$. For $s \in \mathbb{N}$, $p \in [1,\infty)$ and $\lambda \ge 0$, let $M^p_{s,\lambda}$ denote the completion of the space of functions in $C^\infty(\overline{\Omega})$ with compact support in $\overline{\Omega}$ for the norm

$$\|u\|_{M^p_{s,\lambda}} = \sum_{\substack{\beta = (\beta_1,\beta_2,\beta_3) \in \mathbb{N}^3\\ \beta_1+\beta_2+\beta_3 \le s}} \|\langle y \rangle^{\lambda+\beta_1+\beta_2+\beta_3} \partial^\beta u\|_{L^p(\Omega)}.$$

In particular, for $s = 0$, $\|u\|_{M^p_{0,\lambda}} = \|u\|_{L^p_{p\lambda}} := (\int_\Omega |u|^p \langle y \rangle^{p\lambda} \mathrm{d}y)^{\frac{1}{p}}$. We shall mainly use the space $M^p_{1,\lambda}$ (for the vorticity) and $M^p_{2,\lambda}$ (for the velocity) endowed with the respective norms

$$\|u\|_{M^p_{1,\lambda}} = \|\langle y \rangle^\lambda u\|_{L^p(\Omega)} + \sum_{1 \le i \le 3} \|\langle y \rangle^{\lambda+1} \partial_{y_i} u\|_{L^p(\Omega)}, \tag{2.1}$$

$$\|u\|_{M^p_{2,\lambda}} = \|\langle y \rangle^\lambda u\|_{L^p(\Omega)} + \sum_{1 \le i \le 3} \|\langle y \rangle^{\lambda+1} \partial_{y_i} u\|_{L^p(\Omega)} + \sum_{1 \le i,j \le 3} \|\langle y \rangle^{\lambda+2} \partial_{y_j}\partial_{y_i} u\|_{L^p(\Omega)}. \tag{2.2}$$

Let π be a continuous linear extension operator from functions defined in Ω to functions defined in $\mathbb{R}^3$, which maps $C^{k,\alpha}(\overline{\Omega})$ to $C^{k,\alpha}(\mathbb{R}^3)$ for all $k \in \mathbb{N}$ and all $\alpha \in (0,1)$. The construction of such a "universal" extension operator is classical, see *e.g.* ([15], p. 194). We may also ask that π preserves the divergence-free character, see *e.g.* [8].

We introduce some functions ϕ_i, $i = 1,2,3$, φ_i, $i = 1,2,3$, and ψ_j, $j = 1,2,\dots,m$, satisfying

$$\Delta\phi_i = \Delta\varphi_i = \Delta\psi_j = 0 \qquad \text{in } \Omega, \tag{2.3}$$

$$\frac{\partial\phi_i}{\partial n} = n_i, \quad \frac{\partial\varphi_i}{\partial n} = (y \times n)_i, \quad \frac{\partial\psi_j}{\partial n} = \chi_j \qquad \text{on } \partial\Omega, \tag{2.4}$$

$$\lim_{|y|\to+\infty} \nabla\phi_i(y) = 0, \quad \lim_{|y|\to+\infty} \nabla\varphi_i(y) = 0, \quad \lim_{|y|\to+\infty} \nabla\psi_j(y) = 0. \tag{2.5}$$

As the open set Ω and the functions χ_j, $1 \le j \le m$, supporting the control are assumed to be smooth, we infer that the functions $\nabla\phi_i$ ($i = 1,2,3$), the functions $\nabla\varphi_i$ ($i = 1,2,3$) and the functions $\nabla\psi_j$ ($1 \le j \le m$) belong to $H^\infty(\Omega)$. On the other hand, it follows from ([10], Proof of Lem. 2.7) that for all $\alpha = (\alpha_1,\alpha_2,\alpha_3) \in \mathbb{N}^3$ with $\alpha_1+\alpha_2+\alpha_3 \ge 1$, we have

$$|\partial^\alpha\phi_i(y)| + |\partial^\alpha\varphi_i(y)| + |\partial^\alpha\psi_j(y)| \le C\langle y\rangle^{-1-(\alpha_1+\alpha_2+\alpha_3)}, \qquad i = 1,2,3,\ j = 1,2,\dots,m,\ y \in \Omega. \tag{2.6}$$

For notational convenience, in what follows $\int_\Omega f$ (resp. $\int_{\partial\Omega} f$) stands for $\int_\Omega f(y)\mathrm{d}y$ (resp. $\int_{\partial\Omega} f(y)\mathrm{d}\sigma(y)$).

Let us introduce the matrices $M, J, N \in \mathbb{R}^{3\times3}$, defined by

$$M_{i,j} = \int\limits_\Omega \nabla\phi_i \cdot \nabla\phi_j = \int\limits_{\partial\Omega} n_i\phi_j = \int\limits_{\partial\Omega} \frac{\partial\phi_i}{\partial n}\phi_j, \tag{2.7}$$

$$J_{i,j} = \int\limits_\Omega \nabla\varphi_i \cdot \nabla\varphi_j = \int\limits_{\partial\Omega} (y\times n)_i\varphi_j = \int\limits_{\partial\Omega} \frac{\partial\varphi_i}{\partial n}\varphi_j, \tag{2.8}$$

$$N_{i,j} = \int\limits_\Omega \nabla\phi_i \cdot \nabla\varphi_j = \int\limits_{\partial\Omega} n_i\varphi_j = \int\limits_{\partial\Omega} \phi_i(y\times n)_j. \tag{2.9}$$

Next we define the matrix $\mathcal{J} \in \mathbb{R}^{6\times6}$ by

$$\mathcal{J} = \begin{pmatrix} m_0\, Id & 0 \\ 0 & J_0 \end{pmatrix} + \begin{pmatrix} M & N \\ N^* & J \end{pmatrix}. \tag{2.10}$$

It is easy to see that $\mathcal{J}$ is a (symmetric) positive definite matrix.

For a potential flow (*i.e.* without vorticity), the dynamics of (l,r) are given by

$$\begin{pmatrix} l \\ r \end{pmatrix}' = \mathcal{J}^{-1}(Cw' + F(l,r,w)), \tag{2.11}$$

where

$$F(l,r,w) = -\begin{pmatrix} S(r) & 0 \\ S(l) & S(r) \end{pmatrix}\left(\mathcal{J}\begin{pmatrix} l \\ r \end{pmatrix} - Cw\right) - \sum_{p=1}^m w_p \begin{pmatrix} L_p^M l + R_p^M r + W_p^M w \\ L_p^J l + R_p^J r + W_p^J w \end{pmatrix}, \tag{2.12}$$

and

$$C = -\begin{pmatrix} C^M \\ C^J \end{pmatrix}, \tag{2.13}$$

with

$$(C^M)_{i,j} = \int_\Omega \nabla\phi_i \cdot \nabla\psi_j = \int_{\partial\Omega} n_i\psi_j = \int_{\partial\Omega} \phi_i\chi_j, \tag{2.14}$$

$$(C^J)_{i,j} = \int_\Omega \nabla\varphi_i \cdot \nabla\psi_j = \int_{\partial\Omega} (y\times n)_i\psi_j = \int_{\partial\Omega} \varphi_i\chi_j, \tag{2.15}$$

$$(L_p^M)_{i,j} = \int_{\partial\Omega} (\nabla\phi_j)_i\chi_p,\ \ (L_p^J)_{i,j} = \int_{\partial\Omega} (y\times\nabla\phi_j)_i\chi_p, \tag{2.16}$$

$$(R_p^M)_{i,j} = \int_{\partial\Omega} (\nabla\varphi_j)_i\chi_p,\ \ (R_p^J)_{i,j} = \int_{\partial\Omega} (y\times\nabla\varphi_j)_i\chi_p, \tag{2.17}$$

$$(W_p^M)_{i,j} = \int_{\partial\Omega} (\nabla\psi_j)_i\chi_p,\ \ (W_p^J)_{i,j} = \int_{\partial\Omega} (y\times\nabla\psi_j)_i\chi_p. \tag{2.18}$$

We refer the reader to [11] for the derivation of (2.11).

The first main result in this paper is a local existence result.

Theorem 2.1. *Let* $p \in (3,4]$, $\delta \in [0, 1-\frac{3}{p})$, $\alpha \in (0, 1-\frac{3}{p}]$, *and* $T > 0$. *Assume given* $(l_0, r_0) \in \mathbb{R}^3 \times \mathbb{R}^3$ *and* $\omega_0 \in C^{1,\alpha}(\overline{\Omega}) \cap M^p_{1,\delta+2} \cap M^p_{0,\delta+3}$ *with*

$$|\omega_0(y^1) - \omega_0(y^2)| \le \frac{K}{[1+\min(|y^1|,|y^2|)]^\kappa}|y^1-y^2|, \quad \forall (y^1,y^2)\in\Omega^2, \tag{2.19}$$

$$\left|\frac{\partial\omega_0}{\partial y}\right| = O(|y|^{-1}) \qquad \text{as } |y|\to+\infty, \tag{2.20}$$

$$\left|\frac{\partial\omega_0}{\partial y}(y^1) - \frac{\partial\omega_0}{\partial y}(y^2)\right| \le \frac{K}{1+\min(|y^1|,|y^2|)}|y^1-y^2|, \quad \forall (y^1,y^2)\in\Omega^2 \tag{2.21}$$

for some constants $\kappa > 3+\delta+\frac{3}{p}$ *and* $K>0$. *Let also a control input* $w \in C^1([0,T],\mathbb{R}^m)$ *be given. Assume that the initial velocity field* $v_0 \in C^{2,\alpha}(\overline{\Omega})$ *fulfills the following compatibility conditions*

$$\begin{cases} \operatorname{curl} v_0 = \omega_0 & \text{in } \Omega,\\ \operatorname{div} v_0 = 0 & \text{in } \Omega,\\ v_0\cdot n = (l_0 + r_0\times y)\cdot n + \sum_{1\le j\le m} w_j(0)\chi_j(y) & \text{on } \partial\Omega,\\ \lim_{|y|\to+\infty} v_0(y) = 0. \end{cases} \tag{2.22}$$

Then we can find a time $T' \in (0,T]$ *satisfying* $CT' < 1$, *where*

$$C = C(\|\omega_0\|_{C^{1,\alpha}(\overline{\Omega})} + \|\omega_0\|_{M^p_{1,\delta+2}}, |l_0|, |r_0|, \|w\|_{C^1([0,T])})$$

is nondecreasing in all its arguments, and a solution $(v, \mathbf{q}, l, r)$ *of* (1.19)–(1.25) *in the class*

$$v \in C([0,T']; C^{2,\alpha}(\overline{\Omega}) \cap M^p_{2,\delta+1}), \ \ \nabla v \in C([0,T']; L^4_{p(\delta+2)}(\Omega)), \tag{2.23}$$

$$\nabla\mathbf{q} \in C([0,T']; L^2(\Omega)), \tag{2.24}$$

$$\lim_{|y|\to+\infty} \nabla\mathbf{q}(t,y) = 0, \quad \forall t\in[0,T']. \tag{2.25}$$

$$(l,r) \in C^1([0,T'];\mathbb{R}^3\times\mathbb{R}^3). \tag{2.26}$$

Moreover, for $\|w\|_{C^1([0,T])} \le R$ *(*$R>0$ *being any given constant), this solution satisfies*

$$\big\|(l,r)-(\bar{l},\overline{r})\big\|_{L^\infty(0,T')} + \|v-\overline{v}\|_{L^\infty(0,T';C^{2,\alpha}(\overline{\Omega}))} \le C'\Big(\|\omega_0\|_{C^{1,\alpha}(\overline{\Omega})} + \|\omega_0\|_{M^p_{1,\delta+2}} \Big), \tag{2.27}$$

for some constant $C'>0$*, where* $(\bar{l},\overline{r},\overline{v})$ *is the potential solution of* (1.19)–(1.25) *associated with* l_0, r_0*,* $\{w_j\}_{1\le j\le m}$*, and* $\overline{\omega}_0 = 0$*.*

Remark 2.2.

(1) Note that, using the mean-value theorem, the assumption (2.20) implies (2.19) for $\kappa=1$, while $\omega_0\in M^p_{1,\delta+2}$ yields $|\omega_0(y)|\le O(|y|^{-\delta-2})$ as $|y|\to+\infty$ by ([10], Lem. 2.2).

(2) Note that the fluid-structure system considered here is more complicated than those considered in [14, 17], for we have added a control input in the boundary condition (1.21). Moreover, we require the solution to be continuous with respect to the control input in order to apply a perturbative argument at the end of the proof of Theorem 1.1. Inspired by the method developed in [5], it is quite natural to work in Kikuchi's spaces. Here, we shall prove the existence, uniqueness and continuous dependence of the solution with respect to the control input in *one step*, by using the contraction mapping theorem.

Theorem 2.1 will be established by using the contraction mapping principle (*i.e.* the Banach fixed-point theorem). We first define an operator whose fixed-points will give local-in-time solutions.

2.1. The operator

Let $p\in(3,4]$, $\delta\in[0,1-\frac{3}{p})$, $\alpha\in(0,1-\frac{3}{p}]$, and $T>0$. We fix a control input $w\in C^1([0,T];\mathbb{R}^m)$. For $N>0$ and $P>0$ given, we introduce the set

$$\begin{aligned}\mathcal{F} := \Big\{ &(l,r,\omega)\in\mathbb{R}^3\times\mathbb{R}^3\times\big(C^{1,\alpha}(\overline{\Omega})\cap M^p_{1,\delta+2}\big);\ |l-l_0|+|r-r_0|\le N,\\ &\|\omega\|_{C^{1,\alpha}(\overline{\Omega})}+\|\omega\|_{M^p_{1,\delta+2}}\le P,\ \operatorname{div}\omega=0,\quad \int_{\partial\Omega}\omega\cdot n\,\mathrm{d}\sigma=0\Big\}.\end{aligned} \tag{2.28}$$

Then, using Arzela−Ascoli theorem for the restrictions to closed ball centered at the origin of partial derivatives of order one, it is easy to see that $\mathcal{F}$ is a *closed* subset of the Banach space $E:=\mathbb{R}^3\times\mathbb{R}^3\times\big(C^{0,\alpha}(\overline{\Omega})\cap M^p_{0,\delta+2}\big)$ endowed with the norm

$$\|(l,r,\omega)\|_E := |l|+|r|+\|\omega\|_{C^{0,\alpha}(\overline{\Omega})}+\|\omega\|_{L^p_{p(\delta+2)}(\Omega)}.$$

It follows at once that for any $T'>0$, the set

$$\mathcal{C} := \{(l,r,\omega)\in C([0,T'],\mathcal{F});\ (l(0),r(0),\omega(0))=(l_0,r_0,\omega_0)\}$$

is a *closed* subset of the Banach space $C([0,T'];E)$ endowed with the norm $\sup_{t\in[0,T']}\|(l(t),r(t),\omega(t))\|_E$, which is also complete for the equivalent norm

$$|||(l,r,\omega)||| := \|l\|_{L^\infty(0,T')}+\|r\|_{L^\infty(0,T')}+\|\omega\|_{L^\infty(0,T';C^{0,\alpha}(\overline{\Omega}))}+\|\omega\|_{L^\infty(0,T';L^p_{p(\delta+2)}(\Omega))}. \tag{2.29}$$

Therefore, $\mathcal{C}$ is *complete* for the distance associated with the norm $|||\cdot|||$.

Here, we pick

$$P = e^e\cdot\big(C_6\|\pi(\omega_0)\|_{C^{1,\alpha}(\mathbb{R}^3)}+C_7\|\pi(\omega_0)\|_{M^p_{1,\delta+2}(\mathbb{R}^3)}\big), \tag{2.30}$$

where C_6 and C_7 are some universal constants arising in the computations below and that we do not intend to give explicitly, and $\|\cdot\|_{C^{1,\alpha}(\mathbb{R}^3)}$ and $\|\cdot\|_{M^p_{1,\delta+2}(\mathbb{R}^3)}$ are defined as $\|\cdot\|_{C^{1,\alpha}(\overline{\Omega})}$ and $\|\cdot\|_{M^p_{1,\delta+2}}$, respectively.

Let us now define the operator $\mathcal{T}$ on $\mathcal{C}$: to any $(l,r,\omega)\in\mathcal{C}$, we associate

$$\mathcal{T}\,(l,r,\omega) := (\hat{l},\hat{r},\hat{\omega}), \tag{2.31}$$

as follows. First, we introduce the "fluid velocity"

$$v = \eta + \sum_{i=1}^{3} l_i \nabla\phi_i + \sum_{i=1}^{3} r_i \nabla\varphi_i + \sum_{1\le j\le m} w_j(t)\nabla\psi_j, \tag{2.32}$$

where η is the solution to the div-curl system (see *e.g.* [10], Prop. 3.1)

$$\operatorname{curl}\eta = \omega, \quad (t,y)\in(0,T)\times\Omega, \tag{2.33}$$

$$\operatorname{div}\eta = 0, \quad (t,y)\in(0,T)\times\Omega, \tag{2.34}$$

$$\eta\cdot n = 0, \quad (t,y)\in(0,T)\times\partial\Omega, \tag{2.35}$$

$$\lim_{|y|\to+\infty}\eta(t,y) = 0, \quad t\in(0,T). \tag{2.36}$$

Next, we extend the velocity field and the initial vorticity by setting

$$\hat{v}(t,\cdot) := \pi[v(t,\cdot)], \tag{2.37}$$

$$\hat{\omega}_0 := \pi[\omega_0]. \tag{2.38}$$

The flow $\hat{X}$ associated with $\tilde{v} := \hat{v} - l - r\times y$ is defined as the solution to the Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial s}\hat{X}(s;t,y) = \tilde{v}(s,\hat{X}(s;t,y)) = \hat{v}(s,\hat{X}(s;t,y)) - l(s) - r(s)\times\hat{X}(s;t,y), \\ \hat{X}(t;t,y) \quad = y. \end{cases} \tag{2.39}$$

The fact that $\hat{X}$ is defined globally on $[0,T']^2\times\mathbb{R}^3$ follows from the boundedness of $\hat{v}$ (see below (2.50)).

We denote by G the Jacobi matrix of $\hat{X}$, namely

$$G(s;t,y) = \frac{\partial\hat{X}}{\partial y}(s;t,y). \tag{2.40}$$

Differentiating in (2.39) with respect to y_j $(j=1,2,3)$, we see that $G(s;t,y)$ satisfies the following system:

$$\begin{aligned} \frac{\partial G}{\partial s} &= \frac{\partial\hat{v}}{\partial y}(s,\hat{X}(s;t,y))\,G(s;t,y) - r(s)\times G(s;t,y), \\ &= \left(\frac{\partial\hat{v}}{\partial y}(s,\hat{X}(s;t,y)) - S(r(s))\right)G(s;t,y), \\ G(t;t,y) &= Id \text{ (the identity matrix).} \end{aligned} \tag{2.41}$$

We infer from

$$\operatorname{div}(\tilde{v}) = 0 \tag{2.42}$$

that

$$\det\, G(s;t,y) = 1. \tag{2.43}$$

We now define

$$\hat{\omega}(t,y) := G^{-1}(0;t,y)\hat{\omega}_0(\hat{X}(0;t,y)). \tag{2.44}$$

Finally, in order to define the pair $(\hat{l},\hat{r})$, we introduce the function $\mu : [0,T']\times\Omega\to\mathbb{R}$ (defined up to a function of t) which solves

$$\begin{cases} -\Delta\mu = \mathrm{tr}\Big(\nabla v\cdot\nabla v\Big), & \text{in } (0,T)\times\Omega,\\ \dfrac{\partial\mu}{\partial n} = -\displaystyle\sum_{1\le j\le m}\dot{w}_j(t)\chi_j(y) - \Big(\Big((v-l-r\times y)\cdot\nabla\Big)v + r\times v\Big)\cdot n, & \text{on } (0,T)\times\partial\Omega,\\ \lim\limits_{|y|\to+\infty}\nabla\mu(t,y)=0 & \text{in } (0,T). \end{cases} \tag{2.45}$$

Note that $\nabla\mu\in L^2(\Omega)$ if $\nabla v\in L^4(\Omega)\cap C^1(\overline{\Omega})$, and that, by Schauder estimates, $\mu\in C^{2,\alpha}_{\rm loc}(\overline{\Omega})$ if in addition $v\in C^{2,\alpha}(\overline{\Omega})$. Then we define $\hat{l}$ and $\hat{r}$ by

$$\begin{pmatrix}\hat{l}(t)\\ \hat{r}(t)\end{pmatrix} := \begin{pmatrix} l_0\\ r_0\end{pmatrix} + \mathcal{J}^{-1}\int_0^t\left\{\begin{pmatrix}\Big(\int_\Omega\nabla\mu(\tau,y)\cdot\nabla\phi_i(y)\,dy\Big)_{i=1,2,3}\\ \Big(\int_\Omega\nabla\mu(\tau,y)\cdot\nabla\varphi_i(y)\,dy\Big)_{i=1,2,3}\end{pmatrix} - \begin{pmatrix} m_0 r\times l\\ r\times J_0 r\end{pmatrix}\right\}\mathrm{d}\tau. \tag{2.46}$$

This completes the definition of $\mathcal{T}$.

2.2. Fixed-point argument and local-in-time existence

Our first step consists in proving the following result.

Theorem 2.3. *Let $N>0$ and $P>0$ be given. Then there exists some time $T'>0$ such that $\mathcal{T}$ is a contraction in $\mathcal{C}$ for the distance associated with $|||\cdot|||$. Thus, $\mathcal{T}$ has a unique fixed-point in $\mathcal{C}$.*

Proof of Theorem 2.3. Set

$$\overline{N} := N + |l_0| + |r_0|.$$

In the sequel, the various positive constants C_i will depend on the geometry, on $\mathcal{J}$ and on the size of the controls $\|w_i\|_{C^1}$ only (hence, possibly also on π, but not on T, l_0, r_0, ω_0, N, *etc.*).

Step 1. Let $(l,r,\omega)\in\mathcal{C}$, and let v be defined by (2.32). It follows from ([10], Prop. 2.11) that for any $t\in[0,T']$, system (2.33)–(2.36) has a unique solution $\eta(t)\in M^p_{2,\delta+1}$, and that

$$\|\eta(t)\|_{M^p_{2,\delta+1}} \le C\|\omega(t)\|_{M^p_{1,\delta+2}}.$$

On the other hand, using (2.6) and the fact that $0\le\delta<1-\frac{3}{p}$, it is easy to see that $\nabla\phi_i,\nabla\varphi_i,\nabla\psi_j\in M^p_{2,\delta+1}$ for $i=1,2,3$ and $j=1,2,\ldots,m$. It follows that

$$\|v(t)\|_{M^p_{2,\delta+1}} \le C(\|\omega(t)\|_{M^p_{1,\delta+2}} + |l(t)| + |r(t)| + |w(t)|). \tag{2.47}$$

Thus $v\in C([0,T'];M^p_{2,\delta+1})$ with

$$\|v\|_{L^\infty(0,T';M^p_{2,\delta+1})} \le C(\overline{N}+P+1). \tag{2.48}$$

Set

$$\mathcal{N} := \overline{N}+P+1 = N+|l_0|+|r_0|+P+1.$$

Now Schauder estimates combined with the embedding $M^p_{2,\delta+1} \subset C^1_b(\overline{\Omega})$ (see [10], Lem. 2.2) give that

$$\begin{aligned}\|v(t)\|_{C^{2,\alpha}(\overline{\Omega})} &\leq C\left(\|\omega(t)\|_{C^{1,\alpha}(\overline{\Omega})} + \|v(t)\|_{C^{0,\alpha}(\overline{\Omega})} + |l(t)| + |r(t)| + |w(t)|\right)\\ &\leq C\left(\|\omega(t)\|_{C^{1,\alpha}(\overline{\Omega})} + \|v(t)\|_{M^p_{2,\delta+1}} + |l(t)| + |r(t)| + |w(t)|\right)\\ &\leq C\left(\|\omega(t)\|_{C^{1,\alpha}(\overline{\Omega})} + \|\omega(t)\|_{M^p_{1,\delta+2}} + |l(t)| + |r(t)| + |w(t)|\right).\end{aligned}$$

Thus $v \in C([0,T'];C^{2,\alpha}(\overline{\Omega}))$ with

$$\begin{aligned}\|v\|_{L^\infty(0,T';C^{2,\alpha}(\overline{\Omega}))} \leq C\Big(&\|\omega\|_{L^\infty(0,T';C^{1,\alpha}(\overline{\Omega}))} + \|\omega\|_{L^\infty(0,T';M^p_{1,\delta+2})}\\ &+ \|l\|_{L^\infty(0,T')} + \|r\|_{L^\infty(0,T')} + \|w\|_{L^\infty(0,T')}\Big).\end{aligned} \tag{2.49}$$

Therefore, using the continuity of π, we obtain that

$$\|\hat{v}\|_{L^\infty(0,T';C^{2,\alpha}(\mathbb{R}^3))} \leq \|\pi\|\, C\mathcal{N} \leq C_1\mathcal{N}, \tag{2.50}$$

where $\|\pi\|$ denotes the norm of π as an operator in $\mathcal{L}(C^{2,\alpha}(\overline{\Omega}), C^{2,\alpha}(\mathbb{R}^3))$.

Step 2. Let us turn our attention to $\hat{X}$ and $\hat{\omega}$. Taking the scalar product of each term of the first equation in (2.39) by $\hat{X}$ results in

$$|\hat{X}|\frac{\partial|\hat{X}|}{\partial s} = \frac{\partial}{\partial s}(\frac{1}{2}|\hat{X}|^2) = \hat{X}\cdot\frac{\partial\hat{X}}{\partial s} = (\hat{v}(s,\hat{X}) - l(s))\cdot\hat{X} \leq \left(|\hat{v}(s,\hat{X}(s))| + |l(s)|\right)|\hat{X}|.$$

Simplifying by $|\hat{X}|$ and using the second equation in (2.39), we obtain

$$\Big||\hat{X}(s;t,y)| - |y|\Big| \leq CT'\mathcal{N}. \tag{2.51}$$

Thus $\hat{X}(s,t;\cdot) \notin L^\infty(\mathbb{R}^3)$ for all $(s,t) \in [0,T']^2$.

It follows from ([10], Lem. 2.2) that any function $u \in M^p_{1,\delta+2}$ satisfies $|u(x)| = O(|x|^{-\delta-1})$ as $|x| \to +\infty$, and from ([10], Lem. 2.3) that $M^p_{1,\delta+2}$ is an algebra. Let $M^p_{1,\delta+2}(\mathbb{R}^3)$ be defined as $M^p_{1,\delta+2}$ but with functions from $\mathbb{R}^3$ to $\mathbb{R}$. We introduce the space

$$V := M^p_{1,\delta+2}(\mathbb{R}^3)^{3\times 3} \oplus \mathbb{R}^{3\times 3}$$

with norm

$$\|G\|_V := \|G_1\|_{M^p_{1,\delta+2}(\mathbb{R}^3)} + \|G_2\|_{\mathbb{R}^{3\times 3}}$$

if $G = G_1 + G_2$ with $G_1 \in M^p_{1,\delta+2}(\mathbb{R}^3)^{3\times 3}$ and $G_2 \in \mathbb{R}^{3\times 3}$. (Note that G_2 is uniquely determined by G, as it is nothing but the 3×3 matrix of the limits at infinity of the entries of G.) Then it is easy to see that V is a Banach space and an algebra.

Let us check that $(\partial\hat{v}/\partial y)(s,\hat{X}(s;t,y)) \in L^\infty((0,T')^2, V)$. From (2.51) we have that

$$\left|\hat{X}(s;t,y)\right| \leq |y| + CT'\mathcal{N}$$

and proceeding as in ([10], p. 587−588), we infer that for any $u \in M^p_{1,\delta+2}(\mathbb{R}^3)$

$$\|u(\hat{X}(s;t,.))\|_{M^p_{0,\delta+2}(\mathbb{R}^3)} \leq C(1 + [CT'\mathcal{N}]^{p(\delta+2)})\|u\|_{M^p_{0,\delta+2}(\mathbb{R}^3)}. \tag{2.52}$$

On the other hand, using Gronwall's lemma in (2.41), we obtain with (2.50) that

$$\|G(s;t,.)\|_{L^\infty(\Omega)} \le C\exp(CT'\mathcal{N}). \tag{2.53}$$

Since

$$\frac{\partial[u(\hat{X}(s;t,y)]}{\partial y_j} = \sum_{k=1}^{3} \frac{\partial u}{\partial x_k}(\hat{X}(s;t,y))\frac{\partial \hat{X}_k}{\partial y_j}(s;t,y),$$

using (2.52) and (2.53) for $\partial u/\partial y$ (with $\delta+3$ substituted to $\delta+2$), we arrive to

$$\|\frac{\partial}{\partial y}[u(\hat{X}(s;t,y))]\|_{M^p_{0,\delta+3}(\mathbb{R}^3)} \le C\exp(CT'\mathcal{N})\|\frac{\partial u}{\partial y}\|_{M^p_{0,\delta+3}(\mathbb{R}^3)}. \tag{2.54}$$

We infer from (2.48) and (2.50) that

$$\|\frac{\partial \hat{v}}{\partial y}\|_{L^\infty(0,T',M^p_{1,\delta+2}(\mathbb{R}^3))} \le C\mathcal{N}.$$

It follows that $\frac{\partial \hat{v}}{\partial y}(s,\hat{X}(s;t,y)) - S(r(s)) \in L^\infty((0,T')^2,V)$ with

$$\left\|\frac{\partial \hat{v}}{\partial y}(s,\hat{X}(s;t,y)) - S(r(s))\right\|_V \le C\mathcal{N}\exp(CT'\mathcal{N}). \tag{2.55}$$

Solving the linear Cauchy problem (2.41) in the Banach algebra V, we see that $G\in C([0,T']^2;V)$ and (with Gronwall's lemma) that

$$\|G\|_{L^\infty((0,T')^2;V)} \le C_2\exp(C_2T'\mathcal{N}\mathrm{e}^{C_2T'\mathcal{N}}). \tag{2.56}$$

By (2.43), each entry of G^{-1} is a cofactor of G, so that we infer that

$$\left\|G^{-1}\right\|_{L^\infty((0,T')^2;V)} \le C_3\exp(C_3T'\mathcal{N}\mathrm{e}^{C_3T'\mathcal{N}}). \tag{2.57}$$

If $f\in C^1(\Omega,\mathbb{R}^3)$ with $\partial f/\partial y\in C^{0,\alpha}(\Omega,\mathbb{R}^{3\times 3})$ and $g\in C^{1,\alpha}(\Omega,\mathbb{R}^3)$, then

$$|g\circ f|_{0,\alpha} \le C|g|_{0,\alpha}\left\|\frac{\partial f}{\partial y}\right\|^\alpha_{L^\infty}, \tag{2.58}$$

$$\left\|\frac{\partial}{\partial y}(g\circ f)\right\|_{L^\infty} \le C\left\|\frac{\partial g}{\partial y}\right\|_{L^\infty}\left\|\frac{\partial f}{\partial y}\right\|_{L^\infty}, \tag{2.59}$$

$$\left|\frac{\partial}{\partial y}(g\circ f)\right|_{0,\alpha} \le C\left(\left\|\frac{\partial g}{\partial y}\right\|_{L^\infty}\left|\frac{\partial f}{\partial y}\right|_{0,\alpha} + \left\|\frac{\partial f}{\partial y}\right\|^{1+\alpha}_{L^\infty}\left|\frac{\partial g}{\partial y}\right|_{0,\alpha}\right). \tag{2.60}$$

Using (2.40), (2.41), (2.50), (2.53), (2.58) and Gronwall's lemma, we obtain that

$$\|G\|_{L^\infty((0,T')^2;C^{0,\alpha}(\mathbb{R}^3))} \le C\exp(CT'\mathcal{N}\mathrm{e}^{CT'\mathcal{N}}). \tag{2.61}$$

Next, it follows from (2.41), (2.50), (2.59), (2.60) and (2.61) that

$$\|G\|_{L^\infty((0,T')^2;C^{1,\alpha}(\mathbb{R}^3))} \le C_4\exp(C_4T'\mathcal{N}\mathrm{e}^{C_4T'\mathcal{N}}). \tag{2.62}$$

Using again (2.43), we obtain that

$$\left\|G^{-1}\right\|_{L^\infty((0,T')^2;C^{1,\alpha}(\mathbb{R}^3))} \le C_5\exp(C_5T'\mathcal{N}\mathrm{e}^{C_5T'\mathcal{N}}). \tag{2.63}$$

We are in a position to derive the required estimates for $\hat{\omega}$. From (2.58)–(2.60) and (2.62), we infer that

$$\|\hat{\omega}_0(\hat{X}(0;t,.))\|_{L^\infty(0,T';C^{1,\alpha}(\mathbb{R}^3))} \le C\exp(CT'\mathcal{N}\mathrm{e}^{CT'\mathcal{N}})\|\hat{\omega}_0\|_{C^{1,\alpha}(\mathbb{R}^3)}$$

which yields with (2.44) and (2.63)

$$\|\hat{\omega}\|_{L^\infty(0,T';C^{1,\alpha}(\mathbb{R}^3))} \le C_6\exp(C_6T'\mathcal{N}\mathrm{e}^{C_6T'\mathcal{N}})\|\hat{\omega}_0\|_{C^{1,\alpha}(\mathbb{R}^3)}. \tag{2.64}$$

From (2.52)–(2.54), we obtain that

$$\|\hat{\omega}_0(\hat{X}(0;t,.))\|_{L^\infty(0,T';M^p_{1,\delta+2}(\mathbb{R}^3))} \le C\exp(CT'\mathcal{N})\|\hat{\omega}_0\|_{M^p_{1,\delta+2}(\mathbb{R}^3)}$$

which gives with (2.57)

$$\|\hat{\omega}\|_{L^\infty(0,T';M^p_{1,\delta+2}(\mathbb{R}^3))} \le C_7\exp(C_7T'\mathcal{N}\mathrm{e}^{C_7T'\mathcal{N}})\ \|\hat{\omega}_0\|_{M^p_{1,\delta+2}(\mathbb{R}^3)}. \tag{2.65}$$

If we define $C_8 = \max\{C_6, C_7\}$, and take $T' > 0$ such that

$$T' \le \frac{1}{C_8\mathcal{N}}, \tag{2.66}$$

we obtain

$$\begin{aligned}\|\hat{\omega}\|_{L^\infty(0,T';C^{1,\alpha}(\mathbb{R}^3))} + \|\hat{\omega}\|_{L^\infty(0,T';M^p_{1,\delta+2}(\mathbb{R}^3))} &\le e^e\cdot(C_6\|\hat{\omega}_0\|_{C^{1,\alpha}(\mathbb{R}^3)} + C_7\|\hat{\omega}_0\|_{M^p_{1,\delta+2}(\mathbb{R}^3)})\\ &=: P.\end{aligned} \tag{2.67}$$

On the other hand, if we consider any scalar function $\varphi \in C^1(\mathbb{R}^3)$ with compact support, we obtain by using the change of variables $y = \hat{X}(t;0,x)$

$$\begin{aligned}\int_{\mathbb{R}^3}\hat{\omega}(t,y)\cdot\nabla\varphi(y)\mathrm{d}y &= \int_{\mathbb{R}^3}G^{-1}(0;t,y)\hat{\omega}_0(\hat{X}(0;t,y))\cdot\nabla\varphi(y)\mathrm{d}y\\ &= \int_{\mathbb{R}^3}G^{-1}(0;t,\hat{X}(t;0,x))\hat{\omega}_0(x)\cdot\nabla_y\varphi(\hat{X}(t;0,x))\mathrm{d}x,\\ &= \int_{\mathbb{R}^3}G(t;0,x)\hat{\omega}_0(x)\cdot\nabla_y\varphi(\hat{X}(t;0,x))\mathrm{d}x\\ &= \int_{\mathbb{R}^3}\hat{\omega}_0(x)\cdot\nabla_x[\varphi(\hat{X}(t;0,x))]\mathrm{d}x.\end{aligned}$$

Since, by (2.51), the function $\varphi(\hat{X}(t;0,\cdot)) \in C^1(\mathbb{R}^3)$ has a compact support, we infer from $\mathrm{div}(\hat{\omega}_0) = 0$ that

$$\mathrm{div}(\hat{\omega}) = 0 \quad \text{in } \mathbb{R}^3. \tag{2.68}$$

Integrating over Ω, we obtain

$$\int_{\Omega}\hat{\omega}\cdot n\mathrm{d}\sigma = 0. \tag{2.69}$$

On the other hand, $\hat{\omega}(0)_{|\overline{\Omega}} = \omega_0$. Therefore, the condition about $\hat{\omega}$ for $(\hat{l},\hat{r},\hat{\omega})$ to belong to $\mathcal{C}$ is satisfied.

Step 3. Let us turn our attention to $(\hat{l},\hat{r})$. Since $M^p_{1,\delta+2}(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$ and $\nabla\hat{v}(t) \in M^p_{1,\delta+2}(\mathbb{R}^3)$ for all $t \in [0,T']$, we infer from (2.48) that for all $t \in [0,T']$

$$\int_{\mathbb{R}^3}|\nabla\hat{v}|^4\langle y\rangle^{p(\delta+2)}\mathrm{d}y \le \|\nabla\hat{v}\|^{4-p}_{L^\infty(\mathbb{R}^3)}\int_{\mathbb{R}^3}|\nabla\hat{v}|^p\langle y\rangle^{p(\delta+2)}\mathrm{d}y \le C\|\nabla\hat{v}\|^4_{M^p_{1,\delta+2}(\mathbb{R}^3)} < +\infty.$$

Furthermore,

$$\|\nabla \hat{v}\|_{L^\infty(0,T',L^4(\mathbb{R}^3))} \le C\|\nabla \hat{v}\|_{L^\infty(0,T',M^p_{1,\delta+2}(\mathbb{R}^3))} \le C_9\mathcal{N}. \tag{2.70}$$

Using (2.45) and (2.50), we infer that

$$\|\nabla \mu\|_{L^\infty(0,T',L^2(\Omega))} \le C_{10}\mathcal{N}^2. \tag{2.71}$$

From (2.46), we deduce that $(\hat{l},\hat{r}) \in C([0,T'];\mathbb{R}^6)$ with

$$\left\|(\hat{l},\hat{r}) - (l_0,r_0)\right\|_{L^\infty(0,T')} \le C_{11}T'\mathcal{N}^2.$$

On the other hand, $(\hat{l}(0),\hat{r}(0)) = (l_0,r_0)$. Therefore, the condition about $(\hat{l},\hat{r})$ for $\mathcal{T}(l,r,\omega)$ to belong to $\mathcal{C}$ is satisfied provided that

$$T' \le \frac{N}{C_{11}\mathcal{N}^2}\cdot \tag{2.72}$$

Hence for T' satisfying (2.66) and (2.72), one has $\mathcal{T}(\mathcal{C}) \subset \mathcal{C}$.

Note also that, since $\nabla\hat{v}(t) \in M^p_{1,\delta+2}(\mathbb{R}^3)$ for all $t\in[0,T]$, we have

$$\langle y\rangle \nabla \hat{v}(t) \in M^p_{1,\delta+1}(\mathbb{R}^3) \subset C_0(\mathbb{R}^3) \quad \text{for all } t\in[0,T'].$$

Step 4. Now, we prove that the operator $\mathcal{T}$ is a *contraction* in $\mathcal{C}$ for the distance induced by $|||\cdot|||$ for T' small enough.

From now on, the constant C may depend on $\mathcal{N}$, but not on T' or on (l^k,r^k,ω^k).

Assume given $(l^k,r^k,\omega^k)\in\mathcal{C}$, $k=1,2$. Note that (l^1,r^1,ω^1) and (l^2,r^2,ω^2) correspond to the same initial data (l_0,r_0,ω_0) and the same control input ω.

Let us introduce for $k=1,2$

$$\mathcal{T}(l^k,r^k,\omega^k) := (\hat{l}^k,\hat{r}^k,\hat{\omega}^k).$$

Then, for $k=1,2$, $\hat{\omega}^k$ fulfills

$$\hat{\omega}^k(t,y) = A^k(0;t,y)\hat{\omega}_0(\hat{X}^k(0;t,y)), \tag{2.73}$$

where $\hat{\omega}_0 = \pi(w_0)$, $\hat{X}^k$ denotes the solution to

$$\begin{cases} \dfrac{\partial}{\partial s}\hat{X}^k(s;t,y) = \hat{v}^k(s,\hat{X}^k(s;t,y)) - l^k(s) - r^k(s)\times \hat{X}^k(s;t,y), \\ \\ \hat{X}^k(t;t,y) \quad = y, \end{cases} \tag{2.74}$$

$$G^k(s;t,y) := \frac{\partial \hat{X}^k}{\partial y}(s;t,y), \quad G(t;t,y) = Id, \tag{2.75}$$

and $A^k := (G^k)^{-1}$. The velocity $\hat{v}^k = \pi(v^k) : \mathbb{R}^3 \to \mathbb{R}^3$ is the extension of the velocity $v^k : \Omega \to \mathbb{R}^3$ decomposed as

$$v^k = \eta^k + \sum_{i=1}^3 l_i^k \nabla\phi_i + \sum_{i=1}^3 r_i^k\nabla\varphi_i + \sum_{1\le j\le m} w_j(t)\nabla\psi_j,$$

where η^k is the solution of

$$\operatorname{curl}\eta^k = \omega^k, \quad (t,y)\in(0,T)\times\Omega, \tag{2.76}$$

$$\operatorname{div}\eta^k = 0, \quad (t,y)\in(0,T)\times\Omega, \tag{2.77}$$

$$\eta^k\cdot n = 0, \quad (t,y)\in(0,T)\times\partial\Omega, \tag{2.78}$$

$$\lim_{|y|\to+\infty}\eta^k(t,y) = 0, \quad t\in(0,T). \tag{2.79}$$

We introduce the functions

$$v := v^1 - v^2,\;\; \eta := \eta^1 - \eta^2,\;\; r := r^1 - r^2,\;\; l := l^1 - l^2,\;\; \omega := \omega^1 - \omega^2,\;\text{and}\; A := A^1 - A^2. \tag{2.80}$$

Thus v may be written as

$$v = \eta + \sum_{i=1}^{3} l_i \nabla\phi_i + \sum_{i=1}^{3} r_i \nabla\varphi_i,\;\; v(0,y) = 0, \tag{2.81}$$

where η is the solution to the system

$$\operatorname{curl}\eta = \omega, \quad (t,y) \in (0,T)\times\Omega, \tag{2.82}$$

$$\operatorname{div}\eta = 0, \quad (t,y) \in (0,T)\times\Omega, \tag{2.83}$$

$$\eta\cdot n = 0, \quad (t,y) \in (0,T)\times\partial\Omega, \tag{2.84}$$

$$\lim_{|y|\to+\infty} \eta(t,y) = 0, \quad t \in (0,T). \tag{2.85}$$

Step 5. Let $\hat{X} := \hat{X}^1 - \hat{X}^2$. Then

$$\begin{cases} \dfrac{\partial}{\partial s}\hat{X}(s;t,y) = \hat{v}^1(s,\hat{X}^1(s;t,y)) - \hat{v}^1(s,\hat{X}^2(s;t,y)) + \hat{v}(s,\hat{X}^2(s;t,y)) \\ \qquad\qquad -l(s) - r^1(s)\times\hat{X}(s;t,y) - r(s)\times\hat{X}^2(s;t,y), \\ \hat{X}(t;t,y) \;\; = 0, \end{cases} \tag{2.86}$$

where $\hat{v} := \hat{v}^1 - \hat{v}^2 = \pi(v)$.

Taking the scalar product of each term in (2.86) by $\hat{X}$ results in

$$|\hat{X}|\frac{\partial|\hat{X}|}{\partial s} = \frac{\partial}{\partial s}\left(\frac{1}{2}|\hat{X}|^2\right) = \hat{X}\cdot\frac{\partial\hat{X}}{\partial s} = \left(\hat{v}^1(s,\hat{X}^1) - \hat{v}^1(s,\hat{X}^2) + \hat{v}(s,\hat{X}^2) - l - r\times\hat{X}^2\right)\cdot\hat{X}.$$

It follows that

$$\frac{\partial|\hat{X}|}{\partial s} \le C\left(\left\|\frac{\partial\hat{v}^1}{\partial y}\right\|_{L^\infty(\mathbb{R}^3)} |\hat{X}| + \|\hat{v}\|_{L^\infty(\mathbb{R}^3)} + |l| + |r|\cdot|\hat{X}^2|\right).$$

Since

$$\|\hat{v}(s)\|_{L^\infty(\mathbb{R}^3)} \le C\|v(s)\|_{C^{0,\alpha}(\overline{\Omega})} \le C\left(\|\eta(s)\|_{C^{0,\alpha}(\overline{\Omega})} + |l(s)| + |r(s)|\right) \le C \;|||\; (l,r,\omega) |||$$

and

$$|\hat{X}^2| \le |y| + CT'\mathcal{N} \le C\langle y\rangle,$$

we obtain with Gronwall's lemma that for $(s,t,y) \in [0,T']^2\times\mathbb{R}^3$,

$$|\hat{X}(s,t,y)| \le \mathrm{e}^{CT'}\int_0^{T'} C(1+\langle y\rangle) \;|||\; (l,r,\omega) \;|||\; \mathrm{d}t \le CT'\langle y\rangle \;|||\; (l,r,\omega) \;|||\,. \tag{2.87}$$

Step 6. Let us set $A := A^1 - A^2$. (Recall that $A^k = (G^k)^{-1}$ for $k = 1,2$.) Then we notice that

$$\frac{\partial A^k}{\partial s}(s;t,y) = -A^k(s;t,y)\left(\frac{\partial\hat{v}^k}{\partial y}(s,\hat{X}^k(s;t,y)) - S(r^k(s))\right),\quad A^k(t;t,y) = Id.$$

Thus

$$\begin{aligned}\frac{\partial A}{\partial s}(s;t,y) &= -A(s;t,y)\left(\frac{\partial \hat{v}^1}{\partial y}(s,\hat{X}^1(s;t,y)) - S(r^1(s))\right)\\ &\quad -A^2(s;t,y)\left(\frac{\partial \hat{v}^1}{\partial y}(s,\hat{X}^1(s;t,y)) - \frac{\partial \hat{v}^1}{\partial y}(s,\hat{X}^2(s;t,y))\right)\\ &\quad -A^2(s;t,y)\left(\frac{\partial \hat{v}}{\partial y}(s,\hat{X}^2(s;t,y)) - S(r(s))\right), \quad (2.88)\\ A(t;t,y) &= 0. \quad (2.89)\end{aligned}$$

It follows that

$$\left\|\frac{\partial A}{\partial s}(s;t,y)\right\| \le C\left(\|A\|\left(\left\|\frac{\partial \hat{v}^1}{\partial y}\right\|_{L^\infty(\mathbb{R}^3)} + |r^1|\right) + \|A^2\|\cdot\left\|\frac{\partial \hat{v}^1}{\partial y}\right\|_{W^{1,\infty}(\mathbb{R}^3)}|\hat{X}| + \|A^2\|\left(\left\|\frac{\partial \hat{v}}{\partial y}\right\|_{L^\infty(\mathbb{R}^3)} + |r|\right)\right). \quad (2.90)$$

From (2.50), we have that

$$\left\|\frac{\partial \hat{v}^1}{\partial y}(s)\right\|_{W^{1,\infty}(\mathbb{R}^3)} \le \left\|\hat{v}^1(s)\right\|_{C^{2,\alpha}(\mathbb{R}^3)} \le C\mathcal{N} \le C.$$

Clearly,

$$\left\|A^2\right\| + |r^1| \le C\mathcal{N} \le C.$$

On the other hand, it follows from Morrey's inequality that

$$\|\eta\|_{C^{0,1-\frac{3}{p}}(\overline{\Omega})} \le C\|\eta\|_{M^p_{1,\delta+1}} \le C\|\omega\|_{M^p_{0,\delta+2}}.$$

Let

$$|f|_{0,\alpha,\mathbb{R}^3} := \sup\left\{\frac{|f(x)-f(y)|}{|x-y|^\alpha};\ x,y\in\mathbb{R}^3,\ x\ne y\right\}.$$

Then, since $0<\alpha\le 1-\frac{3}{p}$, we have that

$$\begin{aligned}\left\|\frac{\partial \hat{v}}{\partial y}(s)\right\|_{L^\infty(\mathbb{R}^3)} + \left|\frac{\partial \hat{v}}{\partial y}(s)\right|_{0,\alpha,\mathbb{R}^3} &\le C\|v(s)\|_{C^{1,\alpha}(\overline{\Omega})}\\ &\le C\left(\|\eta(s)\|_{C^{1,\alpha}(\overline{\Omega})} + |l(s)| + |r(s)|\right)\\ &\le C\left(\|\eta(s)\|_{C^{0,\alpha}(\overline{\Omega})} + \|\omega(s)\|_{C^{0,\alpha}(\overline{\Omega})} + |l(s)| + |r(s)|\right)\\ &\le C\left(\|\omega(s)\|_{M^p_{0,\delta+2}} + \|\omega(s)\|_{C^{0,\alpha}(\overline{\Omega})} + |l(s)| + |r(s)|\right)\\ &\le C\, \|\!|\, (l,r,\omega)\, |\!\|\,. \quad (2.91)\end{aligned}$$

We infer with (2.87) and (2.90) that

$$\left\|\frac{\partial A}{\partial s}(s;t,y)\right\| \le C\|A\| + C(T'\langle y\rangle + 1)\, \|\!|\, (l,r,\omega)\, |\!\|\,.$$

Since $A(t;t,y)=0$, we obtain by using Gronwall's lemma that for $(s,t,y)\in[0,T']^2\times\mathbb{R}^3$

$$\|A(s;t,y)\| \le CT'\langle y\rangle\, \|\!|\, (l,r,\omega)\, |\!\|\,. \quad (2.92)$$

Step 7. Let $\hat{\omega} := \hat{\omega}^1 - \hat{\omega}^2$. We first give an estimate of $\|\hat{\omega}\|_{L^\infty(\Omega)}$. We write

$$\begin{aligned} |\hat{\omega}| &= \left| A^1(0;t,y)\hat{\omega}_0(\hat{X}^1(0;t,y)) - A^2(0;t,y)\hat{\omega}_0(\hat{X}^2(0;t,y)) \right| \\ &\leq \left| A(0;t,y)\hat{\omega}_0(\hat{X}^1(0;t,y)) \right| + \left| A^2(0;t,y)\left(\hat{\omega}_0(\hat{X}^1(0;t,y)) - \hat{\omega}_0(\hat{X}^2(0;t,y))\right) \right|. \end{aligned} \tag{2.93}$$

Since $\omega_0 \in M^p_{1,\delta+2}$, we have by ([10], Lem. 2.2) that

$$|\hat{\omega}_0(y)| = O(|y|^{-\delta-2}) \ \text{ as } \ |y| \to +\infty, \tag{2.94}$$

so that we infer from (2.51) and (2.92) that

$$\left| A(0;t,y)\hat{\omega}_0(\hat{X}^1(0;t,y)) \right| \leq CT'\langle y\rangle \,|||\, (l,r,\omega) \,|||\, |\hat{\omega}_0(\hat{X}^1(0;t,y)| \leq CT' \,|||\, (l,r,\omega) \,|||\, .$$

On the other hand, by (2.19), (2.63) and (2.87), we have that

$$\left| A^2(0;t,y)\left(\hat{\omega}_0(\hat{X}^1(0;t,y)) - \hat{\omega}_0(\hat{X}^2(0;t,y))\right) \right| \leq \frac{C}{1+\min(|\hat{X}^1|,|\hat{X}^2|)}|\hat{X}| \leq CT' \,|||\, (l,r,\omega)|||,$$

where we used (2.51) to get $1+\min(|\hat{X}^1|,|\hat{X}^2|) \geq C\langle y\rangle$ for $y \in \Omega$ and $t \in [0,T']$. Thus, we have proved that for $T' > 0$ satisfying (2.66) and (2.72), we have

$$\|\hat{\omega}\|_{L^\infty(0,T';L^\infty(\Omega))} \leq CT' \,|||\, (l,r,\omega) \,|||\, . \tag{2.95}$$

Step 8. Let us now estimate the Hölder norm $|\hat{\omega}|_{0,\alpha}$. Note first that it is not clear whether $\hat{X} \in C^{0,\alpha}(\overline{\Omega})$, since it could happen that $\hat{X} \sim \langle y\rangle$ as $|y| \to +\infty$ (and hence, $\hat{X} \notin L^\infty(\Omega)$). Rather, we shall prove that $\langle y\rangle^{-1}\hat{X} \in C^{0,\alpha}(\overline{\Omega})$.

We infer from (2.86) that

$$\begin{aligned} \frac{\partial}{\partial s}\left(\langle y\rangle^{-1}\hat{X}(s;t,y)\right) &= \langle y\rangle^{-1}\Big(\hat{v}^1(s,\hat{X}^1(s;t,y)) - \hat{v}^1(s,\hat{X}^2(s;t,y)) + \hat{v}(s,\hat{X}^2(s;t,y)) \\ &\qquad -l(s) - r^1(s)\times\hat{X}(s;t,y) - r(s)\times\hat{X}^2(s;t,y)\Big), \\ &= \int_0^1 \frac{\partial \hat{v}^1}{\partial y}(s,\hat{X}^2+\sigma\hat{X})\langle y\rangle^{-1}\hat{X}\mathrm{d}\sigma \\ &\qquad +\langle y\rangle^{-1}(\hat{v}(s,\hat{X}^2) - l - r^1\times\hat{X} - r\times\hat{X}^2) \end{aligned}$$

Therefore, using (2.58), we obtain

$$\begin{aligned} \left|\frac{\partial}{\partial s}\left(\frac{\hat{X}}{\langle y\rangle}\right)\right|_{0,\alpha,\mathbb{R}^3} \leq &\left[\left\|\frac{\partial\hat{v}^1}{\partial y}\right\|_{L^\infty(\mathbb{R}^3)}\left|\frac{\hat{X}}{\langle y\rangle}\right|_{0,\alpha,\mathbb{R}^3} + \left|\frac{\partial\hat{v}^1}{\partial y}\right|_{0,\alpha,\mathbb{R}^3}\left(\left\|\frac{\partial\hat{X}^2}{\partial y}\right\|_{L^\infty(\mathbb{R}^3)} + \left\|\frac{\partial\hat{X}^1}{\partial y}\right\|_{L^\infty(\mathbb{R}^3)}\right)^\alpha \left\|\frac{\hat{X}}{\langle y\rangle}\right\|_{L^\infty(\mathbb{R}^3)}\right] \\ &+\left|\frac{\hat{v}(s,\hat{X}^2)}{\langle y\rangle}\right|_{0,\alpha,\mathbb{R}^3} + C\left(1+\left|\frac{\hat{X}^2}{\langle y\rangle}\right|_{0,\alpha,\mathbb{R}^3}\right) |||\, (l,r,\omega) \,||| + C\left|\frac{\hat{X}}{\langle y\rangle}\right|_{0,\alpha,\mathbb{R}^3}. \end{aligned} \tag{2.96}$$

It is clear that

$$\left\|\frac{\partial\hat{v}^1}{\partial y}\right\|_{L^\infty(\mathbb{R}^3)} + \left|\frac{\partial\hat{v}^1}{\partial y}\right|_{0,\alpha,\mathbb{R}^3} + \left\|\frac{\partial\hat{X}^2}{\partial y}\right\|_{L^\infty(\mathbb{R}^3)} + \left\|\frac{\partial\hat{X}^1}{\partial y}\right\|_{L^\infty(\mathbb{R}^3)} \leq C. \tag{2.97}$$

To bound $|\frac{\hat{X}^2}{\langle y\rangle}|_{0,\alpha,\mathbb{R}^3}$, we notice that $\frac{\hat{X}^2}{\langle y\rangle}$ solves the system

$$\frac{\partial}{\partial s}\left(\frac{\hat{X}^2}{\langle y\rangle}\right) = \frac{\hat{v}^2(s,\hat{X}^2)}{\langle y\rangle} - \frac{l^2}{\langle y\rangle} - r^2 \times \frac{\hat{X}^2}{\langle y\rangle},$$
$$\frac{\hat{X}^2}{\langle y\rangle}(t;t,y) = \frac{y}{\langle y\rangle}\cdot$$

Since $|\hat{v}^2(s,\hat{X}^2)|_{0,\alpha,\mathbb{R}^3} \le C|\hat{v}^2|_{0,\alpha}\|\frac{\partial \hat{X}^2}{\partial y}\|^{\alpha}_{L^\infty(\mathbb{R}^3)} \le C$, $\|\hat{v}^2(s,\hat{X}^2)\|_{L^\infty(\Omega)} \le C$, and both $\langle y\rangle^{-1}$ and $\langle y\rangle^{-1}y$ belong to $W^{1,\infty}(\mathbb{R}^3) \subset C^{0,\alpha}(\mathbb{R}^3)$, we obtain with Gronwall's lemma that

$$\left|\frac{\hat{X}^2}{\langle y\rangle}\right|_{0,\alpha,\mathbb{R}^3} \le C. \tag{2.98}$$

On the other hand, we infer from (2.58) and (2.91) that

$$\left|\frac{\hat{v}(s,\hat{X}^2)}{\langle y\rangle}\right|_{0,\alpha,\mathbb{R}^3} \le C\left(\|\hat{v}(s,X^2)\|_{L^\infty(\mathbb{R}^3)} + |\hat{v}(s,\hat{X}^2)|_{0,\alpha,\mathbb{R}^3}\right) \le C \, |||\, (l,r,\omega) \,|||\, .$$

It follows from (2.86), (2.96)–(2.98) and Gronwall's lemma that

$$\left|\frac{\hat{X}}{\langle y\rangle}\right|_{0,\alpha} \le CT' \, |||\, (l,r,\omega) \,|||\, . \tag{2.99}$$

Next, we prove that a similar estimate holds for $|\frac{A}{\langle y\rangle}|_{0,\alpha}$. Writing for $1 \le i,j \le 3$

$$\frac{\partial \hat{v}^1_i}{\partial y_j}(s,\hat{X}^1) - \frac{\partial \hat{v}^1_i}{\partial y_j}(s,\hat{X}^2) = \int_0^1 \frac{\partial}{\partial y}\left(\frac{\partial \hat{v}^1_i}{\partial y_j}\right)(s,\hat{X}^2+\sigma\hat{X})\cdot\hat{X}\,\mathrm{d}\sigma,$$

and using (2.88), we infer that

$$\begin{aligned}
\left|\frac{\partial}{\partial s}\left(\frac{A}{\langle y\rangle}\right)\right|_{0,\alpha} \le{}& C\left|\frac{A}{\langle y\rangle}\right|_{0,\alpha}\left(\left\|\frac{\partial \hat{v}^1}{\partial y}\right\|_{L^\infty(\Omega)} + |r^1|\right) + C\left\|\frac{A}{\langle y\rangle}\right\|_{L^\infty(\Omega)}\left|\frac{\partial \hat{v}^1}{\partial y}\right|_{0,\alpha}\left\|\frac{\partial \hat{X}^1}{\partial y}\right\|^{\alpha}_{L^\infty(\Omega)} \\
&+ C\,\|A^2\|_{L^\infty(\Omega)}\left(|\hat{v}_1|_{2,\alpha}(\left\|\frac{\partial \hat{X}^2}{\partial y}\right\|_{L^\infty(\Omega)} + \|\frac{\partial \hat{X}^1}{\partial y}\|_{L^\infty(\Omega)})^{\alpha}\|\frac{\hat{X}}{\langle y\rangle}\|_{L^\infty(\Omega)}\right. \\
&\left. + \left\|\frac{\partial}{\partial y}\left(\frac{\partial \hat{v}^1}{\partial y}\right)\right\|_{L^\infty(\Omega)}\left|\frac{\hat{X}}{\langle y\rangle}\right|_{0,\alpha}\right) \\
&+ C\,|A^2|_{0,\alpha}\|\hat{v}^1\|_{W^{2,\infty}(\Omega)}\left\|\frac{\hat{X}}{\langle y\rangle}\right\|_{L^\infty(\Omega)} \\
&+ C\left|\frac{A^2}{\langle y\rangle}\right|_{0,\alpha}\left(\left\|\frac{\partial \hat{v}}{\partial y}\right\|_{L^\infty(\Omega)} + |r|\right) + C\left\|\frac{A^2}{\langle y\rangle}\right\|_{L^\infty(\Omega)}\left|\frac{\partial \hat{v}}{\partial y}\right|_{0,\alpha}\left\|\frac{\partial \hat{X}^2}{\partial y}\right\|^{\alpha}_{L^\infty(\Omega)}.
\end{aligned}$$

Then using (2.87), (2.91), (2.92), we infer that

$$\left|\frac{\partial}{\partial s}\left(\frac{A}{\langle y\rangle}\right)\right|_{0,\alpha} \le C|\frac{A}{\langle y\rangle}|_{0,\alpha} + C(1+T') \, |||\, (l,r,\omega) \,|||\, .$$

Therefore, using the fact that $A(t;t,y)=0$, we obtain with Gronwall's lemma that

$$\left|\frac{A}{\langle y\rangle}\right|_{0,\alpha} \leq CT' \;|||\; (l,r,\omega) \;|||\; . \tag{2.100}$$

We are in a position to estimate $|\hat\omega|_{0,\alpha}$. We have

$$\begin{aligned}
|\hat\omega|_{0,\alpha} &\leq \left|A(0;t,y)\hat\omega_0(\hat X^1(0;t,y))\right|_{0,\alpha} + \left|A^2(0;t,y)\left(\hat\omega_0(\hat X^1(0;t,y)) - \hat\omega_0(\hat X^2(0;t,y))\right)\right|_{0,\alpha}\\
&\leq \left|\frac{A}{\langle y\rangle}\right|_{0,\alpha} \|\langle y\rangle\hat\omega_0(\hat X^1)\|_{L^\infty(\Omega)} + \left\|\frac{A}{\langle y\rangle}\right\|_{L^\infty(\Omega)} |\langle y\rangle\hat\omega_0(\hat X^1)|_{0,\alpha}\\
&\quad + |A^2|_{0,\alpha}\|\hat\omega_0(\hat X^1)-\hat\omega_0(\hat X^2)\|_{L^\infty(\Omega)} + \|A^2\|_{L^\infty(\Omega)}|\hat\omega_0(\hat X^1)-\hat\omega_0(\hat X^2)|_{0,\alpha}\\
&\leq C(1+|\langle y\rangle\hat\omega_0(\hat X^1)|_{0,\alpha})T' \;|||\; (l,r,\omega) \;|||\; + C|\hat\omega_0(\hat X^1)-\hat\omega_0(\hat X^2)|_{0,\alpha}
\end{aligned}$$

where we used (2.19), (2.87), (2.92), and (2.100). It remains to estimate $|\hat\omega_0(\hat X^1)-\hat\omega_0(\hat X^2)|_{0,\alpha}$ and $|\langle y\rangle\hat\omega_0(\hat X^1)|_{0,\alpha}$. For the first one, we write

$$\begin{aligned}
|\hat\omega_0(\hat X^1)-\hat\omega_0(\hat X^2)|_{0,\alpha} &= \left|\int_0^1 \frac{\partial\hat\omega_0}{\partial y}(\hat X^2+\sigma\hat X)\hat X \mathrm{d}\sigma\right|_{0,\alpha}\\
&\leq C\left(\sup_{\sigma\in(0,1)} \|\langle y\rangle\frac{\partial\hat\omega_0}{\partial y}(\hat X^2+\sigma\hat X)\|_{L^\infty(\Omega)} \left|\frac{\hat X}{\langle y\rangle}\right|_{0,\alpha}\right.\\
&\qquad \left. + \sup_{\sigma\in(0,1)}\left|\langle y\rangle\frac{\partial\hat\omega_0}{\partial y}(\hat X^2+\sigma\hat X)\right|_{0,\alpha} \|\frac{\hat X}{\langle y\rangle}\|_{L^\infty(\Omega)}\right)\\
&\leq C\left(1+\sup_{\sigma\in(0,1)}\left|\langle y\rangle\frac{\partial\hat\omega_0}{\partial y}(\hat X^2+\sigma\hat X)\right|_{0,\alpha}\right) T' \;|||\; (l,r,\omega)|||
\end{aligned}$$

where we used (2.20), (2.21), (2.87), (2.99), and the fact that (using (2.51) for $\hat X^2$)

$$|\hat X^2+\sigma\hat X| \geq |y| + O(T')\langle y\rangle \geq \frac{1}{2}\langle y\rangle \text{ for } T' \text{ small enough, } |y|>1 \text{ and } \sigma\in(0,1).$$

We aim to prove that

$$\sup_{\substack{\sigma\in(0,1)\\ t\in[0,T']}} \left|\langle y\rangle\frac{\partial\hat\omega_0}{\partial y}(\hat X^2(y)+\sigma\hat X(y))\right|_{0,\alpha} < +\infty, \tag{2.101}$$

where we write $\hat X(y)$ for $\hat X(0;t,y)$, *etc.* We have with (2.20) that

$$\sup_{\substack{\sigma\in(0,1)\\ |y-y'|\geq 1\\ t\in[0,T']}} \frac{\left|\langle y\rangle\frac{\partial\hat\omega_0}{\partial y}(\hat X^2(y)+\sigma\hat X(y)) - \langle y'\rangle\frac{\partial\hat\omega_0}{\partial y}(\hat X^2(y')+\sigma\hat X(y'))\right|}{|y-y'|^\alpha} < +\infty.$$

On the other hand, for $\sigma \in (0,1)$, $|y-y'|<1$, and $t\in[0,T']$,

$$\begin{aligned}
&\frac{\left|\langle y\rangle \frac{\partial \hat\omega_0}{\partial y}(\hat X^2(y)+\sigma \hat X(y)) - \langle y'\rangle \frac{\partial \hat\omega_0}{\partial y}(\hat X^2(y')+\sigma \hat X(y'))\right|}{|y-y'|^\alpha}\\
&\le \left|\frac{\langle y\rangle - \langle y'\rangle}{|y-y'|^\alpha}\frac{\partial \hat\omega_0}{\partial y}(\hat X^2(y)+\sigma\hat X(y))\right| + \langle y'\rangle \frac{\left|\frac{\partial \hat\omega_0}{\partial y}(\hat X^2(y)+\sigma \hat X(y)) - \frac{\partial \hat\omega_0}{\partial y}(\hat X^2(y')+\sigma \hat X(y'))\right|}{|y-y'|^\alpha}\\
&\le C\left\|\frac{\partial\hat\omega_0}{\partial y}\right\|_{L^\infty(\Omega)} + C\left(\left\|\frac{\partial \hat X^2}{\partial y}\right\|_{L^\infty(\Omega)} + \left\|\frac{\partial \hat X}{\partial y}\right\|_{L^\infty(\Omega)}\right)|y-y'|^{1-\alpha}
\end{aligned}$$

where we used (2.21) and the mean value theorem for the last term. This completes the proof of (2.101). We infer that

$$|\hat\omega_0(\hat X^1) - \hat\omega_0(\hat X^2)|_{0,\alpha} \le CT' \,|||\, (l,r,\omega)\,|||\,.$$

We can prove in a very similar way that $|\langle y\rangle \hat\omega_0(\hat X^1)|_{0,\alpha} < +\infty$. We conclude that

$$|\hat\omega(t)|_{0,\alpha} \le CT' \,|||\, (l,r,\omega)|||, \quad t\in[0,T']. \tag{2.102}$$

Step 9. Let us estimate $\|\hat\omega\|_{L^p_{p(\delta+2)}(\Omega)}$. We write

$$\|\hat\omega\|^p_{L^p_{p(\delta+2)}(\Omega)} \le C\left(\int_\Omega |A\hat\omega_0(\hat X^1)|^p\langle y\rangle^{p(\delta+2)}\mathrm{d}y + \int_\Omega |A^2(\hat\omega_0(\hat X^1)-\hat\omega_0(\hat X^2))|^p \langle y\rangle^{p(\delta+2)}\mathrm{d}y\right) =: C(I_1+I_2),$$

where we have written A^1 for $A^1(0;t,y)$, $\hat X^1$ for $\hat X^1(0;t,y)$, *etc.*

Then, using the fact that $\omega_0\in M^0_{p,\delta+3}$ and (2.92), we obtain that

$$\begin{aligned}
I_1 &\le (CT' \,|||\, (l,r,\omega)|||)^p \int_\Omega |\hat\omega_0(\hat X^1(0;t,y)|^p\langle y\rangle^{p(\delta+3)}\mathrm{d}y\\
&\le (CT' \,|||\, (l,r,\omega)|||)^p \int_{\mathbb{R}^3} |\hat\omega_0(x)|^p\langle \hat X^1(t;0,x)\rangle^{p(\delta+3)}\mathrm{d}x\\
&\le (CT' \,|||\, (l,r,\omega)|||)^p \|\hat\omega_0\|^p_{M^p_{0,\delta+3}(\mathbb{R}^3)}.
\end{aligned}$$

Therefore, increasing the value of C if needed, we obtain

$$I_1 \le (CT' \,|||\, (l,r,\omega)|||)^p.$$

For I_2, we infer from (2.19) with $\kappa > 3+\delta+\frac{3}{p}$ that

$$\begin{aligned}
I_2 &\le C\int_\Omega |\hat\omega_0(\hat X^1)-\hat\omega_0(\hat X^2)|^p\langle y\rangle^{p(\delta+2)}\mathrm{d}y\\
&\le C\int_\Omega \left(\frac{|\hat X|}{[1+\min(|\hat X^1|,|\hat X^2|)]^\kappa}\right)^p \langle y\rangle^{p(\delta+2)}\mathrm{d}y\\
&\le (CT' \,|||\, (l,r,\omega)|||)^p \int_{\mathbb{R}^3}\langle y\rangle^{p(\delta+3-\kappa)}\mathrm{d}y\\
&\le (CT' \,|||\, (l,r,\omega)|||)^p.
\end{aligned}$$

We conclude that

$$\|\hat\omega(t)\|_{L^p_{p(\delta+2)}(\Omega)} \le CT' \,|||\, (l,r,\omega)|||, \quad t\in[0,T']. \tag{2.103}$$

Step 10. Let us turn to the estimates of v. Since $\alpha \leq 1 - \frac{3}{p}$, we have $M^p_{1,\delta+1} \subset C^{0,\alpha}(\overline{\Omega})$, and hence

$$\begin{aligned}\|\nabla v\|_{C^{0,\alpha}(\overline{\Omega})} &\leq C(\|\omega\|_{C^{0,\alpha}(\overline{\Omega})} + \|v\|_{C^{0,\alpha}(\overline{\Omega})} + |l| + |r|) \\ &\leq C(\|\omega\|_{C^{0,\alpha}(\overline{\Omega})} + \|v\|_{M^p_{1,\delta+1}} + |l| + |r|) \\ &\leq C(\|\omega\|_{C^{0,\alpha}(\overline{\Omega})} + \|\omega\|_{M^p_{0,\delta+2}} + |l| + |r|) \\ &\leq CT' \left|\!\left|\!\left| (l,r,\omega) \right|\!\right|\!\right| .\end{aligned}$$

It follows that

$$\begin{aligned}\|\nabla v\|_{L^4(\Omega)} &\leq \|\nabla v\|^{\frac{p}{4}}_{L^p(\Omega)} \|\nabla v\|^{1-\frac{p}{4}}_{L^\infty(\Omega)} \\ &\leq C\|v\|^{\frac{p}{4}}_{M^p_{1,\delta+1}} \|\nabla v\|^{1-\frac{p}{4}}_{C^{0,\alpha}(\overline{\Omega})} \\ &\leq C(\|\omega\|_{C^{0,\alpha}(\overline{\Omega})} + \|\omega\|_{M^p_{0,\delta+2}} + |l| + |r|) \\ &\leq CT' \left|\!\left|\!\left| (l,r,\omega) \right|\!\right|\!\right| .\end{aligned}$$

Step 11. We now turn our attention to $\hat{l} := \hat{l}^1 - \hat{l}^2$ and $\hat{r} := \hat{r}^1 - \hat{r}^2$, where for $k = 1, 2$

$$\begin{pmatrix} \hat{l}^k(t) \\ \hat{r}^k(t) \end{pmatrix} := \begin{pmatrix} l_0 \\ r_0 \end{pmatrix} + \mathcal{J}^{-1} \int_0^t \left\{ \begin{pmatrix} \left(\int_\Omega \nabla\mu^k(\tau,y) \cdot \nabla\phi_i(y)\,\mathrm{d}y \right)_{i=1,2,3} \\ \left(\int_\Omega \nabla\mu^k(\tau,y) \cdot \nabla\varphi_i(y)\,\mathrm{d}y \right)_{i=1,2,3} \end{pmatrix} - \begin{pmatrix} m_0 r^k \times l^k \\ r^k \times J_0 r^k \end{pmatrix} \right\} \mathrm{d}\tau, \tag{2.104}$$

and the function $\mu^k : [0,T'] \times \Omega \to \mathbb{R}$ is defined as the solution to the system

$$\begin{cases} -\Delta\mu^k = \operatorname{tr}\Big(\nabla v^k \cdot \nabla v^k\Big), & \text{in } (0,T)\times\Omega, \\ \dfrac{\partial\mu^k}{\partial n} = -\displaystyle\sum_{1\leq j\leq m} \dot{w}_j(t)\chi_j(y) - \Big(\Big((v^k - l^k - r^k \times y)\cdot\nabla\Big)v^k + r^k \times v^k\Big)\cdot n, & \text{on } (0,T)\times\partial\Omega, \\ \lim\limits_{|y|\to\infty} \nabla\mu^k(t,y) = 0 & \text{in } (0,T). \end{cases}$$

Then $\mu := \mu^1 - \mu^2$ satisfies the system

$$\begin{cases} -\Delta\mu = \operatorname{tr}\Big(\nabla(v^1 + v^2)\cdot\nabla v\Big), & \text{in } (0,T)\times\Omega, \\ \dfrac{\partial\mu}{\partial n} = -\Big(\Big((v - l - r\times y)\cdot\nabla\Big)v^1 + r\times v^1\Big)\cdot n & \\ \qquad\quad - \Big(\Big((v^2 - l^2 - r^2\times y)\cdot\nabla\Big)v + r^2\times v\Big)\cdot n & \text{on } (0,T)\times\partial\Omega, \\ \lim\limits_{|y|\to\infty} \nabla\mu(t,y) = 0 & \text{in } (0,T). \end{cases}$$

It follows that

$$\begin{aligned}\|\nabla\mu\|_{L^2(\Omega)} &\leq C\left(\|\nabla v^1\|_{L^4(\Omega)} + \|\nabla v^2\|_{L^4(\Omega)}\right)\|\nabla v\|_{L^4(\Omega)} + C(\|v\|_{C^{0,\alpha}(\overline{\Omega})} + \|\nabla v\|_{C^{0,\alpha}(\overline{\Omega})} + |l| + |r|) \\ &\leq CT' \left|\!\left|\!\left| (l,r,\omega) \right|\!\right|\!\right| .\end{aligned} \tag{2.105}$$

We infer from (2.104) that $(\hat{l},\hat{r})$ satisfies

$$\begin{pmatrix} \hat{l}(t) \\ \hat{r}(t) \end{pmatrix} = \mathcal{J}^{-1} \int_0^t \left\{ \begin{pmatrix} \left(\int_\Omega \nabla\mu(\tau,y) \cdot \nabla\phi_i(y)\,\mathrm{d}y \right)_{i=1,2,3} \\ \left(\int_\Omega \nabla\mu(\tau,y) \cdot \nabla\varphi_i(y)\,\mathrm{d}y \right)_{i=1,2,3} \end{pmatrix} - \begin{pmatrix} m_0(r\times l^1 + r^2\times l) \\ r\times J_0 r^1 + r^2 \times J_0 r \end{pmatrix} \right\} \mathrm{d}\tau,$$

and hence, with (2.105),

$$|\hat{l}(t)| + |\hat{r}(t)| \leq CT' \, ||| \, (l, r, \omega) |||, \quad t \in [0, T']. \tag{2.106}$$

Gathering together (2.95), (2.102), (2.103) and (2.106), we obtain

$$|||(\hat{l}, \hat{r}, \hat{\omega})||| \leq CT' \, ||| \, (l, r, \omega) |||, \quad t \in [0, T']. \tag{2.107}$$

Thus, for $T' < 1/C$, we have that

$$|||\mathcal{T} \, (l^1, r^1, \omega^1) - \mathcal{T} \, (l^2, r^2, \omega^2)||| \leq k \, ||| \, (l^1, r^1, \omega^1) - (l^2, r^2, \omega^2)|||$$

for some constant $k \in (0, 1)$, *i.e.* $\mathcal{T}$ is a contraction in $\mathcal{C}$. The Proof of Theorem 2.3 is complete. □

2.3. Existence of a solution of system (1.19)–(1.25)

Let us now check that the fixed-point (l, r, ω) given in Theorem 2.3 yields a solution of (1.19)–(1.25). Let v and μ be given by (2.32)–(2.36) and (2.45), respectively. Since $(l, r, \omega) \in \mathcal{C} \subset C([0, T'], \mathcal{F})$, then (2.23) holds, $\nabla \mu \in C([0, T'], L^2(\Omega))$ and hence, with (2.46), (2.26) holds as well. Let us set

$$\mathbf{q} := \mu - \sum_{i=1}^{3} \dot{l}_i \phi_i - \sum_{i=1}^{3} \dot{r}_i \varphi_i. \tag{2.108}$$

Then (2.24) holds and we have for a.e. $t \in (0, T')$, $q(t, .) \in C^{2,\alpha}_{\text{loc}}(\overline{\Omega})$ and $\lim_{|y| \to +\infty} \nabla \mathbf{q}(t, y) = 0$.

Proposition 2.4. *Let T' be as in Theorem 2.3 and let (l, r, ω) denote the corresponding fixed-point of $\mathcal{T}$ in $\mathcal{C}$. Then $(v, \mathbf{q}, l, r)$ is a solution of (1.19)–(1.25) in $(0, T')$.*

Proof of Proposition 2.4. Let

$$f := ((v - l - r \times y) \cdot \nabla) \, v + r \times v. \tag{2.109}$$

Then we have that $f(t, .) \in C^{1,\alpha}_{\text{loc}}(\overline{\Omega})$ for all $t \in [0, T']$. On the other hand, since $v \in C([0, T'], M^p_{2,\delta+1})$, we have that $|v| \leq C \langle y \rangle^{-1-\delta}$ and $|\nabla v| \leq C \langle y \rangle^{-2-\delta}$, so that

$$|f(t, y)| \leq C \langle y \rangle^{-1-\delta}.$$

The divergence of f is given by

$$\begin{aligned}
\text{div } f &= \text{div} \Big(((v - l - r \times y) \cdot \nabla) v + r \times v \Big) \\
&= \partial_i \Big(v_j \partial_j v_i \Big) - \partial_i \Big(l_j \partial_j v_i \Big) - \partial_i \Big((r \times y)_j \partial_j v_i \Big) + \text{div} \Big(r \times v \Big) \\
&= (\partial_i v_j) \, (\partial_j v_i) - (r \times \partial_i y)_j \partial_j v_i + \text{div} \Big(r \times v \Big) \\
&= \text{tr} \Big(\nabla v \cdot \nabla v \Big) - \partial_j \Big((r \times v_i \partial_i y)_j \Big) + \text{div} \Big(r \times v \Big) \\
&= \text{tr} \Big(\nabla v \cdot \nabla v \Big) - \text{div} \Big(r \times v \Big) + \text{div} \Big(r \times v \Big) \\
&= \text{tr} \Big(\nabla v \cdot \nabla v \Big) \\
&= -\Delta \mu,
\end{aligned}$$

where we used Einstein's convention of repeated indices and the fact that $\text{div}\,(v) = 0$. Therefore, using (2.3) and (2.108), we obtain

$$\text{div}\,(f + \nabla \mathbf{q}) = 0. \tag{2.110}$$

Now we turn our attention to the curl of f. Define $\tilde{v} := v - l - r \times y$. Then

$$\operatorname{curl} \tilde{v} = \omega - 2r. \tag{2.111}$$

We shall use the following identities (see *e.g.* [10])

$$\operatorname{curl}((v \cdot \nabla)v) = (v \cdot \nabla)\operatorname{curl}(v) - (\operatorname{curl}(v) \cdot \nabla)v + \operatorname{div}(v)\operatorname{curl}(v), \tag{2.112}$$
$$\operatorname{curl}(r \times v) = \operatorname{div}(v)r - (r \cdot \nabla)v. \tag{2.113}$$

Applying the operator curl to f and using (2.112)–(2.113), we obtain

$$\begin{aligned}
\operatorname{curl} f &= \operatorname{curl}((\tilde{v} \cdot \nabla)\tilde{v}) + \operatorname{curl}((\tilde{v} \cdot \nabla)(l + r \times y)) + \operatorname{curl}(r \times v)\\
&= \operatorname{curl}\,((\tilde{v} \cdot \nabla)\tilde{v}) + \operatorname{curl}(r \times \tilde{v}) + \operatorname{curl}(r \times v)\\
&= \operatorname{curl}\,((\tilde{v} \cdot \nabla)\tilde{v}) + \operatorname{div}(\tilde{v})r - (r \cdot \nabla)\tilde{v} + \operatorname{div}(v)r - (r \cdot \nabla)v\\
&= \operatorname{curl}\,((\tilde{v} \cdot \nabla)\tilde{v}) - (r \cdot \nabla)\tilde{v} - (r \cdot \nabla)v\\
&= (\tilde{v} \cdot \nabla)\operatorname{curl}(\tilde{v}) - (\operatorname{curl}(\tilde{v}) \cdot \nabla)\tilde{v} - (r \cdot \nabla)\tilde{v} - (r \cdot \nabla)v\\
&= (\tilde{v} \cdot \nabla)(\omega - 2r) - ((\omega - 2r) \cdot \nabla)\tilde{v} - (r \cdot \nabla)\tilde{v} - (r \cdot \nabla)v\\
&= (\tilde{v} \cdot \nabla)\omega - \omega \cdot \nabla\tilde{v}.
\end{aligned}$$

Using (2.44), we see that ω satisfies

$$\frac{\partial \omega}{\partial t} + (\tilde{v} \cdot \nabla)\omega - (\omega \cdot \nabla)\tilde{v} = 0, \quad t \in (0, T'). \tag{2.114}$$

It follows that

$$\operatorname{curl}\, f + \frac{\partial \omega}{\partial t} = 0. \tag{2.115}$$

On other hand, using (2.45) we obtain that

$$f \cdot n = -\frac{\partial \mu}{\partial n} - \sum_{1 \le j \le m} \dot{w}_j(t)\chi_j(y) \tag{2.116}$$
$$= -\frac{\partial q}{\partial n} - (\dot{l} + \dot{r} \times y) \cdot n - \sum_{1 \le j \le m} \dot{w}_j(t)\chi_j(y). \tag{2.117}$$

Introduce now the function

$$F(t, y) := v(t, y) - v_0(y) + \int_0^t (f(s, y) + \nabla \mathbf{q}(s, y))\mathrm{d}s. \tag{2.118}$$

Then $F(t, .) \in C^{1,\alpha}_{\mathrm{loc}}(\overline{\Omega})$ for all $t \in [0, T']$. On the other hand, it follows from (2.110), (2.115) and (2.117) that

$$\begin{aligned}
\operatorname{div}\, F &= 0 \quad \text{in } \Omega,\\
\operatorname{curl}\, F &= 0 \quad \text{in } \Omega,\\
F \cdot n &= 0 \quad \text{on } \partial\Omega,\\
\lim_{|y| \to +\infty} F(t, y) &= 0.
\end{aligned}$$

Then we infer from ([10], Lem. 2.7) that $F \equiv 0$. Taking into account the definition of F, this implies that $v \in C^1([0, T']; C^{1,\alpha}_{\mathrm{loc}}(\overline{\Omega}))$ with (1.19) satisfied together with $v(0, .) = v_0$. Using (2.32)–(2.36), we see that the equations (1.20)–(1.22) are satisfied. Finally, equations (1.23)–(1.25) hold by (2.46) and (2.108). □

2.4. Proof of the estimate (2.27)

The potential solution $(\bar{l}, \bar{r}, \bar{v})$ of (1.19)–(1.25) associated with $l_0, r_0, \{w_j\}_{1\le j\le m}$, and $\overline{\omega}_0 = 0$ is obtained in the following way. Since $\overline{\omega}_0 = 0$, $\pi(\overline{\omega}_0) = 0$ in $\mathbb{R}^3$, and hence with (2.44) the vorticity $\overline{\omega}$ is null. Then we infer from (2.33)–(2.36) that $\overline{\eta} = 0$ and from (2.32) that

$$\overline{v} = \sum_{i=1}^{3} \bar{l}_i \nabla \phi_i + \sum_{i=1}^{3} \overline{r}_i \nabla \varphi_i + \sum_{1\le j\le m} w_j(t) \nabla \psi_j. \tag{2.119}$$

It follows from ([11], Prop. 2.3) that $(\bar{l}, \overline{r})$ satisfies the ODE (2.11), whose solution is unique.

Consider a solution (l, r, ω) associated with the same $l_0, r_0, \{w_j\}_{1\le j\le m}$ as for $(\bar{l}, \overline{r}, \overline{\omega})$, but with an initial vorticity ω_0 not necessarily null. It follows from (2.28)–(2.30) that for all $t \in [0, T']$

$$\|\omega(t)\|_{C^{1,\alpha}(\overline{\Omega})} + \|\omega(t)\|_{M^p_{1,\delta+2}} \le P = \mathrm{e}^e \cdot \left(C_6 \|\pi(\omega_0)\|_{C^{1,\alpha}(\mathbb{R}^3)} + C_7 \|\pi(\omega_0)\|_{M^p_{1,\delta+2}(\mathbb{R}^3)} \right).$$

Now, from (2.32) and (2.33)–(2.36), we infer that for all $t \in [0, T']$

$$\|v(t) - \overline{v}(t)\|_{C^{2,\alpha}(\overline{\Omega})} + \|\nabla v(t) - \nabla \overline{v}(t)\|_{L^4(\Omega)} \le C \left(P + |(l(t) - \bar{l}(t), r(t) - \overline{r}(t))| \right).$$

Combined with (2.45), this yields

$$\|\nabla \mu(t) - \nabla \overline{\mu}(t)\|_{L^2(\Omega)} \le C \left(P + |(l(t) - \bar{l}(t), r(t) - \overline{r}(t))| \right).$$

Using (2.46), we obtain

$$|(\dot{l}(t) - \dot{\bar{l}}(t), \dot{r}(t) - \dot{\overline{r}}(t))| \le C \left(P + |(l(t) - \bar{l}(t), r(t) - \overline{r}(t))| \right).$$

Then (2.27) follows by using Gronwall's lemma. The proof of Theorem 2.1 is complete. □

2.5. Uniqueness and continuity with respect to the control

The following result is concerned with the uniqueness of the solution $(l, r, v, \mathbf{q})$ of (1.19)–(1.25), when the vorticity $\omega = \operatorname{curl} v$ satisfies

$$\omega(t, y) = G^{-1}(0; t, y) \pi(\omega_0)(\hat{X}(0; t, y)) \tag{2.120}$$

where $G(s; t, y) = (\partial \hat{X}/\partial y)(s; t, y)$ and the flow $\hat{X}$ is defined on $[0, T']^2 \times \mathbb{R}^3$ by

$$\begin{cases} \dfrac{\partial}{\partial s} \hat{X}(s; t, y) = \pi(v)(s, \hat{X}(s; t, y)) - l(s) - r(s) \times \hat{X}(s; t, y), \\ \hat{X}(t; t, y) \quad = y. \end{cases} \tag{2.121}$$

Proposition 2.5. *Let l_0, r_0, ω_0, v_0 and T' be as in Theorem 2.1. Then the solution $(v, \mathbf{q}, l, r, \omega)$ of (1.19)–(1.25) and (2.120)–(2.121) is unique in the class (2.23)–(2.26) ($\mathbf{q}$ being unique up to the addition of an arbitrary function of time). On the other hand, for any given initial data (l_0, r_0, ω_0) as above and any $R > 0$, the map $w \in \mathcal{B} := \{w \in C^1([0, T'], \mathbb{R}^m);\ \|w\|_{C^1([0,T])} \le R\} \mapsto (l, r) \in C^0([0, T'], \mathbb{R}^6)$ is continuous.*

Proof. Let $(v, \mathbf{q}, l, r)$ be a solution of (1.19)–(1.25) in the class (2.23)–(2.26). Then we can expand v in the form (2.32) with η as in (2.33)–(2.36). Then it is well-known that the vorticity $\omega = \operatorname{curl} v$ satisfies the equation (2.114) with $\tilde{v} = v - l - r \times y$, and that it is given by (2.120) "away" from the rigid body. We assume that it is given by (2.120) everywhere, even on $\partial\Omega$. Roughly speaking, this amounts to specifying the tangent components of the vorticity on the inflow section. Let us show that the pair (l, r) satisfies (2.46). Let μ be as in (2.108) and let f be as in (2.109). Then by (1.19) and the computations above, we have that

$$-\Delta\mu = -\Delta q = \text{ div } f = \text{tr } (\nabla v \cdot \nabla v),$$

and

$$\begin{aligned}
\frac{\partial \mu}{\partial n} &= \frac{\partial q}{\partial n} + \sum_{i=1}^{3} \dot{l}_i n_i + \sum_{i=1}^{3} \dot{r}_i (y \times n)_i \\
&= -\left(\frac{\partial v}{\partial t} + f\right) \cdot n + \dot{l} \cdot n + \dot{r} \cdot (y \times n) \\
&= -\frac{\partial}{\partial t}\left([l + r \times y] \cdot n + \sum_{1 \le j \le m} w_j(t)\chi_j(y)\right) - f \cdot n + \dot{l} \cdot n + \dot{r} \cdot (y \times n) \\
&= -\sum_{1 \le j \le m} \dot{w}_j(t)\chi_j(y) - \big((v - l - r \times y) \cdot \nabla v + r \times v\big) \cdot n.
\end{aligned}$$

Thus μ solves (2.45). Integrating in (1.23)–(1.24) and using (2.108), we arrive to (2.46) with $(\hat{l}, \hat{r}) = (l, r)$. Thus (l, r, ω) is a fixed-point of $\mathcal{T}$. As there is (for T' small enough) *only one* fixed-point of $\mathcal{T}$ by the contraction mapping theorem, we infer that (l, r, ω) is unique. Then η is unique by (2.33)–(2.36), and v is unique by (2.32). Finally, ∇q is unique by (1.19) and q is unique (up to the addition of an arbitrary function of time).

Let us proceed with the continuity with respect to the control. Assume given some initial data (l_0, r_0, ω_0) as above and pick any number $R > 0$. Let

$$\mathcal{B} := \{w \in C^1([0, T], \mathbb{R}^m); \|w\|_{C^1([0,T])} \le R\}.$$

Assume that the constants C_8 and C_{11} are suitably chosen to be convenient for all $w \in \mathcal{B}$, and pick a time $T' > 0$ convenient for all $w \in \mathcal{B}$. Then
(i) for T' small enough, we have for $w \in \mathcal{B}$ and $(l^i, r^i, \omega^i) \in \mathcal{C}$, $i = 1, 2$,

$$|||\mathcal{T}\ (l^1, r^1, \omega^1) - \mathcal{T}\ (l^2, r^2, \omega^2)||| \le k\ |||\ (l^1, r^1, \omega^1) - (l^2, r^2, \omega^2)|||,$$

for some constant $k \in (0, 1)$;
(ii) for given $(l, r, \omega) \in \mathcal{C}$, the map $w \in \mathcal{B} \mapsto (\hat{l}, \hat{r}, \hat{\omega}) \in \mathcal{C}$ is continuous.

Indeed, the map $w \in \mathcal{B} \mapsto v \in C([0, T'], C^{2,\alpha}(\overline{\Omega}) \cap M^p_{2,\delta+1})$ is clearly continuous (using (2.32) and (2.33)–(2.36)), and hence the map $w \in \mathcal{B} \mapsto (l, r) \in C^1([0, T'], \mathbb{R}^6)$ is continuous (by (2.45)–(2.46)). Finally, using the assumption $\omega_0 \in M^p_{0,\delta+3}$, (2.28), (2.114), Aubin–Lions' lemma and the continuity of v, one can see (as *e.g.* in [5]) that the map $w \in \mathcal{B} \mapsto \omega \in C([0, T'], C^{0,\alpha}(\overline{\Omega}) \cap L^p_{p(\delta+2)}(\Omega))$ is continuous.

It follows again from the contraction mapping theorem (for a map depending on a parameter) that the map which associates with $w \in \mathcal{B}$ the fixed-point $(l, r, \omega) \in \mathcal{C}$ is continuous. □

3. Proof of the main result

We are now in a position to prove the main result in this paper. Let T_0, P, N, K and R be some given positive numbers. Then by Theorem 2.1, there exists a time $T = T(T_0, P, N, K, R) \in (0, T_0]$ such that system (1.19)–(1.25) has a solution $(v, \mathbf{q}, l, r)$ for $t \in [0, T]$, with $(l, r, \omega) \in C([0, T], \mathcal{F})$, provided that $|(l_0, r_0)| \le 1$, $\|w\|_{C^1([0,T_0])} \le R$ and ω_0 satisfies (2.19)–(2.21) and

$$\|\omega_0\|_{C^{1,\alpha}(\overline{\Omega})} + \|\omega_0\|_{M^p_{1,\delta+2}} \le P, \quad \text{div } \omega_0 = 0, \qquad \int_{\partial\Omega} \omega_0 \cdot n \, \mathrm{d}\sigma = 0.$$

Let Π be a (continuous and linear) extension operator from $C^1([0,T])$ to $C^1([0,T_0])$ and pick $\delta := R/\|\Pi\|$. Then $\|\Pi(w)\|_{C^1([0,T_0])} \le R$ if $\|w\|_{C^1([0,T])} \le \delta$. In particular, using assumption (H) for the time T, we have that $\|\Pi(w)\|_{C^1([0,T_0])} \le R$ if $w = W(h_0, \boldsymbol{q_0}, l_0, r_0, h_T, \boldsymbol{q_T}, l_T, r_T)$ for $|(h_0, \boldsymbol{q_0}, l_0, r_0, h_T, \boldsymbol{q_T}, l_T, r_T)| < \eta$ with $\eta > 0$ small enough. Then system (1.19)−(1.25) has a solution defined for $t \in [0,T]$ corresponding to (l_0, r_0, ω_0, w) as above, and also a potential solution corresponding to the same data (l_0, r_0, w) and to $\bar\omega_0 \equiv 0$.

Let ω_0 be as in the statement of Theorem 1.1, and write $a_0 = (h_0, \boldsymbol{q_0})$, $b_0 = (l_0, r_0)$, $a_T = (h_T, \boldsymbol{q_T})$, and $b_T = (l_T, r_T)$. Let $a(t) := (h(t), \boldsymbol{q}(t))$ and $b(t) = (l(t), r(t))$. The proof is done in two steps. In the first step, we prove the result for $||\omega_0||_{C^{1,\alpha}(\overline{\Omega})}$, $||\omega_0||_{M^p_{1,\delta+2}}$, $|l_0|$, $|r_0|$, $|l_T|$ and $|r_T|$ small enough, and in the second step, we remove this assumption by performing a scaling in time.

Step 1. Let the map W be as in the assumption (H) for the time T. We may pick a number $\eta_1 \in (0,1)$ such that $w = W(a_0, b_0, a_T, b_T)$ is defined for $|(a_0,b_0)| \le \eta_1$ and $|(a_T, b_T)| \le \eta_1$, with

$$||w||_{C^1([0,T])} \le \delta.$$

Pick any initial state $(a_0, b_0) = (h_0, \boldsymbol{q}_0, l_0, r_0)$ with $|(a_0,b_0)| \le \eta_1$. For any given (a_T, b_T, v_0) with $|(a_T,b_T)| \le \eta_1$, we denote by $(h, \boldsymbol{q}, l, r, v, \mathbf{q})$ the solution of (1.19)–(1.25) and (1.30) corresponding to the velocity v_0 and to the control $w = W(a_0, b_0, a_T, b_T)$, and by $(\overline{h}, \overline{\boldsymbol{q}}, \overline{l}, \overline{r}, \overline{v}, \overline{\mathbf{q}})$ the solution corresponding to (a_0, b_0) together with the velocity $\overline{v}_0$ which solves

$$\begin{aligned} \operatorname{curl} \overline{v}_0 &= 0, && \text{in } \Omega,\\ \operatorname{div} \overline{v}_0 &= 0, && \text{in } \Omega,\\ \overline{v}_0 \cdot n &= (l_0 + r_0 \times y)\cdot n, && \text{on } \partial\Omega,\\ \lim_{|y|\to\infty} \overline{v}_0(y) &= 0, \end{aligned}$$

and to the (same) control w. From (2.27) we infer that there exists some constant $C_1 > 0$ such that

$$||(l - \overline{l}, r - \overline{r})||_{L^\infty(0,T)} \le C_1 \Big(\|\omega_0\|_{C^{1,\alpha}(\overline{\Omega})} + \|\omega_0\|_{M^p_{1,\delta+2}} \Big), \tag{3.1}$$

whenever

$$|(l_0, r_0)| \le 1, \quad ||\omega_0||_{C^{1,\alpha}(\overline{\Omega})} + ||\omega_0||_{M^p_{1,\delta+2}} \le P, \quad \text{and} \quad ||w||_{C^1([0,T])} \le \delta. \tag{3.2}$$

Combined to the equations

$$\begin{cases} h'(t) = (1 - |\boldsymbol{q}|^2) l + 2\sqrt{1-|\boldsymbol{q}|^2}\, \boldsymbol{q} \times l + (l \cdot \boldsymbol{q})\boldsymbol{q} - \boldsymbol{q} \times l \times \boldsymbol{q},\\ \boldsymbol{q}'(t) = \frac12 (\sqrt{1-|\boldsymbol{q}|^2}\, r + \boldsymbol{q} \times r),\\ \overline{h}'(t) = (1 - |\overline{\boldsymbol{q}}|^2) \overline{l} + 2\sqrt{1-|\overline{\boldsymbol{q}}|^2}\, \overline{\boldsymbol{q}} \times \overline{l} + (\overline{l} \cdot \overline{\boldsymbol{q}})\overline{\boldsymbol{q}} - \overline{\boldsymbol{q}} \times \overline{l} \times \overline{\boldsymbol{q}},\\ \overline{\boldsymbol{q}}'(t) = \frac12 (\sqrt{1-|\overline{\boldsymbol{q}}|^2}\, \overline{r} + \overline{\boldsymbol{q}} \times \overline{r}),\\ h(0) = \overline{h}(0) = h_0, \quad \boldsymbol{q}(0) = \overline{\boldsymbol{q}}(0) = \boldsymbol{q}_0, \end{cases}$$

this gives for some constant $C_2 > 0$

$$||(h - \overline{h}, \boldsymbol{q} - \overline{\boldsymbol{q}})||_{L^\infty(0,T)} \le C_2 \left(||\omega_0||_{C^{1,\alpha}(\overline{\Omega})} + ||\omega_0||_{M^p_{1,\delta+2}} \right), \tag{3.3}$$

provided that (3.2) holds. Let $f : \overline{B} = \{x \in \mathbb{R}^{12};\ |x| \le 1\} \to \mathbb{R}^{12}$ be defined by

$$f(x_T) = \frac{1}{\eta_1}(a(T), b(T)) = \frac{1}{\eta_1}(h(T), \boldsymbol{q}(T), l(T), r(T))$$

where $(a_T, b_T) =: \eta_1 x_T$.

We notice that f is continuous, by virtue of Proposition 2.5 and (1.30). Pick any $\varepsilon \in (0,1)$. From (3.1) and (3.3), we deduce that for

$$||\omega_0||_{C^{1,\alpha}(\overline{\Omega})} + ||\omega_0||_{M^p_{1,\delta+2}} < \nu \tag{3.4}$$

with $\nu > 0$ small enough, we have that

$$|f(x_T) - x_T| < \varepsilon, \qquad \text{for } |x_T| \leq 1.$$

We need the following topological result ([5], Lem. 4.1).

Lemma 3.1. *Let $B = \{x \in \mathbb{R}^n;\ |x| < 1\}$ and $S = \partial B$. Let $f : \overline{B} \to \mathbb{R}^n$ be a continuous map such that for some constant $\varepsilon \in (0,1)$*

$$|f(x) - x| \leq \varepsilon \qquad \forall x \in S. \tag{3.5}$$

Then

$$(1-\varepsilon)B \subset f(\overline{B}). \tag{3.6}$$

Thus, we infer from Lemma 3.1 that if $(a_0, b_0, a_T, b_T) \in \mathbb{R}^{24}$ is such that

$$|(a_0, b_0)| < \eta_1, \quad |(a_T, b_T)| < \eta_2 := \eta_1(1-\varepsilon),$$

and (3.4) is satisfied, then there exists a control $w = W(a_0, b_0, \tilde{a}_T, \tilde{b}_T)$ for which the solution of (1.19)–(1.25) and (1.30) satisfies $(h(T), q(T), l(T), r(T)) = (a(T), b(T)) = (a_T, b_T)$.

Step 2. To drop the assumptions $|b_0| < \eta_1$, $|b_T| < \eta_2$, and (3.4) (corresponding to a given time $T \in (0, T_0]$), we use a scaling in time introduced in [1] for the control of Euler equations. Let (a_0, b_0), (a_T, b_T), and v_0 be given data with

$$|a_0| < \eta_2 \ \text{ and } \ |a_T| < \eta_2.$$

We set $b_0^\lambda := \lambda b_0$, $b_T^\lambda := \lambda b_T$, and $v_0^\lambda := \lambda v_0$. Then for $\lambda > 0$ small enough, we have that

$$|(a_0, b_0^\lambda)| < \eta_2, \quad |(a_T, b_T^\lambda)| < \eta_2,$$

and $\omega_0^\lambda := \operatorname{curl} v_0^\lambda$ satisfies

$$||\omega_0^\lambda||_{C^{1,\alpha}(\overline{\Omega})} + ||\omega_0^\lambda||_{M^p_{1,\delta+2}} < \nu.$$

By Step 1, there exists some trajectory (a^λ, b^λ) for the underwater vehicle connecting (a_0, b_0^λ) at $t=0$ to (a_T, b_T^λ) at $t = T$, with corresponding fluid velocity v^λ, pressure $\mathbf{q}^\lambda$, and control w^λ. Let us set

$$\begin{aligned} a(t) &:= a^\lambda(\lambda^{-1}t), \\ b(t) &:= \lambda^{-1} b^\lambda(\lambda^{-1}t), \\ v(t,y) &:= \lambda^{-1} v^\lambda(\lambda^{-1}t, y), \\ \mathbf{q}(t,y) &:= \lambda^{-2}\mathbf{q}^\lambda(\lambda^{-1}t, y), \\ w(t) &:= \lambda^{-1} w^\lambda(\lambda^{-1}t), \end{aligned}$$

for $y \in \Omega$ and $0 \leq t \leq T_\lambda := \lambda T \in (0, T_0]$. Then (a,b) is a trajectory for the underwater vehicle connecting (a_0, b_0) at $t = 0$ to (a_T, b_T) at $t = T_\lambda$ and corresponding to the initial fluid velocity v_0. □

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