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CONTROLLABLE LEAD TIME, SERVICE LEVEL CONSTRAINT, AND TRANSPORTATION DISCOUNTS IN A CONTINUOUS REVIEW INVENTORY MODEL

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Abstract. This paper studies two models based on the distribution of lead time demand. The first model assumes a normally distributed lead time demand and the second assumes that there is no specific distribution for lead time demand, but it is with known mean and standard deviation. The continuous-review inventory model is used for both cases. Transportation cost is dependent on the ordered quantity *i.e.*, how much quantity buyer orders for delivery, based on that, a transportation discount is used to reduce the total cost. Service level constraint is included in this model to avoid backorder cost. Two efficient lemmas are established to obtain the optimum solution of the model. The expected value of additional information (EVAI) is calculated to show the excess amount needed for the distribution free case. Some numerical examples and sensitivity analysis are given to illustrate the model.

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1. Introduction

Lead time denotes the time gap between the placing of an order and collecting the same ordered quantity. Most of research articles related to lead time stated that lead time is zero *i.e.* while customer orders any product from retailer; it is assumed that customers instantly obtained that ordered item. In general, this is not possible except JIT (Just-in-Time) case. Lead time may not be zero always. In recent scenario, lead time and demand are associated with each other. If lead time increases, some customers will wait for the ordered products, but others are impatience for waiting. For this reason, either these customers will cancel their orders or retailer will have to fulfill their orders within the normal delivery time. While lead time rises, demand of product decreases. Thus, it is very reasonable to consider demand of products as lead time dependent. Wang [32] surveyed some convenient formulas for developing the probability distribution of lead time demand while demand of products

Keywords. Controllable lead time, service level constraint, lead time crashing cost, transportation discounts, distribution free approach.

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follows a probability distribution of any arbitrary shape. Snyder *et al.* [28] calculated a lead time demand (LTD) to control inventory by exponential smoothing. They found some formulae for measuring means and variances of LTD. Those formulae allowed the opportunity to implement methods such that safety stocks were adjusted to changes in trend or in season.

Yang *et al.* [34] discussed an inventory model with time value of money and variable lead time. That was a mixed inventory model, in which the distribution of lead time demand is normal by assuming the time value of money. Rossetti and Ünlü [21] addressed a continuous review inventory model to examine the robustness of lead time demand. Those models were compared to each other by considering a known demand process and then the expected value of additional information related with the different models were calculated. Finally, those models were investigated by a large sample of simulated demand conditions. Sarkar and Majumder [22] developed an integrated vendor-buyer supply chain model with variable lead time. They considered a normally distributed lead time demand and the distribution free lead time demand along with reduction of setup cost of the vendor. Mekhtiev [14] deduced a supply chain model for investigating (LTD) lead time demand distributions. He measured LTD distributions analytically for smaller networks by utilizing numerical integration for larger networks. The validation of the developed framework has been conducted by comparing the obtained formalism-based LTD distribution for two test networks against simulation.

By considering present market situation, it can be found that there are several competitions among many manufacturing companies. To promote their products as best to customers, they always provide better facility in comparison to other companies. In this case, service level of products performs a major role for promoting and selling their products. Customers are always more sensitive to buy any item if they can get more service facility. Products with more service level increases demand of the product and *vice-versa*. Therefore, service level constraint acts as a major contributor to increase the profit of vendor and buyer. Therefore, demand of products can vary with the service level of any product. In addition, safety stock is also plays an important character to save the companies from backorder. Chen and Krass [5] formulated some inventory models where stockout cost is replaced by a minimal service level constraint (SLC). They discussed that the minimal service level approach had virtue of calculating the computation of an optimal ordering policy as optimal reorder level was obtained by the minimal SLC and demand distributions.

Lee *et al.* [12] considered a service level constraint instead of having a stockout cost in the objective function in their model. In their study, the backorder rate is lead time dependent. They assumed the lead time demand as a mixture of normal distributions by assuming a service level constraint. Jha and Shanker [9] formulated an integrated vendor-buyer integrated system. In place of shortage cost in inventory systems, a service level constraint was incorporated in their model. They also studied about stochastic demand which is assumed as normally distributed. In addition, lead time is controllable, which can be shortened at an added cost. Taleizadeh *et al.* [30] described a multi-product single-machine production system under economic production quantity (EPQ) model with a single machine, which has limited production capacity and service rate constraint for the company. They investigated optimal production quantity, allowable shortage level, and the period length of each product in such a way that expected total cost was minimized. Sarkar *et al.* [25] derived a continuous-review inventory model by involving the joint idea of setup cost reduction and quality improvement. They added a service level constraint in their model and used a distribution free approach to minimize the total system.

In reality, it is noted that generally lead time does not follow any specific type of distributions, but it plays an important role in inventory system mainly in the industries. Many research works discussed about lead time which follows some continuous distributions. Lead time may not be continuous always; it can vary due to several reasons like limited stock of products. It will be more applicable to the real life if lead time is assumed as discrete rather than as continuous. Larger lead times may effect to vendor and buyer's business strategy. This is because if lead time increases during filling demands, product's demand decreases automatically. It is essential for vendor or buyer to reduce the lead time for increasing their profits. It is very difficult to obtain lead time as zero, but it can be minimized. This can be done by using some lead time crashing cost. Therefore, chances to receive more orders for vendor and buyer will increase. This concept helps them to gain more profits over the market strategy. Chang *et al.* [4] worked with the lead time and ordering cost reduction problem in a single-vendor single-buyer integrated inventory model. They also assumed that the lead time can be shortened as a crashing cost which depends on the lead time length and the ordering lot size.

Hoque and Goyal [7] developed a heuristic solution procedure for an integrated inventory system under controllable lead time with equal or unequal sized batch shipments between a vendor and a buyer. They minimized the total set up cost, inventory holding cost and lead time crashing cost for an integrated inventory system under controllable lead time. By considering controllable lead time, Arkan and Hejazi [2] formulated a two-echelon supply chain model. They incorporated controllable ordering cost for single-buyer single-supplier model. There was an implementation method to share extra savings between the two parties in order to achieve a win-win outcome between them. To minimize the total cost of supply chain, they utilized a coordination mechanism by using credit period. Yi and Sarker [35] studied an operational policy for an integrated inventory system under consignment stock policy with controllable lead time and buyers' space limitation. They proved how to enable the decision maker of an integrated vendor–buyer system under consignment stock policy with controlable lead time and buyer's space limitation. Priyan and Uthayakumar [18] studied two-echelon and multi-product inventory model in which the distributor had limited space capacity and limited budget to purchase all products with controllable lead time and lead time crashing cost. Moon *et al.* [17] derived a distribution free continuous-review inventory system where fill rate was assumed as service level. They applied a negative exponential crashing cost function with related lead time in their model such that the total cost is reduced.

Transportation cost incurs the additional charge during shipment of any product. To minimize the total cost function for vendor and buyer, it is really helpful to reduce this cost. Transportation cost of an item is dependent with the size of the ordered quantity. Larger ordered quantity can reduce the transportation cost. This point of view will be cleared from the following example. First a buyer made an order for 200 quantities and received the same. Next, he ordered for 50 quantities and received the same quantities. For the first case, buyer has to pay small transportation charge per product compared to the second case. That means, transportation cost per product can be reduced if larger quantities of products are ordered. This is possible only when the demand of the product increases. As demand increases, ordered quantity is also increased and this helps to reduce the transportation cost. Most of the former research papers defined transportation cost as constant. But, this cost can be variable due to different routing path and shipment size. Adlakha *et al.* [1] investigated a simple heuristic MFL (more-for-less) algorithm to determine the demand destinations and the FCTPs (fixed-charge transportation problems).

Tsao and Lu [31] surveyed an integrated facility location and inventory allocation problem by providing two types of transportation cost discounts such as quantity discounts for inbound transportation cost and distance discounts for outbound transportation cost. In addition, they also utilized an approximation procedure to obtain distance discount calculation details and formulated an algorithm to determine the aforementioned supply chain management (SCM) problems. Lee and Fu [11] discovered an integrated production quantity model with transportation cost which is a fitted power function of delivery quantity by considering actual shipping rate data. Besides, they included a coordinated policy by scheduling single setup at the producer with multi-delivery to the buyer. Jha and Shanker [10] developed a coordinated two-phase iterative approach to solve an integrated inventory problem with transportation in a divergent supply chain under a service level constraint. Priyan and Uthayakumar [19] discussed a single-vendor single-buyer supply chain model for the ordering cost reduction and they discussed the effect of transportation cost discounts. Besides, they considered that transportation cost as a function of shipment lot size, which was taken to be in an all-unit-discount cost format to determine the proposed inventory problem. See Table 1 for contribution of different authors.

Scarf [27] first gave the idea of the distribution free approach in his study: a min-max solution in a newsvendor problem. This idea solves the problem which arises due to the use of a specified distribution function. Gallego and Moon [6] simplified the Scarf's ordering rule in more consistent way. Moon and Gallego [16] also worked on the distribution free procedures for some inventory models. Moon and Choi [15] extended this approach with a service level constraint. Wu *et al.* [33] developed their studies on computational algorithm procedure of optimal inventory policy involving a negative exponential cost and variable lead time demand. Sarkar and Moon [23] discussed their model on improved quality, setup cost reduction, and variable backorder rates in an imperfect production

TABLE 1. Contribution of the different authors.

Authors	Lead time dependent demand	Service level constraint	Lead time crashing cost	Transportation cost
Wang [32]	✓			
Snyder *et al.* [28]	✓			
Yang *et al.* [34]	✓			
Rossetti and Ünlü [21]	✓			
Sarkar and Majumder [22]			✓	
Mekhtiev [14]	✓			
Chen and Krass [5]		✓		
Lee *et al.* [12]		✓		
Jha and Shanker [9]		✓		
Taleizadeh *et al.* [30]		✓		
Chang *et al.* [4]			✓	
Hoque and Goyal [7]			✓	
Arkan and Hejazi [2]			✓	
Yi and Sarker [35]			✓	
Priyan and Uthayakumar [18]			✓	
Moon *et al.* [17]	✓	✓		
Adlakha *et al.* [1]				✓
Tsao and Lu [31]				✓
Lee and Fu [11]				✓
Jha and Shanker [10]		✓		✓
Priyan and Uthayakumar [19]				✓
Moon and Choi [15]		✓		
Sarkar an Moon [23]	✓		✓	
Sarkar *et al.* [26]		✓		
Sarkar and Mahaptra [24]	✓		✓	
This paper	✓	✓	✓	✓

process. Sarkar *et al.* [26] studied in quality improvement and backorder price discount under controllable lead time in an inventory model. Sarkar and Mahaptra [24] extended distribution free strategies in periodic review fuzzy inventory model with variable lead time and fuzzy demand.

Lin *et al.* [13] proposed a hybrid approach for applied fuzzy set theory which comprises applied interpretive structural modeling to build a hierarchical structure and uses the analytic network process to analyze the dependence relations. Iwao and Kusukawa [8] proposed an optimal production planning model for remanufacturing of parts in used products with quality classification errors made by the collection trader. Further, Asghari *et al.* [3] formulated a dynamic nonlinear programming model of reverse logistics network design with the aim of managing the used products allocation by coordinating the collection centers and recovery facilities to warrant economic efficiency. They also proposed a heuristic method based tabu search procedure to solve the model. Qiang and Peng [20] described a two-in-one methodology utilizing simulation optimization technique and data envelopment analysis (DEA) in measuring an accurate efficiency score. They developed a novel methodology known as Data Collection Budget Allocation-Data Envelopment Analysis (DCBA-DEA). Soh *et al.* [29] investigated the factors that influence the use of public bus transport (PBT) in Malaysia. Their results show that the twin factors consisting of perceived satisfaction of and perceived importance by users towards safety and comfort of facilities and services offered by PBT are significantly related to intention to use those services. Thus, transportation discount is an important issue everywhere.

The present study is to develop a continuous review inventory model with normally distributed lead time demand, service level constraints, and transportation discounts. Then, in the second model, the model relaxes the assumption of normal distribution and uses the distribution free approach with known mean and standard deviation. In this paper, lead time is reduced by adding some crashing cost. Service level constraint is assumed in this model to avoid backordering cost. This paper is designed in the following manner: Section 2 delivers mathematical model with notation and assumptions with normal distribution. Section 3 delivers mathematical model with distribution free approach. Section 4 serves numerical examples to illustrate this model. An algorithm is there for this case. To see the effects of changing of parameters sensitivity analysis has also been performed. Finally, Section 5 gives conclusion and future extensions.

2. Mathematical model

Following notation are used in the entire model.

Notation

Q	lot size (units)
L	length of lead time (weeks)
k	safety factor
L_i	length of lead time with components $i = 1, 2, \ldots, n$ (weeks)
h	holding cost (\$/unit/unit time)
r	reorder point
A	fixed ordering cost (\$/order)
D	demand (units/year)
X	lead time demand (random variable) (units)
μ	mean of lead time demand
σ	standard deviation of lead time demand
T	cycle time
$E[X - r]^+$	expected shortage quantity at the end of the cycle
c_i	crashing cost with components $i = 1, 2, \ldots, n$ (\$/unit time/day)
u_i	minimum duration with components $i = 1, 2, \ldots, n$ (days)
v_i	normal duration with components $i = 1, 2, \ldots, n$ (days)
C_i	transportation cost, $i = 1, 2, \ldots, B$

Assumptions
To develop this model, following assumptions are considered

1. This paper deals with a single type of product.
2. Demand D is assumed to be constant.
3. There are n mutually independent components of lead time L and for reducing lead time, every lead time has a different crashing cost. Let the ith component of this lead time has a normal duration v_i, the minimum duration u_i and crashing cost c_i per unit time with $c_1 \leqslant c_2 \leqslant c_3 \leqslant \ldots \leqslant c_n$.
4. Let $L_0 = \sum_{j=1}^{n} v_j$ and L_i be the length of lead time with components $1, 2, \ldots, i$ crashed to their minimum duration. Then, L_i can be written in the form $L_i = L_0 - \sum_{j=1}^{i} (v_j - u_j)$. The lead time crashing cost per cycle is $R(L)$, can be expressed as $R(L) = c_i(L_i - L) + \sum_{j=1}^{i-1} (v_j - u_j)$ for $i = 1, 2, 3, \ldots, n$.
5. The reorder point $r =$ the expected demand during lead time + safety stock (S_s) and $S_s = k \times \sigma\sqrt{L}$, k is safety factor.
6. X follows a normal distribution with mean DL and standard deviation $\sigma\sqrt{L}$.

Formulation of model
This paper used the concept of Moon *et al.*'s [17] model with stochastic lead time, crashing cost and transportation cost reduction, which is a continuous-review inventory model of distribution free type. Lead time is an important factor for market demand. First, this model considers a normal distributed lead time demand.

We consider the cost equation from Moon and Choi [15] as $C_N(Q,r) = \frac{AD}{Q} + h(\frac{Q}{2} + r - \mu)$, '$N$' denotes for normal distribution.

Now, we introduce the crashing cost for total cost reduction and variable lead time as described in Wu *et al.* [33]. Crashing cost is dependent upon the lead time, *i.e.*, every lead time has different and different crashing cost. Using crashing cost, the total cost expression is now in the form

$$C_N\left(Q,r,L\right) = \frac{AD}{Q} + h\left(\frac{Q}{2} + r - \mu L\right) + \frac{D}{Q}R(L).$$

This paper introduces the concept of transportation cost reduction. This is not constant type cost; this cost depends on demand function directly. In literature, transportation cost is taken as fixed *i.e.* independent of any shipment or ordering quantity. In many cases, transportation cost is introduced here as a demand-dependent price *i.e.* unit transportation cost is depending on the ordering quantity (Priyan and Uthayakumar [19]).

$$\begin{aligned} TC(Q) = C_0D, Q \in [M_0, M_1) \\ C_1D, Q \in [M_1, M_2) \\ C_2D, Q \in [M_2, M_3) \\ \dots \\ C_BD, Q \in [M_B, \infty), \end{aligned}$$

where $C_0 > C_1 > C_2 \ldots > C_B$, C_i is the transportation cost per unit. M_i refers to the range of Q ($M_0 = 0$), *i.e.* Q should lie in some specified range, otherwise $TC(Q)$ cannot be defined. Using $TC(Q)$ the cost function can be rewritten as:

$$C_N\left(Q,r,L\right) = \frac{AD}{Q} + h\left(\frac{Q}{2} + r - \mu L\right) + \frac{D}{Q}R(L) + TC(Q).$$

Now, we minimize $C_N(Q,r,L)$ subject to a specified fill rate. The fill rate is defined as the partial demand satisfied directly from inventory. β is the measurement of service level, thus β can be written as:

$$\beta = 1 - \frac{E[X-r]^+}{Q},$$

which reduces to

$$E[X-r]^+ = (1-\beta)\,Q.$$

We can write the cost function as follows:

$$C_N\left(Q,k,L\right) = \frac{D}{Q}(A + R\left(L\right)) + h\left(\frac{Q}{2} + k\sigma\sqrt{L}\right) + TC(Q), \quad \text{where} \quad r - \mu L = k\sigma\sqrt{L}.$$

We use here a normal distribution with mean DL and standard deviation $\sigma\sqrt{L}$ as in Sarkar and Moon [23].

$$\begin{aligned} E[X-r]^+ &= \int_r^\infty (X-r)\,\mathrm{d}F\left(x\right) \\ &= \sigma\sqrt{L}\left\{\varphi\left(k\right) - k\left(1 - \Phi\left(k\right)\right)\right\} \\ &= \sigma\sqrt{L}\psi(k), \end{aligned}$$

where $(k) = \varphi(k) - k(1-\Phi(k))$, $\varphi(k)$ and $\Phi(k)$ are the standard normal distribution function and the probability density function of the normal distribution, respectively.

Therefore,

$$\sigma\sqrt{L}\psi\left(k\right)=\left(1-\beta\right)Q$$

i.e., $k=\frac{\varphi(k)\sigma\sqrt{L}-(1-\beta)Q}{\sigma\sqrt{L}(1-\Phi(k))}$.

Therefore, the cost function is transformed into the following expression after using the normal distribution.

$$C_N\left(Q,k,L\right)=\frac{D}{Q}(A+R\left(L\right))+h\left(\frac{Q}{2}+\frac{\varphi(k)\sigma\sqrt{L}-\left(1-\beta\right)Q}{\left(1-\Phi\left(k\right)\right)}\right)+TC(Q).$$

We now minimize the cost function $C_N(Q,k,L)$ with respect to L. We solve the cost equation for the decision variable L as

$$\frac{\partial C_N\left(Q,k,L\right)}{\partial L}=0$$

i.e. $\frac{h\sigma\varphi(k)}{2\sqrt{L}(1-\Phi(k))}-\frac{Dc_i}{Q}=0.$

Now, we investigate the second order derivative of $C_N(Q,k,L)$ with respect to L as

$$\frac{\partial^2 C_N\left(Q,k,L\right)}{\partial L^2}=-\frac{h\sigma\varphi(k)}{4(1-\Phi\left(k\right))}L^{-3/2}<0.$$

It can be seen that $C_N(Q,k,L)$ is concave function with respect to L, not a convex function. Thus, for a fixed value of k, the minimum optimum cost would be obtained at the end point $[L_i,L_{i-1}]$ (Refer to Sarkar *et al.* [26]).

We minimize $C_N(Q,k,L)$ over $Q>0$. $C_N(Q,k,L)$ becomes $C_N(Q)$ only. Solving the following equation for minimization,

$$\frac{\partial C_N\left(Q,\,k,L\right)}{\partial Q}=0$$

i.e., $hQ^2\left[\frac{1}{2}-\frac{1-\beta}{1-\Phi(k)}\right]-D(A+R(L))=0$

i.e., $Q=\left[\frac{2D(1-\Phi(k))(A+R(L))}{h(2\beta-1-\Phi(k))}\right]^{1/2}$.

To obtain the sufficient condition for minimization, second order derivative of $C_N(Q,k,L)$ with respect to Q is as follows:

$$\frac{\partial^2 C_N\left(Q,\,k,L\right)}{\partial Q^2}=\frac{2D(A+R(L))}{Q^3}>0.$$

As $\frac{\partial^2 C_N(Q,k,L)}{\partial Q^2}>0$, for $Q>0$, $C_N(Q,k,L)$ is a convex function in Q.

Hence, we get the optimum values of decision variables Q and k as

$$Q_\beta=\left[\frac{2D(1-\Phi\left(k\right))(A+R(L))}{h(2\beta-1-\Phi\left(k\right))}\right]^{1/2}$$

$$k_\beta=\frac{\varphi(k)\sigma\sqrt{L}-\left(1-\beta\right)Q}{\sigma\sqrt{L}(1-\Phi\left(k\right))}.$$

Based on the optimization, we obtain the lemma.

Lemma 2.1. *The cost function $C_N(Q,k,L)$ attains it's the minimum value at $Q_\beta=\left[\frac{2D(1-\Phi(k))(A+R(L))}{h(2\beta-1-\Phi(k))}\right]^{1/2}$, under normal distributed demand, if $\frac{2D(A+R(L))}{Q^3}>$ with fixed L.*

2.1. Solution algorithm

Here, the algorithm has been developed to obtain the numerical solution of this model.

Step 1. Set the value of n and i.

Step 2. Give input values of parameters $D,A,h,\sigma,\beta,k[n]$ and values for L.

Step 3. Set $v[n]$, $u[n]$ and $c[n]$, $n=1,2,3,\ldots$ from given set of data.

Step 4. Find the value of L_0 from $L_0 = \sum_{j=1}^{n} v_j$ using Steps 1 and 3.

Step 5. Compute L_n from $L_n = L_0 - \sum_{j=1}^{n}(v_j - u_j)$ using Steps 1, Step 3, and Step 4.

Step 6. Compute $R(L) = c_n(L_n - L) + \sum_{j=1}^{n-1}(v_j - u_j)$ using Step 1, Step 4, and Step 5.

Step 7. Find the value of $\Phi(k)$ and $\varphi(k)$ from normal table.

Step 8 Obtain the value of Q from the equation $Q = [\frac{2D(1-\Phi(k))(A+R(L))}{h(2\beta-1-\Phi(k))}]^{1/2}$ using Step 1, Step 2, Step 6, and Step 7.

Step 9. Evaluate the value of k from $k = \frac{\varphi(k)\sigma\sqrt{L}-(1-\beta)Q}{\sigma\sqrt{L}(1-\Phi(k))}$ using Step 2, Step 7, and Step 8.

Step 10. Set the value of $TC(Q)$ according to the value of Q from Step 8.

Step 11 Determine the value of the cost equation $C_N(Q,k,L) = \frac{D}{Q}(A+R(L))+h(\frac{Q}{2}+\frac{\sigma^2L-4Q^2(1-\beta)^2}{4Q(1-\beta)})+TC(Q)$ using Step 2, Step 6, Step 8, and Step 10.

Step 12. Repeat Step 1–Step 11 with the change of L and i until get an optimum result.

Step 13. End.

3. Distribution free approach

In many cases, it is very difficult to collect the information about lead time demand. Without lead time demand distribution, is not possible to calculate exact shortage amount. Thus, Scarf [27] first developed a min-max distribution free approach to solve this problem without any distribution for lead time demand with known mean and standard deviation. But, that proof was very difficult to understand for the industry people or the researchers. Thus, Gallego and Moon [6] simplified the ordering rule by Scarf [27]. Since then, this is the best approach to derive the optimum cost function with the help of distribution free lead time demand.

Here, this model uses the distribution free lead time with known mean and standard deviation and uses the lemma by Gallego and Moon [6]. Now we minimize $C_N(Q,k,L)$ subject to a specified fill rate. The fill rate is defined as the partial demand satisfied directly from inventory. As before, β can be obtained as

$$E[X-r]^{+} = (1-\beta)\,Q.$$

Thus, the cost function can be written as follows:

$$C(Q,k,L) = \frac{AD}{Q} + h\left(\frac{Q}{2}+k\sigma\sqrt{L}\right) + \frac{D}{Q}R(L) + TC(Q), \quad \text{where} \quad r - DL = k\sigma\sqrt{L}.$$

i.e., $= \frac{D}{Q}(A+R(L)) + h\left(\frac{Q}{2}+k\sigma\sqrt{L}\right) + TC(Q).$

This cost function is now minimized with the help of distribution free approach subject to a specified fill rate. We obtain the minimum cost at the optimal (Q,k,L) for the least favorable distribution function F in $\mathbf{F}$.

To obtain the least favorable distribution F in $\mathbf{F}$, we use the following lemma as per Gallego and Moon [15]:

$$E[X-r]^{+} \leqslant \frac{\sqrt{\sigma^2L+(r-\mu L)^2}-(r-\mu L)}{2} \quad \text{for any} \quad F\in\mathbf{F}$$
$$\leqslant \frac{\sigma\sqrt{L}\left(\sqrt{(1+k^2)}-k\right)}{2}.$$

And thus, $\frac{\sigma\sqrt{L}(\sqrt{(1+k^2)}-k)}{2} = (1-\beta)Q$

i.e., $1-\beta = \frac{\sigma\sqrt{L}\left(\sqrt{(k^2+1)}-k\right)}{2Q}$,

which gives $k = \frac{\sigma^2L-4Q^2(1-\beta)^2}{4Q\sigma\sqrt{L}(1-\beta)}$.

Using this k, the cost function reduced in the following form

$$C\left(Q,L\right)=\frac{D}{Q}(A+R\left(L\right))+h\left(\frac{Q}{2}+\frac{\sigma^2L-4Q^2{(1-\beta)}^2}{4Q\left(1-\beta\right)}\right)+TC(Q)$$
$$=\frac{D}{Q}(A+c_i\left(L_i-L\right)+\sum_{j=1}^{i-1}(v_j-u_j))+h\left(\frac{Q}{2}+\frac{\sigma^2L-4Q^2{(1-\beta)}^2}{4Q\left(1-\beta\right)}\right)+TC(Q).$$

We solve the cost equation for decision variable L as

$$\frac{\partial C(Q,L)}{\partial L}=0$$

i.e. $-\frac{Dc_i}{Q}+\frac{h\sigma^2}{4Q(1-\beta)}=0\cdot$

To obtain the sufficient condition of optimization, one can find the second order derivative of $C(Q,L)$ with respect of L, one can obtain

$$\frac{\partial C^2(Q,L)}{\partial L^2}=0.$$

Thus, $C(Q,L)$ neither convex nor concave. Therefore, the minim optimum cost would be obtained at the end point of $[L_i,L_{i-1}]$.

Now we minimize $C(Q,L)$ over $Q>0$. Now solving the following equation

$$\frac{\partial C(Q,L)}{\partial Q}=0$$

i.e., $Q=\sqrt{\frac{4D(1-\beta)\{A+R(L)\}+h\sigma^2L}{2h(1-\beta)(2\beta-1)}}\cdot$

For sufficient condition, the 2nd order derivative of $C(Q,L)$ with respect to Q is as follows:

$$\frac{\partial C^2\left(Q,L\right)}{\partial Q^2}=\frac{2D(A+R\left(L\right))}{Q^3}+\frac{h\sigma^2L}{2Q^3(1-\beta)}>0.$$

As $\frac{\partial^2C(Q,L)}{\partial Q^2}>0$, for $Q>0$, $C(Q,L)$ is a convex function in Q.

Hence, we get the following decision variables Q and k as

$$Q_\beta=\sqrt{\frac{4D\left(1-\beta\right)\left\{A+R\left(L\right)\right\}+h\sigma^2L}{2h\left(1-\beta\right)\left(2\beta-1\right)}}\cdot$$
$$k_\beta=\frac{\sigma^2L-4Q^2{(1-\beta)}^2}{4Q\sigma\sqrt{L}\left(1-\beta\right)}\cdot$$

and hence the lemma.

Lemma 3.1. *The cost function $C(Q,L)$ attains the minimum value at $Q_\beta=\sqrt{\frac{4D(1-\beta)\{A+R(L)\}+h\sigma^2L}{2h(1-\beta)(2\beta-1)}}$, under distribution free lead time demand, if $\frac{2D(A+R(L))}{Q^3}+\frac{h\sigma^2L}{2Q^3(1-\beta)}>0$ with fixed L.*

3.1. Solution algorithm

Here, the algorithm has been developed to obtain the numerical solution of this model. The algorithm is as follows:

Step 1. Set the value of n and i.

Step 2. Give input values of parameters D,A,h,σ,β and set values for L.

Step 3. Set $v[n]$, $u[n]$ and $c[n]$, $n=1,2,3,\ldots$ from given set of data.

Step 4. Find the value of L_0 from $L_0=\sum_{j=1}^{n}v_j$ using Step 1 and Step 3.

Step 5. Compute L_n from $L_n = L_0 - \sum_{j=1}^{n}(v_j - u_j)$ using Step 1, Step 3, and Step 4.

Step 6. Compute $R(L) = c_n(L_n - L) + \sum_{j=1}^{n-1}(v_j - u_j)$ using Step 1, Step 4, and Step 5.

Step 7. Obtain the value of Q from the equation $Q = \sqrt{\frac{4D(1-\beta)\{A+R(L)\}+h\sigma^2 L}{2h(1-\beta)(2\beta-1)}}$ using Step 2 and Step 6.

Step 8. Calculate the value of $k = \frac{\sigma^2 L - 4Q^2(1-\beta)^2}{4Q\sigma\sqrt{L}(1-\beta)}$ from the Step 2 and Step 7.

Step 9. Set the value of $TC(Q)$ according to the value of Q from Step 7.

Step 10. Find the value of the cost equation $C(Q,k,L) = \frac{D}{Q}(A + R(L)) + h\left(\frac{Q}{2} + \frac{\sigma^2 L - 4Q^2(1-\beta)^2}{4Q(1-\beta)}\right) + TC(Q)$ using Step 2, Step 6, Step 7, and Step 9.

Step 11. Repeat Step 6–Step 10 with respect to i and L, until getting minimum cost.

Step 12. End.

4. Numerical example

4.1. Example 1

This is for normal distributed model. In this numerical example, we used the data from Moon *et al.* [17]. The values are as follows: $D = 600$; $A = \$200$/per order; $h = \$20$/unit/year; $\sigma = 6$; lead time data has three components, giving in Table 2.

Table 2. Lead time data.

Lead time Component i	Normal duration v_i (days)	Minimum duration u_i (days)	Unit crashing cost c_i (\$/day)
1	20	6	0.4
2	20	6	1.2
3	16	9	5.0

Table 3. Transportation cost structure.

Range	Unit Transportation Cost
$0 \leqslant Q < 100$	0.2
$100 \leqslant Q < 200$	0.15
$200 \leqslant Q < 300$	0.1
$300 \leqslant Q$	0.05

Table 4. Solution table of $C_N(Q_\beta, k_\beta, L)$.

L	β	$C_N(Q_\beta, k_\beta, L)$	Q_β	k_β
3	0.98	2607.25	126.84	0.15
4	0.98	2447.59	117.90	0.19
$\underline{6}^*$	$\underline{0.98}^*$	$\underline{2380.56}^*$	$\underline{113.36}^*$	$\underline{0.23}^*$
8	0.98	2367.71	111.80	0.25

- ➢ The underlined numbers indicate the optimum values for normal distribution model. The values with * symbol indicate the optimum cost, lot size, safety stock, lead time.
- ➢ At $L = 8$, proposed function obtain the minimum cost. As crashing cost for $L = 8$ is Zero, it can be exempled from optimum cost consideration. So, the next minimum cost is considered as the optimum cost *i.e.* minimum cost.

4.2. Example 2

This example is for distribution free model. All data are given in Example 1. Lead time data and transportation cost reduction table is same as Example 1.

TABLE 5. Solution table of $C(Q_\beta, k_\beta, L)$.

L	β	$C(Q_\beta, k_\beta, L)$	Q_β	k_β
3	0.98	2729.56	137.48	0.68
$\underline{4^{**}}$	$\underline{0.98^{**}}$	$\underline{2640.78^{**}}$	$\underline{132.85^{**}}$	$\underline{0.91^{**}}$
6	0.98	2699.72	135.92	1.17
8	0.98	2805.29	141.42	1.33

The underlined numbers indicate the optimum values for distribution free model. The values with ** symbol indicate the optimum cost, lot size, safety stock, lead time.

4.3. Evaluation of EVAI

This section compares these two models *i.e.*, normally distributed and distribution free model. Since distribution free approach does not follow any particular distribution function, some additional information is needed. For distribution free, $(Q_\beta, k_\beta, L) = (132.85^{**}, 0.91^{**}, 4)$ and for normal $(Q_\beta, k_\beta, L) = (113.36, 0.23, 6)$ are the optimal solutions. Therefore, the additional cost is $C_N(132.85, 0.91, 4) - C_N(113.36, 0.23, 6) = \$2458.50$–$\$2380.56 = \$77.94$. This amount is the additional cost for distribution free model. This amount is called the expected value of additional information (EVAI). Buyer has to pay this extra cost due to non-distributed demand approach [16].

4.4. Sensitivity analysis

This section gives a sensitivity analysis of both two models. Sensitivity analysis revealed the changes of the cost with respect to the different parameters. One can define which parameter is more sensible to the cost.

Sensitivity analysis for Example 1.
Sensitivity analysis is performed on the key parameter of the optimum values of the example of this normally distributed model and is presented in Table 6.

TABLE 6. Sensitivity analysis for Example 1.

Parameters	Changes (in %)	$C(Q_\beta, k_\beta, L)$
A	–50	–28.07
	–25	–10.65
	+25	+10.53
	+50	+17.57
h	–50	–29.14
	–25	–13.43
	+25	+11.97
	+50	+19.35

Sensitivity analysis is performed here to find out themeasure of sensitiveness of the parameters such as A and h respectively. Effects of these parameters are discussed below:

- ➢ These parameters are more sensitive in negative rather than positive percentage change. In addition, while ordering cost increases the total cost also increases. On the other hand, total cost reduces when ordering cost decreases. Thus, it can be concluded that, profit is affected by raising ordering cost A. The positive change is not similar with negative change.

➢ Studying sensitivity analysis for h, it is found that this model is more sensitive in negative for holding cost h than positive percentage change.Total cost is influenced by the parameter h, *i.e.*, whenever this cost raises, total cost increases automatically.

Sensitivity analysis for Example 2.
Sensitivity analysis is performed for the key parameter of the optimum values for Example 2 and is presented in Table 7.

TABLE 7. Sensitivity analysis is for Example 2.

Parameters	Changes (in %)	$C(Q_\beta, k_\beta, L)$
	–50	–18.96
A	–25	–8.97
	+25	+8.20
	+50	+15.80
	–50	–32.02
h	–25	–15.19
	+25	+14.23
	+50	+27.83

Effects of these parameters are discussed below:

➢ Distribution free model is more sensitive for A in negative change than positive change. In comparison with normal case, sensitiveness of A is more in normally distributed demand in both positive and negative changes than distribution free demand. That is, cost is further affected in normal demand pattern than distribution free demand pattern when ordering cost increased or decreased.

➢ Sensitivity analysis for h is more sensitive in distribution free model in negative change than positive change. In comparison with normal demand, distribution free case is more sensitive in both positive and negative case. Total cost is influenced by h is more than normal case.

5. Conclusions

This paper developed a continuous review inventory model with stochastic lead time. The model used a normally distributed lead time first and then used a distribution free approach for lead time demand with known mean and standard deviation. The transportation discount was considered to reduce the total transportation cost. Transportation cost reduction and lead time crashing cost were used here; a service level was used to avoid backorder. The total cost was minimized with respect to the decision variables, simultaneously. From the numerical experiments, it was found that we saved more from the exiting literature. The expected value of additional information (EVAI) was calculated and it gave the amount needed to obtain the additional information. Based on that, the industry manager may decide which policy they would prefer to use. In future, this paper can be extended by consider uncertainty in cost parameters and another type of lead time crashing cost function for this model. Further, it can also be extended using green supply chain management by Lin *et al.* [13] andIwao and Kusukawa [8] and for dynamic nonlinear programming using reverse logistics by Asghari *et al.* [3].

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