Journal de la Société Française de Statistique
Vol. 154 No. 1 (2013)

Statistical Procedures for the Selection of a Multidimensional Meta-elliptical Distribution

Titre: Procédures statistiques pour le choix d'une loi méta-elliptique multidimensionnelle

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Abstract: Meta-elliptical distributions are multivariate statistical models in which the dependence structure is governed by an elliptical copula and where the marginal distributions are arbitrary. In this paper, goodness-of-fit tests are proposed for the construction of an appropriate meta-elliptical model for multidimensional data. While the choice of the marginal distributions can be guided by classical goodness-of-fit testing, how to select an adequate elliptical copula is less clear. In order to fill this gap, formal copula goodness-of-fit methodologies are developed here around the *radial part* that characterizes an elliptical distribution. The key idea consists in estimating its univariate distribution function from a pseudo-sample derived from the original multivariate observations. Then, a Cramér–von Mises distance between this non-parametric estimator and the expected parametric version under the null hypothesis is used as a test statistic. An approximate p-value is obtained from an application of the parametric bootstrap. The method is extended to the case where the elliptical generator has unknown parameters using a minimum-distance criterion. While a careful investigation of the asymptotic behavior of the tests is not presented here, Monte–Carlo simulations indicate that the methods have good sample properties in terms of size and power. The techniques are illustrated on the Danish fire insurance, Upper Mississippi river, Oil currency and Uranium exploration data sets.

Résumé : Les lois méta-elliptiques sont des modèles statistiques multivariés dans lesquels la structure de dépendance est gouvernée par une copule elliptique et où les distributions marginales sont arbitraires. Dans cet article, des tests d'adéquation sont proposés afin de construire un modèle méta-elliptique approprié pour des données multidimensionnelles. Alors que le choix des marges peut se faire via des tests d'adéquation classiques, comment sélectionner une copule elliptique adéquate est moins clair. Pour combler ce manque, des méthodes d'adéquation formelles sont développées ici autour de la *partie radiale* qui caractérise une loi elliptique. L'idée centrale consiste à estimer sa fonction de répartition univariée à partir d'un pseudo-échantillon qui découle des observations multivariées originales. Ensuite, on utilise comme statistique de test la distance de Cramér–von Mises entre cet estimateur non-paramétrique et sa version attendue sous l'hypothèse nulle. Une p-valeur approximative est obtenue d'une application du bootstrap paramétrique. La méthode est généralisée au cas où le générateur elliptique est à paramètres inconnus en utilisant un critère à distance minimale. Bien que le comportement asymptotique des tests ne soit pas étudié ici, des simulations Monte–Carlo indiquent que les méthodes possèdent de belles propriétés échantillonnales en termes de seuil et de puissance. Les techniques sont illustrées sur les jeux de données "Danish fire insurance", "Upper Mississippi river", "Oil currency" et "Uranium exploration".

Keywords: Copula, goodness-of-fit test, meta-elliptical distributions, minimum-distance method, parametric bootstrap

Mots-clés : Copule, tests d'adéquation, lois méta-elliptiques, méthode à distance minimale, bootstrap paramétrique

AMS 2000 subject classifications: 62G10, 62H15

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1. Introduction

Following Cambanis et al. (1981), a random vector $\mathbf{X} \in \mathbb{R}^d$ is said to be elliptically contoured if its associated characteristic function is of the form

$$\mathrm{E}\left(e^{i\mathbf{t}^\top \mathbf{X}}\right) = \exp\left(i\mathbf{t}^\top \mu\right) \Upsilon\left(\frac{1}{2}\mathbf{t}^\top \Sigma \mathbf{t}\right), \tag{1}$$

where $\mathbf{t} = (t_1, \ldots, t_d)^\top$, $\mu \in \mathbb{R}^d$ is a mean vector and $\Sigma \in \mathbb{R}^{d\times d}$ is a positive definite matrix; the function $\Upsilon : \mathbb{R}^+ \to \mathbb{R}$ is called the *characteristic generator* and is such that $\upsilon(t_1, \ldots, t_d) = \Upsilon(t_1^2 + \cdots + t_d^2)$ is a d-variate characteristic function. This class of multidimensional distributions was first introduced by Kelker (1970) as a generalization of the classical Normal law, and comprises in particular the spherical laws when $\mu = (0, \ldots, 0)^\top \in \mathbb{R}^d$ and Σ is the identity matrix. The most notorious members of the general elliptical family are the multivariate Normal and Student distributions. These models have proven useful for statistical modeling in situations where alternatives to normality were needed, especially in finance and hydrology. Their success lies mainly in that they allow for heterogeneous levels of dependence for the pairs, via the elements of a covariance matrix, and several kinds of tail behaviors by means of the elliptical generator. Among the many applications of elliptical distributions, one can cite Landsman and Valdez (2003) for the computation of tail conditional expectations, and Owen and Rabinovitch (1983) in the theory of portfolio choice.

Even if elliptical models are quite flexible, they nevertheless have the disadvantage that all marginal distributions have the same analytical form up to location and scale factors. This can be quite restrictive in situations where, for example, marginal tail behaviors are of a different nature, some components being heavy tailed and others having light tails. A model-building strategy that allows much more flexibility is to consider the *copula* of elliptically contoured distributions combined with any choice of the marginal distributions. Of course, the starting point of this approach is Sklar's Theorem, who ensures that if the marginal distributions of a d-variate random vector $\mathbf{X} = (X_1, \ldots, X_d)^\top$ are continuous, then there exists a unique copula $C_{\mathbf{X}} : [0,1]^d \to [0,1]$ such that for each $\mathbf{x} = (x_1, \ldots, x_d)^\top \in \mathbb{R}^d$,

$$\mathrm{P}(\mathbf{X} \le \mathbf{x}) = C_{\mathbf{X}}\left\{\mathrm{P}(X_1 \le x_1), \ldots, \mathrm{P}(X_d \le x_d)\right\}. \tag{2}$$

When $\mathbf{X}$ is elliptically contoured, the function $C_{\mathbf{X}}$ is called an *elliptical copula*. These dependence functions are the key elements of the multivariate meta-elliptical distributions, which are the multivariate probability laws whose underlying copula is elliptical. In other words, a random vector $\mathbf{Y} = (Y_1, \ldots, Y_d)^\top \in \mathbb{R}^d$ is said to be meta-elliptically distributed with marginal distributions $F_1, \ldots, F_d$ if its copula $C_{\mathbf{Y}}$ belongs to the family of elliptical copulas. These models thus provide a general framework where one can select arbitrary margins and where the dependence structure is governed by a correlation matrix and an elliptical generator. As enlightened by Genest et al. (2007), "Meta-elliptical models are a good compromise between convenience and flexibility." They appear in several contexts of multivariate statistical analysis. For example, Abdous et al. (2005) studied their dependence properties in the bivariate case, Landsman (2009) used them to model capital allocation and Wang et al. (2010) in multivariate modeling of hydrological data.

Despite the numerous successes in the application of meta-elliptical models, the problem of selecting an appropriate elliptical copula is still an open problem. Up until now, only the methodology based on the empirical copula process (see Genest et al., 2009) and on Kendall's process (see Genest et al., 2006) have been suggested for the selection of a meta-elliptical model. This has been done by Genest et al. (2007), where trivariate hydrological data were modeled. However, the power properties of these tests have not been investigated for elliptical copulas. Furthermore, these methods have been considered solely in cases where the elliptical generator is entirely known, *i.e.* no parameter needs to be estimated.

In this paper, statistical tools for the construction of an appropriate meta-elliptical model are proposed and their properties in small samples are investigated. While the selection of suitable marginal distributions can be made by standard univariate goodness-of-fit methods (see Durbin, 1973, for example), how to thoroughly choose an elliptical copula isn't as clear. In order to fill this gap, a special focus is put here on the investigation of new goodness-of-fit tests specifically designed to assess the quality of the fit of a given elliptical copula family on multivariate data. The key idea exploits a characterization of elliptically contoured random vectors via their so-called *radial part*. Both the cases where the radial part has an entirely known distribution and a distribution with unknown parameters are explored. The latter situation enables to test, for example, for a Student dependence structure without having to specify the number of degrees of freedom. A graphical tool based on sample counterparts of the radial part is also proposed.

This paper is organized as follows. In Section 2, standard results on elliptical and meta-elliptical families of distributions are reviewed; many members, including the Normal, Student and Pearson type II distributions are described. Section 3 concerns the development of a goodness-of-fit procedure for selecting an appropriate elliptical copula; simulation results indicate that the method works well. In Section 4, the framework is extended in order to consider cases where the elliptical generator has unknown parameters; the behavior of the proposed minimum-distance test statistic in small samples is numerically studied as well. Section 5 reports the results of the statistical analysis of four data sets in the light of the new tests. Final remarks are given in Section 6, especially around the routes that should be followed in order to obtain the asymptotic behavior of the newly introduced test statistics and to thoroughly validate the use of the parametric bootstrap method. In view of the simulation results, one has all the reasons to believe that the test statistics converge weakly and that the bootstrap is asymptotically valid.

2. The meta-elliptical family of dependence functions

2.1. Elliptically contoured distributions

Let $\mathbf{X} \in \mathbb{R}^d$ be elliptically contoured, *i.e.* its characteristic function has the form (1). Following the same line of work as Schoenberg (1938) on spherically symmetric distributions, Cambanis et al. (1981) obtained that $\mathbf{X}$ admits the stochastic representation

$$\mathbf{X} \stackrel{d}{=} \mu + \mathscr{G} A \mathscr{U}, \tag{3}$$

where $\mu \in \mathbb{R}^d$ is the mean vector, the *radial part* $\mathscr{G}$ is a positive random variable, $A \in \mathbb{R}^{d \times d}$ is a (fixed) matrix such that $AA^\top = \Sigma$ and $\mathscr{U}$ is a random vector uniformly distributed on the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$, *i.e.*

$$\mathbb{S}^{d-1} = \left\{ \mathbf{u} = (u_1, \ldots, u_d)^\top \in \mathbb{R}^d : \mathbf{u}^\top \mathbf{u} = 1 \right\}.$$

Note that if the distribution of $\mathscr{G}$ is standardized in such a way that $\mathrm{E}(\mathscr{G}^2) = d$, one has $\mathrm{E}\{(\mathbf{X}-\mu)(\mathbf{X}-\mu)^\top\} = \Sigma$, so that Σ corresponds to the covariance matrix of $\mathbf{X}$. From equation (3),

$$\left\|A^{-1}(\mathbf{X}-\mu)\right\|^2 = \mathscr{G}^2 \|\mathscr{U}\| = \mathscr{G}^2, \tag{4}$$

where $\|\cdot\|$ is the usual euclidian norm in $\mathbb{R}^d$. Since in practice, one generally observes *copies* of the random vector $\mathbf{X}$, representation (4) enables to *recover* the radial part $\mathscr{G}$ that characterizes an elliptical distribution. This simple observation is at the base of the statistical methodologies developed in this paper. When $\mathscr{G}$ has a density, the density of $\mathbf{X}$ exists and is of the form

$$h_{\mu,\Sigma,g}(x_1,\ldots,x_d) = |\Sigma|^{-1/2} g\left\{\frac{1}{2}(\mathbf{x}-\mu)^\top \Sigma^{-1}(\mathbf{x}-\mu)\right\}, \tag{5}$$

in terms of a *density generator* $g : \mathbb{R}^+ \to \mathbb{R}^+$ standardized in such a way that

$$\int_0^\infty \gamma^{\frac{d}{2}-1} g(\gamma)\,\mathrm{d}\gamma = \frac{\Gamma(d/2)}{(2\pi)^{d/2}}.$$

In the sequel, one notes $\mathbf{X} \sim \mathscr{E}(\mu,\Sigma,g)$ whenever $\mathbf{X}$ admits the representation (3).

The marginal distributions of elliptically contoured random vectors belong to the same location-scale family whose standard cdf, which will be denoted Q_g in the sequel, is part of the univariate elliptical family. Using the transformation in polar coordinates described in the Appendix of the article by Landsman and Valdez (2003), one deduces that the density associated to Q_g is

$$q_g(x) = \frac{\pi^{d/2}}{\Gamma\left(\frac{d-1}{2}\right)} \int_{x^2}^\infty \left(\gamma - x^2\right)^{\frac{d-3}{2}} g\left(\frac{\gamma}{2}\right) \mathrm{d}\gamma.$$

It is clear from (5) that elliptical distributions are symmetric about μ. This formula also entails that the density of $A^{-1}(\mathbf{X}-\mu)$ is $g(\mathbf{x}^\top\mathbf{x}/2)$, from which it follows that the density and distribution functions of the squared radial part $\mathscr{G}^2$ are respectively

$$\psi_{\mathscr{G}^2}(\gamma) = \frac{\pi^{d/2}}{\Gamma(d/2)} \gamma^{\frac{d}{2}-1} g\left(\frac{\gamma}{2}\right) \quad \text{and} \quad \Psi_{\mathscr{G}^2}(\gamma) = \frac{\pi^{d/2}}{\Gamma(d/2)} \int_0^\gamma s^{\frac{d}{2}-1} g\left(\frac{s}{2}\right) \mathrm{d}s.$$

Some of the most popular elliptical distributions are now described; see Fang et al. (1990) for more details.

Example 1 (Normal and Student distributions)**.** *The multivariate Normal distribution, noted N, arises when*

$$g(\gamma) = \frac{1}{(2\pi)^{d/2}} \exp(-\gamma).$$

In that case, $\mathscr{G}^2$ is chi-squared distributed with d degrees of freedom and the marginal distribution Q_g in the standard case corresponds to that of the $\mathscr{N}(0,1)$ law, namely

$$Q_g(x) = \Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-s^2/2}\,\mathrm{d}s.$$

A generalization of the Normal distribution often used in robustness studies of Gaussian-based statistical methods (see Bilodeau and Brenner, 1999, for instance) is the multivariate Student distribution. In that case, the elliptical generator is proportional to

$$g_\nu(\gamma) = \left(1 + \frac{\gamma}{\nu}\right)^{-\frac{\nu}{2}-1},$$

where $\nu > 0$ is often referred to as the number of degrees of freedom. The law of $d\mathscr{G}^2$ is then the Fisher–Snedecor distribution with d and ν degrees of freedom. The Student distribution will be noted T_ν for the remaining of the paper. The case $\nu = 1$ corresponds to the Cauchy distribution.

Example 2 (Normal variance mixture models)**.** *The N and T_ν distributions can be viewed as particular cases of the general normal variance mixture models described, for example, by Klüppelberg and Kuhn (2009). Random variables in this class admit the stochastic representation $\mathbf{X} = \sqrt{W}\, A\, \mathbf{Z}$, where $\mathbf{Z}$ follows a d-dimensional standard Normal distribution and W is a non-negative random variable. One can then show that $\mathscr{G}^2 = W\, \mathbf{Z}^\top \mathbf{Z}/\mathrm{E}(W)$, so that $\mathscr{G}^2 \stackrel{d}{=} W \chi^2_d/\mathrm{E}(W)$. The Normal and Student laws appear when $W \equiv 1$ and $1/W \stackrel{d}{=} \chi^2_\nu/\nu$, respectively. Other popular models also possess this stochastic representation. For example, one recovers the multivariate Laplace distribution when W is an exponential random variable; for more details, see e.g. Eltoft et al. (2006).*

Example 3 (Pearson type II)**.** *An elliptical distribution is said to be unbounded if any subset of $\mathbb{R}^d$ has non-null probability; this is the case, in particular, for the Normal and Student distributions. The bounded case happens whenever the elliptical generator is such that $g(\gamma) = 0$ for all γ exceeding some constant $K > 0$. This property is shared by the Pearson type II distribution (referred to as Pe_θ in the sequel) whose generator is*

$$g_\theta(\gamma) = \frac{\Gamma\left(\frac{d}{2} + \theta + 1\right)}{\pi^{d/2}\Gamma(\theta+1)} (1 - 2\gamma)^\theta, \quad \gamma \in [0, 1/2],$$

where $\theta > -1$. In that case,

$$q_g(x) = \frac{\Gamma\left(\frac{d}{2} + \theta + 1\right)}{\sqrt{\pi}\,\Gamma\left(\frac{d}{2} + \theta + \frac{1}{2}\right)} \left(1 - x^2\right)^{\frac{d-1}{2}+\theta}, \quad x \in [-1, 1],$$

and one obtains that $Q_g(x) = \{1 + \mathrm{sign}(x) B(x^2)\}/2$, $x \in [-1,1]$, where B is the cdf of the $\mathrm{Beta}\left(\frac{1}{2}, \frac{d+1}{2} + \theta\right)$ distribution. As a consequence, $Q_g^{-1}(u) = \mathrm{sign}(2u-1)\sqrt{B^{-1}(|1-2u|)}$. One can also show that the distribution of $\mathscr{G}^2$ is $\mathrm{Beta}\left(\frac{d}{2}, \theta + 1\right)$.

Example 4 (Exponential power family)**.** *Consider an elliptical generator proportional to*

$$g_{\theta_1,\theta_2}(\gamma) = \exp\left(-\theta_1\, \gamma^{\theta_2}\right),$$

where $\theta_1, \theta_2 > 0$. One recovers the Normal distribution when $\theta_1 = \theta_2 = 1$ and Laplace's distribution when $\theta_1 = \sqrt{2}$, $\theta_2 = 1/2$. Another interesting sub-model is Kotz's distribution (see Kotz, 1975) that arises when $\theta_1 = \theta$ and $\theta_2 = 1$, i.e. $g_\theta(\gamma) \propto \exp(-\theta\,\gamma)$. It is worth noting that θ_1 only acts here as a scale parameter, so that it has no influence on the underlying copula. The copula associated with g_{θ_1,θ_2} is then the same as the one generated by $g_\theta(\gamma) \propto \exp(-\gamma^\theta)$. In that case, $\mathscr{G}^2 \stackrel{d}{=} Y^{1/\theta}$, where Y is Gamma distributed with parameters $d/(2\theta)$ and 2^θ.

2.2. Elliptical copulas and meta-elliptical models

Elliptical copulas are simply the dependence structures that one can extract from the distributions of elliptically contoured random vectors. Since copulas are invariant under strictly increasing transformations of the individual variables, the copula of $\mathbf{X} = (X_1,\ldots,X_d)^\top \sim \mathscr{E}(\mu,\Sigma,g)$ is the same as the copula of $\widetilde{\mathbf{X}} \sim \mathscr{E}(\mathbf{0},R,g)$, where R is the correlation matrix associated to Σ. Hence, without loss of generality, one can assume that $\mathbf{X} \sim \mathscr{E}(\mathbf{0},R,g)$ in the sequel. Considering the *inverse* version of Sklar's representation in (2), the copula of $\mathbf{X}$ is then

$$C_{\mathbf{X}}(\mathbf{u}) = \mathrm{P}\left\{\mathbf{X} \le \mathbf{Q}_g^{-1}(\mathbf{u})\right\}, \quad \mathbf{u} = (u_1,\ldots,u_d)^\top,$$

where

$$\mathbf{Q}_g^{-1}(\mathbf{u}) = \left(Q_g^{-1}(u_1),\ldots,Q_g^{-1}(u_d)\right)^\top$$

is the vector of componentwise inverses of the marginal distributions. From (5), one then deduces that the copula of $\mathbf{X}$ is

$$C_{R,g}(\mathbf{u}) = |R|^{-1/2} \int_{-\infty}^{Q_g^{-1}(u_1)} \cdots \int_{-\infty}^{Q_g^{-1}(u_d)} g\left(\frac{1}{2}\mathbf{x}^\top R^{-1}\mathbf{x}\right) \mathrm{d}\mathbf{x}, \tag{6}$$

from which it follows that the density of an elliptical copula is of the form

$$c_{R,g}(\mathbf{u}) = |R|^{-1/2}\, g\left(\frac{1}{2}\widetilde{\mathbf{x}}^\top R^{-1}\widetilde{\mathbf{x}}\right),$$

where $\widetilde{\mathbf{x}} = \mathbf{Q}_g^{-1}(\mathbf{u})$. In Figure 1, one can see the copula density plots (*i.e.* the normalized ranks) of the realization of 5 000 pairs from the T_1, T_3, T_9, N, Pe_1 and Pe_5 elliptical distributions.

Elliptical copulas lead directly to the meta-elliptical family of models. Indeed, a random vector $\mathbf{Y} = (Y_1,\ldots,Y_d)^\top$ is said to have a meta-elliptical distribution with correlation matrix R, generator g and marginals $\mathbf{F} = (F_1,\ldots,F_d)^\top$ if its underlying copula is $C_{R,g}$; it will be noted $\mathbf{Y} \sim \mathscr{ME}(R,g,\mathbf{F})$ in the sequel. The meta-Gaussian distribution seems to have been first considered by Krzysztofowicz and Kelly (1996) and Kelly and Krzysztofowicz (1997); the idea was extended to general meta-elliptical distributions by Fang et al. (2002).

The multidimensional probability integral transformation ensures that $\mathbf{F}(\mathbf{Y}) \sim C_{R,g}$, from which it follows that

$$\widetilde{\mathbf{X}} = \mathbf{Q}_g^{-1} \circ \mathbf{F}(\mathbf{Y}) \sim \mathscr{E}(\mathbf{0},R,g). \tag{7}$$

Therefore, a particular case of equation (4) with $\mu = (0,\ldots,0)^\top \in \mathbb{R}^d$ yields

$$\widetilde{\mathscr{G}}^2 = \widetilde{\mathbf{X}}^\top R^{-1}\widetilde{\mathbf{X}} \sim \Psi_{\mathscr{G}^2}. \tag{8}$$

Equation (8) is at the basis of the goodness-of-fit procedures developed in the next two sections.

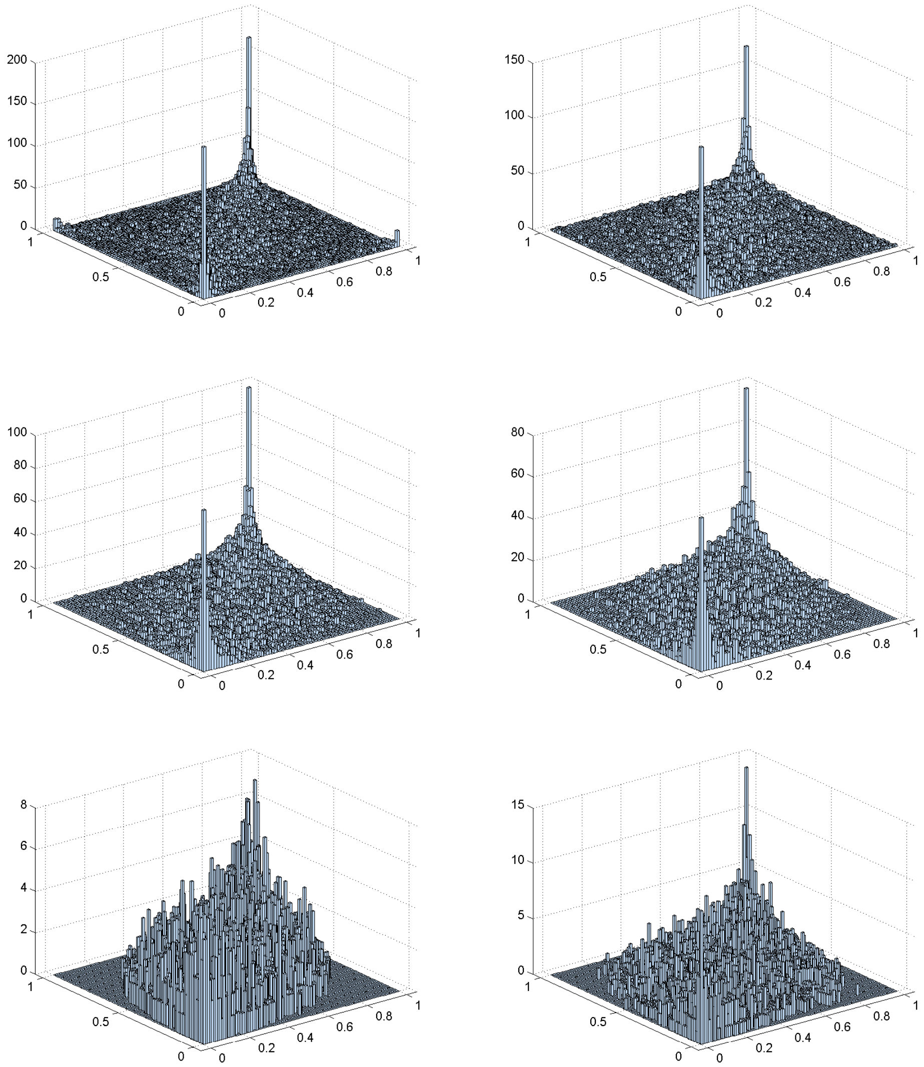

Figure 1: Three-dimensional histograms of the realizations of 5 000 pairs from the T_1 (upper left panel), T_3 (upper right panel), T_9 (middle left panel), Normal (middle right panel), Pe_1 (lower left panel) and Pe_5 (lower right panel) copulas when $\tau = .5$.

3. Goodness-of-fit procedure in the case of a fixed generator

3.1. Context

Let $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$, where $\mathbf{Y}_i = (Y_{i1}, \ldots, Y_{id})^\top$, be independent copies of a random vector $\mathbf{Y} \in \mathbb{R}^d$. The aim of this section is to develop a statistical method for the null and alternative hypotheses

$$\mathscr{H}_0 : \mathbf{Y} \sim \mathscr{M}\mathscr{E}(R, g, \mathbf{F}) \quad \text{and} \quad \mathscr{H}_1 : \mathbf{Y} \nsim \mathscr{M}\mathscr{E}(R, g, \mathbf{F}), \tag{9}$$

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where R and $\mathbf{F}$ are unknown, while the elliptical generator g is entirely known. This setting encompasses the cases where one wants to test for a meta-Gaussian distribution or a meta-Student distribution with a fixed number of degrees of freedom. The assumption of a fixed generator is relaxed in Section 4 in order to allow for parametric generators $g = g_\theta$, where $\theta \in \Theta$ is unknown.

The fact that the vector $\mathbf{F}$ is undetermined enables to infer on the dependence structure of $\mathbf{Y}$ regardless of the marginal behavior of its components. In general, non-parametric estimation of the marginal distributions uses empirical distribution functions, *i.e.* $\mathbf{F}$ is estimated by $\mathbf{F}_n = (F_{n1}, \ldots, F_{nd})^\top$, where for each $j \in \{1, \ldots, d\}$,

$$F_{nj}(y) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(Y_{ij} \leq y).$$

A model-free estimation of the correlation matrix R can also be accomplished. This is described in the next subsection.

3.2. Estimation of the correlation matrix

An interesting feature of elliptical copulas is the fact that the correlation matrix can be estimated independently of the form of the elliptical generator g. The idea is to exploit a relationship that exists between the entries of R and the value of Kendall's measure of association for each of the $d(d-1)/2$ pairs. Recall that Kendall's tau for a random pair (Y_1, Y_2) is defined by

$$\tau(Y_1, Y_2) = \mathrm{P}\{(Y_{11} - Y_{21})(Y_{12} - Y_{22}) > 0\} - \mathrm{P}\{(Y_{11} - Y_{21})(Y_{12} - Y_{22}) < 0\},$$

where (Y_{11}, Y_{12}) and (Y_{21}, Y_{22}) are independent copies of (Y_1, Y_2). For a meta-elliptically contoured random vector $\mathbf{Y} \sim \mathscr{ME}(R, g, \mathbf{F})$, one deduces from Lindskog et al. (2003) and Fang et al. (2002) that

$$\tau(Y_k, Y_\ell) = \frac{2}{\pi} \sin^{-1}(R_{k\ell})$$

for each (k, ℓ) such that $k < \ell \in \{1, \ldots, d\}$. Then, if $\tau_{n,k\ell}$ is an empirical version of $\tau(Y_k, Y_\ell)$, the correlation matrix R can easily be estimated by $R_n = (R_n)_{k\ell}$, where

$$(R_n)_{k\ell} = \sin\left(\frac{\pi}{2} \tau_{n,k\ell}\right). \tag{10}$$

Usually, $\tau_{n,k\ell}$ is the U-statistic defined by

$$\tau_{n,k\ell} = -1 + \frac{4}{n(n-1)} \sum_{i<j} \left\{\mathbb{I}\left(Y_{ki} < Y_{kj}, Y_{\ell i} < Y_{\ell j}\right) + \mathbb{I}\left(Y_{ki} > Y_{kj}, Y_{\ell i} > Y_{\ell j}\right)\right\},$$

which is unbiased for $\tau(Y_k, Y_\ell)$ and asymptotically Normal as $n \to \infty$; see the monograph by Lee (1990) for a thorough overview of the theory of U-statistics. The estimation method in (10) was employed by Genest et al. (2007) and Klüppelberg and Kuhn (2009). As a generalization of the large-sample behavior of Kendall's tau for a single pair, Klüppelberg and Kuhn (2009) obtained that

$$\sqrt{n}(R_n - R) \rightsquigarrow \mathscr{N}_{\frac{d(d-1)}{2}}(\mathbf{0}, \Lambda)$$

for some asymptotic variance-covariance matrix Λ that depends on the parameters R and g of the underlying elliptical copula $C_{R,g}$. In Table 1, the results of a simulation study that aims to evaluate the efficiency of R_n as an estimator of R are presented. To this end, let $S \in \mathbb{R}^{d\times d}$ be a symmetric matrix and define the mean of its off-diagonal elements by

$$\mathscr{L}(S) = \frac{2}{d(d-1)} \sum_{k<\ell\in\{1,\ldots,d\}} S_{k\ell}.$$

The mean-squared error criterion that was used in the investigation is

$$\mathrm{MSE}_R(R_n) = \mathrm{E}\left\{(\mathscr{L}(R_n) - \mathscr{L}(R))^2\right\}$$

and has been estimated by

$$\widehat{\mathrm{MSE}}_R(R_n) = \frac{1}{N}\sum_{i=1}^{N} \{\mathscr{L}(R_{ni}) - \mathscr{L}(R)\}^2,$$

where for each $i \in \{1,\ldots,N\}$, the sample correlation matrix R_{ni} is computed from a random sample of size n from the $\mathscr{E}(\mathbf{0},R,g)$ distribution for a given elliptical model generated by g. In our empirical study, only the equi-correlated case has been considered, *i.e.* $R_{k\ell} = \rho$ for all $k < \ell \in \{1,\ldots,d\}$. Note that the entries in Table 1 correspond to the estimated *standardized* MSE, *i.e.* $n \times \widehat{\mathrm{MSE}}_R(R_n)$. One can see that the standardized MSE's

(i) are quite equivalent for $n = 100$ and $n = 250$;

(ii) decrease monotonically as the strength of the pairwise dependence coefficient ρ increases when $d \in \{2,3\}$; for $d \in \{4,5\}$, they are equivalent when $\rho \in \{1/4,1/2\}$ and about half lower when $\rho = 3/4$;

(iii) tend to be lower as the dimension d increases;

(iv) depend on the elliptical generator : the lowest values appear for the Pearson type II distribution, while under the Student model, they are higher for low values of ν and tend to the values for the Normal distribution when $\nu \in \{6,9\}$, as was expected.

3.3. Test statistic and parametric bootstrap

Under the null hypothesis $\mathscr{H}_0$ stated in equation (9), each component of the random sample $\mathbf{Y}_1,\ldots,\mathbf{Y}_n$ follows a $\mathscr{ME}(R,g,\mathbf{F})$ distribution. Hence, from equation (7), one has for each $i \in \{1,\ldots,n\}$ that

$$\widetilde{\mathbf{X}}_i = \mathbf{Q}_g^{-1} \circ \mathbf{F}(\mathbf{Y}_i) \sim \mathscr{E}(\mathbf{0},R,g).$$

However, since the vector of the marginal distributions $\mathbf{F}$ is unknown, $\widetilde{\mathbf{X}}_1,\ldots,\widetilde{\mathbf{X}}_n$ are unobservable. One would rather work with the *pseudo-sample* $\mathbf{X}_{1,n},\ldots,\mathbf{X}_{n,n}$, where for each $i \in \{1,\ldots,n\}$,

$$\mathbf{X}_{i,n} = \mathbf{Q}_g^{-1} \circ \mathbf{F}_n(\mathbf{Y}_i). \tag{11}$$

Note that

$$\mathbf{F}_n(\mathbf{Y}_i) = \left(\frac{\mathrm{Rank}(Y_{i1})}{n},\ldots,\frac{\mathrm{Rank}(Y_{id})}{n}\right)^\top,$$

TABLE 1. *Estimation, based on 1 000 replicates, of n times the mean-squared error $\widehat{\text{MSE}}_R(R_n)$ of R_n for estimating the correlation matrix R in the equi-correlated case*

		$d=2$		$d=3$		$d=4$		$d=5$	
Model	ρ	$n=100$	$n=250$	$n=100$	$n=250$	$n=100$	$n=250$	$n=100$	$n=250$
	1/4	1.0703	1.0388	0.4800	0.4477	0.3118	0.2739	0.2464	0.2471
N	1/2	0.6267	0.6701	0.3920	0.3792	0.2860	0.2897	0.2582	0.2561
	3/4	0.2342	0.2303	0.1517	0.1555	0.1240	0.1362	0.1130	0.1218
	1/4	1.7105	1.7624	0.7896	0.7802	0.5111	0.5279	0.3864	0.3918
T_1	1/2	1.2420	1.1652	0.6402	0.6761	0.5114	0.4782	0.4091	0.3926
	3/4	0.5136	0.4710	0.3236	0.3305	0.2633	0.2638	0.2328	0.2321
	1/4	1.2011	1.2695	0.5831	0.5946	0.3716	0.3887	0.3179	0.2808
T_3	1/2	0.8690	0.8866	0.4808	0.4804	0.3757	0.3640	0.3184	0.3296
	3/4	0.3356	0.3499	0.2224	0.2143	0.1904	0.1831	0.1544	0.1463
	1/4	1.0920	1.1576	0.5013	0.4969	0.3350	0.3560	0.2830	0.2778
T_6	1/2	0.7941	0.7371	0.4605	0.4266	0.3107	0.3270	0.2872	0.3062
	3/4	0.2717	0.2670	0.1742	0.1944	0.1464	0.1568	0.1519	0.1384
	1/4	1.0349	1.1108	0.5090	0.5119	0.3330	0.3376	0.2620	0.2718
T_9	1/2	0.7330	0.6892	0.4164	0.4142	0.3161	0.3035	0.2757	0.2602
	3/4	0.2664	0.2666	0.1722	0.1744	0.1468	0.1581	0.1409	0.1372
	1/4	0.8081	0.7833	0.3983	0.3792	0.2906	0.2838	0.2189	0.2359
Pe_1	1/2	0.5037	0.5175	0.3058	0.3027	0.2662	0.2599	0.2224	0.2094
	3/4	0.1864	0.1580	0.1365	0.1260	0.1146	0.1133	0.1051	0.1021
	1/4	0.9074	0.8476	0.4072	0.3966	0.2755	0.2744	0.2326	0.2134
Pe_2	1/2	0.5513	0.5486	0.3335	0.3450	0.2582	0.2664	0.2381	0.2273
	3/4	0.2077	0.1769	0.1343	0.1374	0.1230	0.1059	0.1032	0.1003
	1/4	0.9727	0.9236	0.4414	0.4502	0.2666	0.2787	0.2204	0.2458
Pe_5	1/2	0.6237	0.5920	0.3748	0.3450	0.3120	0.2752	0.2201	0.2439
	3/4	0.2071	0.2044	0.1351	0.1339	0.1293	0.1160	0.1114	0.1067

where $\text{Rank}(Y_{ij})$ is the rank of Y_{ij} among $Y_{1j},\dots,Y_{nj}$; it is then clear that the upcoming statistical methods are entirely rank-based. Since $\mathbf{F}_n$ is a uniformly consistent estimator of $\mathbf{F}$, it is expected that $\mathbf{X}_{i,n}$ follows *approximately* a $\mathscr{E}(\mathbf{0},R,g)$ distribution. The construction of sample counterparts of the random variable $\mathscr{G}^2$ adds another level of complexity since R must be estimated, too. Starting from equation (8) and admitting that

$$\mathscr{G}_{i,n}^2 = \mathbf{X}_{i,n}^\top R_n^{-1} \mathbf{X}_{i,n} \tag{12}$$

for each $i \in \{1,\dots,n\}$, one then has a pseudo-sample $\mathscr{G}_{1,n}^2,\dots,\mathscr{G}_{n,n}^2$ that should behave asymptotically like the random variable $\mathscr{G}^2$. A non-parametric sample version of $\Psi_{\mathscr{G}^2}$ would then be

$$\Psi_n(\gamma) = \frac{1}{n}\sum_{i=1}^n \mathbb{I}\left(\mathscr{G}_{i,n}^2 \le \gamma\right).$$

Showing thoroughly that Ψ_n is indeed a good estimator of $\Psi_{\mathscr{G}^2}$ and that $\sqrt{n}(\Psi_n - \Psi_{\mathscr{G}^2})$ converges weakly to some limiting process are highly non-trivial problems that possibly require asymptotic tools developed by van der Vaart and Wellner (2007) for empirical processes indexed by estimated functions. Some remarks about these theoretical aspects, as well as other related issues, are given in Section 6.

Reformulating the null and alternative hypotheses as $\mathscr{H}_0 : \mathscr{G}^2 \sim \Psi_{\mathscr{G}^2}$ and $\mathscr{H}_1 : \mathscr{G}^2 \nsim \Psi_{\mathscr{G}^2}$, where $\Psi_{\mathscr{G}^2}$ is associated to the $\mathscr{E}(\mathbf{0}, R, g)$ distribution, a naturel test consist in rejecting $\mathscr{H}_0$ for large values of the Cramér–von Mises distance between Ψ_n and $\Psi_{\mathscr{G}^2}$, namely

$$V_n = n \int_0^\infty \{\Psi_n(\gamma) - \Psi_{\mathscr{G}^2}(\gamma)\}^2 \, \psi_{\mathscr{G}^2}(\gamma) \, \mathrm{d}\gamma.$$

One can easily show that

$$V_n = \frac{n}{3} + \sum_{i=1}^n \Psi_{\mathscr{G}^2}\left(\mathscr{G}^2_{i,n}\right) - \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n \left\{\Psi_{\mathscr{G}^2}\left(\mathscr{G}^2_{i,n} \vee \mathscr{G}^2_{j,n}\right)\right\}^2.$$

Assuming that the underlying law that generates the data is meta-elliptical, a test based on V_n will be consistent since $\mathscr{G}^2$ characterizes elliptical distributions. Note that a test statistic similar to V_n was proposed by Malevergne and Sornette (2003) in the special case of the Normal copula, where $\Psi_{\mathscr{G}^2}$ is the cdf of the chi-squared distribution with d degrees of freedom. Checking the Normal copula hypothesis could also possibly be done using a rank-based version of the normality test developed by Huffer and Park (2007).

The p-value of the test based on V_n will be computed by an application of the parametric bootstrap method described next. Showing the asymptotic validity, as $n \to \infty$, of the algorithm below could probably be obtained from results by Genest and Rémillard (2008). Based on the simulation results presented in subsection 3.4, however, there is every reason to believe that the method works well.

Algorithm 1. *Given a random sample $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$, the parametric bootstrap under the null hypothesis $\mathscr{H}_0$ of a meta-elliptical distribution $\mathscr{ME}(R, g, \mathbf{F})$ consists in*

(1) computing the sample correlation matrix R_n and the test statistic V_n from $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$;

(2) generating, for a sufficiently large $M \in \mathbb{N}$, independent random samples

$$\left(\mathbf{Y}_1^{(1)}, \ldots, \mathbf{Y}_n^{(1)}\right), \ldots, \left(\mathbf{Y}_1^{(M)}, \ldots, \mathbf{Y}_n^{(M)}\right)$$

from the $\mathscr{E}(0, R_n, g)$ distribution, and computing the test statistics $V_n^{(1)}, \ldots, V_n^{(M)}$ associated to each of these samples;

(3) obtaining the approximate p-value

$$p_{n,M} = \frac{1}{M} \sum_{h=1}^M \mathbb{I}\left(V_n^{(h)} > V_n\right).$$

An approximate confidence interval for $\Psi_{\mathscr{G}^2}$ can be built from Algorithm 1 and the fact that $\sqrt{n}(\Psi_n - \Psi_{\mathscr{G}^2})$ converges weakly. Suppose one is seeking for a $q_\alpha \in \mathbb{R}^+$ such that

$$\lim_{n\to\infty} \mathrm{P}\left(\sqrt{n}\,\|\Psi_n - \Psi_{\mathscr{G}^2}\|_\infty \le q_\alpha\right) = 1 - \alpha,$$

where $1-\alpha$ is the desired confidence level. Take $\widehat{q}_\alpha$ as the $100\times(1-\alpha)$-th percentile of

$$\sqrt{n}\left\|\Psi_n^{(1)}-\Psi_{\mathscr{G}^2}\right\|_\infty,\ldots,\sqrt{n}\left\|\Psi_n^{(M)}-\Psi_{\mathscr{G}^2}\right\|_\infty,$$

where $\Psi_n^{(h)}$ is the empirical distribution function of $\mathscr{G}_{1,n}^{(h)},\ldots,\mathscr{G}_{n,n}^{(h)}$ computed from the parametric bootstrap sample $\mathbf{Y}_1^{(h)},\ldots,\mathbf{Y}_n^{(h)}$. Then, the confidence band is defined for each $x\in\mathbb{R}^+$ by

$$\mathbb{CB}_\alpha(x)=\left[\Psi_n(x)-\frac{\widehat{q}_\alpha}{\sqrt{n}},\ \Psi_n(x)+\frac{\widehat{q}_\alpha}{\sqrt{n}}\right].$$

Remark 1. *It is possible to extend the statistical method in this section to cases when the form of $\Psi_{\mathscr{G}^2}$ in the model $\mathscr{E}(\mathbf{0},R,g)$ is not explicit. The idea is to replace the test statistic V_n by*

$$V_{n,N}=n\int_0^\infty\{\Psi_n(x)-\Psi_N(x)\}^2\,\mathrm{d}x,$$

where Ψ_N is the empirical distribution function of $\mathscr{G}_{1,N}^2,\ldots,\mathscr{G}_{N,N}^2$, where for each $i\in\{1,\ldots,N\}$, $\mathscr{G}_{i,N}^2=\mathbf{Y}_i^\top\mathbf{Y}_i$ and $\mathbf{Y}_i\sim\mathscr{E}(\mathbf{0},I_d,g)$, with $I_d\in\mathbb{R}^{d\times d}$ the identity matrix. Note that one could also use the weight function $\mathrm{d}\Psi_N$ in the definition of $V_{n,N}$, but $\mathrm{d}x$ is chosen here for computational convenience. In that case, one obtains a simple formula, namely

$$V_{n,N}=\frac{2}{N}\sum_{i=1}^n\sum_{j=1}^N\left(\mathscr{G}_{i,n}^2\vee\mathscr{G}_{j,N}^2\right)-\frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n\left(\mathscr{G}_{i,n}^2\vee\mathscr{G}_{j,n}^2\right)-\frac{1}{n}\sum_{i=1}^N\sum_{j=1}^N\left(\mathscr{G}_{i,N}^2\vee\mathscr{G}_{j,N}^2\right).$$

The p-value can then be computed from a slight modification of Algorithm 1.

3.4. Investigation of the size and power of the test

The size and power of the test based on V_n have been investigated with the help of Monte-Carlo simulations. The elliptical copulas considered under the null hypothesis are those extracted from the T_1, T_3, T_6, T_9, N, Pe_1, Pe_2 and Pe_5 distributions. The probability of rejecting the null hypothesis is always estimated from 1 000 replicates and $M=1\ 000$ bootstrap samples. In addition to the eight elliptical models listed above, the non-elliptical Clayton (CL) and Gumbel–Hougaard (GH) copulas were used as alternatives; see Nelsen (2006) for more details on these two models. For the elliptical distributions, the parameter $\rho\in\{1/4,1/2,3/4\}$ corresponds to the correlation coefficient, *i.e.* $R_{21}=R_{12}=\rho$, while for the Clayton and Gumbel–Hougaard models, ρ corresponds to Spearman's rank correlation coefficient. The latter depends on the copula C of a bivariate population via $\rho_S=12\int_0^1\int_0^1\{C(u,v)-uv\}\mathrm{d}u\mathrm{d}v$.

The results for the bivariate case when $n=100$ are reported in Table 2. One first sees that the test keeps its 5% nominal level quite well under most of the considered scenarios. The power of the test is also generally very good, which shows that the method discriminates well between the various elliptical models. As expected, similar models are harder to distinguish, for example T_3 vs T_6, T_9 vs N and Pe_1 vs Pe_2. It is to note that the value of ρ has little influence on the power results. The alternative model which is the most easily rejected is T_1, even when the null hypothesis is the closely related T_3 distribution. As expected, two Student distributions are harder to distinguish when their associated number of degrees of freedom are relatively large, since then they are both close to the Normal distribution. Finally, the probability of rejecting the null hypothesis when the data come from the Clayton or the Gumbel–Hougaard copulas are generally high when the null hypothesis is one of the Pearson type II distributions. For some reason, these departures from the null hypothesis are not well detected when $\mathscr{H}_0$ is the T_6 or T_9 model. The results for $n = 250$, not reported here, show that the power increases as n increases, as expected; see Bellerive (2012).

TABLE 2. *Percentages of rejection, as estimated from 1 000 replicates, of the goodness-of-fit tests for bivariate meta-elliptical models when* $n = 100$

Model under $\mathscr{H}_1$	ρ	Model under $\mathscr{H}_0$: T_1	T_3	T_6	T_9	N	Pe_1	Pe_2	Pe_5
T_1	1/4	5.9	84.6	98.1	99.0	99.9	100.0	100.0	100.0
	1/2	5.2	80.0	96.2	97.8	99.7	100.0	100.0	99.9
	3/4	5.5	68.8	91.1	92.8	98.2	100.0	99.7	99.2
T_3	1/4	75.3	3.0	21.2	31.8	59.7	96.6	90.7	79.7
	1/2	69.3	6.6	21.3	29.7	58.8	95.3	85.3	80.2
	3/4	59.4	5.3	17.4	30.0	54.7	95.1	86.0	75.4
T_6	1/4	94.0	8.1	4.7	8.8	23.3	80.1	64.2	42.8
	1/2	92.2	6.7	4.9	6.9	24.9	81.3	61.7	45.0
	3/4	85.5	6.3	5.6	6.4	23.1	79.3	60.8	43.1
T_9	1/4	97.2	10.6	4.5	5.1	15.8	72.6	52.6	31.7
	1/2	96.1	10.4	4.4	4.9	14.6	70.4	47.0	27.7
	3/4	93.9	7.0	4.4	5.1	13.0	69.5	47.8	28.3
N	1/4	99.5	25.7	8.3	3.8	5.5	44.2	23.7	11.3
	1/2	99.6	25.7	6.0	4.4	4.4	44.5	19.7	10.2
	3/4	98.4	17.0	5.7	5.1	4.9	43.7	18.5	10.8
Pe_1	1/4	100.0	81.1	45.3	34.3	14.8	4.8	5.8	7.6
	1/2	100.0	79.0	40.8	29.6	14.3	5.8	6.3	6.6
	3/4	100.0	80.1	37.0	29.5	14.4	4.4	6.4	7.5
Pe_2	1/4	100.0	62.7	25.4	15.9	7.2	10.7	6.0	3.1
	1/2	100.0	64.5	24.3	17.5	6.8	10.7	4.2	6.4
	3/4	100.0	59.0	21.4	15.6	6.9	10.2	4.9	3.4
Pe_5	1/4	99.9	41.6	13.9	7.7	4.4	22.7	11.8	5.2
	1/2	99.8	42.6	12.8	7.1	4.4	22.3	9.4	7.1
	3/4	100.0	39.5	11.5	7.3	4.4	24.5	8.4	5.1
CL	1/4	99.5	19.7	5.2	4.3	6.4	50.3	31.8	13.7
	1/2	98.9	18.0	3.1	5.1	8.3	57.8	36.2	20.1
	3/4	97.9	13.2	4.4	4.2	10.4	63.4	37.7	22.9
GH	1/4	98.3	11.5	3.5	4.1	12.8	68.4	43.9	26.9
	1/2	91.5	6.2	4.2	6.1	17.4	76.2	53.3	35.6
	3/4	57.3	6.2	4.1	7.8	12.8	76.1	53.5	39.7

The multivariate case has also been considered. In Table 3, the results are presented for $d \in \{3,4\}$ and the same alternatives that were used for the results presented in Table 2, except that the Clayton and Gumbel–Hougaard copulas are now excluded. Only the case when $n = 100$ and R is equi-correlated are presented here. Many of the comments stated for the results in Table 2 also apply here: the tests keep their 5% nominal level quite well, and the probabilities of rejection are similar for the values of ρ that were considered. For a given scenario under $\mathscr{H}_1$, the power tends to be higher in dimension $d = 4$ compared to $d = 3$. Finally note that the T_1 model is highly rejected for all the null hypotheses.

4. Goodness-of-fit procedure in the case of a parametric generator

4.1. A minimum-distance method

Suppose that the generator of an elliptical distribution depends on some unknown parameter $\theta \in \Theta \subseteq \mathbb{R}^p$, *i.e.* $g = g_\theta$. In that case, the null and alternative hypotheses are

$$\mathscr{H}_0^\star : \mathbf{Y} \sim \mathscr{M}\mathscr{E}(R, g_\theta, \mathbf{F}) \quad \text{and} \quad \mathscr{H}_1^\star : \mathbf{Y} \nsim \mathscr{M}\mathscr{E}(R, g_\theta, \mathbf{F}),$$

where R, θ and $\mathbf{F}$ are unknown. In what follows, one writes Q_θ for the marginal distributions of the elliptical law generated by g_θ. A *parametric* version of (11) is then given by

$$\mathbf{X}_{i,n}(\theta) = \mathbf{Q}_\theta^{-1} \circ \mathbf{F}_n(\mathbf{Y}_i),$$

where $\mathbf{Q}_\theta^{-1} = (Q_\theta^{-1}, \ldots, Q_\theta^{-1})^\top$, yielding the parametric pseudo-sample $\mathbf{X}_{1,n}(\theta), \ldots, \mathbf{X}_{n,n}(\theta)$. Similarly, a parametric version of (12) is

$$\mathscr{G}_{i,n}^2(\theta) = \mathbf{X}_{i,n}(\theta)^\top R_n^{-1} \mathbf{X}_{i,n}(\theta),$$

where R_n is the estimator described in subsection 3.2. For a fixed value of θ, an empirical version of the distribution $\Psi_{\mathscr{G}^2}(\gamma, \theta)$ of the squared radial part $\mathscr{G}^2$ under the model $\mathscr{E}(\mathbf{0}, R, g_\theta)$ is

$$\Psi_n(\gamma, \theta) = \frac{1}{n} \sum_{i=1}^n \mathbb{I}\left\{\mathscr{G}_{i,n}^2(\theta) \le \gamma\right\}.$$

Since $\theta \in \Theta$ is unknown, the proposed goodness-of-fit procedure will be based on the minimum-distance statistic

$$W_n = \inf_{\theta \in \Theta} W_n(\theta),$$

where

$$\begin{aligned} W_n(\theta) &= n \int_0^\infty \{\Psi_n(\gamma, \theta) - \Psi_{\mathscr{G}^2}(\gamma, \theta)\}^2 \psi_{\mathscr{G}^2}(\gamma, \theta)\, \mathrm{d}\gamma \\ &= \frac{n}{3} + \sum_{i=1}^n \Psi_{\mathscr{G}^2}\left\{\mathscr{G}_{i,n}^2(\theta), \theta\right\} - \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n \left[\Psi_{\mathscr{G}^2}\left\{\mathscr{G}_{i,n}^2(\theta) \vee \mathscr{G}_{j,n}^2(\theta)\right\}, \theta\right]^2 \end{aligned}$$

is the Cramér–von Mises distance between $F_n(\gamma, \theta)$ and $F_{\mathscr{G}^2}(\gamma, \theta)$. In practice, W_n will be approximated on a grid $(\theta_1, \ldots, \theta_T)$ of Θ in such a way that

$$W_n \approx \min_{\theta_1, \ldots, \theta_T \in \Theta} W_n(\theta).$$

Table 3: Percentages of rejection, as estimated from 1 000 replicates, of the goodness-of-fit tests for meta-elliptical models in the equi-correlated case for $d \in \{3,4\}$ and $n = 100$

Model under $\mathscr{H}_1$	ρ	$d=3$, Model under $\mathscr{H}_0$: T_1	T_3	T_6	T_9	N	Pe_1	Pe_2	Pe_5	$d=4$, Model under $\mathscr{H}_0$: T_1	T_3	T_6	T_9	N	Pe_1	Pe_2	Pe_5
T_1	1/4	3.8	97.0	100.0	100.0	100.0	100.0	100.0	100.0	5.4	99.3	100.0	100.0	100.0	100.0	100.0	100.0
	1/2	4.5	95.2	100.0	100.0	100.0	100.0	100.0	100.0	4.0	98.7	100.0	100.0	100.0	100.0	100.0	100.0
	3/4	4.2	88.3	98.6	99.6	100.0	100.0	100.0	100.0	3.2	94.5	99.7	100.0	100.0	100.0	100.0	100.0
T_3	1/4	98.3	5.1	37.8	58.2	91.0	100.0	99.9	98.1	99.8	4.9	49.8	72.8	98.3	100.0	100.0	99.9
	1/2	96.7	3.7	38.7	55.4	90.4	99.9	99.7	98.0	99.8	3.3	46.0	70.8	98.0	100.0	100.0	99.9
	3/4	93.3	4.1	26.2	46.8	84.6	99.9	99.8	97.4	99.2	4.3	37.0	61.9	95.7	100.0	100.0	99.6
T_6	1/4	100.0	16.6	4.9	12.1	48.0	98.5	95.5	82.3	100.0	29.8	5.2	13.5	69.0	99.7	98.7	93.3
	1/2	99.9	17.0	5.3	11.0	46.6	96.7	94.3	81.7	100.0	30.8	6.0	13.0	66.8	99.5	99.3	94.2
	3/4	99.7	14.8	4.8	10.7	44.8	97.2	91.9	77.0	100.0	30.7	4.8	10.5	58.7	99.8	98.6	89.4
T_9	1/4	100.0	30.8	5.6	6.0	27.3	94.4	86.3	60.9	100.0	61.2	6.5	4.4	42.3	98.1	94.4	78.2
	1/2	100.0	32.7	4.1	5.1	26.0	93.2	87.3	62.6	100.0	57.0	5.2	4.4	41.2	98.3	95.7	76.2
	3/4	100.0	27.8	5.0	6.1	23.9	93.8	82.2	57.2	100.0	55.8	8.2	4.7	33.0	98.0	95.2	72.5
N	1/4	100.0	75.1	23.6	10.2	5.0	63.6	44.0	17.5	100.0	98.2	47.5	16.7	5.4	72.3	50.7	19.8
	1/2	100.0	76.9	19.9	7.8	5.4	65.9	44.3	19.5	100.0	97.0	44.9	18.8	6.4	74.5	57.9	18.2
	3/4	100.0	68.9	19.0	10.4	3.7	65.7	38.8	16.4	100.0	96.6	43.5	17.9	5.2	74.6	54.0	20.7
Pe_1	1/4	100.0	100.0	97.5	89.2	51.0	5.8	9.9	22.4	100.0	100.0	99.9	99.5	77.7	4.2	14.1	43.1
	1/2	100.0	99.8	96.6	85.7	46.8	5.6	8.7	20.4	100.0	100.0	100.0	99.7	82.6	3.7	13.9	37.0
	3/4	100.0	100.0	95.5	84.3	47.1	4.8	9.4	21.6	100.0	100.0	99.9	99.5	77.4	5.9	10.4	33.3
Pe_2	1/4	100.0	99.6	82.7	65.2	21.6	8.5	5.6	6.5	100.0	100.0	98.8	93.2	41.1	4.8	4.1	14.2
	1/2	100.0	99.8	80.8	62.5	23.2	7.8	5.4	8.9	100.0	100.0	98.9	92.5	44.1	5.6	4.8	12.6
	3/4	100.0	99.6	82.3	59.9	21.6	6.2	5.5	7.5	100.0	100.0	98.5	92.8	43.8	5.5	5.3	12.4
Pe_5	1/4	100.0	95.5	53.3	29.2	5.8	28.2	14.1	6.0	100.0	100.0	88.1	65.0	13.2	24.7	11.8	4.5
	1/2	100.0	95.6	52.1	30.1	6.9	28.7	12.8	6.2	100.0	100.0	88.7	60.8	12.2	26.0	14.1	4.5
	3/4	100.0	95.3	50.4	28.0	6.3	26.4	11.9	5.3	100.0	99.9	88.0	61.5	13.3	26.0	11.2	5.4

An estimator of θ that is implicit in the definition of W_n is

$$\theta_n = \arg\min_{\theta\in\Theta} W_n(\theta). \tag{13}$$

In order to compute a p-value for the test based on W_n, an extension of the parametric bootstrap described in Algorithm 1 is needed; it is described next.

Algorithm 2. *Given a random sample $\mathbf{Y}_1,\ldots,\mathbf{Y}_n$, the parametric bootstrap under the null hypothesis $\mathscr{H}_0^\star$ of a meta-elliptical distribution $\mathscr{M}\mathscr{E}(R,g_\theta,\mathbf{F})$ consists in*

(1) computing the sample correlation matrix R_n, the estimator θ_n and the test statistic W_n from $\mathbf{Y}_1,\ldots,\mathbf{Y}_n$;

(2) generating, for a sufficiently large $M\in\mathbb{N}$, independent random samples

$$\left(\mathbf{Y}_1^{(1)},\ldots,\mathbf{Y}_n^{(1)}\right),\ldots,\left(\mathbf{Y}_1^{(M)},\ldots,\mathbf{Y}_n^{(M)}\right)$$

from the $\mathscr{E}(0,R_n,g_{\theta_n})$ distribution, and computing the test statistics $W_n^{(1)},\ldots,W_n^{(M)}$ associated to each of these samples;

(3) obtaining the approximate p-value

$$p^\star_{n,M} = \frac{1}{M}\sum_{h=1}^{M}\mathbb{I}\left(W_n^{(h)} > W_n\right).$$

In the light of the empirical results presented in the next subsection, this algorithm works well.

4.2. Investigation of the power of the test

The power of the test based on W_n has been investigated when $\mathscr{H}_0^\star$ is either the Student or the Pearson type II copula. The alternatives are Student copulas with $\nu\in\{1,3,6,9\}$ degrees of freedom and Pearson type II copulas with $\theta\in\{1,2,5\}$. The results for $n\in\{50,100\}$ and $d\in\{2,3\}$ are in Table 4. When $d=3$, the elliptical models under which the data are simulated are restricted to the equi-correlated case $R_{12}=R_{13}=R_{23}=1/2$.

One can say that the tests keep their 5% nominal level very well when the null hypothesis is a Student copula. The tests are not as good under Pearson type II alternatives when $n=50$, but it improves markedly when $n=100$. Concerning the power of the tests, it obviously increases as the sample size increases. It is very high when testing for a Pearson type II copula under Student alternatives, both in the bivariate and in the trivariate case. Of course, the power decreases as the number of degrees of freedom increases, since then it gets closer to the limiting Normal case. The power of the test is not as good when testing for a Student copula under Pearson type II alternatives. It is significantly higher in dimension $d=3$ compared to $d=2$.

Results not presented here show that under Normal alternatives, the rejection rates are not as close to 5% as one could expect. It can be explained by the fact that the test statistic W_n is evaluated on a grid, so strictly speaking, the Normal case which occurs as the parameter value tends to infinity is not considered. It is recommended to test the Normal copula hypothesis using V_n in a first step, and then test for a general Student and / or Pearson type II dependence structures only in case it is rejected.

TABLE 4. *Percentages of rejection, as estimated from 1 000 replicates, of the goodness-of-fit tests for the parametric meta-elliptical models T and* Pe *in the equi-correlated case with $\rho = 1/2$ for $d \in \{2,3\}$ when $n \in \{50,100\}$*

	$n = 50$				$n = 100$			
	$d = 2$		$d = 3$		$d = 2$		$d = 3$	
Model	Model under $\mathscr{H}_0$		Model under $\mathscr{H}_0$		Model under $\mathscr{H}_0$		Model under $\mathscr{H}_0$	
under $\mathscr{H}_1$	T	Pe	T	Pe	T	Pe	T	Pe
T_1	6.0	95.8	6.0	98.7	4.7	99.5	6.1	100.0
T_3	4.2	58.1	6.6	66.2	8.0	71.1	5.0	96.7
T_6	4.7	35.4	8.8	34.2	3.3	41.4	4.6	70.6
T_9	5.7	25.3	7.7	23.3	3.8	25.8	3.5	54.6
Pe_1	15.3	8.0	54.1	10.2	33.4	6.4	88.9	9.1
Pe_2	11.4	7.7	32.4	7.3	19.5	4.1	62.5	3.9
Pe_5	7.9	8.0	19.2	6.5	9.8	4.4	29.8	6.0

5. Illustrations on real data

5.1. *The Danish fire insurance data*

These data consists of $n = 2\,167$ insurance claims relative to fire losses collected at the Copenhagen Reinsurance Company covering the years 1980–1990. This data set has been considered by Rytgaard (1996) and Embrechts et al. (1997), among others. Here, an observation refers to a loss of property and a loss of contents. When only the non-zero values are taken into account, this results in a data set of $n = 604$ pairs. The scatter plot and density plot of the normalized ranks are presented on Figure 2.

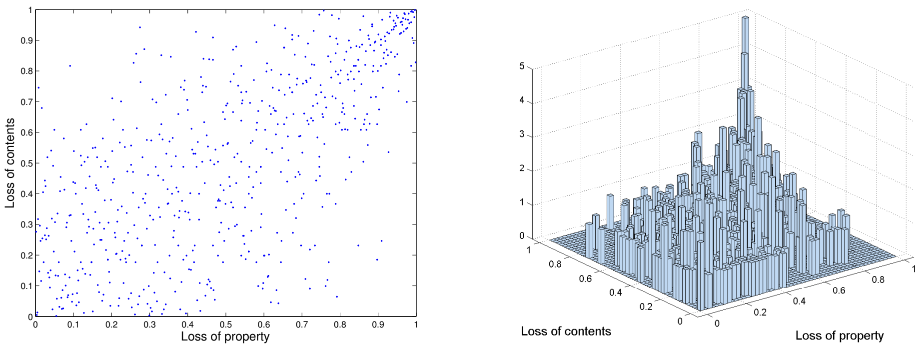

Figure 2: Scatterplot (left panel) and density plot (right panel) of the normalized ranks for the Danish insurance data set.

In order to find a suitable elliptical copula for these data, the test based on the statistic V_n has been applied for various models; the results of the analysis are presented in the left part of Table 5. From the entries therein, the T_6, T_9 et Normal models are not rejected at the 5% significance level. These results are somewhat confirmed by the test based on W_n, where the Student model

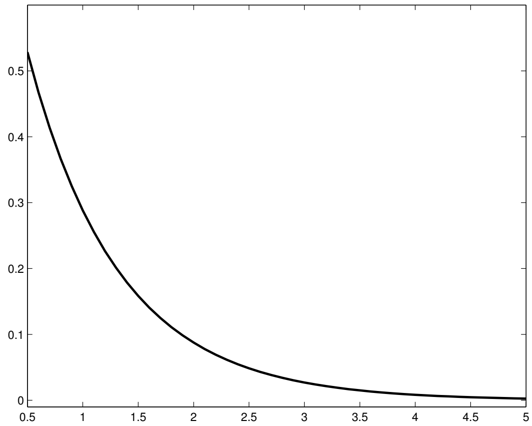

Figure 3: Probability that both claims in the Danish fire insurance data set exceed x_0 (in millions of Euros).

is highly accepted ($W_n = 0.0208$, $p^{\star}_{n,M} = 0.979$, $\theta_n = 10.75$), while the Pearson type II copula is also accepted, but with a much smaller p-value ($p^{\star}_{n,M} = 0.088$). It the latter case, the fact that $\theta_n = 15$ indicates that a Normal dependence structure would probably be suitable.

TABLE 5. *Results of the goodness-of-fit tests on the Danish fire insurance and Upper Mississippi river data sets for the null hypothesis of a meta-elliptical distribution with a fixed generator (p-values and critical values estimated from* $M = 1\,000$ *parametric bootstrap samples)*

Model	Danish fire insurance data			Upper Mississippi river data		
under $\mathscr{H}_0$	V_n	p-value	Critical value	V_n	p-value	Critical value
T_1	1.7674	< 0.01	0.1228	13.5651	0.0520	14.1875
T_3	0.2074	0.0040	0.1252	9.2496	0.0800	11.8633
T_6	0.0400	0.7300	0.1178	7.6496	0.1040	13.7071
T_9	0.0220	0.9760	0.1224	7.1185	0.1040	10.5601
T_{15}	0.0233	0.9760	0.1149	6.7834	0.0400	6.2213
N	0.0516	0.5810	0.1176	6.7896	0.0760	8.3766
Pe_1	0.4882	< 0.01	0.1223	9.9542	0.0680	11.8050
Pe_2	0.2853	< 0.01	0.1235	8.4941	0.1080	11.8820
Pe_5	0.1370	0.0270	0.1176	7.7326	0.0560	9.6842
Pe_{10}	0.0866	0.1960	0.1240	7.9703	0.0800	9.4569

In order to achieve a complete modeling, appropriate marginal distributions must be formally selected once a suitable copula has been chosen. This step can be accomplished using a minimum-distance approach similar to that employed in Section 4 for the choice of a parametric elliptical copula. To describe the method briefly, let $F(y,\beta)$ be a univariate distribution function, where $\beta \in \mathbb{B}$, and suppose one wants to test for $\mathscr{H}_0 : F \in \{F(\cdot,\beta); \beta \in \mathbb{B}\}$ and $\mathscr{H}_1 : F \notin \{F(\cdot,\beta); \beta \in \mathbb{B}\}$

on the basis of $Y_1, \ldots, Y_n$ i.i.d. F. First define

$$S_n(\beta) = n \int_{\mathbb{R}} \{F_n(y) - F(y, \beta)\}^2 \, \mathrm{d}F(y, \beta),$$

and consider the goodness-of-fit test statistic and estimator

$$S_n = \inf_{\beta \in \mathbb{B}} S_n(\beta) \quad \text{and} \quad \beta_n = \arg\min_{\beta \in \mathbb{B}} S_n(\beta).$$

The null hypothesis can be rejected whenever S_n exceeds a critical value estimated from an application of the parametric bootstrap. See Boos (1981) and Parr and Schucany (1982) for more details about this minimum-distance goodness-of-fit method in the univariate case.

Of a particular interest for insurers is the probability that the two claims exceed some large value x_0 simultaneously, *i.e.* $\mathrm{P}(X > x_0, Y > x_0)$. From Sklar's Theorem, one can write

$$\mathrm{P}(X > x_0, Y > x_0) = 1 - F_X(x_0) - F_Y(x_0) + C\{F_X(x_0), F_Y(x_0)\},$$

where F_X, F_Y are the marginal distributions. In view of the histograms of both variables (not presented here), exponential distributions could possibly be appropriate models for the individual random variables. Considering a bivariate Normal copula C_ρ for modeling the dependence in the pair, which is radially symmetric, the probability is estimated by

$$\mathrm{P}(X > x_0, Y > x_0) = C_{\rho_n}\left(e^{-x_0/\hat{\lambda}_X}, e^{-x_0/\hat{\lambda}_Y}\right),$$

where $\rho_n = 0.67$, $\hat{\lambda}_X = 2.35$ and $\hat{\lambda}_Y = 0.86$. The graphic of this probability as a function of x_0 is presented in Figure 3. As one can see, the probability of both claims exceeding five million Euros is very close to 0 under this model. Of course, another choice of a copula and / or margins would have an influence on this probability.

5.2. The Upper Mississippi river data

The Mississippi river crosses the United States (US) from the North (state of Minnesota) to the South (state of Louisiana), where it discharges water into the Gulf of Mexico. This river is part of the Jefferson–Missouri–Mississippi system that drains about 40% of the US territory. Here, a data set built by Ghizzoni et al. (2012) will be analyzed in the light of our goodness-of-fit methods for the selection of an elliptical copula. The latter is based on a large database of simultaneous measures of discharge taken at $d = 18$ stations on the upper part of the Mississippi river from the years 1943 to 2008. The goal of Ghizzoni et al. (2012) was to fit either the Student copula (combined with log-normal and generalized extreme-value marginals) or the skew Student distribution to $n = 89$ selected events. However, no formal test was applied in order to validate or discard these models. Using a two-step maximum likelihood approach called *the inference for margins* method (see Joe, 2005), where some chosen parametric marginal distributions are first estimated and then *plugged* into the full likelihood, they obtained $\widehat{\nu} = 9.43$ for the number of degrees of freedom of the Student copula.

The results of the statistical analysis based on the test statistic V_n are presented in the right part of Table 5. Since the estimated correlation matrix R_n was singular, a modified version was used instead. Indeed, two among the eighteen eigenvalues in the diagonal matrix D, such that $R_n = VDV^\top$, were slightly negative. This problem has been overcome by using a variation of the method proposed by Higham (2002) that consists in replacing D by $\widetilde{D}$ in which the negative eigenvalues are replaced by some small value, in this case .01. The resulting matrix $\widetilde{R}_n = V\widetilde{D}V^\top$ is positive definite and appears to be the closest, with respect to the euclidian norm, to R_n in the space of positive definite matrices. The results in Table 5 are quite congruent with those of Ghizzoni et al. (2012). In fact, all the T_ν copulas that were tested were accepted at the 5% level (except the T_{15} copula); however, the p-values are only slightly larger than .05. These findings are also to be shaded by the fact that all the Pearson type II copulas were also accepted. It just illustrates the general difficulty of using goodness-of-fit methods to distinguish between various models in small data sets. Such disappointing results were also obtained by Genest et al. (2007) for a trivariate data set of size $n = 47$. When testing for the hypothesis of a Student copula model with an unknown number of degrees of freedom, the minimum-distance statistic yields $W_n = 7.65$, $p^\star_{n,M} = 0.14$ and $\theta_n = 3.5$; for the parametric Pearson type II model, $W_n = 7.11$, $p^\star_{n,M} = 0.09$ and $\theta_n = 17.00$. Once again, both models are accepted by a small amount, in conformity with the results for a fixed generator.

5.3. *The Oil currency data*

This data set consists of daily log-returns of the oil price, Standard & Poor's 500, and of six currency exchange rates from May 1985 to June 2004. It has been analyzed by Klüppelberg and Kuhn (2009) to illustrate their newly introduced *copula structure analysis*. When considering the last $n = 904$ observations, these authors concluded to the rejection of the Normal copula and to the acceptance of a Student dependence structure using a goodness-of-fit test introduced by Berg and Bakken (2007). Based on our results presented on the left part of Table 6, the T_{15} copula is clearly the right choice. Indeed, all other models considered are strongly rejected. When performing the minimum-distance test for a global Student dependence structure, one obtains $W_n = 0.0264$, $p^\star_{n,M} = 1$ and $\theta_n = 16.00$. These conclusions agree with that observed within the graphics of Ψ_n versus $\Psi_{\mathscr{G}^2}$. Those when the null hypothesis is the N, T_6, T_{16} and Pe_2 copulas are presented in Figure 4.

5.4. *The Cook & Johnson data set revisited*

As a final illustration, the Uranium exploration data set originally considered by Cook and Johnson (1981, 1986) was analyzed. These $d = 7$ dimensional data consist of 655 chemical analyses from water samples collected from the Montrose quadrangle of western Colorado (USA). Concentrations were measured for uranium, lithium, cobalt, potassium, caesium, scandium and titanium. A pair-by-pair copula modeling has been considered by Genest et al. (2006) for many Archimedean copula families.

In view of results in the right part of Table 6, the Student copulas with $\nu = 6$ and $\nu = 9$ degrees of freedom are not rejected at the 5% level, while the other models are clearly rejected. Again, these results are confirmed by the test based on W_n, where the parametric Student model is accepted ($W_n = 0.0197$, $p^\star_{n,M} = 0.995$, $\theta_n = 7.75$) and the Pearson type II copula is rejected ($W_n = 4.3232$, $p^\star_{n,M} < 0.01$). A T_8 copula would then be suitable for these data.

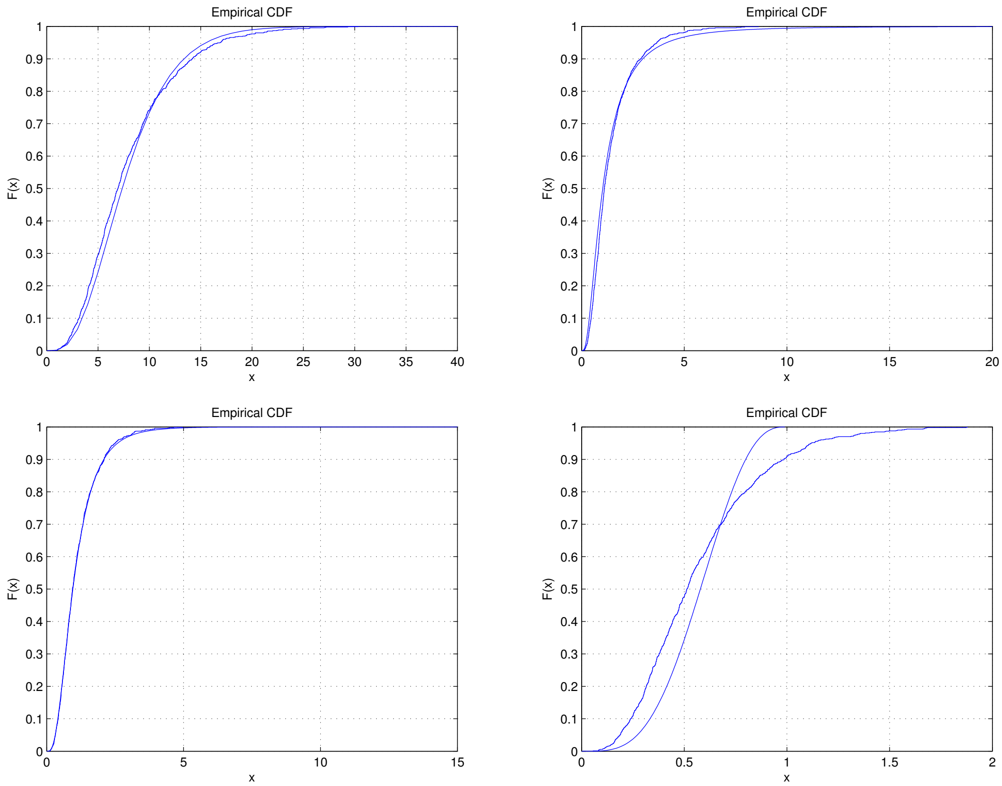

Figure 4: Plot of Ψ_n (dashed lines) and $\Psi_{\mathscr{G}^2}$ for the N (upper left panel), T_6 (upper right panel), T_{16} (lower left panel) and Pe_2 (lower right panel) elliptical copulas for the Oil currency data set.

TABLE 6. *Results of the goodness-of-fit tests on the Oil currency and Uranium exploration data sets for the null hypothesis of a meta-elliptical distribution with a fixed generator (p-values and critical values estimated from $M = 250$ parametric bootstrap samples)*

Model	Oil currency data			Uranium exploration data		
under $\mathscr{H}_0$	V_n	p-value	Critical value	V_n	p-value	Critical value
T_1	45.1691	< 0.01	0.2922	22.1950	< 0.01	0.3361
T_3	8.6763	< 0.01	0.2480	2.7615	< 0.01	0.2225
T_6	1.7985	< 0.01	0.2127	0.1422	0.1810	0.2052
T_9	0.4727	< 0.01	0.2179	0.0526	0.8670	0.2029
T_{15}	0.0308	1.0000	0.2368	0.4561	< 0.01	0.2256
N	1.0803	< 0.01	0.3109	2.2810	< 0.01	0.2617
Pe_1	14.0350	< 0.01	0.9329	13.3178	< 0.01	0.7454
Pe_2	9.7005	< 0.01	0.7094	9.8694	< 0.01	0.5670
Pe_5	5.1667	< 0.01	0.5022	6.0796	< 0.01	0.4297
Pe_{10}	3.1627	< 0.01	0.3912	4.3359	< 0.01	0.3485

6. Final remarks

In this work, statistical methodologies for the selection of an appropriate elliptical copula have been developed. The test statistics that have been proposed are constructed from pseudo-copies of the non-observable i.i.d. random variables $\mathscr{G}_1^2,\ldots,\mathscr{G}_n^2$, where $\mathscr{G}_i^2 = \mathbf{X}_i^\top R^{-1}\mathbf{X}_i$, with $\mathbf{X}_i = \mathbf{Q}_g^{-1}\circ \mathbf{F}(\mathbf{Y}_i)$. To this end, the original observations $\mathbf{Y}_1,\ldots,\mathbf{Y}_n$ were transformed to the scale of a given elliptical distribution $\mathscr{E}(\mathbf{0},R,g)$ via $\mathbf{X}_{i,n} = \mathbf{Q}_g^{-1}\circ \mathbf{F}_n(\mathbf{Y}_i)$ and a version of $\mathscr{G}_i^2$ was defined as $\mathscr{G}_{i,n}^2 = \mathbf{X}_{i,n}^\top R_n^{-1}\mathbf{X}_{i,n}$. Obtaining the asymptotic behavior of statistical methods based on such data dependent random variables is a very challenging problem that requires a careful treatment.

The first step would be to investigate the asymptotic behavior of $\sqrt{n}(\Psi_n - \Psi_{\mathscr{G}^2})$. This could possibly be done in the light of results by van der Vaart and Wellner (2007) on empirical processes indexed by estimated functions. Following their main idea, write $\sqrt{n}(\Psi_n - \Psi_{\mathscr{G}^2}) = A_{n1} + A_{n2} + A_{n3}$, where

$$\begin{aligned} A_{n1}(\gamma) &= \frac{1}{\sqrt{n}}\sum_{i=1}^n \left\{\mathbb{I}\left(\mathscr{G}_i^2 \le \gamma\right) - \Psi_{\mathscr{G}^2}(\gamma)\right\}, \\ A_{n2}(\gamma) &= \frac{1}{\sqrt{n}}\sum_{i=1}^n \left\{\mathbb{I}\left(\mathscr{G}_{i,n}^2 \le \gamma\right) - \mathrm{P}\left(\mathscr{G}_{i,n}^2 \le \gamma\right)\right\} - \frac{1}{\sqrt{n}}\sum_{i=1}^n \left\{\mathbb{I}\left(\mathscr{G}_i^2 \le \gamma\right) - \mathrm{P}\left(\mathscr{G}_i^2 \le \gamma\right)\right\} \end{aligned}$$

and $A_{n3}(\gamma) = \sqrt{n}\left\{\mathrm{P}\left(\mathscr{G}_{i,n}^2 \le \gamma\right) - \mathrm{P}\left(\mathscr{G}_i^2 \le \gamma\right)\right\}$. From standard theory, A_{n1} converges weakly to a $\Psi_{\mathscr{G}^2}$-Brownian bridge. Regularity conditions on $\mathrm{d}\Psi_{\mathscr{G}^2}$ and on the conditional distribution of $\mathbf{Y}$ given $\mathscr{G}^2 = \mathbf{X}^\top R^{-1}\mathbf{X}$ with $\mathbf{X} = \mathbf{Q}_g^{-1}\circ\mathbf{F}(\mathbf{Y})$ will be needed in order that $\sup_{\gamma\in\mathbb{R}^+}|A_{n2}(\gamma)| \to 0$ in probability. The asymptotic behavior of A_{n3} will be a consequence of the Hadamard differentiability of the map $\Phi(\mathbf{F},R) = \mathrm{P}\{\mathbf{Q}_g^{-1}\circ\mathbf{F}(\mathbf{Y})^\top R^{-1}\mathbf{Q}_g^{-1}\circ\mathbf{F}(\mathbf{Y}) \le \gamma\}$.

The asymptotic behavior of $W_n = \inf_{\theta\in\Theta} W_n(\theta)$ could probably be derived from the general results by Pollard (1980) on minimum-distance statistics. Letting θ_0 be the *true* parameter value and assuming that the weak convergence of $\sqrt{n}\{\Psi_n(\gamma,\theta_0) - \Psi_{\mathscr{G}^2}(\gamma,\theta_0)\}$ to some limiting process holds, regularity conditions on $\partial\Psi_{\mathscr{G}^2}(\gamma,\theta)/\partial\theta$ should entail the weak convergence of W_n. Finally, the validity of the parametric bootstrap method for the computation of p-values should follow from arguments similar as those in Genest and Rémillard (2008) under regularity conditions on the estimator R_n.

Acknowledgements

We would like to thank two referees for their valuable comments that improved the quality of this paper. We also wish to thank Giorgio Roth for providing us with the Upper Mississippi river data set, and Stephan Haug for sending us the Oil currency data set. This research was supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC) and the Fonds québécois de recherche Nature et technologie (FQRNT).

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