JOURNAL

DE

MATHÉMATIQUES

PURES ET APPLIQUÉES

FONDÉ EN 1836 ET PUBLIÉ JUSQU'EN 1874

PAR JOSEPH LIOUVILLE

I. J. SCHWATT

Finite expressions for the Bernoulli Numbers obtained by the actual Expansion of Trigonometric Functions by Maclaurins Theorem

Journal de mathématiques pures et appliquées 9e série, tome 11 (1932), p. 143-151.

<http://www.numdam.org/item?id=JMPA_1932_9_11__143_0>

Finite expressions for the Bernoulli numbers obtained by the actual expansion of Trigonometric Functions by Maclaurins Theorem;

By I. J. SCHWATT

(in Philadelphia).

1. In the following methods are given by which the expansions of $x\cot x$, $\tan x$ and $x\operatorname{cosec} x$ are obtained by Maclaurins theorem, rendering the Bernoulli numbers as coefficients of the terms of the expansions and as finite expressions. The methods and the results are believed to be new [1].

---

[1] The expansion

$$(a) \qquad \frac{x}{e^x - 1} = 1 - \frac{1}{2}x + \sum_{n=1}^{\infty}(-1)^{n-1} B_n \frac{x^{2n}}{(2n)!},$$

and the one related to it

$$(b) \qquad x\cot x = 1 - \sum_{n=1}^{\infty} 2^{2n} B_n \frac{x^{2n}}{(2n)!}$$

are used, as a rule, to define the Bernoulli numbers.

But nowhere in mathematical literature could the author find the expansion of the first members of (a) and (b) the terms involving finite expressions for the Bernoulli numbers. The writers on the subject seem to be satisfied to show that the first few terms contain the Bernoulli numbers, but do not obtain the general term of the expansion and in a form as to involve a finite expression for the Bernoulli nnmber. In support of these statements we quote herewith a few

2. We shall first expand

$$y = x \cot x \tag{1}$$

by Maclaurins theorem.

Then

$$y = 1 + \sum_{n=1}^{\infty} \frac{d^n}{dx^n} y \Bigg]_{x=0} \frac{x^n}{n!}. \tag{2}$$

Now

$$\frac{d^n}{dx^n} y \Bigg]_{x=0} = \frac{d^n}{dx^n} \left(ix + \frac{2ix}{e^{2ix} - 1} \right) \Bigg]_{x=0}, \qquad (n \geqq 1). \tag{3}$$

But

$$\frac{d^n}{dx^n} (ix) \Bigg]_{x=0} = i \frac{1 - (-1)^{\left[\frac{2}{n+2}\right]}}{2} x \frac{1 - (-1)^{\left[\frac{2}{n+2}\right]}}{2} \Bigg]_{x=0} \tag{4}$$

and

$$\frac{d^n}{dx^n} \frac{2ix}{e^{2ix} - 1} \Bigg]_{x=0} = (2i)^n \frac{d^n}{dx^n} \frac{x}{e^x - 1} \Bigg]_{x=0}. \tag{5}$$

Applying (4) and (5) to (3), then since y is real, n must be even and we have

$$\frac{d^{2n}}{dx^{2n}} y \Bigg]_{x=0} = (-1)^n 2^{2n} \frac{d^{2n}}{dx^{2n}} \frac{x}{e^x - 1} \Bigg]_{x=0}. \qquad (n \leqq 1). \tag{6}$$

---

representative authors on the subject : BERTRAND, *Traité de Calcul différentiel et de Calcul intégral*, 1864, vol. I, p. 305-306, 347-351, 389-390. — WORPITZKY, *Lehrbuch der Differential und Integralrechnung*, 1880, p. 278-279, 522. — BOOLE, *A Treatise on the Calculus of finite Differences*, 1880, p. 90 and 108. — TANNERY, *Introduction à la Théorie des fonctions d'une variable*, 1886, p. 355-356. — CHRYSTAL, *Text Book of Algebra*, 1889, vol. II, p. 204-209, 339-340. — SCHLÖMILCH, *Compendium der Höheren Analysis*, vol. I, p. 240–245; vol. II, p. 211. — JORDAN, *Cours d'Analyse*, 1909, vol. I, p. 266. — BROMWICH, *An Introduction to the Theory of Infinite series*, 1908, p. 232-236. — GODEFROY, *Théorie élémentaire des séries*, 1903, p. 118. — EDWARDS, *Differential calculus*, 1906, p. 105-106, 499. — DE LA VALLÉE POUSSIN, *Cours d'Analyse infinitésimale*, 1928, vol. II, p. 69 and 339. — COURANT, *Vorlesungen über Differential und Integralrechnung*, 1927, p. 343-344.

3. To find

$$\frac{d^{2n}}{dx^{2n}}\frac{x}{e^x-1}\Big]_{x=0}$$

we proceed as follows.

From

$$\frac{x}{e^x-1}=\frac{x}{e^x+1}+\frac{2x}{e^{2x}-1},$$

we obtain

(7) $$\frac{d^{2n}}{dx^{2n}}\frac{x}{e^x-1}\Big]_{x=0}=-\frac{1}{2^{2n}-1}\frac{d^{2n}}{dx^{2n}}\frac{x}{e^x+1}\Big]_{x=0}.$$

To evaluate the second member of (7) we have by Leibnitz's theorem

(8) $$\frac{d^{2n}}{dx^{2n}}\frac{x}{e^x+1}\Big]_{x=0}=\sum_{k=0}^{\infty}\binom{2n}{k}\frac{d^{2n-k}}{dx^{2n-k}}x\frac{d^k}{dx^k}\frac{1}{e^x+1}\Big]_{x=0}.$$

$$=x\frac{d^{2n}}{dx^{2n}}\frac{1}{e^x+1}\Big]_{x=0}+2n\frac{d^{2n-1}}{dx^{2n-1}}\frac{1}{e^x+1}\Big]_{x=0}$$

(9) $$=2n\frac{d^{2n-1}}{dx^{2n-1}}\frac{1}{e^x+1}\Big]_{x=0}.$$

Letting now in

(10) $$\frac{d^{2n-1}}{dx^{2n-1}}y=\sum_{k=0}^{2n-1}\frac{(-1)^k}{k!}\sum_{\alpha=0}^{k}(-1)^\alpha\binom{k}{\alpha}u^{k-\alpha}\frac{d^{2n-1}}{dx^{2n-1}}u^\alpha\frac{d^k}{du^k}y \quad (^1).$$

$y=\frac{1}{e^x+1}$ and $u=e^x$, then (9) becomes

(11) $$\frac{d^{2n-1}}{dx^{2n-1}}\frac{x}{e^x+1}\Big]_{x=0}=2n\sum_{k=0}^{2n-1}\frac{1}{2^{k+1}}\sum_{\alpha=0}^{k}(-1)^\alpha\binom{k}{\alpha}\alpha^{2n-1}.$$

by means of which we obtain from (7)

(12) $$\frac{d^{2n}}{dx^{2n}}\frac{x}{e^x-1}\Big]_{x=0}=-\frac{n}{2^{2n}-1}\sum_{k=1}^{2n-1}\frac{1}{2^k}\sum_{\alpha=1}^{k}(-1)^\alpha\binom{k}{\alpha}\alpha^{2n-1}, \qquad (n\geq 1).$$

---

(1) The authors book, : *An Introduction to the Operation with Series.* The Press of the University of Pennsylvania, Chap. I (83), p. 12.

4. That for $x=0$ the odd derivatives of $\frac{x}{e^x-1}$, except the first vanisch, can also be shown as follows.

We shall prove that

$$(13) \qquad S_1=\sum_{k=0}^{n-1}\frac{1}{2^k}\sum_{\alpha=0}^{k}(-1)^{\alpha}\binom{k}{\alpha}\alpha^{n-1}=0,$$

if $n-1$ is even, that is, if n is odd, except if $n=1$, in which case $S_1=1$.

$$(14) \qquad S_2=\sum_{\alpha=0}^{k}(-1)^{\alpha}\binom{k}{\alpha}\alpha^{n-1}=\frac{d^{n-1}}{dx^{n-1}}(1-e^x)^k\Big]_{x=0}$$

and since $S_2=0$, if $k>n-1$, we may write

$$(15) \qquad S_1=\sum_{k=1}^{\infty}\frac{d^{n-1}}{dx^{n-1}}\left(\frac{1-e^x}{2}\right)^k\Bigg]_{x=0}=\frac{d^{n-1}}{dx^{n-1}}\,\frac{1-e^x}{1+e^x}\Bigg]_{x=0}.$$

But $\frac{1-e^x}{1+e^x}$ being an odd function of x, we conclude that $S_1=0$, if $n-1$ is even. Therefore in S_1, n must be even.

5. Applying (12) to (6) we obtain from (2)

$$(16) \qquad x\cot x=1-\sum_{n=1}^{\infty}(-1)^n\frac{2^{2n}n}{2^{2n}-1}\,\frac{x^{2n}}{(2n)!}\sum_{k=1}^{2n-1}\frac{1}{2^k}\sum_{\alpha=1}^{k}(-1)^{\alpha}\binom{k}{\alpha}\alpha^{2n-1}.$$

Comparing (16) with (*b*) in foot note (1) we have

$$(17) \qquad B_n=(-1)^n\frac{n}{2^{2n}-1}\sum_{k=1}^{2n-1}\frac{1}{2^k}\sum_{\alpha=1}^{k}(-1)^{\alpha}\binom{k}{\alpha}\alpha^{2n-1},$$

the n^{th} Bernoulli numbers.

6. We shall next find the expansion of

$$y=\tan x$$

in powers of x.

Now

$$y = \frac{2i}{e^{2ix}+1} - i,$$

and

(19) $$\frac{d^n}{dx^n}y\Big]_{x=0} = (2i)^{n+1}\frac{d^n}{dx^n}\frac{1}{e^x+1}\Big]_{x=0} - \frac{1-(-1)^{\left[\frac{1}{n+1}\right]}}{2}\,i,$$

in which, since y is real, n must be odd.

We then have

(20) $$\frac{d^{2n-1}}{dx^{2n-1}}y\Big]_{x=0} = (-1)^n 2^{2n}\frac{d^{2n-1}}{dx^{2n-1}}\frac{1}{e^x+1}\Big]_{x=0}.$$

Therefore

$$y = \sum_{n=1}^{\infty}\frac{d^{2n-1}}{dx^{2n-1}}y\Big]_{x=0}\frac{x^{2n-1}}{(2n-1)!},$$

(21) $$= \sum_{n=1}^{\infty}(-1)^n 2^{2n-1}\frac{x^{2n-1}}{(2n-1)!}\sum_{k=1}^{2n-1}\frac{1}{2^k}\sum_{\alpha=1}^{k}(-1)^\alpha\binom{k}{\alpha}\alpha^{2n-1};$$

which by means of (17) changes to

(22) $$\tan x = \sum_{n=1}^{\infty}(2^{2n}-1)\frac{2^{2n-1}}{n}B_n\frac{x^{2n-1}}{(2n-1)!}, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

This result can also be obtained from

$$\frac{x}{2}\tan\frac{x}{2} = \frac{x}{2}\cot\frac{x}{2} - x\cot x$$

and by the use of (6) in foot note 1.

7. By the methods given above the expansion

(23) $$y = x\,\mathrm{cosec}\,x$$

is derived.

We may write

(24) $$y = \frac{ix}{e^{2ix}+1} + \frac{ix}{e^{2ix}-1},$$

then

$$(25)\qquad \frac{d^n}{dx^n}y\Big]_{x=0}=i^n\left[\frac{d^n}{dx^n}\frac{x}{e^x+1}-\frac{1}{2^n-1}\frac{d^n}{dx^n}\frac{x}{e^x+1}\right]_{x=0},$$

$$=2i^n\left[n\,\frac{2^{n-1}-1}{2^n-1}\,\frac{d^{n-1}}{dx^{n-1}}\,\frac{1}{e^x+1}\right]_{x=0},$$

and since y is real, n must be even.

Therefore

$$(26)\qquad x\,\mathrm{cosec}\,x=1+\sum_{n=1}^{\infty}2(2^{2n-1}-1)B_n\frac{x^{2n}}{(2n)!},\qquad(-\pi<x<\pi).$$

This result can also be obtained by writing $\frac{x}{2}$ in place of x in the expansion of $\tan x$ and adding to it the expansion of $\cot x$.

8. We shall now develop methods for the expansion of $\tan x$, $x\,\mathrm{cosec}\,x$ and $x\cot x$, which are different from those given above and which will render new forms for B_n.

9. We have

$$(27)\qquad \mathrm{tang}\,x=\frac{\sin x}{\sqrt{1-\sin^2 x}}=\sum_{k=0}^{\infty}(-1)^k\binom{-\frac{1}{2}}{k}\sin^{2k+1}x,\qquad -\frac{\pi}{2}<x<\frac{\pi}{2}.$$

Now

$$(28)\qquad \sin^{2k+1}x=\frac{(-1)^{k+1}i}{2^{2k+1}}(e^{ix}-e^{-ix})^{2k+1}$$

$$=\frac{(-1)^{k+1}i}{2^{2k+1}}\sum_{\alpha=0}^{2k+1}(-1)^{\alpha}\binom{2k+1}{\alpha}e^{(2k+1-2\alpha)ix}$$

$$=\frac{(-1)^{k+1}}{2^{2k+1}}\sum_{\alpha=0}^{\infty}i^{n+1}\frac{x^n}{n!}\sum_{n=0}^{2k+1}(-1)^{\alpha}\binom{2k+1}{\alpha}(2k+1-2\alpha)^n$$

and since $\sin^{2k+1}x$ is real, n must be odd and (28) changes to

$$(29)\qquad \sin^{2k+1}x=\frac{(-1)^{k+1}}{2^{2k+1}}\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n+1}}{(2n+1)!}$$

$$\times\sum_{n=0}^{2k+1}(-1)^{\alpha}\binom{2k+1}{\alpha}(2k+1-2\alpha)^{2n+1}.$$

But

$$(30)\quad S=\sum_{\alpha=0}^{2k+1}(-1)^\alpha\binom{2k+1}{\alpha}(2k+1-2\alpha)^{2n+1}=0, \qquad if\, n<k \quad (^1),$$

hence $n=k$ is the lower limit of n in (29).

To reduce S we denote the expression under the summation sign by P_α, then

$$(31)\qquad S=\sum_{\alpha=0}^{k}P_\alpha+\sum_{\alpha=k+1}^{2k+1}P_\alpha.$$

Letting in the second summation $2k+1-\alpha=\alpha'$, we find

$$(32)\qquad S=2\sum_{\alpha=0}^{k}P_\alpha.$$

Letting $k-\alpha=\alpha'$, we have

$$(33)\qquad S=2(-1)^k\sum_{\alpha=0}^{k}(-1)^\alpha\binom{2k+1}{k-\alpha}(2\alpha+1)^{2n+1}.$$

Applying (33) to (29) gives

$$(34)\quad \sin^{2k+1}x=\frac{1}{2^{2k}}\sum_{n=k}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\sum_{\alpha=0}^{k}(-1)^\alpha\binom{2k+1}{k-\alpha}(2\alpha+1)^{2n+1},$$

by means of which (27) becomes

$$(35)\qquad \operatorname{tang}x=\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}}\binom{-\frac{1}{2}}{k}\sum_{n=k}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$
$$\times\sum_{\alpha=k}^{k}(-1)^\alpha\binom{2k+1}{k-\alpha}(2\alpha+1)^{2n+1},$$

and since

$$(36)\qquad \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}A_{n,k}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A_{n,k};$$

---

(1) For, $\sum_{\beta=0}^{\gamma}(-1)^\beta(\gamma\beta)\beta^p=\frac{d^p}{dx^p}[(1-e^x)^\gamma]_{x=0}=0$, if $p<\gamma$.

and

(37) $$\binom{-\frac{1}{2}}{k}=\frac{(-1)^k}{2^{2k}}\binom{2k}{k},$$

we obtain

(38) $$\tan x=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}\sum_{k=0}^{n-1}\frac{1}{2^{4k}}\binom{2k}{k}$$
$$\times\sum_{\alpha=0}^{k}(-1)^{\alpha}\binom{2k+1}{k-\alpha}(2\alpha+1)^{2n-1}$$

and

(39) $$B_n=\frac{(-1)^{n-1}n}{2^{2n-1}(2^{2n}-1)}\sum_{k=0}^{n-1}\frac{1}{2^{4k}}\binom{2k}{k}\sum_{\alpha=0}^{k}(-1)^{\alpha}\binom{2k+1}{k-\alpha}(2\alpha+1)^{2n-1}.$$

which is a form different from (17).

10. The following devise leads to expansions of $x\operatorname{cosec}x$ and $x\cot x$ and to expressions for B_n different from those obtained above.

Letting

(40) $$x=\sin^{-1}y,$$

then

(41) $$x\operatorname{cosec}x=\frac{\sin^{-1}y}{y}=\sum_{k=0}^{\infty}(-1)^k\binom{-\frac{1}{2}}{k}\frac{y^{2k}}{2k+1}$$
$$=1+\sum_{k=1}^{\infty}(-1)^k\binom{-\frac{1}{2}}{k}\frac{\sin^{2k}x}{2k+1}.$$

Following the method by which (34) was derived we obtain

(42) $$\sin^{2k}x=\frac{1}{2^{2k-1}}\sum_{n=k}^{\infty}(-1)^n 2^{2n}\frac{x^{2n}}{(2n)!}\sum_{\alpha=1}^{k}(-1)^{\alpha}\binom{2k}{k-\alpha}\alpha^{2n}.$$

Therefore

(43) $$x\operatorname{cosec}x=1+\sum_{n=k}^{\infty}(-1)^n 2^{2n}\frac{x^{2n}}{(2n)!}$$
$$\times\sum_{k=1}^{n}\frac{1}{2^{4k-1}}\binom{2k}{k}\frac{1}{2k+1}\sum_{\alpha=0}^{k}(-1)^{\alpha}\binom{2k}{k-\alpha}\alpha^{2n}$$

and

$$(44)\qquad B_n = (-1)^n \frac{2^{2n}}{2^{2n-1}-1}\sum_{k=1}^{n}\frac{1}{2^{4k}}\binom{2k}{k}\frac{1}{2k+1}\sum_{\alpha=1}^{k}(-1)^{\alpha}\binom{2k}{k-\alpha}\alpha^{2n}.$$

11. By means of (40) we have

$$(45)\qquad x\cot x = \frac{1}{y}(1-y^2)f(y).$$

where

$$(46)\qquad f(y) = (1-y^2)^{-\frac{1}{2}}\sin^{-1} y$$

$$= \sum_{k=0}^{\infty}\binom{-\frac{1}{2}}{k}\frac{1}{2k+1}\sum_{n=0}^{\infty}(-1)^n\binom{-\frac{1}{2}}{k}y^{2k+2n+1}.$$

Letting $k+n=n'$, then

$$(47)\qquad f(y) = \sum_{k=0}^{\infty}\binom{-\frac{1}{2}}{k}\frac{1}{2k+1}\sum_{n=k}^{\infty}(-1)^n\binom{-\frac{1}{2}}{n-k}y^{2n+1}.$$

Applying (36) to (47) and the result to (45) gives

$$(48)\qquad x\cot x = (1-y^2)\sum_{n=0}^{\infty}(-1)^n y^{2n}\sum_{k=0}^{n}\binom{-\frac{1}{2}}{k}\binom{-\frac{1}{2}}{n-k}\frac{1}{2k+1},$$

which by means of

$$(49)\qquad \sum_{k=0}^{n}\binom{-\frac{1}{2}}{k}\binom{-\frac{1}{2}}{n-k}\frac{1}{2k+1} = \frac{(-1)^n 2^{2n}}{(2n+1)\binom{2n}{n}}$$

and the expansion of $\sin^{2k} x$ in (42) changes to

$$(50)\qquad x\cot x = 1 - \sum_{n=1}^{\infty}(-1)^n 2^{2n}\frac{x^{2n}}{(2n)!}\sum_{k=1}^{n}\frac{k!(k-1)!}{(2k+1)!}\sum_{\alpha=1}^{k}(-1)^{\alpha}\binom{2k}{k-\alpha}\alpha^{2n}.$$

Therefore

$$(51)\qquad B_n = (-1)^n\sum_{k=1}^{n}\frac{k!(k-1)!}{(2k+1)!}\sum_{\alpha=1}^{k}(-1)^{\alpha}\binom{2k}{k-\alpha}\alpha^{2n}.$$