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Differential Properties of Functions of a Complex Variable which are Invariant under Linear Transformations

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Differential Properties of Fonctions of a Complex Variable which are Invariant under Linear Transformations;

BY E.-J. WILCZYNSKI.

PART II (1).

VI. — Cogredients.

The expression

$$p = \frac{a_1}{a_2},$$

which whe have found for the pole of the osculating linear function, was derived under the assumption that the point of contact was $z = 0$. But it is easy to derive a more general formula. Let $w = f(\zeta)$ be a function of ζ analytic in the neighborhood of $\zeta = z$, and let its expansion at this point be

$$a_0 + a_1(\zeta - z) + a_2(\zeta - z)^2 + \ldots.$$

(1) La première Partie a paru dans ce journal, en Tome I de la neuvième série, 1922.

If we denote again by p the pole of the osculating linear function of $w = f(\zeta)$, the point of contact being $\zeta = z$, we find

$$p - z = \frac{a_1}{a_2}$$

so that

(107)
$$p = z + \frac{a_1}{a_2} = z + 2\frac{w'}{w''}$$

is the general expression for the pole of the linear function which osculates the function $w = f(\zeta)$ at the point $\zeta = z$. In this formula a_1, a_2 are the coefficients of $\zeta - z$ and $(\zeta - z)^2$ in the expansion of $f(\zeta)$ in powers of $\zeta - z$, and w' and w'' are the values of $f'(\zeta)$ and $f''(\zeta)$ for $\zeta = z$.

If we subject the function $w = f(\zeta)$ to any transformation of the group

$$\overline{w} = w, \qquad \overline{\zeta} = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta},$$

where α, β, γ, δ are constants, the point of contact z and the pole p of the corresponding osculating linear function of $w = f(\zeta)$ will be transformed into a new point $\overline{z}$ and the pole $\overline{p}$ of the osculating linear function of the function $w = \overline{f}(\overline{\zeta})$. Moreover, it may be verified that

$$\overline{p} = \frac{\alpha p + \beta}{\gamma p + \delta}.$$

We express this by saying that p is *a cogredient* of z with respect to the function $w = f(\zeta)$, and evidently we have obtained in Articles 4 and 5 *a* number of other cogredients.

The general expressions of these cogredients may be obtained as follows.

Let

$$\gamma = f(\alpha_0, \alpha_1, \alpha_2, \ldots)$$

be the expression for any cogredient when the origin $\zeta = 0$ is the point of contact, the quantities α_0, α_1, α_2, ... being the coefficients of the expansion of $w = f(\zeta)$ at $\zeta = 0$. Then

$$c = z + f(a_0, a_1, a_2, \ldots)$$

will be the expression of the corresponding cogredient when the point $\zeta = z$ is the point of contact, the quantities $a_0, a_1, a_2, \ldots$ being the coefficients of the expansion $w = f(\zeta)$ in a series of powers of $\zeta - z$.

VII. — Interpretation of the integral invariant φ.

Let us apply the remark just made to formulae (103). We find the following formulae for the singular points a and b, of the osculating logarithm of $w = f(\zeta)$, the point of contact being $\zeta = z$:

$$(108) \qquad \frac{2}{a-z} = \frac{w''}{w'} - \sqrt{2\{w, z\}}, \qquad \frac{2}{b-z} = \frac{w''}{w'} + \sqrt{2\{w, z\}},$$

whence

$$(109) \qquad \sqrt{\{w, z\}} = \frac{1}{\sqrt{2}}\left(\frac{1}{b-z} - \frac{1}{a-z}\right).$$

We may therefore write

$$(110) \qquad \varphi = \int_{z_0}^{\zeta} \sqrt{\{w, z\}}\, dz = \frac{1}{\sqrt{2}}\int_{z_0}^{\zeta}\left(\frac{1}{b-z} - \frac{1}{a-z}\right) dz.$$

We may express this as follows.

Given an analytic function $w = f(\zeta)$. Let us select a curve C of finite length in the ζ plane, at all of whose points $f(\zeta)$ is analytic and $f'(\zeta)$ different from zero. Let a and b be the singular points of the logarithmic function which osculates $f(\zeta)$ at $\zeta = z$. Then the value of the integral

$$\varphi = \frac{1}{\sqrt{2}}\int_C \left(\frac{1}{b-z} - \frac{1}{a-z}\right) dz$$

extended over the curve C, will remain unchanged if all of the points of the ζ plane are subjected to the same linear transformation.

If we represent the variables a, b, z by the points A, B, Z of the ζ plane, we may write

$$(111) \qquad \varphi = \frac{1}{\sqrt{2}}\int_C \left(\frac{1}{ZB} - \frac{1}{ZA}\right) dz$$

in terms of the vectors ZA and ZB.

We may also express the integral φ in terms of the poles, p_1 and p_2, of the singular penosculating quadratics. In fact, by the method of Art. 6, we find the generalized expressions

$$p_1 - z = \frac{a_2 + \sqrt{a_2^2 - a_1 a_3}}{a_3}, \qquad p_2 - z = \frac{a_2 - \sqrt{a_2^2 - a_1 a_3}}{a_3},$$

whence

$$\frac{1}{p_2 - z} - \frac{1}{p_1 - z} = \frac{2\sqrt{a_2^2 - a_1 a_3}}{a_1} = \frac{2i}{\sqrt{6}}\sqrt{\{w, z\}},$$

so that we find *the expression*

$$(112) \qquad \varphi = \frac{\sqrt{6}}{2i}\int_C \left(\frac{1}{p_2 - z} - \frac{1}{p_1 - z}\right) dz,$$

for the integral invariant φ, *which is quite analogous to* (110).

Both of these formulae for φ may be used to advantage. But they cannot be regarded as altogether satisfactory as interpretations of the integral φ from our point of view. For, although the points z, a, b, p_1, p_2 which occur in these integrals are defined invariantly, they occur in combinations such as $a - z$ which are not invariant under linear transformations of the independent variable.

We now proceed to obtain a new expression for φ which is free from this objection. Let us divide the curve C, of finite length L, into n pieces, by means of points

$$z_0, \quad z_1, \quad z_2, \quad \ldots, \quad z_{k-1}, \quad z_k, \quad \ldots, \quad z_{n-1}, \quad z_n = Z,$$

where z_0 and Z denote the end points, as is customary when defining a line integral. We shall put

$$z_{k+1} = z_k + \delta z_k,$$

and assume that all of the quantities δz_k approach zero, uniformly, as infinitesimals of the first order, when n grows beyond bound, and that

$$\lim \sum |\delta z_k| = \mathfrak{L}.$$

Let a_k and b_k be the singular points of the logarithmic function which osculates $f(\zeta)$ at $\zeta = z_k$. We proceed to calculate the double-ratio of a_k, b_k, z_k, z_{k+1}. We find

$$(a_k, b_k, z_k, z_{k+1}) = \frac{z_k - a_k}{z_k - b_k} : \frac{z_{k+1} - a_k}{z_{k+1} - b_k} = \frac{z_k - a_k}{z_k - b_k}\,\frac{z_k - b_k + \delta z_k}{z_k - a_k + \delta z_k}$$

$$= \frac{1 + \dfrac{\delta z_k}{z_k - b_k}}{1 + \dfrac{\delta z_k}{z_k - a_k}},$$

and this differs from

$$1 + \left[\frac{1}{a_k - z_k} - \frac{1}{b_k - z_k}\right]\delta z_k$$

only by an infinitesimal of order higher than the first. Let us put

(113) $$\sigma = \{w, z\}.$$

Then we may write, making use of (109),

(114) $$(a_k, b_k, z_k, z_{k+1}) = 1 - \sqrt{2\sigma(z_k)}\,\delta z_k + \varepsilon_k\,\delta z_k,$$

where

(115) $$\lim_{n \to \infty} \varepsilon_k = 0.$$

Thus we have

(116) $$\begin{cases} (a_0, b_0, z_0, z_1) &= 1 - \sqrt{2\sigma(z_0)}\,\delta z_0 + \varepsilon_0\,\delta z_0, \\ (a_1, b_1, z_1, z_2) &= 1 - \sqrt{2\sigma(z_1)}\,\delta z_1 + \varepsilon_1\,\delta z_1, \\ (a_2, b_2, z_2, z_3) &= 1 - \sqrt{2\sigma(z_2)}\,\delta z_2 + \varepsilon_2\,\delta z_2, \\ \dots\dots\dots\dots\dots\dots\dots & \\ (a_{n-1}, b_{n-1}, z_{n-1}, Z) &= 1 - \sqrt{2\sigma(z_{n-1})}\,\delta z_{n-1} + \varepsilon_{n-1}\,\delta z_{n-1}. \end{cases}$$

Now there exists a unique linear transformation which converts any three distinct points into any three others. Denote by T_1 the linear transformation which converts

$$a_1,\ b_1,\ z_1 \quad \text{into} \quad a_0,\ b_0,\ z_1$$

respectively. Since T_1 does not alter double ratios, we may write in place of the second equation of (116)

(117) $$(a_0, b_0, z_1, z_2') = 1 - \sqrt{2\sigma(z_1)}\,\delta z_1 + \varepsilon_1\,\delta z_1,$$

where z_2' is the point which corresponds to z_2 in the transformation T_1.

But if A, B, C, D, E are five elements, whe have the fundamental double-ratio equation

$$(118)\qquad (A, B, C, D)(A, B, D, E) = (A, B, C, E).$$

Consequently we deduce, from (117) and the first equation of system (116),

$$(119)\quad (a_0, b_0, z_0, z_2') = \left[1 - \sqrt{2\sigma(z_0)}\,\delta z_0\right]\left[1 - \sqrt{2\sigma(z_1)}\,\delta z_1\right] + \varepsilon_0'\,\delta z_0 + \varepsilon_1'\,\delta z_1,$$

where

$$\lim_{n\to\infty} \varepsilon_0' = \lim_{n\to\infty} \varepsilon_1' = 0.$$

Let T_2 be the linear transformation which transforms

$$a_2,\quad b_2,\quad z_2 \qquad \text{into} \qquad a_0,\quad b_0,\quad z_2',$$

respectively, and let z_3' be the point which corresponds, by means of T_2 to z_3. Then we find, from (116)

$$(a_0, b_0, z_2', z_3') = 1 - \sqrt{2\sigma(z_2)}\,\delta z_2 + \varepsilon_2\,\delta z_2.$$

If we multiply both members of this equation by the corresponding members of (119), and make use of (118), we find

$$(a_0, b_0, z_0, z_3') = \left[1 - \sqrt{2\sigma(z_0)}\,\delta z_0\right]\left[1 - \sqrt{2\sigma(z_1)}\,\delta z_1\right]\left[1 - \sqrt{2\sigma(z_2)}\,\delta z_2\right]$$
$$+ \varepsilon_0''\,\delta z_0 + \varepsilon_1''\,\delta z_1 + \varepsilon_2''\,\delta z_2,$$
$$\lim_{n\to\infty} \varepsilon_0'' = \lim_{n\to\infty} \varepsilon_1'' = \lim_{n\to\infty} \varepsilon_2'' = 0.$$

In general, let T_i be the linear transformation which converts

$$a_i,\quad b_i,\quad z_i \qquad \text{into} \qquad a_0,\quad b_0,\quad z_i',$$

where z_i' is the point obtained from z_i by means of T_{i-1}. We obtain finally

$$(120)\qquad (a_0, b_0, z_0, Z') = \prod_{i=0}^{n-1}\left[1 - \sqrt{2\sigma(z_i)}\,\delta z_i\right] + \sum_{i=0}^{n-1} \varepsilon_i^{(n-1)}\,\delta z_i,$$

where Z' is obtained from Z by means of T_n, and where

$$\lim_{n\to\infty} \varepsilon_i^{(n-1)} = 0.$$

We now proceed to let n grow beyond bound. We have assumed that the curve C is of finite length L, that the function $w = f(\zeta)$ is analytic in the neighborhood of every point of C and that $f'(\zeta)$ is different from zero at all points of C. Under these assumptions the sum which occurs in the right member of (120) will approach the limit zero, the transformation T_n will tend toward a limit T, and the infinite product will converge. Thus we find

$$k = (a_0, b_0, z_0, \zeta) = \lim_{n \to \infty} \prod_{i=0}^{n-1} \left[1 - \sqrt{2\sigma(z_i)}\, \delta z_i\right], \tag{121}$$

where ζ is obtained from Z by means of the transformation

$$T = \lim_{n \to \infty} T_n.$$

It is noteworthy that we have obtained this cross-ratio k by means of an infinite product which is the multiplicative analogon of a definite integral.

We now propose to establish a relation between k and φ. Let us think of k as a function of Z,

$$k = k(Z).$$

Let us extend the curve C to $Z + h$ by means of an arc which satisfies the assumptions which we have made for C. Then we shall have

$$\frac{k(Z+h)}{k(Z)} = 1 - \sqrt{2\sigma(Z)}\, h + \varepsilon h,$$

where ε approaches zero with h, and therefore

$$\log k(Z+h) - \log k(Z) = -h\sqrt{2\sigma(Z)} + \varepsilon' h,$$

where

$$\lim_{h \to 0} \varepsilon' = 0.$$

Consequently we find

$$\frac{d \log k(Z)}{dZ} = \lim_{h \to 0} \frac{\log k(Z+h) - \log k(Z)}{h} = -\sqrt{2\sigma(Z)}$$

so that

$$k = e^{-\sqrt{2}\varphi}, \qquad \varphi = -\frac{1}{\sqrt{2}} \log k, \tag{122}$$

since for $Z = z_0$, k reduces to unity and φ to zero.

The analytic form of the correspondence T between ζ and Z is given by (121) combined with (122), or explicitly by

$$\frac{\zeta - a_0}{\zeta - b_0} = \frac{z_0 - a_0}{z_0 - b_0} e^{\sqrt{2}\int_{z_0}^{Z} \sqrt{\{w, z\}}\, dz}. \tag{123}$$

Every analytic function $w = f(z)$ determines a transformation T *of curves in the z – plane, which has just been defined geometrically and whose analytic expression is given by* (123).

Let R be a simply connected region in the z — plane, such that $f(z)$ is uniform in R and has no essential singularities in R. Let us assume further that $\{w, z\}$ is different from zero at all points of R, and let C be a closed curve all of whose points are in R.

It involves no essential restriction to assume that $z = 0$ is a point of R. If $z = 0$ is an ordinary point for the function $f(z)$, we have

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots,$$

and $\{w, z\}$ will also be expressed by an ordinary power series in z, whose constant term is

$$\frac{6(a_1 a_3 - a_2^2)}{a_1^2},$$

provided $a_1 \neq 0$. Since $\{w, z\}$ is supposed to be different from zero at all points of R, $a_1 a_3 - a_2^2$ is not zero, and therefore the integral φ, taken around a circle of sufficiently small radius with such a point as center, is equal to zero.

Suppose however that

$$a_1 = a_2 = \ldots = a_{m-1} = 0 \qquad (a_m \neq 0).$$

Then

$$\begin{aligned} w &= f(z) = a_0 + a_m z^m + a_{m+1} z^{m+1} + \ldots, \\ w' &= m a_m z^{m-1} + (m+1) a_{m+1} z^m + \ldots, \\ w'' &= m(m-1) a_m z^{m-2} + \ldots, \\ w''' &= m(m-1)(m-2) a_m z^{m-3} + \ldots, \end{aligned}$$

so that

$$\frac{w''}{w'} = (m-1) z^{-1}[1 + P_1(z)], \qquad \frac{w'''}{w'} = (m-1)(m-2) z^{-2}[1 + P_2(z)],$$

$$\{w, z\} = -2(m^2 - 1) z^{-2}[1 + P(z)],$$

$$\sqrt{\{w, z\}} = \pm i\sqrt{2}\sqrt{m^2 - 1}\, z^{-1}[1 + Q(z)],$$

where P_1, P_2, P, and Q represent power series which vanish for $z = 0$. Since $\{w, z\}$ remains different from zero for all points in R, we may choose that determination of the square root for which

$$\sqrt{\{w, z\}} = i\sqrt{2(m^2-1)}\,[1 + Q(z)]\,z^{-1}.$$

Whe see that, in this case, the integral φ has $z = 0$ as a logarithmic singularity, and the value of the integral taken, in the positive sense, around a small curve which encloses $z = 0$, will be

$$-2\sqrt{2}\,\pi\sqrt{m^2-1}.$$

Let us suppose finally that $z = 0$ is a pole of $w = f(z)$, of multiplicity n. Then $z = 0$ will be *a* zero of $\frac{1}{w}$, of order n. But $\{w, z\}$ is equal to $\left\{\frac{1}{w}, z\right\}$. Consequently the integral will, in this case, be equal to

$$-2\sqrt{2}\,\pi\sqrt{n^2-1}.$$

We now easily deduce the following consequence from (123).

Let R *be a simply connected region in the* z *— plane such that* $w = f(z)$ *is uniform in* R, *and has no essential singularities in* R. *Moreover let* $\{w, z\}$ *be different from zero at all points of* R. *If* R *contains no points for which the equation* $f(z) = k$ *has a multiple solution,* k *being a finite number or* ∞, *then the transformation* T *will define* ζ *as a uniform function of* Z *for all points* Z *in* R. *If* R *does contain such points,* ζ *may be a many-valued function of* Z *in the region, but all of its branches will be connected by linear substitutions.*

It only remains to note the fact that our method of defining φ by means of an infinite product, may be applied without essential change if we make use of p_1 and p_2, the poles of the singular penosculating quadratics, in place of a and b. The corresponding formulae may of course be obtained directly from the equations of this article by making use of the relations betwee the points z, a, b, p_1, p_2 which were discovered in Art. 3.

VIII. — Introduction of φ as independent variable.

Having recognized the importance of the integral invariant φ, and having explored some of its properties, its seems natural to make use of φ as independent variable in all of the formulae which involve invariant relations of the function $w=f(z)$. In fact we have already done this in our discussion of the intrinsic equation

We first recall the following formulae, due to Cayley and very easy to verify, for the transformation of Schwarzian derivatives such as $\{s, x\}$.

If we transform the dependent variable by putting $s=\mathrm{F}(\mathrm{S})$, we find

$$\{s, x\}=\left(\frac{d\mathrm{S}}{dx}\right)^2\{s, \mathrm{S}\}+\{\mathrm{S}, x\}. \tag{124}$$

Transformation of the independent variable is governed by the formula

$$\{s, x\}=\left(\frac{d\mathrm{X}}{dx}\right)^2[\{s, \mathrm{X}\}-\{x, \mathrm{X}\}], \tag{125}$$

and if we transform both variables simultaneously, we find

$$\{s, x\}=\left(\frac{d\mathrm{S}}{dx}\right)^2\{s, \mathrm{S}\}-\left(\frac{d\mathrm{X}}{dx}\right)^2\{x, \mathrm{X}\}+\left(\frac{d\mathrm{X}}{dx}\right)^2\{\mathrm{S}, \mathrm{X}\}. \tag{126}$$

In particular we find the formula

$$\{s, x\}=-\left(\frac{ds}{dx}\right)^2\{x, s\} \tag{127}$$

for interchanging the two variables. Finally we note the following familiar equations for *linear* transformations with constant coefficients:

$$\left\{\frac{as+b}{cs+d}, x\right\}=\{s, x\}, \tag{128}$$

$$\left\{s, \frac{\alpha x+\beta}{\gamma x+\delta}\right\}=\frac{(\gamma x+\delta)^4}{(\alpha\delta-\beta\gamma)^2}\{s, x\}, \tag{129}$$

$$\left\{\frac{as+b}{cs+d}, \frac{\alpha x+\beta}{\gamma x+\delta}\right\}=\frac{(\gamma x+\delta)^4}{(\alpha\delta-\beta\gamma)^2}\{s, x\}. \tag{130}$$

We are studying a function $w = f(z)$, and we propose to introduce

(131) $$z_1 = \varphi = \int_{z_0}^{z} \sqrt{\{w, z\}}\, dz$$

as a new independent variable, so that w becomes a function of z_1. According to (125) we find

(132) $$\{w, z\} = \left(\frac{dz_1}{dz}\right)^2 [\{w, z_1\} - \{z, z_1\}]$$
$$= \{w, z\} [\{w, z_1\} - \{z, z_1\}],$$

whence

(133) $$\{w, z_1\} - \{z, z_1\} = 1 \quad \text{if} \quad \{w, z\} \neq 0,$$

a formula which we have already used in Art. **5**.

We may also write (132) as follows, if we make use of (127),

(134) $$\{w, z\} = \{w, z\}\{w, z_1\} + \{z_1, z\},$$

whence

(135) $$\{w, z\} = \frac{\{z_1, z\}}{1 - \{w, z_1\}}, \qquad \{w, z_1\} = 1 - \frac{\{z_1, z\}}{\{w, z\}},$$

provided again that $\{w, z\} \neq 0$ and $\{w, z_1\} \neq 1$.

Let us assume further that $\{w, z_1\} \neq 0$. We may then repeat this transformation, putting

(136) $$z_2 = \int \sqrt{\{w, z_1\}}\, dz_1.$$

We find

(137) $$\begin{cases} \{w, z_2\} - \{z_1, z_2\} = 1 \quad \text{if} \quad \{w, z_1\} \neq 0, \\ \{w, z_1\} = \{w, z_1\}\{w, z_2\} + \{z_2, z_1\}, \end{cases}$$

whence

(138) $$\{w, z_1\} = \frac{\{z_2, z_1\}}{1 - \{w, z_2\}}, \qquad \{w, z_2\} = 1 - \frac{\{z_2, z_1\}}{\{w, z_1\}},$$

or by combining (135) and (138),

(139) $$\{w, z\} = \frac{\{z_1, z\}}{1 - \dfrac{\{z_2, z_1\}}{1 - \{w, z_2\}}}, \qquad \{w, z_2\} = 1 - \frac{\{z_2, z_1\}}{1 - \dfrac{\{z_1, z\}}{\{w, z\}}}.$$

Let us continue in this way. *Assume that all of the Schwarzians* $\{w, z\}$, $\{w, z_1\}$, $\{w, z_2\}$, $\ldots$, $\{w, z_{k-1}\}$ *are different from zero and from unity, and put*

$$(140) \qquad z_i = \int \sqrt{\{w, z_{i-1}\}}\, dz_{i-1}, \qquad z_0 = z \qquad (i = 1, 2, \ldots, k).$$

We find

$$(141) \qquad \begin{cases} \{w, z_i\} - \{z_{i-1}, z_i\} = 1, \\ \{w, z_{i-1}\}[1 - \{w, z_i\}] = \{z_i, z_{i-1}\}, \end{cases}$$

and obtain therefore the following two forms for the relation between $\{w, z\}$ *and* $\{w, z_k\}$:

$$(142) \qquad \{w, z\} = \cfrac{\{z_1, z\}}{1 - \cfrac{\{z_2, z_1\}}{1 - \cfrac{\{z_3, z_2\}}{1 - \cfrac{\ddots}{1 - \cfrac{\{z_k, z_{k-1}\}}{\{w, z_k\}}}}}},$$

and

$$(143) \qquad \{w, z_k\} = 1 - \cfrac{\{z_k, z_{k-1}\}}{1 - \cfrac{\{z_{k-1}, z_{k-2}\}}{1 - \cfrac{\ddots}{1 - \cfrac{\{z_1, z\}}{\{w, z\}}}}}.$$

Let us investigate the corresponding question for the variable k which was introduced in Art. **7**, namely

$$(122) \qquad k = e^{-\sqrt{2}\varphi} = e^{-\sqrt{2} z_1}.$$

If we put $S = \log s$, in (124), we find

$$(144) \qquad \{e^S, x\} = \{S, x\} + \left(\frac{dS}{dx}\right)^2 \{e^S, S\} = \{S, x\} - \frac{1}{2}\left(\frac{dS}{dx}\right)^2.$$

Therefore we find from (122),

$$\{k, z\} = \{\varphi, z\} - \frac{1}{2}\left[\frac{d(-\sqrt{2}\varphi)}{dz}\right]^2 = \{\varphi, z\} - \{w, z\},$$

or

$$(145) \qquad \{\varphi, z\} = \{w, z\} + \{k, z\}.$$

From (125) we find

$$\{w, k\} = \left(\frac{d\varphi}{dk}\right)^2 [\{w, \varphi\} - \{k, \varphi\}],$$

and from (122) we find

$$\frac{dk}{d\varphi} = -\sqrt{2}\,k, \qquad \{k, \varphi\} = -1,$$

so that

(146) $$\{w, \varphi\} = \{k, \varphi\} + 2k^2\{w, k\} = 2k^2\{w, k\} - 1.$$

According to (134) we have

$$\{w, z\}[1 - \{w, \varphi\}] = \{\varphi, z\}.$$

If we substitute in this equation the values (145) and (146) for $\{w, \varphi\}$ and $\{\varphi, z\}$, we find

(147) $$\{w, z\} = \frac{\{k, z\}}{1 - 2k^2\{w, k\}}.$$

Let us use the notation k_1 in place of k and let us repeat the transformation by putting

$$k_2 = e^{-\sqrt{2}t_2}, \qquad t_2 = \int \sqrt{\{w, k_1\}}\, dk_1,$$

so that

$$\{w, k_1\} = \frac{\{k_2, k_1\}}{1 - 2k_2^2\{w, k_2\}}.$$

If we continue in this way, we find

(148) $$\{w, z\} = \cfrac{\{k, z\}}{1 - \cfrac{2k_1^2\{k_2, k_1\}}{1 - \cfrac{2k_2^2\{k_3, k_2\}}{1 - \ddots \cfrac{}{1 - 2k_n^2\{w, k_n\}}}}}.$$

The cases when the continued fractions (142) or (143) terminate are of special interest. We shall discuss the simplest cases of this sort.

If $\{z_1, z\} = 0$, (134) shows that either $\{w, z\} = 0$ or $\{w, z\} \neq 0$, $\{w, z_1\} = 1$. In the first case w is a linear function of z. In the second case we observe first that z_1 is not a constant, since

$$z_1 = \int \sqrt{\{w, z\}}\, dz, \qquad \{w, z\} \neq 0.$$

Since $\{z_1, z\} = 0$, z_1 is a non-constant linear function of z. Since

$$\{w, z_1\} = 1,$$

we have therefore

$$(149) \qquad w = \frac{\alpha e^{i\sqrt{2} z_1} + \beta}{\gamma e^{i\sqrt{2} z_1} + \delta}, \qquad z_1 = \frac{a z + b}{c z + d}, \qquad \begin{array}{l} ad - bc \neq 0, \\ \alpha\delta - \beta\gamma \neq 0, \end{array}$$

where α, β, γ, δ, a, b, c, d are constants. These are the functions which may be obtained from an exponential function $w = e^z$ by linear transformation of both z and w.

To find the intrinsic equation of these functions we observe that

$$\frac{dz_1}{dz} = \sqrt{\{w, z\}} = \frac{ad - bc}{(cz + d)^2}, \qquad \frac{dw}{dz_1} = \frac{i\sqrt{2}(\alpha\delta - \beta\gamma)\, e^{i\sqrt{2} z_1}}{(\gamma e^{i\sqrt{2} z_1} + \delta)^2},$$

$$\frac{dw}{dz} = \frac{dw}{dz_1}\frac{dz_1}{dz} = \frac{dw}{dz_1}\sqrt{\{w, z\}},$$

so that

$$\theta = \frac{\sqrt{\{w, z\}}}{w'} = \frac{(\gamma e^{i\sqrt{2} z_1} + \delta)^2}{i\sqrt{2}(\alpha\delta - \beta\gamma)\, e^{i\sqrt{2} z_1}},$$

and of course

$$z_1 = \varphi + \text{const.}$$

The resulting intrinsic equations are of the form

$$(150) \qquad \theta = \left(l e^{\frac{1}{2} i\sqrt{2}\varphi} + m e^{-\frac{1}{2} i\sqrt{2}\varphi}\right)^2.$$

Let us suppose next that $\{z_1, z\} \neq 0$, $\{z_2, z_1\} = 0$. Then we have, from (141),

$$(151) \qquad \{w, z_2\} - \{z_1, z_2\} = 1, \qquad \{w, z_1\}\left[1 - \{w, z_2\}\right] = 0,$$

so that either $\{w, z_1\} = 0$ or $\{w, z_1\} \neq 0$, $\{w, z_2\} = 1$.

In the former case we have from (134) and (133),

$$(152) \qquad \{w, z_1\} = 0, \qquad \{w, z\} = \{z_1, z\}, \qquad \{z, z_1\} = -1,$$

and these conditions imply again $\{z_2, z_1\} = 0$. But from (152)

we find

$$w = \frac{a z_1 + b}{c z_1 + d}, \qquad z = \frac{\alpha e^{\sqrt{2} z_1} + \beta}{\gamma e^{\sqrt{2} z_1} + \delta},$$

giving the general class of functions

(153) $$\frac{Aw + B}{Cw + D} = \log \frac{A' z + B'}{C' z + D'}.$$

To find the intrinsic equation of these functions, we put

$$\{w, z\} = \sigma.$$

Then the second equation of (152) becomes

(154) $$\frac{\sigma''}{\sigma} - \frac{5}{4}\left(\frac{\sigma'}{\sigma}\right)^2 = 2\sigma.$$

Since

$$\theta = \frac{\sqrt{\sigma}}{w'}, \qquad \sigma = (w')^2 \theta^2,$$

we find

(155) $$\frac{\sigma''}{\sigma} - \frac{5}{4}\left(\frac{\sigma'}{\sigma}\right)^2 = 2(w')^2\theta^2 + 2\frac{\theta''}{\theta} - 3\left(\frac{\theta'}{\theta}\right)^2 - 2\frac{w''}{w'}\frac{\theta'}{\theta}.$$

If we introduce w as independent variable in place of z, this formula becomes

(156) $$\frac{\sigma''}{\sigma} - \frac{5}{4}\left(\frac{\sigma'}{\sigma}\right)^2 = 2(w')^2\left[\theta^2 + \frac{1}{\theta}\frac{d^2\theta}{dw^2} - \frac{3}{2\theta^2}\left(\frac{d\theta}{dw}\right)^2\right],$$

and if we use φ as independent variable,

(157) $$\frac{\sigma''}{\sigma} - \frac{5}{4}\left(\frac{\sigma'}{\sigma}\right)^2 = 2(w')^2\left[\theta^2 + \theta\frac{d^2\theta}{d\varphi^2} - \frac{1}{2}\left(\frac{d\theta}{d\varphi}\right)^2\right].$$

Finally, if we again make use of the notation

(158) $$I = -\left[\frac{d^2 \log\theta}{d\varphi^2} + \frac{1}{2}\left(\frac{d\log\theta}{d\varphi}\right)^2\right],$$

as in Art. 5, we may write

(159) $$\frac{\sigma''}{\sigma} - \frac{5}{4}\left(\frac{\sigma'}{\sigma}\right)^2 = 2(w')^2\theta^2(1 - I).$$

The functions which are now studying satisfy the condition (154), that is, $I = 0$. This condition is easy to integrate, and so we find

$$\theta = k(\varphi + c)^2, \tag{160}$$

k and c being constants, as the intrinsic equation for any function of this sort.

Thus, if $\{z_1, z\} \neq 0$, $\{z_2, z_1\} = 0$, the function $w = f(z)$ is either a logarithmic function of the form (153) with an intrinsic equation of form (160) or else

$$\{w, z_1\} \neq 0, \qquad \{w, z_2\} = 1, \qquad \{z_2, z_1\} = 0.$$

We find therefore

$$w = \frac{a e^{i\sqrt{2} z_1} + b}{c e^{i\sqrt{2} z_1} + d}, \qquad z_2 = \frac{\alpha z_1 + \beta}{\gamma z_1 + \delta}, \tag{161}$$

and it only remains to find the relation between z_1 and z. According to (133) we have

$$\{z, z_1\} = \{w, z_1\} - 1,$$

and

$$\{w, z_1\} = \left\{ e^{i\sqrt{2}\frac{\alpha z_1 + \beta}{\gamma z_1 + \delta}}, z_1 \right\} = -\frac{1}{2}\left[\frac{d}{dz_1}\left(i\sqrt{2}\frac{\alpha z_1 + \beta}{\gamma z_1 + \delta}\right)\right]^2 = \frac{(\alpha\delta - \beta\gamma)^2}{(\gamma z_1 + \delta)^4},$$

according to (128) and (144). Consequently the relation between z and z_1 will be obtained from the differential equation

$$\{z, z_1\} = \frac{(\alpha\delta - \beta\gamma)^2}{(\gamma z_1 + \delta)^4} - 1. \tag{162}$$

Of course this differential equation may easily be reduced to an equation of the Riccati form

$$\frac{d\zeta}{dz_1} - \frac{1}{2}\zeta^2 = \frac{(\alpha\delta - \beta\gamma)^2}{(\gamma z_1 + \delta)^4} - 1, \qquad \zeta = \frac{\dfrac{d^2 z}{dz_1^2}}{\dfrac{dz}{dz_1}}, \tag{163}$$

or else to a linear equation of the second order, namely

$$\frac{d^2 y}{dz_1^2} + \frac{1}{2}\left[\frac{(\alpha\delta - \beta\gamma)^2}{(\gamma z_1 + \delta)^4} - 1\right] y = 0. \tag{164}$$

If y_1 and y_2 are linearly independent solutions of (164), we may write

$$z = \frac{Ay_1 + By_2}{Cy_1 + Dy_2}$$

as the general solution of (163).

The intrinsic equations of these functions follow at once form (161). We have

$$\theta = \frac{1}{\frac{dw}{dz_1}}, \qquad \varphi = z_1 + \text{const.},$$

and find therefore

(165)
$$\theta = \frac{(\gamma\varphi+\delta)^2\left[c\,e^{\frac{i}{\sqrt{2}}\frac{\alpha\varphi+\beta}{\gamma\varphi+\delta}} + d\,e^{-\frac{i}{\sqrt{2}}\frac{\alpha\varphi+\beta}{\gamma\varphi+\delta}}\right]^2}{i\sqrt{2}(ad-bc)(\alpha\delta-\beta\gamma)}.$$

We now return to the general theory. We wish to find the effect of the transformation from z to z_1 upon θ and φ. We have

$$\theta = \frac{\sqrt{\{w, z\}}}{w'}, \qquad z_1 = \varphi = \int \sqrt{\{w, z\}}\,dz,$$

and we put similarly

(166)
$$\theta_1 = \frac{\sqrt{\{w, z_1\}}}{\frac{dw}{dz_1}}, \qquad \varphi_1 = \int \sqrt{\{w, z_1\}}\,dz_1 = z_2.$$

We have found

$$\{w, z_1\} = 1 - \frac{\{z_1, z\}}{\{w, z\}} = 1 - \frac{\{z_1, z\}}{\sigma},$$

if we again put $\{w, z\} = \sigma$, and

$$\{z_1, z\} = \frac{1}{2}\left[\frac{\sigma''}{\sigma} - \frac{5}{4}\left(\frac{\sigma'}{\sigma}\right)^2\right] = (w')^2\theta^2(1-\mathrm{I}),$$

according to (159). Consequently we find

(167)
$$\{w, z_1\} = \mathrm{I},$$

and therefore we find *the formulae*

(168)
$$\theta_1 = \theta\sqrt{\mathrm{I}}, \qquad \varphi_1 = z_2 = \int \sqrt{\mathrm{I}}\,d\varphi,$$

which enable us to find the intrinsic equation of w as function of z_1 when the intrinsic equation of $w = f(z)$ is given.

Let us apply the these formulae to the intrinsic equation

$$\theta = a\varphi + b, \tag{169}$$

where a and b are constants. We find

$$I = \frac{a^2}{2(a\varphi + b)^2}, \qquad \theta_1 = \frac{a}{\sqrt{2}}, \qquad \varphi_1 = \frac{1}{\sqrt{2}}\log(a\varphi + b).$$

Since θ_1 is a constant, w is a logarithmic function of φ. In fact

$$w = \int \frac{d\varphi}{a\varphi + b} = \frac{1}{a}\log k(a\varphi + b), \qquad \theta = \frac{1}{k}e^{aw}.$$

We also have

$$\{z, w\} = -\theta^2 = -\frac{1}{k^2}e^{2aw}.$$

Therefore z must be a quotient of two independent solutions of

$$\frac{d^2 z}{dw^2} - \frac{1}{2k^2}e^{2aw}z = 0.$$

If we put

$$e^{2aw} = x, \qquad \frac{1}{8a^2k^2} = m,$$

this equation becomes

$$x\frac{d^2 z}{dx^2} + \frac{dz}{dx} - mz = 0,$$

which has the series

$$z_1 = \sum_{k=0}^{\infty} \frac{m^k z^k}{(k!)^2} \tag{170}$$

as one solution and

$$z_2 = z_1\int\frac{dx}{xz_1^2}$$

as a second independent one. We shall therefore obtain a function with a linear intrinsic equation by putting

$$z = \int\frac{dx}{xz_1^2}, \qquad x = e^{2aw}, \tag{171}$$

and then inverting this relation for w as function of z. The most general function of this sort will result, of course, if we replace z by a linear fractional function of z.

This example is only one of several in wich Bessel functions or other closely related functions make their appearance.

We return once more to our general theory. From (168) we find

$$\frac{d\log\theta_1}{d\varphi_1} = \frac{1}{\sqrt{I}}\left(\frac{d\log\theta}{d\varphi} + \frac{1}{2}\frac{d\log I}{d\varphi}\right),$$

$$\frac{d^2\log\theta_1}{d\varphi_1^2} = \frac{1}{I}\left(\frac{d^2\log\theta}{d\varphi^2} + \frac{1}{2}\frac{d^2\log I}{d\varphi^2}\right) - \frac{1}{2I^2}\frac{dI}{d\varphi}\left(\frac{d\log\theta}{d\varphi} + \frac{1}{2}\frac{d\log I}{d\varphi}\right).$$

If we put

(172) $$I_1 = -\left[\frac{d^2\log\theta_1}{d\varphi_1^2} + \frac{1}{2}\left(\frac{d\log\theta_1}{d\varphi_1}\right)^2\right],$$

we find therefore

(173) $$I I_1 = I - \frac{1}{2}\left[\frac{d^2\log I}{d\varphi^2} - \frac{1}{4}\left(\frac{d\log I}{d\varphi}\right)^2\right].$$

Thus if I is a constant, an important special case which we shall consider more fully later, I_1 will be equal to 1.

9. *Correspondences defined by the osculating linear function.* — The simplest cogredient which we have found is the pole of the osculating linear function. If z is the point of contact, we have the formula

$$p = z + \frac{a_1}{a_2} = z + 2\frac{w'}{w''}$$

for this point. We now proceed to study the question : as z changes its position in the z-plane, how will p move? Of course, the above equation contains the answer to this question since $w = f(z)$ is a given function of z.

In order to find $\frac{dp}{dz}$ it suffices to differentiate the expression for p. This is done most conveniently by making use of the formula

$$a'_k = (k+1)\alpha_k,$$

which we have already employed.

We find

$$p' = 1 + \frac{a_2 a'_1 - a_1 a'_2}{a_2^2} = 1 + \frac{2a_2^2 - 3a_1 a_3}{a_2^2} = \frac{3(a_2^2 - a_1 a_3)}{a_2^2},$$

and therefore

$$(174) \qquad p' = \frac{3(a_2^2 - a_1 a_3)}{a_1^2}(p - z)^2 = -\frac{1}{2}\{w, z\}(p - z)^2.$$

On account of the relations between z, p, and b, where a and b are the singularities of the osculating logarithm, we may also write

$$(175) \qquad p' = -\left(\frac{a - p}{a - z}\right)^2 = -\left(\frac{b - p}{b - z}\right)^2.$$

The zero of the osculating linear function was given by

$$(176) \qquad e = z + \frac{a_0 a_1}{a_0 a_2 - a_1^2} = z + \frac{2ww'}{ww'' - 2(w')^2}.$$

We find

$$(177) \qquad e' = -\frac{1}{2}\{w, z\}(e - z)^2,$$

so that p and e are solutions of the same Riccati equation

$$(178) \qquad \frac{d\lambda}{dz} = -\frac{1}{2}\{w, z\}(\lambda - z)^2.$$

Let us denote by l the point where the osculating linear function assumes the given value k. It is a simple matter to write down the expression for l, and to verify that l is also a solution of (178). We note the familiar fact that the cross-ratio of any four solutions of the same Riccati equation is a constant, and obtain the following theorem.

Let k_1, k_2, k_3, k_4 be any four constants, and let l_1, l_2, l_3, l_4 be the four points in which the linear function, which osculates $w = f(z)$ at the point z, assumes the values k_1, k_2, k_3, k_4 respectively. If z moves in any way in the z plane, the four points l_1, l_2, l_3, l_4 will move in such a way as to keep the cross-ratio (l_1, l_2, l_3, l_4) constant and equal to (k_1, k_2, k_3, k_4).

Equations (174) and (177) also show us that *the pole or the zero of the osculating linear function will be a fixed point, that is, the* *same point for all positions of z, if and only if w is itself a linear function of z.* The same remark applies to the point l.

If we introduce w or φ as independent variable, in place of z, we may write

$$(179) \qquad \frac{dp}{dw} = -\frac{1}{2}\theta^2(p-z)^2, \qquad \frac{dp}{d\varphi} = -\frac{1}{2}\theta(p-z)^2,$$

giving rise to the new expressions

$$(180) \qquad \theta^2 = \frac{-2\frac{dp}{dw}}{(p-z)^2}, \qquad \varphi = \int \frac{i\sqrt{2}\sqrt{\frac{dp}{dw}}\,dw}{p-z},$$

for θ and φ.

If w is given as function of z, we obtain p as a function of z by operations involving differentiations only. If p is given as function of z, w can be found by two quadratures, namely,

$$(181) \qquad w = c_0 + c_1 \int e^{2\int \frac{dz}{p-z}} dz.$$

It happens frequently, in the theory of linear differential equations, in the theory of automorphic functions, and in many problems of differential geometry, that $\{w, z\}$ or θ is given as a function of z, or of w, or of φ. The equations (174) and (179) will then be of use in connection with the determination of the corresponding function w. Thus, if $\{w, z\}$ is given as function of z, the Riccati equation (174) will determine p, and w may then be found from (181).

Of course, if one solution, say p, of (178) is known, all other solutions may be found by quadratures. If we apply the familiar formulae of the theory of the Riccati equation, we obtain the following result. *If p is one solution of (178), the general solution will be*

$$(182) \qquad \lambda = p + \frac{e^{2\int \frac{p'\,dz}{p-z}}}{c - \int \frac{p'}{(p-z)^2} e^{2\int \frac{p'\,dz}{p-z}} dz},$$

where c is an arbitrary constant.

We may write

$$e - z = \frac{\dfrac{a_1}{a_2}}{1 - \dfrac{a_1}{a_0}\dfrac{a_1}{a_2}} = \frac{p - z}{1 - \dfrac{a_1}{a_0}(p - z)},$$

whence

$$\frac{w'}{w} = \frac{e - p}{(e - z)(p - z)} = \frac{1}{p - z} - \frac{1}{e - z} = \frac{1}{z - e} - \frac{1}{z - p}.$$

If we differentiate both members, making use of (174) and (172), we find

$$\frac{d^2 \log w}{dz^2} = -\frac{1}{(z - e)^2} + \frac{1}{(z - p)^2},$$

whence follows the theorem :

If e and p represent the zero and the pole of the linear function which osculates $w = f(z)$ at the point z, e and p will, in general, be non constant functions of z. But the formulae

(183)
$$\left\{\begin{aligned} \log w &= \int\left(\frac{1}{z - e} - \frac{1}{z - p}\right) dz, \\ \frac{d \log w}{dz} &= \frac{1}{z - e} - \frac{1}{z - p}, \\ \frac{d^2 \log w}{dz^2} &= -\frac{1}{(z - e)^2} + \frac{1}{(z - p)^2}, \end{aligned}\right.$$

will hold, just as though e and p were constants.

Of course it is understood that, in the first of these equations, the path of integration is specified.

We proceed to make some simple applications. Let $w = e^{az}$. Then

(184)
$$p = z + \frac{2}{a}, \qquad e = z - \frac{2}{a}.$$

Consequently, *the zero and pole of the osculating linear function of an exponential function e^{az} are collinear with the point of contact. They are situated at equal distances on opposite sides of the point of contact, and the mutual distances of the three points remains constant for all positions of the point of contact.*

We may generalize this theorem by subjecting z to a linear transformation. To aid us in making the generalisation we remark that, according to (184), there exists a parabolic linear transformation, namely,

$$z'' = z' + \frac{2}{a},$$

which makes correspond to $z' = z$ and $z' = e$, the points $z'' = p$ and $z'' = z$ respectively. The generalized theorem is as follows.

Consider a function of the form $w = e^{a\frac{z-\lambda}{z-\mu}}$. If p and ε are the pole and the zero of the osculating linear function whose point of contact is z, the circle determined by p, ε, and z will pass through μ, and all of the circles obtained in this way, for different values of z, will have a common tangent at μ. The pairs (ε, p) and (z, μ) will be harmonic. The parabolic linear substitution which has μ as its only double point and which makes p correspond to z, will also make z correspond to ε.

In both of these cases, whenever z describes a circle, p and e will also describe circles. We now ask the general question; how shall we find the most general function $w = f(z)$ such that, when z describes any circle in the z plane the corresponding locus for p is also a circle?

For such functions we must have

$$(185) \qquad p = \frac{\alpha z + \beta}{\gamma z + \delta},$$

where α, β, γ, δ are constants, and therefore

$$\frac{w''}{w'} = \frac{2}{p - z} = \frac{2(\gamma z + \delta)}{-\gamma z^2 + (\alpha - \delta) z + \beta}.$$

Assume

$$\gamma \neq 0, \qquad (\alpha - \delta)^2 + 4\beta\gamma \neq 0.$$

Then we may write

$$\frac{w''}{w'} = \frac{A_1}{z - a_1} + \frac{A_2}{z - a_2},$$

where a_1 and a_2 are the two finite distinct zeros of $-\gamma z^2 + (\alpha - \delta) z + \beta$ which exist in this case, and where we have the equations

$$A_1 + A_2 = -2, \qquad A_1 a_2 + A_2 a_1 = 2\frac{\delta}{\gamma},$$

for A_1 and A_2. We write

$$(186) \qquad A_1 = -1 + \lambda, \qquad A_2 = -1 - \lambda, \qquad \lambda = \frac{a_1 + a_2 + 2\frac{\delta}{\gamma}}{a_2 - a_1},$$

and obtain

$$w = k \int (z - a_1)^{-1+\lambda} (z - a_2)^{-1-\lambda} dz + l,$$

or

$$(187) \qquad w - l = \frac{k}{a_1 - a_2} \left(\frac{z - a_1}{z - a_2} \right)^{\lambda},$$

where k and l are arbitrary constants, and where λ is determined by the coefficients α, β, γ, δ which occur in (185). Evidently we may also regard λ as being assigned in advance, the quantities α, β, γ, δ being determined subject to this condition.

We find

$$\theta = \frac{1}{k} \sqrt{\frac{1 - \lambda^2}{1}} (a_1 - a_2) \left(\frac{z - a_1}{z - a_2} \right)^{-\lambda}, \qquad \varphi = \sqrt{\frac{1 - \lambda^2}{2}} \log \frac{z - a_1}{z - a_2} + \varphi_0,$$

where the constant φ_0 depends upon the choice of the lower limit of the integral φ, and may be equated to zero if we take specifically

$$\varphi = \int_{\infty}^{z} \sqrt{\{w, z\}}\, dz.$$

Consequently the intrinsic equation of power functions of the form (187) is

$$(188) \qquad \theta = \frac{1 - a_2}{k} \sqrt{\frac{1 - \lambda^2}{2}} e^{-\lambda \sqrt{\frac{2}{1 - \lambda^2}} (\varphi - \varphi_0)}.$$

In a form more convenient for future reference, we may state this result as follows.

The intrinsic equation of a power function of the form

$$w = l + m \left(\frac{z - a_1}{z - a_2} \right)^{\lambda}, \qquad \lambda^2 \neq 0, \neq 1, \quad a_2 \neq a_1$$

is

$$\theta = a e^{-\sqrt{\frac{2\lambda^2}{1 - \lambda^2}} \varphi}.$$

Let us consider now the case

$$\gamma \neq 0, \qquad (\alpha-\delta)^2+4\beta\gamma=0.$$

We may write

$$\frac{w''}{w'}=\frac{-2\gamma(\gamma z+\delta)}{\gamma^2(z-a)^2}=-\frac{2}{z-a}+\frac{b}{(z-a)^2},$$

whence

$$w=l+m\,e^{-\frac{b}{z-a}}.$$

These are the exponential functions discussed previously, and may be regarded as limiting cases of the functions just obtained.

Finally if $\gamma=0$, $\alpha-\delta\neq 0$, we have

$$\frac{w''}{w'}=\frac{2\delta}{(\alpha-\delta)z+\beta}, \qquad \delta\neq 0,$$

which may be written

$$\frac{w''}{w'}=\frac{A}{z-a},$$

whence

$$w=l+m(z-a)^{A+1} \qquad \text{if} \qquad A\neq -1,$$

which is again of the form (187) except for differences removable by a linear transformation on z. If $A=-1$ we find a logarithmic function.

The case $\gamma=\alpha-\delta=0$ is also easily disposed of.

Thus *the functions, for wich p and z are connected by a linear relation, are the power functions of the form*

$$l+m\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)^{\lambda}$$

and the limiting cases in which they become exponentials or logarithms.

These same functions also have the further property that the singularities, a and b, of the osculating logarithm are connected linearly, with each other and with the point of contact. Consequently when z describes a circle, p, a, and b also describe circles.

This property of the points a and b may be deduced easily by making use of the general formulae for a and b. But we shall find an independent proof of this statement later.

The solution of the corresponding problem about the zero of the osculating linear function is immediate, since e will be the pole of the osculating linear function for the reciprocal function $\frac{1}{w}$. Therefore the functions for which e and z are connected by a linear relation are of the type

$$\frac{1}{l + m\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right)^{\lambda}}.$$

We now return to the general theory. We may regard the equation connecting p and w as defining a new function of z, namely

$$w_1 = p = z + 2\frac{w'}{w''},$$

and we may consider the pole $p_1 = w_2$ of *its* osculating linear function, so that

$$w_2 = p_1 = z + 2\frac{w'_1}{w''_1}.$$

If we continue in this way, we obtain a suite of functions, w, w_1, w_2, etc. The following two questions present themselves at once; when will the suite be a terminating one, and when will it be periodic?

The suite will terminate if and only if one of the functions of the suite, say w_k, has a fixed point for the pole of its osculating linear function, that is, if and only if w_k is a linear function,

$$w_k = \frac{\alpha z + \beta}{\gamma z + \delta}.$$

We may then find w_{k-1} by means of two quadratures,

$$w_{k-1} = c_{k-1.0} + c_{k-1.1}\int e^{2\int\frac{dz}{w_k - z}}dz.$$

To determine w_{k-2} we have a similar formula. Thus we obtain finally w as a result of $2k$ quadratures.

The same formulae are, of course, applicable to the case where w_k is any assigned function of z.

The simplest case of a periodic suite is given by $w_1 = w$. In that case w must satisfy the differential equation

$$(189) \qquad w = z + 2\frac{w'}{w''}, \qquad \text{or} \qquad \frac{w''}{w'} = \frac{2}{w - z},$$

which has a first integral of the form

$$(190\,a) \qquad w' - \log w' = \log k(w - z)^2,$$

or

$$(190\,b) \qquad k(w - z)^2 w' e^{-w'} = 1,$$

where k is an arbitrary constant.

Let us determine the intrinsic equation of such a function. We have

$$\frac{w''}{w'} = \frac{2}{w - z}, \qquad \frac{w'''}{w'} - \left(\frac{w''}{w'}\right)^2 = -\frac{2(w' - 1)}{(w - z)^2},$$

so that

$$\{w, z\} = -\frac{2w'}{(w - z)^2}, \qquad \theta^2 = -\frac{2}{w'(w - z)^2},$$

whence

$$\varphi = \pm\int \frac{i\sqrt{2w'}\,dz}{w - z} = \pm\frac{1}{2}i\sqrt{2}\int\frac{w''\,dz}{\sqrt{w'}} = \pm i\sqrt{2w'} + \varphi_0,$$

so that

$$w' = -\frac{1}{2}(\varphi - \varphi_0)^2.$$

From (190 b) we have

$$k(w - z)^2 w' = e^{w'}$$

so that

$$\theta^2 = -2k\,e^{-w'} = -2k\,e^{\frac{1}{2}(\varphi - \varphi_0)^2}.$$

If we write $\sqrt{-2k} = a$, $\varphi_0 = -b$ we see that

$$(191) \qquad \theta = a\,e^{\frac{1}{4}(\varphi + b)^2}$$

is the intrinsic equation of a function $w = f(z)$ which has the property that, for every z, the pole of the osculating linear function is given by w.

Of course, as in all cases, we may equate b to zero, the lower limit of the integral invariant φ being selected accordingly. We then find

$$(192)\qquad \begin{cases} w = \dfrac{1}{a}\displaystyle\int e^{-\frac{1}{4}\varphi^2}\,d\varphi, & \{w, \varphi\} = -\dfrac{1}{2} - \dfrac{1}{8}\varphi^2, \\ \{z, \varphi\} = \{w, \varphi\} - 1 = -\dfrac{3}{2} - \dfrac{1}{8}\varphi^2. \end{cases}$$

The effective determination of these functions depends upon integrating this Schwarzian equation for z as a function of φ. If we apply the method of Art. **4**, according to wich we may replace this problem by an integral equation, we find that the kernel function is equal to

$$K(\varphi, \psi) = e^{\frac{1}{8}(\varphi^2+\psi^2)}\int_{\psi}^{\varphi} e^{-\frac{1}{4}\rho^2}\,d\rho.$$

By means of either method, we see that *both w and z will be uniform functions of φ*. Thus the integral invariant φ is a *uniformizing variable* for functions of this class.

10. *Correspondences defined by the osculating logarithm.* — We now pass to the consideration of some analogous questions connected with the cogredients a and b, the singular points of the osculating logarithm. We have found the equations

$$(193)\qquad \begin{cases} \dfrac{1}{a-z} + \dfrac{1}{b-z} = \dfrac{2}{p-z} = \dfrac{2a_2}{a_1}, \\ -\dfrac{1}{a-z} + \dfrac{1}{b-z} = \sqrt{2}\,a_1\theta, \end{cases}$$

in Art. **5**, whence

$$(194)\qquad \begin{cases} \dfrac{2}{a-z} = \dfrac{2a_2}{a_1} - a_1\theta\sqrt{2}, \\ \dfrac{2}{b-z} = \dfrac{2a_2}{a_1} + a_1\theta\sqrt{2}. \end{cases}$$

If we differentiate the first of these equations, we find

$$-\frac{2}{(a-z)^2}\left(\frac{da}{dz} - 1\right) = \frac{6a_3}{a_1} - \frac{4a_2^2}{a_1^2} - 2\sqrt{2}\,a_2\theta - a_1\sqrt{2}\,\theta',$$

which gives rise to the formula

$$(195)\qquad \frac{da}{dz} = \frac{1}{\sqrt{2}}\,w'\theta'(a-z)^2,$$

and similarly

(196) $$\frac{db}{dz} = -\frac{1}{\sqrt{2}} w'\theta'(b-z)^2.$$

The equation

(197) $$\frac{da}{db} = -\left(\frac{a-z}{b-z}\right)^2,$$

which follows from (195) and (196), is especially simple and frequently useful. It is fundamental when we attempt to determine a function $w = f(z)$ for which the relation between a and b has been arbitrarily prescribed.

In this connection we note the following formulae, which follow from (193), (195) and (196).

Let a and b denote the singular points of the osculating logarithm of $w = f(z)$. In general a and b are non-constant functions of z. But the equations

(198) $$\begin{cases} w = \displaystyle\int^z \frac{1}{\theta\sqrt{2}}\left[\frac{1}{z-a} - \frac{1}{z-b}\right] dz, \\ \dfrac{dw}{dz} = \dfrac{1}{\theta\sqrt{2}}\left[\dfrac{1}{z-a} - \dfrac{1}{z-b}\right], \\ \dfrac{d^2w}{dz^2} = \dfrac{1}{\theta\sqrt{2}}\left[-\dfrac{1}{(z-a)^2} + \dfrac{1}{(z-b)^2}\right], \end{cases}$$

hold, just as though a, b, and θ were constants.

As equations (195) and (196) show, *a and b will be fixed points, provided that they are defined at all, if and only if $w = f(z)$ is itself a logarithmic function of z.*

For the exponential function $w = e^{kz}$ we find again a result of noteworthy simplicity. We have in this case

$$a - z = \frac{1}{k}(1+i), \qquad b - z = \frac{1}{k}(1-i), \qquad a - b = \frac{2}{k}i.$$

Thus, *in the case of the exponential function, the triangle azb is a right isosceles triangle, right angled at z, and its sides are of constant length.* The corresponding theorem about exponentials of the form $e^{k\frac{z-\lambda}{z-\mu}}$ may be obtained from this by projective generalization.

In these cases, just mentioned, both a and b describe circles whenever z moves on a circle. We shall solve the more general problem : *to determine those functions $w=f(z)$, for which a and b are connected by a linear relation with constant coefficients, so that whenever a describes a circle, b will also describe a circle.*

Let

$$mab+na+pb+q=0, \tag{199}$$

where m, n, p, and q are constants, and

$$np-mq\neq 0,$$

be the given relation between a and b. We may assume

$$np-mq=1 \tag{200}$$

without any restriction of generality. We find from (199)

$$(mb+n)\,da+(ma+p)\,db=0, \tag{201}$$

and from (197)

$$(b-z)^2\,da+(a-z)^2\,db=0. \tag{202}$$

If da and db are not both zero, that is, if our function $w=f(z)$ does not reduce to a logarithmic function, the consistency of (201) and (202) requires

$$(mb+n)(a-z)^2-(ma+p)(b-z)^2=0. \tag{203}$$

We may write (199) as follows

$$(mb+n)(ma+p)=np-mq=1,$$

as a result of which (203) becomes, after multiplication by $mb+n$,

$$(mb+n)^2(a-z)^2=(b-z)^2,$$

whence

$$(mb+n)(a-z)=\pm(b-z).$$

We may choose the signs of m and n, consistent with (200), in such a way as to have

$$(mb+n)(a-z)=b-z.$$

Therefore, *the existence of a linear relation between a and b implies the existence of linear relations between a and z, and between b and z, namely*

$$(204)\qquad \begin{cases} maz + (n+1)a + (p-1)z + q = 0, \\ mbz + (p+1)b + (n-1)z + q = 0. \end{cases}$$

If we compute $\frac{da}{dz}$ and $\frac{db}{dz}$ from these equations, and substitute these values in (195) and (196), we find

$$w'\theta' = \frac{\sqrt{2}(n-p)}{[mz^2 + (n+p)z + q]^2}.$$

On the other hand we have from (198)

$$w'\theta = \frac{1}{\sqrt{2}}\left[\frac{1}{z-a} - \frac{1}{z-b}\right] = \frac{1}{\sqrt{2}}\,\frac{n-p}{mz^2 + (n+p)z + q},$$

so that we find

$$\frac{\theta'}{\theta} = \frac{2}{mz^2 + (n+p)z + q} = \frac{2}{m(z-r)(z-s)} = \frac{2}{m(r-s)}\left[\frac{1}{z-r} - \frac{1}{z-s}\right],$$

if we denote by $z - r$ and $z - s$ the factors, supposed distinct, of $mz^2 + (n+p)z + q$. But this leads us back to the functions considered in Art. 9, for which there exists a linear relation between z and p. Thus, *the functions of the form*

$$l + m\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right)^{\lambda},$$

and the exponentials and logarithms which arise from them as limiting cases, are the only ones which have the property in question.

The same class of functions is obtained in answer to the following question. We know that for any analytic function, the two point-pairs (p_1, p_2) and (q_1, q_2) are harmonic, so that the cross-ratio (p_1, p_2, q_1, q_2) is constant and, in fact, equal to -1. But the two cross-ratios

$$(205)\qquad \begin{cases} \lambda_1 = (p_1, p_2, z, q_1) = \dfrac{z-p_1}{z-p_2}\,\dfrac{q_1-p_2}{q_1-p_1}, \\[2ex] \lambda_2 = (p_1, p_2, z, q_2) = \dfrac{z-p_1}{z-p_2}\,\dfrac{q_2-p_2}{q_2-p_1}, \end{cases}$$

are, in general, not constant. Let us determine those functions for which one of these cross-ratios is a contant.

We find at once

$$\lambda_1 + \lambda_2 = 0,$$

so that *the constancy of one of the two cross-ratios*, λ_1 *or* λ_2, *implies that of the other*.

To compute the expressions for λ_1 and λ_2 in terms of a_0, a_1, a_2, a_3, a_4, we may assume the point of contact z to be the origin. Then we have, according to Art. 4,

$$(206) \qquad q_1 q_2 = \frac{A}{C}, \qquad q_1 + q_2 = \frac{B}{C},$$

where

$$(207) \qquad A = a_1 a_3 - a_2^2, \qquad B = a_1 a_4 - a_2 a_3, \qquad C = a_2 a_4 - a_3^2,$$

and

$$(208) \qquad p_1 = \frac{1}{a_3}(a_2 + i\sqrt{A}), \qquad p_2 = \frac{1}{a_3}(a_2 - i\sqrt{A}).$$

Consequently we find from (205), putting $z = 0$,

$$(209) \qquad \lambda_1 \lambda_2 = \frac{p_1^2}{p_2^2} \frac{A - Bp_2 + Cp_2^2}{A - Bp_1 + Cp_1^2} = \frac{P + iQ\sqrt{A}}{P - iQ\sqrt{A}},$$

where

$$(210) \qquad P = a_2 Q + a_1(Ca_1 - Aa_3), \qquad Q = 2Aa_2 - Ba_1,$$

so that

$$(211) \qquad \left(\frac{1 - \lambda_1\lambda_2}{1 + \lambda_1\lambda_2}\right)^2 = -\frac{AQ^2}{P^2}.$$

In the case under consideration $\frac{AQ^2}{P^2}$ *must be a constant.*

If we put $|w, z| = \sigma$, we find

$$(212) \qquad A = \frac{1}{6} a_1^2 \sigma, \qquad B = \frac{1}{6} a_1 a_2 \sigma + \frac{1}{24} a_1^2 \sigma',$$

$$C = -\frac{1}{36} a_1^2 \sigma^2 + \frac{1}{24} a_1 a_2 \sigma',$$

so that

$$(213) \qquad P = -\frac{1}{18} a_1^4 \sigma^2, \qquad Q = \frac{1}{24} a_1^2 (4a_2\sigma - a_1\sigma').$$

Consequently the above condition reduces to

$$(214) \qquad \frac{a_1\sigma^{\frac{3}{2}}}{4a_2\sigma - a_1\sigma'} = l = \text{const.}$$

But we have

$$\frac{\sqrt{\sigma}}{a_1} = \theta, \qquad \frac{1}{2}\frac{\sigma'}{\sigma} - 2\frac{a_2}{a_1} = \frac{\theta'}{\theta}.$$

If we substitute these values in (214) and integrate, we find

$$\theta = \frac{2l}{w - a},$$

where a is an arbitrary constant. Therefore we find further

$$-\{z, w\} = \frac{\{w, z\}}{(w')^2} = \theta^2 = \frac{4l^2}{(w - a)^2}.$$

But we have

$$\{w^k, w\} = \frac{1 - k^2}{2w^2}.$$

Consequently one solution of the above differential equation for z, is

$$z = (w - a)^k,$$

where

$$k^2 = 1 + 8l^2.$$

Since l was an arbitrary constant, the exponent k may have any value. Moreover, the differential equation is invariant under linear transformations upon z. We obtain again, therefore, the functions of the form

$$a + \left(\frac{\alpha z + \beta}{\gamma z + \delta}\right)^r$$

with an arbitrary constant exponent r, in harmony with our original statement.

These same functions arise also if we demand that the cross-ratio (z, p, q_1, q_2) shall be constant, but we refrain from giving the proof. In the particular case when $Q = 0$, we find $(z, p, q_1, q_2) = -1$, and the function w reduces to a logarithmic function.

XI. — Functions for which the quadratic satellite of the point of contact is fixed.

The quadratic satellite τ of the point z was defined in Art. 4 as the harmonic conjugate of z with respect to q_1 and q_2, the poles of the osculating quadratic. We propose to obtain the formula for $\frac{d\tau}{dz}$ and, in particular, discuss briefly the cases when τ is a constant.

We have

$$\tau = z + \frac{2q_1q_2}{q_1+q_2} = z + 2\frac{\mathrm{A}}{\mathrm{B}} = z + 8\frac{\mathrm{A}}{\mathrm{A}'}; \tag{215}$$

where we are using again the notations (207) and the further relation

$$\mathrm{A}' = 4\mathrm{B}. \tag{216}$$

If we differentiate (215), we find

$$\tau' = \frac{9(\mathrm{A}')^2 - 8\mathrm{AA}''}{(\mathrm{A}')^2}, \qquad \text{assuming} \qquad \mathrm{A}' \neq 0. \tag{217}$$

This formula may be written in a number of other ways, namely

$$\left(\frac{\mathrm{A}'}{\mathrm{A}}\right)^2 \tau' = -32(w')^2\theta^2 - 16\left[\frac{\theta''}{\theta} - \frac{5}{4}\left(\frac{\theta'}{\theta}\right)^2\right] + 16\frac{w''}{w'}\frac{\theta'}{\theta}, \tag{218}$$

or

$$\left(\frac{d\log \mathrm{A}}{dw}\right)^2 \frac{d\tau}{dz} = -16\left[\frac{1}{\theta}\frac{d^2\theta}{dw^2} - \frac{5}{4\theta^2}\left(\frac{d\theta}{dw}\right)^2 + 2\theta^2\right], \tag{219}$$

which may be written

$$\left(\frac{d\log \mathrm{A}}{dw}\right)^3 \frac{d\tau}{dz} = -4\left[8\theta^2 - 4\mathrm{I}\theta^2 + \left(\frac{1}{\theta}\frac{d\theta}{dw}\right)^2\right], \tag{220}$$

since

$$\mathrm{I} = -\frac{1}{\theta^2}\left[\frac{d^2\log\theta}{dw^2} - \frac{1}{2}\left(\frac{d\log\theta}{dw}\right)^2\right] = -\left[\frac{d^2\log\theta}{d\varphi^2} + \frac{1}{2}\left(\frac{d\log\theta}{d\varphi}\right)^2\right]. \tag{221}$$

Finally we may write

$$\left(\frac{d\log \mathrm{A}}{dw}\right)^2 \frac{d\tau}{dz} = -4\theta^2\left[8 - 4\mathrm{I} + \left(\frac{d\log\theta}{d\varphi}\right)^2\right]. \tag{222}$$

If τ is a fixed point, we have $\tau' = 0$. Let us assume that $\theta \not\equiv 0$, so that $w = f(z)$ is not a linear function, and also that A is not a constant. Then we must have

$$8 + \left(\frac{d \log \theta}{d\varphi}\right)^2 - 4\mathrm{I} = 0. \tag{223}$$

The integration of this equation will give us the intrinsic equations of the functions under consideration.

Let us put

$$u = \frac{d \log \theta}{d\varphi}, \qquad \mathrm{I} = -\left(\frac{du}{d\varphi} + \frac{1}{2} u^2\right), \tag{224}$$

so that (223) becomes

$$\frac{du}{d\varphi} + \frac{3}{4} u^2 + 2 = 0. \tag{225}$$

This equation has two constant solutions, namely

$$u = \pm 2\sqrt{\frac{2}{3}}\, i,$$

leading to the intrinsic equations

$$\theta = k e^{\pm 2\sqrt{\frac{2}{3}} i \varphi}. \tag{226}$$

According to (188) the corresponding function is a power function whose exponent λ is equal to ± 2. That is, the functions which correspond to the intrinsic equations of form (226) are rational functions of z of degree two, whose two poles coincide. It is quite evident that quadratic functions of this sort must be solutions of our problem; the fixed point τ in this case coincides with the two coincident poles of the function.

If u is not a constant, we conclude from (225) by integration, that

$$u = \frac{2}{3}\sqrt{6} \cot \frac{\sqrt{6}}{2}(\varphi - m), \tag{227}$$

and

$$\theta = n \sin^{\frac{4}{3}} \frac{\sqrt{6}}{2}(\varphi - m), \tag{228}$$

where m and n are arbitrary constants, and where m may be equated to zero without essential loss of generality.

Let us return to equation (217). If $A' \neq 0$, $\tau' = 0$, we have

$$9\frac{A'}{A} - 8\frac{A''}{A'} = 0,$$

whence follows

$$A = (kz + l)^{-8}, \tag{229}$$

where k and l are arbitrary constants, and this formula also applies to the case $A' = 0$ by putting $k = 0$. If now we make a linear transformation of the independent variable

$$\bar{z} = \frac{\alpha z + \beta}{\gamma z + \delta},$$

and compute the resulting value of A, which we shall call $\bar{A}$, we find

$$\bar{A} = \frac{(\gamma z + \delta)^8 A}{(\alpha\delta - \beta\gamma)^2} = \frac{(\gamma z + \delta)^8 (kz + l)^{-8}}{(\alpha\delta - \beta\gamma)^2}. \tag{230}$$

Since α, β, γ, δ are at our disposal, we may clearly choose them in such a way as to make $\bar{A}$ become a constant, any nonvanishing constant in fact if $A \neq 0$, that is, if ω is not a linear function of z. Equation (215) tells us that this transformation has the effect of removing the fixed point τ to infinity.

Let us then assume that this transformation has been made, so that

$$A = a_1 a_3 - a_2^2 = a = \text{const.}, \tag{231}$$

and therefore

$$B - \frac{1}{4}A' = a_1 a_4 - a_2 a_3 = 0. \tag{232}$$

These equations may be written

$$\omega'\omega''' - \frac{3}{2}(\omega'')^2 = a, \qquad \omega'\omega^{iv} - 2\omega''\omega''' = 0. \tag{233}$$

These equations are satisfied by $\omega''' = 0$, $\omega'' = \pm\sqrt{-\frac{2}{3}a}$, which corresponds to the special case mentioned above, when w is a quadratic function with coincident poles. In all other cases, we may write in place of the second equation (233),

$$\frac{w^{(4)}}{w'''} - 2\frac{w''}{w'} = 0,$$

whence

$$w''' = b(w')^2, \qquad b \neq 0.$$

If we substitute this value for w''' in (233), we find

$$(w'')^2 = \frac{2}{3}[b(w')^3 - a].$$

We saw that it involved no essential restriction of generality to assume any convenient value for the constant a. Let us therefore put $a = b$, so that we obtain

$$\left(\frac{dw'}{dz}\right)^2 = \frac{2}{3}a[(w')^3 - 1] = \frac{a}{6}[4(w')^3 - 4]. \tag{234}$$

Consequently w' is a doubly periodic function of z, which is easily expressible in terms of the Weierstrass $\wp$ function, with the invariants $g_2 = 0$, $g_3 = 4$, namely

$$w' = \wp\left(\sqrt{\frac{a}{6}}z + k\right), \tag{235}$$

where k is an arbitrary constant. Since $g_2 = 0$, these elliptic functions belong to the equi-anharmonic case. But we have

$$\wp(u) = -\frac{d\zeta(u)}{du} = -\frac{d^2 \log \sigma(u)}{du^2}, \tag{236}$$

where ζ and σ are the Weierstrassian ζ and σ functions. Therefore we find from (235)

$$w = -\sqrt{\frac{6}{a}}\zeta\left(\sqrt{\frac{a}{6}}z + k\right) + l. \tag{237}$$

We have obtained the following result. *The functions for which the quadratic satellite of the point of contact is a fixed point, are either quadratic functions with coincident poles, or else they can be obtained from a linear combination like $m\zeta(u) + l$, where $\zeta(u)$*

is an equi-anharmonic Weierstrass ζ function, by equating u to any linear function of z.

XII. — Enlargement of the group.

So far we have been engaged in studying the effect, upon the function $w = f(z)$, of linear transformations of the independent variable z. If we consider instead linear transformations of the dependent variable w, we obtain nothing essentially new. For we may regard all questions of this king as being connected with linear transformations of the *independent* variable for the *inverse* function $z = f^{-1}(w)$. Moreover, the analytic form which these results would assume may also be regarded as familiar, since the Schwarzian derivative will be the fundamental differential invariant for all such transformations.

We do obtain something essentially new however, if we consider the effect of linear transformations upon both variables at the same time. Let us, therefore, consider the group of transformations of the form

$$(238) \qquad \bar{z} = \frac{\alpha z + \beta}{\gamma z + \delta}, \qquad \bar{w} = \frac{aw + b}{cw + d},$$

where α, β, γ, δ, a, b, c, d are arbitrary constants. This group is clearly a six-parameter group.

The integral

$$(239) \qquad \varphi = \int \sqrt{\{w, z\}}\, dz$$

evidently maintains its invariant character under all of the transformations of our enlarged group. The same thing is *not* true however of θ. We have, from (238),

$$(240) \qquad d\bar{z} = \frac{\alpha\delta - \beta\gamma}{(\gamma z + \delta)^2} dz, \qquad d\bar{w} = \frac{ad - bc}{(cw + d)^2} dw,$$

and from (130)

$$(241) \qquad \{\bar{w}, \bar{z}\} = \frac{(\gamma z + \delta)^4}{(\alpha\delta - \beta\gamma)^2} \{w, z\}.$$

According to (240) we have

$$\frac{d\overline{w}}{dz} = \frac{ad - bc}{\alpha\delta - \beta\gamma}\left(\frac{\gamma z + \delta}{cw + d}\right)^2 \frac{dw}{dz},$$

so that

$$(242) \qquad \overline{\theta} = \pm \frac{(cw + d)^2}{ad - bc}\,\theta.$$

From this equation we find

$$\frac{d\log\overline{\theta}}{d\overline{w}} = \left[\frac{d\log\theta}{dw} + \frac{2c}{cw + d}\right]\frac{(cw + d)^2}{ad - bc},$$

$$\frac{d^2\log\overline{\theta}}{d\overline{w}^2} = \left[\frac{d^2\log\theta}{dw^2} + \frac{2c}{cw + d}\frac{d\log\theta}{dw} + \frac{2c^2}{(cw + d)^2}\right]\frac{(cw + d)^4}{(ad - bc)^2},$$

whence we conclude that the function

$$(242) \qquad I = -\frac{1}{\theta^2}\left[\frac{d^2\log\theta}{dw^2} - \frac{1}{2}\left(\frac{d\log\theta}{dw}\right)^2\right]$$

is an absolute differential invariant of the enlarged group (238). This function has already presented itself to our attention, and we recall two other forms in which is may be written :

$$(243) \qquad I = -\left[\frac{d^2\log\theta}{d\varphi^2} + \frac{1}{2}\left(\frac{d\log\theta}{d\varphi}\right)^2\right],$$

and

$$(244) \qquad I = -\frac{1}{(w')^2\theta^2}\left[\frac{d^2\log\theta}{dz^2} - \frac{1}{2}\left(\frac{d\log\theta}{dz}\right)^2 - \frac{w''}{w'}\frac{\theta'}{\theta}\right].$$

We observe that I is a differential invariant of the fifth order and that there exists no differential invariant of the combined transformations (238) which is of lower order. We express this briefly by speaking of I as the *fundamental differential hyperinvariant* of $w = f(z)$.

We shall speak of the relation which expresses I as a function of φ, as the *hyperintrinsic equation* of the function $w = f(z)$. This hyperintrinsic equation will be the same for all functions which can be obtained from $w = f(z)$ by all of the transformations of our six-parameter group. Moreover it is evident that, whenever I is given as an arbitrary analytic function of φ, we can always assert the existence of infinitely many functions $w = f(z)$ which correspond to the given hyperintrinsic equation, and that all of these functions are transformable into each other by means of transformations of the group (238). We can even be more specific in our statement. If we have given the hyperintrinsic equation

$$(245) \qquad I = G(\varphi)$$

and if we put

$$(246) \qquad u = \frac{d \log \theta}{d\varphi},$$

we find from (245)

$$(247) \qquad \frac{du}{d\varphi} + \frac{1}{2} u^2 = -I = -G(\varphi),$$

so that u may be found by integrating an equation of the Riccati type. We then find φ from (246) by a quadrature. Thus, *if the hyperintrinsic equation of a function is given, its intrinsic equation may be found by integrating an equation of the Riccati form, and a quadrature.* The methods for obtaining the function $w = f(z)$ itself have been discussed previously.

However, the most symetrical solution of our problem is as follows. *To find the functions $w = f(z)$ whose hyperintrinsic equation is $I = G(\varphi)$, integrate the two Schwarzian differential equations*

$$(248) \qquad \{w, \varphi\} = I, \qquad \{z, \varphi\} = I - 1$$

and then eliminate φ.

The familiar connection between Schwarzian equations and linear differential equations of the second order enables us to draw some simple but far-reaching conclusions.

Let W_1 and W_2 be two linearly independent solutions of the linear differential equation

$$(249) \qquad \frac{d^2 W}{d\varphi^2} + \frac{1}{2} I W = 0.$$

Then $\frac{W_2}{W_1}$ will be a solution of the first equation (248) and any solution of the latter equation will be a linear fractional function of $\frac{W_2}{W_1}$. Similarly, any linear function of $\frac{Z_2}{Z_1}$, where Z_1 and Z_2 are independent solutions of

$$\frac{d^2Z}{d\varphi^2} + \frac{1}{2}(I - 1)Z = 0, \tag{250}$$

will be a solution of the second equation (248).

Let us consider now the case when $I = G(\varphi)$ is an integral, rational or transcendental, function of φ. Then w_1, w_2, z_1, z_2 will also be integral functions of φ, and therefore w and z will be uniform meromorphic functions of φ. Of course, in special cases, these functions may even be holomorphic. We have obtained the following theorem.

If a function $w = f(z)$ has the property that its fundamental, hyperinvariant I is an integral (rational or transcendental) function of the integral invariant φ, then this integral invariant is a uniformizing variable for the functional relation $w = f(z)$. More specifically, w and z will both be holomorphic or meromorphic functions of φ.

In all cases the two linear differential equations (249) and (250) are very closely related. They have the same singular points, and at each of these singular points, the canonical fundamental solutions have the same exponents.

Let us denote the inverse function of $w = f(z)$ by $z = f^{-1}(w)$, and let $\bar\varphi$, $\bar\theta$, $\bar I$ be the corresponding invariants. We shall then have

$$\{z, \bar\varphi\} = \bar I, \quad \{w, \bar\varphi\} = \bar I - 1, \tag{251}$$

and

$$d\bar\varphi^2 = \{z, w\}\, dw^2 = -\{w, z\}\, dz^2 = -d\varphi^2,$$

so that

$$d\bar\varphi^2 + d\varphi^2 = 0. \tag{252}$$

Consequently we find

$$\{z, \bar\varphi\} = \{z, \varphi\}\left(\frac{d\varphi}{d\bar\varphi}\right)^2 = -\{z, \varphi\}, \qquad \{w, \bar\varphi\} = -\{w, \varphi\}. \tag{253}$$

From these equations, we conclude

$$-(s, \varphi) = \bar{I} = 1 - I, \qquad -(w, \varphi) = \bar{I} - 1 = -I,$$

whence

$$I + \bar{I} = 1. \tag{254}$$

Therefore, *the fundamental hyperinvariants of two inverse functions have unity for their sum.* Consequently the inverse of any function whose hyperintrinsic equation is $I = G(\varphi)$, will have a hyperintrinsic equation of the form $1 - \bar{I} = G(\pm i\varphi)$.

We proceed to determine those functions for which the hyperinvariant I is a constant. In that case the Riccati equation (247) is easily integrated. It becomes

$$2\frac{du}{d\varphi} = -(u^2 + \lambda^2), \tag{255}$$

if we put

$$I = \frac{1}{2}\lambda^2 = \text{const.} \tag{256}$$

We conclude

$$d\varphi = \frac{-2\,du}{u^2 + \lambda^2}, \tag{257}$$

unless u should be a constant, equal to $\pm i\lambda$, in which case the right member would be indeterminate. This special case, $u = \pm i\lambda$, leads to the intrinsic equation

$$\rho = e^{\pm i\lambda\varphi + \mu}, \tag{258}$$

which we have discussed many times, and to which correspond the power functions of the form

$$w = l + \left(\frac{\alpha z + \beta}{\gamma z + \delta}\right)^r.$$

If u is not a constant, the integration of (257) gives

$$\frac{d\log\rho}{d\varphi} = u = -\lambda\tan\frac{\lambda}{2}(\varphi + \mu), \tag{259}$$

where μ is a constant. A further integration gives

$$\theta = \nu \cos^2 \frac{\lambda}{2}(\varphi + \mu), \tag{260}$$

where ν is a further constant, as *the intrinsic equation of all of those functions for which* I *is a constant while u is not a constant.*

The corresponding functions are again power functions, but of the more general form defined by

$$\frac{aw + b}{cw + d} = \left(\frac{\alpha z + \beta}{\gamma z + \delta}\right)^r, \tag{261}$$

where the exponent r is connected with the hyperinvariant I *by means of the equation*

$$\mathrm{I} = \frac{r^2}{r^2 - 1}, \qquad r^2 = \frac{1}{\mathrm{I} - 1}. \tag{262}$$

This may be proved either by starting from the intrinsic equation (260) and integrating it, or more easily, by observing that I will take the constant value given by (262) when we put $w = z^r$, and remembering, that all functions obtainable from $w = z^r$ by the transformations of our six parameter group, have the same hyperintrinsic equation.

Formula (261) becomes unintelligible when I is equal to zero or unity. However in these cases we see directly, from (248), that either

$$\frac{aw + b}{cw + d} = \log \frac{\alpha z + \beta}{\gamma z + \delta}, \tag{263}$$

or

$$\frac{aw + b}{cw + d} = e^{\frac{\alpha z + \beta}{\gamma z + \delta}}, \tag{264}$$

and these functions may be regarded as included in (261) as limiting cases.

If $r = \pm 2$, w is a rational quadratic function of z, so that we find

$$\mathrm{I} = \frac{4}{3} \tag{265}$$

as the hyperintrinsic equation of all quadratic functions. The intrinsic equation of a quadratic function is either of the form

$$\theta = e^{\pm 2i\sqrt{\frac{2}{3}}(\varphi+\mu)}, \tag{266}$$

or

$$\theta = \nu \cos^2 \sqrt{\frac{2}{3}}(\varphi + \mu). \tag{267}$$

The quadratic functions which correspond to (266) *have coincident poles, while those which correspond to* (267) *have distinct poles.*

The differential equation of all quadratic functions, which we found in Art. 4 in the form $c_3 = 0$, is of course equivalent to (265), and may therefore be written in any of the following forms :

$$\tag{268} \left\{ \begin{aligned} &\frac{d^2 \log \theta}{d\varphi^2} + \frac{1}{2}\left(\frac{d \log \theta}{d\varphi}\right)^2 + \frac{4}{3} = 0, \\ &\frac{d^2 \theta}{dw^2} - \frac{3}{2\theta}\left(\frac{d\theta}{dw}\right)^2 + \frac{4}{3}\theta^3 = 0, \\ &\frac{\theta''}{\theta} - \frac{3}{2}\left(\frac{\theta'}{\theta}\right)^2 + \frac{4}{3}\theta^2(w')^2 - \frac{\theta'}{\theta}\frac{w''}{w'} = 0, \\ &\frac{\sigma''}{\sigma} - \frac{5}{4}\left(\frac{\sigma'}{\sigma}\right)^2 + \frac{2}{3}\sigma = 0, \\ &\sigma = \{w, z\}. \end{aligned} \right.$$

We shall henceforth speak of the functions defined by (261) as *general* power functions, and use the word *special* power function for those of the form

$$w = l + \left(\frac{\alpha z + \beta}{\gamma z + \delta}\right)^r.$$

If a, c, α and γ are all different from zero, we may write the defining equation (261), of a general power function, in the form

$$\frac{w - \lambda}{w - \mu} = M\left(\frac{z - \rho}{z - \sigma}\right)^r, \tag{269}$$

and, unless r is an integer, w will be a multiform function of z, which has $z = \rho$ and $z = \sigma$ as its only branch points. The inverse function will be a multiform function unless $\frac{1}{r}$ is an integer, and will have λ

and μ as its only branch points. If not all of the numbers a, c, α, γ are different from zero, the corresponding branch point is at infinity, and the usual changes in the form of (269) should be made.

We shall use the term *branchpoint* for each of the points ρ, σ, λ, μ in all cases, even if r or $\frac{1}{r}$ is an integer. We may then characterize the *special* power functions as being those for which the inverse function has one of its branchpoints at infinity.

XIII. — The osculating power function.

We have studied in detail the functions for which I is a constant. There are number of other interesting cases in which the equations (248) admit of explicit integration. In general those cases in which the corresponding Riccati equations are integrable by quadratures, lead to functions connected with Bessel functions. But we prefer not to develop this theory any farther, at present. We shall show instead, how to determine the osculating power function, that is, the function of the form

$$\frac{w-\lambda}{w-\mu} = M\left(\frac{z-\rho}{z-\sigma}\right)^r, \tag{269}$$

which has contact of the fifth order with a given function $w = f(z)$ at a given point. Of course, we may again, without essential loss of generality, assume that the origin $z = 0$ is the point of contact.

Let us write (269) as follows

$$W = MZ^r, \qquad Z = \frac{z-\rho}{z-\sigma}, \qquad W = \frac{w-\lambda}{w-\mu}, \tag{270}$$

so that

$$\frac{dw}{dz} = \frac{Mr(\rho-\sigma)}{\lambda-\mu} Z^{r-1}\left(\frac{w-\mu}{z-\sigma}\right)^2, \tag{271}$$

and

$$\frac{\dfrac{d^2w}{dz^2}}{\dfrac{dw}{dz}} = \frac{(r-1)(\rho-\sigma)}{Z(z-\sigma)^2} + \frac{2\dfrac{dw}{dz}}{w-\mu} - \frac{2}{z-\sigma}. \tag{272}$$

From the general properties of the Schwarzian derivative we have,

$$(273)\quad \begin{cases} \{w, z\} = \{W, z\} = \{Z^r, z\} = \left(\dfrac{dZ}{dz}\right)^2 \{Z^r, Z\} = \dfrac{(\rho-\sigma)^2}{(z-\sigma)^4}\{Z^r, Z\}, \\ \{Z^r, Z\} = -\dfrac{1}{2}(r^2-1)Z^{-2}, \end{cases}$$

so that

$$(274)\qquad \{w, z\} = \frac{(1-r^2)(\rho-\sigma)^2}{2(z-\rho)^2(z-\sigma)^2},$$

and

$$(275)\qquad \frac{\{w, z\}'}{\{w, z\}} = -\frac{2}{z-\rho} - \frac{2}{z-\sigma}.$$

If (269) defines that power function which osculates the function

$$(276)\qquad w = a_0 + a_1 z + a_2 z^2 + \ldots$$

at $z = 0$, the expressions (271) to (275) must reduce for $z = 0$ to the corresponding expressions formed for the function (276), which we now refer to as the function w. Therefore we must have

$$(277)\quad \begin{cases} \{w, z\}_0 = \theta_0^2 (w'_0)^2 = \dfrac{1}{2}(1-r^2)\left(\dfrac{\rho-\sigma}{\rho\sigma}\right)^2; \\ \dfrac{\theta'_0}{\theta_0} + \dfrac{w''_0}{w'_0} = \dfrac{1}{\rho} + \dfrac{1}{\sigma}, \end{cases}$$

and, of course, the hyperinvariant I must assume the same value for $z = 0$ for both functions, so that

$$(278)\qquad r^2 = \frac{I_0}{I_0 - 1}.$$

We may re-write (277) as follows:

$$\left(\frac{1}{\rho} - \frac{1}{\sigma}\right)^2 = 2(1 - I_0)(w'_0\theta_0)^2,$$

$$\frac{1}{\rho} + \frac{1}{\sigma} = \frac{w''_0}{w'_0} + \frac{\theta'_0}{\theta_0},$$

so that we find the equations

$$(279)\quad \begin{cases} \dfrac{2}{\rho} = \dfrac{w''_0}{w'_0} + \dfrac{\theta'_0}{\theta_0} + w'_0\theta_0\sqrt{2(1-I_0)}, \\ \dfrac{2}{\sigma} = \dfrac{w''_0}{w'_0} + \dfrac{\theta'_0}{\theta_0} - w'_0\theta_0\sqrt{2(1-I_0)}. \end{cases}$$

where a change in the determination of the square root would be equivalent to a mere change of notation. If we substitute these values in (270), (271) and (272), we find the equations

$$(280)\qquad \begin{cases} \dfrac{2w'_0}{w_0-\mu} = -\dfrac{\theta'_0}{\theta_0} + w'_0\theta_0\sqrt{-2I_0}, \\[2ex] \dfrac{2w'_0}{w_0-\lambda} = -\dfrac{\theta'_0}{\theta_0} - w'_0\theta_0\sqrt{-2I_0}, \end{cases}$$

for the determination of λ and μ, the branchpoints of the inverse of the osculating power function, and finally

$$(281)\qquad M = \frac{w_0-\lambda}{w_0-\mu}\left(\frac{\sigma}{\rho}\right)^r,$$

or

$$(282)\quad M = \frac{-\dfrac{\theta'_0}{\theta_0} + w'_0\theta_0\sqrt{-2I_0}}{-\dfrac{\theta'_0}{\theta_0} - w'_0\theta_0\sqrt{-2I_0}}\left[\frac{\dfrac{w''_0}{w'_0} + \dfrac{\theta'_0}{\theta_0} + w'_0\theta_0\sqrt{2(1-I_0)}}{\dfrac{w''_0}{w'_0} + \dfrac{\theta'_0}{\theta_0} - w'_0\theta_0\sqrt{2(1-I_0)}}\right]^{\sqrt{\frac{I_0}{I_0-1}}}.$$

Let us omit the index o in our further formulae. The quadratic satellite of $z = 0$ was given by

$$z = \frac{2q_1q_2}{q_1+q_2} = 2\frac{a_1a_3-a_2^2}{a_1a_4-a_2a_3} = \frac{8A}{A'},$$

where

$$A = a_1a_3 - a_2^2 = \frac{1}{6}(w')^2\{w, z\} = \frac{1}{6}(w')^4\theta^2,$$

so that

$$\frac{A'}{A} = 4\frac{w''}{w'} + 2\frac{\theta'}{\theta},$$

and consequently

$$\frac{4}{z} = 2\frac{w''}{w'} + \frac{\theta'}{\theta}.$$

We also have

$$\frac{1}{p} = \frac{1}{2}\frac{w''}{w'}$$

if p is the pole of the osculating linear function, so that

$$\frac{4}{z} - \frac{2}{p} = \frac{w''}{w'} + \frac{\theta'}{\theta}.$$

Consequently we find, from (277),

$$(283)\qquad \frac{1}{\rho}+\frac{1}{\sigma}=\frac{4}{\tau}+\frac{2}{p}.$$

Let ψ be the harmonic conjugate of $z=0$ with respect to ρ and σ, so that

$$(284)\qquad \frac{1}{\rho}+\frac{1}{\sigma}=\frac{2}{\psi}.$$

We find, from (283) and (284),

$$(285)\qquad \frac{1}{p}+\frac{1}{\psi}=\frac{2}{\tau},$$

and therefore the following theorem.

Construct the harmonic conjugate ψ, of the pole of the osculating linear function, with respect to the point of contact and its quadratic satellite. Then the branchpoints, ρ and σ, of the osculating power function will be harmonic conjugates of each other with respect to ψ and the point of contact.

If the point of contact is a general point z, instead of $z=0$, we have in place of (279),

$$(286)\qquad \begin{cases} \dfrac{2}{\rho-z}=\dfrac{w''}{w'}+\dfrac{\vartheta'}{\vartheta}+w'\vartheta\sqrt{2(1-I)}, \\[2ex] \dfrac{2}{\sigma-z}=\dfrac{w''}{w'}+\dfrac{\vartheta'}{\vartheta}+w'\vartheta\sqrt{2(1-I)}, \end{cases}$$

whence

$$(287)\qquad w'=\frac{1}{\vartheta\sqrt{2(1-I)}}\left(\frac{1}{z-\sigma}-\frac{1}{z-\rho}\right),$$

and

$$(288)\qquad w=\int\frac{1}{\vartheta\sqrt{2(1-I)}}\left(\frac{1}{z-\sigma}-\frac{1}{z-\rho}\right)dz,$$

a formula which, like the corresponding integral formulae for w in terms of p and e, or in terms of a and b, is capable of interesting applications.

We wish to study however, the variation of ρ and σ with z. Differentiation of (286) gives

$$-\frac{2}{(\rho-z)^2}(\rho'-1)=\frac{w'''}{w'}-\left(\frac{w''}{w'}\right)^2+\frac{d^2\log\theta}{dz^2}+w''\theta\sqrt{2(1-I)}$$
$$+w'\frac{d\theta}{dz}\sqrt{2(1-I)}+\frac{1}{2}w'\theta\left(\frac{-2I'}{\sqrt{2(1-I)}}\right),$$

whence

$$-\frac{4\rho'}{(\rho-z)^2}=2(w')^2\theta^2 I+2\frac{d^2\log\theta}{dz^2}-\left(\frac{\theta'}{\theta}\right)^2-2\frac{w''}{w'}\frac{\theta'}{\theta}-\frac{2w'\theta I'}{\sqrt{2(1-I)}}.$$

If we make use of the equation which defines I, this reduces to

$$(289)\qquad \frac{1}{(\rho-z)^2}\frac{d\rho}{dz}=\frac{w'\theta I'}{2\sqrt{2(1-I)}},$$

and similarly

$$(290)\qquad \frac{1}{(\sigma-z)^2}\frac{d\sigma}{dz}=-\frac{w'\theta I'}{2\sqrt{2(1-I)}}.$$

If we leave aside the cases $w'=0$ and $\theta=0$, in which ρ and σ are not defined, we may say, therefore, that *neither of the branchpoints of the osculating power function of $w=f(z)$ can be a fixed point, unles $f(z)$ is itself a power function.*

If we introduce φ as independant variable, we find

$$(291)\qquad \frac{1}{(\rho-z)^2}\frac{d\rho}{d\varphi}=\frac{w'\theta\frac{dI}{d\varphi}}{2\sqrt{2(1-I)}},\qquad \frac{1}{(\sigma-z)^2}\frac{d\sigma}{d\varphi}=-\frac{w'\theta\frac{dI}{d\varphi}}{2\sqrt{2(1-I)}}.$$

Finally we note the formulae

$$(292)\qquad \frac{1}{(\rho-z)^2}\frac{d\rho}{d\varphi}+\frac{1}{(\sigma-z)^2}\frac{d\sigma}{d\varphi}=0,\qquad \frac{d\rho}{d\sigma}=-\left(\frac{\rho-z}{\sigma-z}\right)^2.$$

Let us consider *those functions for which ρ and σ are connected by a linear relation with constant coefficients*

$$(293)\qquad m\rho\sigma+n\rho+p\sigma+q=0,$$

where we may assume

$$(294)\qquad np-mq=1.$$

We find, as in the corresponding investigation for the singular points of the osculating logarithm (Art. **10**), that *there will then exist linear relations with constant coefficients also between* ρ *and* z, *and between* σ *and* z, namely

$$(295)\qquad \begin{cases} m\rho z + (n+1)\rho + (p-1)z + q = 0, \\ m\sigma z + (p+1)\sigma + (n-1)z + q = 0. \end{cases}$$

From the first of these relations, we find

$$[mz + n + 1]\,d\rho + [m\rho + p - 1]\,dz = 0,$$

whence

$$\frac{d\rho}{dz} = \frac{n-p}{(mz+n+1)^2}, \qquad \rho - z = -\frac{mz^2 + (n+p)z + q}{mz + n + 1},$$

so that we find, from (289),

$$(296)\qquad \frac{n-p}{[mz^2 + (n+p)z + q]^2} = \frac{w'\theta I'}{2\sqrt{2(1-I)}}.$$

From (287) we find

$$(297)\qquad \frac{n-p}{mz^2 + (n+p)z + q} = -w'\theta\sqrt{2(1-I)}.$$

Elimination of z gives

$$(298)\qquad (1-I)^{-\frac{3}{2}}dI = \frac{4\sqrt{2}}{n-p}\,d\varphi,$$

and therefore

$$(299)\qquad \frac{2\sqrt{2}}{n-p}(\varphi - \varphi_0)\sqrt{(1-I)} = 1$$

as the hyperintrinsic equation of these functions, when $n - p \neq 0$.

The case $n = p$ leads to $\frac{d\rho}{dz} = 0$ and therefore corresponds to the case $I =$ const., when the function reduces to a power function.

We also find from (296) and (297), by division,

$$(300)\qquad \frac{I'}{4(1-I)} = -\frac{1}{mz^2 + (n+p)z + q}.$$

If $(n+p)^2 - 4mq \neq 0$, and if we denote by $z - r$ and $z - s$ the linear factors of $mz^2 + (n+p)z + q$, distinct under this hypothesis,

we find

(301) $$1 - I = C e^{\left(\frac{z-r}{z-s}\right)^{\frac{4}{m(r-s)}}}, \qquad r \neq s, \qquad m \neq 0.$$

If $(n+p)^2 - 4mq = 0$, and $m \neq 0$, we may write

$$mz^2 + (n+p)z + q = m(z-r)^2,$$

and we find instead

(302) $$1 - I = C e^{-\frac{4}{m(z-r)}}.$$

Finally, if $m = 0$, we find

(303) $$1 - I = C(z-k)^{\frac{4}{n+p}}, \qquad k = \frac{-q}{n+p}, \qquad n + p \neq 0,$$

and if $n + p$ is also equal to zero,

(304) $$1 - I = C e^{\frac{4z}{q}}.$$