

COMPUTING THE 2-BLOCKS OF DIRECTED GRAPHS

RAED JABERI¹

Abstract. Let G be a directed graph. A *2-directed block* in G is a maximal vertex set $C^{2d} \subseteq V$ with $|C^{2d}| \geq 2$ such that for each pair of distinct vertices $x, y \in C^{2d}$, there exist two vertex-disjoint paths from x to y and two vertex-disjoint paths from y to x in G . In this paper we present two algorithms for computing the 2-directed blocks of G in $O(\min\{m, (t_{sap} + t_{sb})n\}n)$ time, where t_{sap} is the number of the strong articulation points of G and t_{sb} is the number of the strong bridges of G . Furthermore, we study two related concepts: the 2-strong blocks and the 2-edge blocks of G . We give two algorithms for computing the 2-strong blocks of G in $O(\min\{m, t_{sap}n\}n)$ time and we show that the 2-edge blocks of G can be computed in $O(\min\{m, t_{sb}n\}n)$ time. In this paper we also study some optimization problems related to the strong articulation points and the 2-blocks of a directed graph. Given a strongly connected graph $G = (V, E)$, we want to find a minimum strongly connected spanning subgraph $G^* = (V, E^*)$ of G such that the strong articulation points of G coincide with the strong articulation points of G^* . We show that there is a linear time $17/3$ approximation algorithm for this NP-hard problem. We also consider the problem of finding a minimum strongly connected spanning subgraph with the same 2-blocks in a strongly connected graph G . We present approximation algorithms for three versions of this problem, depending on the type of 2-blocks.

Mathematics Subject Classification. 05C85, 05C20, 68W25.

Keywords and phrases. Directed graphs, strong articulation points, strong bridges, 2-blocks, graph algorithms, approximation algorithms.

¹ Faculty of Computer Science and Automation, Technische Universität Ilmenau, 98693 Ilmenau, Germany. raed.jaberi@tu-ilmenau.de

1. INTRODUCTION

Let $G = (V, E)$ be a directed graph with $|V| = n$ vertices and $|E| = m$ edges. A *strong articulation point* (SAP) of G is a vertex whose removal increases the number of strongly connected components (SCCs) of G . A *strong bridge* of G is an edge whose removal increases the number of SCCs of G . We use t_{sap} to denote the number of the strong articulation points (SAPs) of G and t_{sb} to denote the number of the strong bridges of G . A directed graph $G = (V, E)$ is said to be *k-vertex-connected* if it has at least $k+1$ vertices and the induced subgraph on $V \setminus X$ is strongly connected for every $X \subsetneq V$ with $|X| < k$. Thus, a strongly connected digraph $G = (V, E)$ is 2-vertex-connected if and only if it has at least 3 vertices and it contains no SAPs. The *2-vertex-connected components* of a strongly connected graph G are its maximal 2-vertex-connected subgraphs. The concept was defined in [6]. For more information see [20]. A strongly connected graph G is called *2-edge connected* if it contains no strong bridges.

In 2010, Georgiadis [12] gave a linear time algorithm to test whether a strongly connected graph G is 2-vertex-connected or not. Later, Italiano *et al.* [20] gave a linear time algorithm for the same problem which is faster in practice than the algorithm of Georgiadis [12]. Furthermore, Italiano *et al.* [20] presented a linear time algorithm for finding all the SAPs of a directed graph G . They also gave two linear time algorithms for calculating all the strong bridges of a directed graph G . In 2014, Jaberri [21] presented algorithms for computing the 2-vertex-connected components of directed graphs in $O(nm)$ time. The concept of 2-vertex-connected components is not ideal because there are directed graphs in which many vertices are well connected with each other but they lie in distinct 2-vertex-connected components or in no 2-vertex-connected component. This is illustrated in Figure 1.

In this paper we study alternative concepts similar to the k -blocks of undirected graphs which were defined in [4] as follows. A *k-block* in an undirected graph $G = (V, E)$ is a maximal vertex set $U \subseteq V$ with $|U| \geq k$ such that no set $X \subseteq V$ with $|X| < k$ separates any two vertices of $U \setminus X$ in the undirected graph G . In 2013, Carmesin *et al.* [4] showed that there exists an $O(\min\{k, \sqrt{n}\}n^4)$ -time algorithm that calculates all the *k-blocks* in an undirected graph. The 2-blocks in an undirected graph G are similar to the 2-vertex connected components of the undirected graph G , which can be found in linear time using Tarjan's algorithm [28]. In this paper we introduce and study three new concepts: the 2-directed blocks, the 2-strong blocks, and the 2-edge blocks of directed graphs. A *2-directed block* in G is a maximal vertex set $C^{2d} \subseteq V$ with $|C^{2d}| \geq 2$ such that for each pair of distinct vertices $x, y \in C^{2d}$, there exist two vertex-disjoint paths from x to y and two vertex-disjoint paths from y to x in G . Since 2-vertex-connected components are 2-vertex-connected, they must have at least a linear number of edges. In contrast to, the subgraphs induced by the 2-directed blocks may have few or no edges at all, for example the 2-directed block $\{1, 4\}$ in Figure 1. Of course, they may also have many edges, like the subgraph induced by the 2-directed block $\{8, 7, 9, 11, 10, 12\}$ in Figure 1. A *2-strong block* in G is a maximal vertex set

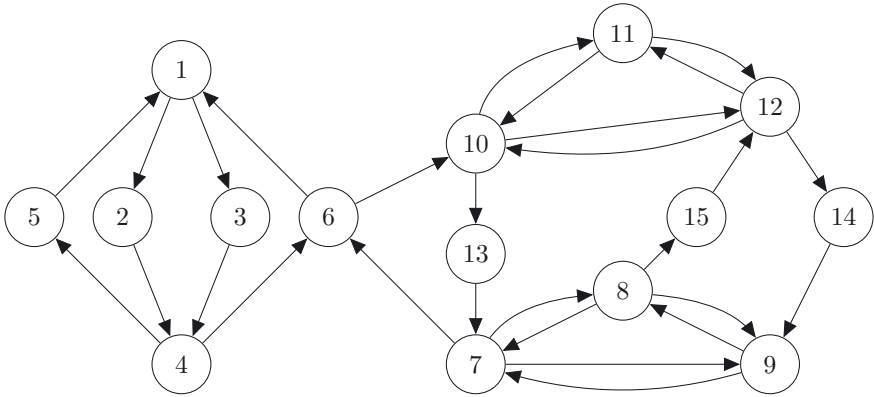


FIGURE 1. A strongly connected graph G . The 2-vertex-connected components of G are $\{7, 8, 9\}$, $\{10, 11, 12\}$. The vertices 8, 11 lie in distinct 2-vertex-connected components of G but there are two vertex-disjoint paths from 8 to 11 and two vertex-disjoint paths from 11 to 8 in G . Notice that the vertices 1, 4 do not lie in any 2-vertex-connected component of G but there exist two vertex-disjoint paths from 1 to 4 and two vertex-disjoint paths from 4 to 1 in G .

$C^{2s} \subseteq V$ with $|C^{2s}| \geq 2$ such that for each pair of distinct vertices $x, y \in C^{2s}$ and for each vertex $z \in V \setminus \{x, y\}$, the vertices x and y lie in the same SCC of the graph $G \setminus \{z\}$. A 2-edge block in G is a maximal vertex set $C^{2e} \subseteq V$ with $|C^{2e}| \geq 2$ such that for each pair of distinct vertices $x, y \in C^{2e}$, there are two edge-disjoint paths from x to y and two edge-disjoint paths from y to x in G . These concepts capture the idea that it is difficult to separate vertices in a block in slightly different ways, and very different from the concept of 2-vertex-connected components. Our new concepts are illustrated in Figure 2.

In this paper we also study some optimization problems related to the SAPs and the 2-blocks of a directed graph. First, we consider the following problem, denoted by MS-SAPs: Given a strongly connected graph $G = (V, E)$, the MS-SAPs problem consists in finding a minimum strongly connected spanning subgraph (MSCSS) $G^* = (V, E^*)$ of G such that the SAPs of G coincide with the SAPs of G^* . Moreover, we consider the problem of finding a MSCSS with the same 2-blocks, defined as follows. Given a strongly connected graph $G = (V, E)$, the goal is to find a subset $E^* \subseteq E$ of minimum size such that $G^* = (V, E^*)$ is strongly connected and the 2-blocks of G coincide with the 2-blocks of $G^* = (V, E^*)$. There are three versions of this problem, depending on the type of 2-blocks: MSCSS with the same 2-directed blocks (denoted by MS-2DBs), MSCSS with the same 2-strong blocks (denoted by MS-2SBs), and MSCSS with the same 2-edge blocks (denoted by MS-2EBs). The analogous problems of MS-2DBs and MS-2SBs for undirected graphs can be reduced to the problem of finding a minimum-size 2-vertex-connected spanning

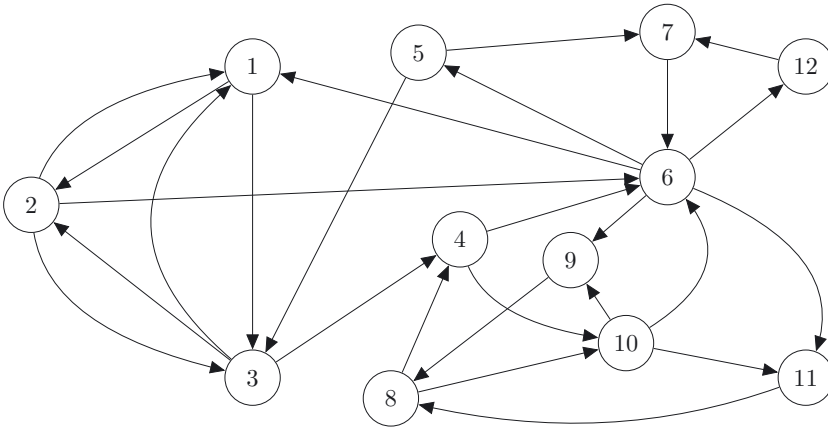


FIGURE 2. A strongly connected graph G , which contains one 2-vertex-connected component $\{1, 2, 3\}$, two 2-directed blocks $\{6, 1, 2, 3\}, \{8, 10, 6, 4\}$, four 2-strong blocks $\{6, 1, 2, 3\}, \{9, 8\}, \{8, 10, 6, 4\}, \{7, 6\}$, and one 2-edge block $\{1, 2, 3, 4, 6, 8, 10\}$. Notice that the 2-vertex-connected component $\{1, 2, 3\}$ is a subset of the 2-directed block $\{6, 1, 2, 3\}$. We shall also see that each 2-directed block is a subset of a 2-strong block.

subgraph of an undirected graph, which has been studied in [30]. Let $G = (V, E)$ be a directed graph. Menger’s Theorem for vertex connectivity in directed graphs can be formulated as follows [2]. Let x, y be a pair of distinct vertices in G such that $(x, y) \notin E$. Then the maximum number of vertex-disjoint paths from x to y in G is equal to the minimum number of vertices different from x and y whose removal from G destroys all the paths from x to y . This theorem implies that V is a 2-directed block if and only if G is 2-vertex connected. Moreover, by definition, V is a 2-strong block if and only if G is 2-vertex connected. Thus, the problem of finding a minimum-size 2-vertex connected spanning subgraph of a directed graph G is a special case of the problems MS-SAPs, MS-2SBs and MS-2DBs when G is 2-vertex-connected. Menger’s Theorem for edge connectivity in directed graphs can be formulated as follows [2]. Let v, w be two vertices in G . Then the maximum number of edge-disjoint paths from v to w in G equals the minimum number of edges whose removal destroys all the paths from v to w . The problem of finding a minimum-cardinality 2-edge connected spanning subgraph of a directed graph $G = (V, E)$ is a special case of the MS-2EBs problem when G is 2-edge-connected since, by Menger’s Theorem for edge connectivity, V is a 2-edge block if and only if G is 2-edge connected. Therefore, by results from [10], the problems MS-SAPs, MS-2SBs, MS-2EBs, and MS-2DBs are NP-hard.

Let G be a directed graph. In this paper, we present two algorithms for computing the 2-directed blocks of G in $O(\min\{m, (t_{sap} + t_{sb})n\}n)$ time. We also present two algorithms for computing the 2-strong blocks of G in $O(\min\{m, t_{sap}n\}n)$ time

and we show that the 2-edge blocks of G can be computed in $O(\min\{m, t_{sb}n\}n)$ time. Furthermore, we elaborate a linear time $17/3$ approximation algorithm for the MS-SAPs problem. We also present a $(2t_{sap} + 17/3)$ approximation algorithm for the MS-2SBs problem and a $(2t_{sb} + 4)$ approximation algorithm for the MS-2EBs problem. Moreover, we prove that there exists a $(2(t_{sap} + t_{sb}) + 29/3)$ approximation algorithm for the MS-2DBs problem.

RELATED AND SUBSEQUENT WORK

In independent work, Georgiadis *et al.* [16] studied 2-edge blocks and gave linear time algorithms for finding them. This is better than our results in Section 5. In 2015, their algorithms [16] were published in SODA [18]. Recently, the same authors [17] gave linear time algorithms for finding 2-directed blocks and 2-strong blocks, improving on our results in Sections 3, 4 and 6.

2. GRAPH TERMINOLOGY AND NOTATION

In this section we recall some basic definitions [20, 23, 25]. A *flowgraph* $G(v) = (V, E, v)$ is a directed graph with $|V| = n$ vertices, $|E| = m$ edges, and a distinguished start vertex $v \in V$ such that every vertex $w \in V$ is reachable from v . For a flowgraph $G(v) = (V, E, v)$, the *dominance relation* of $G(v)$ is defined as follows: a vertex $u \in V$ is a *dominator* of vertex $w \in V$ if every path from v to w includes u . By $\text{dom}(w)$ we denote the set of dominators of vertex w . Obviously, the set of dominators of the start vertex in $G(v)$ is $\text{dom}(v) = \{v\}$. For every vertex $w \in V$ with $w \neq v$, $\{v, w\}$ is a subset of $\text{dom}(w)$; we call w, v the *trivial dominators* of w . A vertex u is a *non-trivial dominator* in $G(v)$ if there is some $w \notin \{v, u\}$ such that $u \in \text{dom}(w) \setminus \{v\}$. The set of all non-trivial dominators is denoted by $D(v)$. The dominance relation is transitive. A vertex $u \in V$ is an *immediate dominator* of vertex $w \in V \setminus \{v\}$ in $G(v)$ if $u \in \text{dom}(w) \setminus \{w\}$ and all elements of $\text{dom}(w) \setminus \{w\}$ are dominators of u . Every vertex w of $G(v)$ except the start vertex v has a unique immediate dominator. The edges (u, w) where u is the immediate dominator of w form a tree with root v , called the *dominator tree* of $G(v)$, denoted by $DT(v)$. Figure 3 illustrates an example of a flowgraph and its dominator tree. Two spanning trees T and T' of $G(v)$ are called independent if for every vertex $w \in V \setminus \{v\}$, the paths from v to w in T and T' contain only $\text{dom}(w)$ in common [15]. An edge (x, y) is an *edge dominator* of vertex w if every path from v to w in $G(v)$ contains edge (x, y) . Let $G = (V, E)$ be a directed graph. Let $F \subseteq V \times V$ be a set of pairs of vertices. We use $G \setminus F$ to denote the directed graph obtained from G by deleting all edges in $E \cap F$ from G . Let U be a subset of V . We use $G \setminus U$ to denote the directed graph obtained from G by removing all the vertices in U and their incident edges. The reversal graph of G is the directed graph $G^R = (V, E^R)$, where $E^R = \{(w, u) \mid (u, w) \in E\}$. Let v be a vertex in G . By $D^R(v)$ we denote the set of all non-trivial dominators in the flowgraph $G^R(v) = (V, E^R, v)$. Let $G = (V, E)$ be an undirected graph. A block of G is a maximal connected subgraph of G that

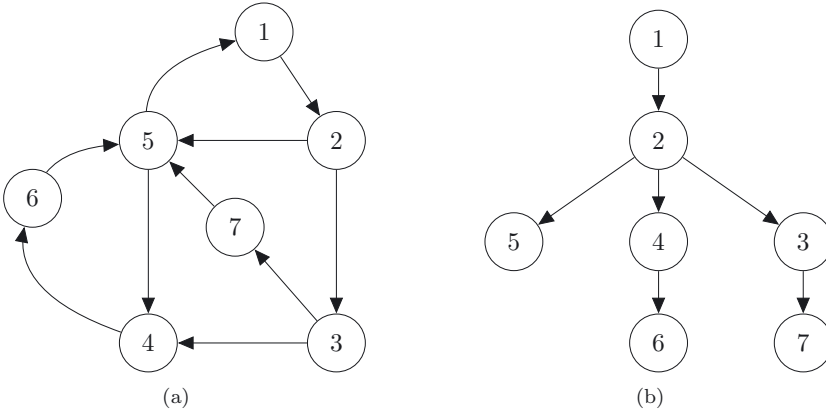


FIGURE 3. (a) A flowgraph $G(1)$. (b) The dominator tree of $G(1)$.

contains no articulation points. An undirected graph G is called chordal if every cycle of length at least 4 has a chord [11, 27].

3. COMPUTING 2-DIRECTED BLOCKS

In this section we present our first algorithm for computing the 2-directed blocks of directed graphs. Our second algorithm will be described in Section 6. We consider only strongly connected graphs since the 2-directed blocks of a directed graph are the union of the 2-directed blocks of its SCCs. Let $G = (V, E)$ be a strongly connected graph. For distinct vertices $x, y \in V$, we write $x \overset{2}{\rightsquigarrow} y$ if there exist two vertex-disjoint paths from x to y in G , and we write $x \overset{2}{\rightsquigarrow\leftarrow} y$ if $x \overset{2}{\rightsquigarrow} y$ and $y \overset{2}{\rightsquigarrow} x$. A 2-directed block in G is a maximal vertex set $C^{2d} \subseteq V$ with $|C^{2d}| \geq 2$ such that for each pair of distinct vertices $x, y \in C^{2d}$, we have $x \overset{2}{\rightsquigarrow\leftarrow} y$.

Lemma 3.1. *Let $G = (V, E)$ be a strongly connected graph and let x, y be distinct vertices in G . Then $x \overset{2}{\rightsquigarrow\leftarrow} y$ if and only if for each vertex $w \in V \setminus \{x, y\}$ the vertices x, y lie in the same SCC of $G \setminus \{w\}$ and in the same SCC of $G \setminus \{(x, y), (y, x)\}$.*

Proof. “ \Leftarrow ”: Without loss of generality, it is sufficient to show that there are two vertex-disjoint paths from x to y in G . We consider two cases.

- (1) $(x, y) \notin E$. Let $w \in V \setminus \{x, y\}$. Since the vertices x, y lie in the same SCC of $G \setminus \{w\}$, there exists a path from x to y in $G \setminus \{w\}$. Thus, one can not interrupt all paths from x to y by removing w from G . Since x and y are not adjacent, by Menger’s Theorem for vertex connectivity [2] we have $x \overset{2}{\rightsquigarrow} y$.
- (2) $(x, y) \in E$. Since x, y lie in the same SCC of $G \setminus \{(x, y), (y, x)\}$, there is a path p_1 from x to y in $G \setminus \{(x, y)\}$. Thus, there are two vertex-disjoint paths p_1 and $p_2 = (x, y)$ from x to y in G .

“ \Rightarrow ”: We know there are two vertex-disjoint paths p_1 and p_2 from x to y in G . We must show that in $G \setminus \{w\}$ and in $G \setminus \{(x, y)\}$ there is a path from x to y . Since at most one of p_1 and p_2 contains w and at most one of p_1 and p_2 is edge (x, y) , the claim follows. \square

Lemma 3.2. *Let $G = (V, E)$ be a strongly connected graph and let x, y be distinct vertices in G such that $x \overset{2}{\rightsquigarrow} y$. Then the vertices x, y lie in the same SCC of $G \setminus \{e\}$ for any edge $e \in E$.*

Proof. There exist two vertex-disjoint paths p_1, p_2 from x to y and two vertex-disjoint paths p_3, p_4 from y to x in G since $x \overset{2}{\rightsquigarrow} y$. The paths p_1, p_2 are edge-disjoint and the paths p_3, p_4 are also edge-disjoint. Hence, there exist a path from x to y and a path from y to x in $G \setminus \{e\}$ for any edge $e \in E$. \square

Lemma 3.3 shows that 2-directed blocks intersect in at most one vertex. (2-vertex-connected components have the same property, see [6, 21]).

Lemma 3.3. *Let C_1^{2d}, C_2^{2d} be distinct 2-directed blocks in a strongly connected graph $G = (V, E)$. Then C_1^{2d} and C_2^{2d} have at most one vertex in common.*

Proof. Suppose for a contradiction that $|C_1^{2d} \cap C_2^{2d}| > 1$. By renaming we can assume that there are at least two vertices $v \in C_1^{2d}, w \in C_2^{2d}$ with $v, w \notin C_1^{2d} \cap C_2^{2d}$ such that there are no two vertex-disjoint paths from v to w in G . We consider two cases.

- (1) $(v, w) \notin E$. By Menger’s Theorem [2] there is some vertex $s \in V \setminus \{v, w\}$ such that s lies on all paths from v to w . Let z be a vertex in $(C_1^{2d} \cap C_2^{2d}) \setminus \{s\}$. Since C_1^{2d} and C_2^{2d} are 2-directed blocks, there is a path from v to z in $G \setminus \{s\}$ and a path from z to w in $G \setminus \{s\}$, hence there is a path from v to w in $G \setminus \{s\}$, which is a contradiction.
- (2) $(v, w) \in E$. In this case there is no path from v to w in $G \setminus \{(v, w)\}$. Let u be a vertex in $C_1^{2d} \cap C_2^{2d}$. But, again by the definition of 2-directed blocks, there are paths from v to u and from u to w in $G \setminus \{(v, w)\}$, a contradiction. \square

Next we note that 2-directed blocks can not form cycles in the following sense.

Lemma 3.4. *Let $G = (V, E)$ be a strongly connected graph and let v_0, v_1, \dots, v_l be distinct vertices of G such that $v_0 \overset{2}{\rightsquigarrow} v_l$ and $v_{i-1} \overset{2}{\rightsquigarrow} v_i$ for $i \in \{1, 2, \dots, l\}$. Then all the vertices v_0, v_1, \dots, v_l lie in the same 2-directed block of G .*

Proof. Suppose for a contradiction that there exist two vertices v_r, v_q with $r, q \in \{0, 1, \dots, l\}$ such that v_r, v_q lie in distinct 2-directed blocks of G and $r < q$. By renaming, we may assume that there do not exist two vertex-disjoint paths from v_r to v_q in G . We consider two cases.

- (1) $(v_r, v_q) \notin E$. In this case, all the paths from v_r to v_q contain a vertex $s \in V \setminus \{v_r, v_q\}$. Therefore, there is no path from v_r to v_q in $G \setminus \{s\}$. There are two cases to consider.

- (a) $s \notin \{v_{r+1}, v_{r+2}, \dots, v_{q-1}\}$. In this case, for each $i \in \{r+1, r+2, \dots, q\}$, there is a path from v_{i-1} to v_i in $G \setminus \{s\}$ by Lemma 3.1, a contradiction.
- (b) $s \in \{v_{r+1}, v_{r+2}, \dots, v_{q-1}\}$. Then by Lemma 3.1, there are paths from v_r to v_{r-1}, \dots from v_1 to v_0 , from v_0 to v_l , from v_l to v_{l-1}, \dots from v_{q+1} to v_q in $G \setminus \{s\}$, again a contradiction.
- (2) $(v_r, v_q) \in E$. By Lemma 3.2, for each $i \in \{r+1, r+2, \dots, q\}$, the vertices v_{i-1}, v_i lie in the same SCC of $G \setminus \{(v_r, v_q)\}$. Therefore, there exists a path p_1 from v_r to v_q in $G \setminus \{(v_r, v_q)\}$. Consequently, there are two vertex-disjoint paths p_1 and $p_2 = (v_r, v_q)$ from v_r to v_q in G , but this is a contradiction. \square

We construct the 2-directed block graph $G^{2d} = (V^{2d}, E^{2d})$ of a strongly connected graph $G = (V, E)$ as follows. It has a vertex v_i for every 2-directed block C_i^{2d} and all vertices w that lie in the intersection of (at least) two 2-directed blocks. For each pair of distinct 2-directed blocks C_i^{2d}, C_j^{2d} with $C_i^{2d} \cap C_j^{2d} = \{w\}$, we add two undirected edges $(v_i, w), (w, v_j)$ to E^{2d} .

Lemma 3.5. *Let $G = (V, E)$ be a strongly connected graph. Then the 2-directed block graph $G^{2d} = (V^{2d}, E^{2d})$ of G is a forest.*

Proof. This follows from Lemma 3.4. \square

Now we turn to algorithm for finding the 2-directed blocks. Algorithm 3.1 describes our first algorithm for computing all the 2-directed blocks of a strongly connected graph G .

Algorithm 3.1.

Input: A strongly connected graph $G = (V, E)$.

Output: The 2-directed blocks of G .

- 1 **if** G is 2-vertex-connected **then**
- 2 Output V .
- 3 **else**
- 4 Let A be an $n \times n$ matrix.
- 5 Initialize A with 0s.
- 6 **for** each ordered pair $(v, w) \in V \times V$ **do**
- 7 **if** there are two vertex-disjoint paths from v to w in G **then**
- 8 $A[v, w] \leftarrow 1$.
- 9 Construct undirected graph $G^* = (V, E^*)$ as follows.
- 10 **for** each pair $(v, w) \in V \times V$ **do**
- 11 **if** $A[v, w] = 1$ and $A[w, v] = 1$ **then**
- 12 Add the undirected edge (v, w) to E^* .
- 13 Compute the blocks of size > 1 of $G^* = (V, E^*)$ and output them.

Lemma 3.6. *Algorithm 3.1 calculates 2-directed blocks.*

Proof. If G is 2-vertex connected, then V is a 2-directed block. Let $G = (V, E)$ be a strongly connected graph which is not 2-vertex connected. For any vertices $v, w \in V$, $v \overset{2}{\rightsquigarrow} w$ if and only if $A[v, w] = 1$ and $A[w, v] = 1$ in line 11. Hence, $v \overset{2}{\rightsquigarrow} w$ if

and only if $(v, w) \in E^*$. Let x, y be two vertices that do not lie in the same block of G^* . Then (x, y) can not be in E^* . Hence, the vertices x, y do not lie in the same 2-directed block of G . Let B be a block of G^* containing v, w with $(v, w) \in E^*$. There are two cases to consider.

- (1) $B = \{v, w\}$. Then $v \overset{2}{\rightsquigarrow} w$ and $\{v, w\}$ is a 2-directed block. (If there were some z such that v, w, z are in the same 2-directed block, we would have the triangle $(v, z), (z, w), (w, v)$ in G^* , hence z would be in the same block as v, w .)
- (2) B contains other vertices. We show that all these vertices are in the same 2-directed block. If $z \in V \setminus \{v, w\}$ is in B , then z, v lie on one simple cycle in G^* . By Lemma 3.4, the vertices z, v lie in the same 2-directed block. \square

It remains to describe Procedure 3.1 that implements steps 6–8 of Algorithm 3.1.

Procedure 3.1.

Purpose: Check if there are two vertex disjoint paths.

Input: A strongly connected graph $G = (V, E)$.

Output: Matrix A .

```

1  for each vertex  $v \in V$  do
2     $E' \leftarrow E$ .
3     $V' \leftarrow V$ .
4    for each edge  $e = (v, w) \in E$  do
5       $E' \leftarrow E' \setminus \{(v, w)\}$ .
6       $V' \leftarrow V' \cup \{u_e\}$ .
7       $E' \leftarrow E' \cup \{(v, u_e), (u_e, w)\}$ .
8    Compute the dominator tree  $DT'(v)$  of the flowgraph  $G'(v) = (V', E', v)$ .
9    for each direct successor  $w$  of  $v$  in  $DT'(v)$  do
10   if  $w \in V$  then
11    $A[v, w] \leftarrow 1$ .
```

For each vertex $v \in V$, we construct a directed graph $G' = (V', E')$ from G as follows. For each edge $(v, w) \in E$, we remove this edge (v, w) and we add a new vertex u_e and two new edges $(v, u_e), (u_e, w)$ to G' . Then we compute the dominator tree $DT'(v)$ of the flowgraph $G'(v) = (V', E', v)$. For each direct successor w of v in $DT'(v)$ such that $w \in V$, line 11 sets $A[v, w]$ to 1. The correctness of Procedure 3.1 follows from the following lemma.

Lemma 3.7. *Let $G = (V, E)$ be a strongly connected graph and let v, w be two distinct vertices in G . Then $v \overset{2}{\rightsquigarrow} w$ in $G(v)$ if and only if v is the immediate dominator of w in the flowgraph $G'(v) = (V', E', v)$.*

Proof. “ \Rightarrow ” Assume that $v \overset{2}{\rightsquigarrow} w$ in $G(v)$. Then there are two vertex-disjoint paths $p_1 = (v = v_1, v_2, \dots, v_t = w)$ and $p_2 = (v = u_1, u_2, \dots, u_l = w)$ from v to w in $G(v)$. In lines 4–7 of Procedure 3.1, the edge $x = (v_1, v_2)$ is replaced by two edges $(v_1, v_x), (v_x, v_2)$ and the edge $y = (u_1, u_2)$ is replaced by two edges

$(u_1, u_y), (u_y, u_2)$. Since $v_x \neq u_y$, there exist two vertex-disjoint paths from v to w in $G'(v)$. Therefore, v is the immediate dominator of w in the flowgraph $G'(v)$.

“ \Leftarrow ” We prove the contrapositive. Assume that $v \overset{2}{\rightsquigarrow} w$ in $G(v)$ is not true. Then there is some vertex $x \in V \setminus \{v, w\}$ such that all paths from v to w in $G(v)$ contain x . Then x is a non-trivial dominator of w in $G(v)$. Thus, v is not the immediate dominator of w in $G(v)$. Let $p = (v = v_1, v_2, \dots, v_t = w)$ be a simple path from v to w in $G(v)$. In lines 4–7 of Procedure 3.1, $e = (v_1, v_2)$ is replaced by $(v_1, u_e), (u_e, v_2)$. Hence, the path p corresponds to the simple path $(v = v_1, u_e, v_2, \dots, v_t = w)$ in $G'(v)$. Since $u_e \neq x$, the vertex x is a non-trivial dominator of w in $G'(v)$. Therefore, v is not the immediate dominator of w in $G'(v)$. \square

Remark 3.8. Procedure 3.1 checks in polynomial time whether there are two vertex-disjoint paths from v to w in G . It may be worth noticing that problems of a similar flavor are NP-complete: Fortune *et al.* [8] proved it is NP-complete to check if there are vertex-disjoint (arc-disjoint) paths from s_1 to t_1 and from s_2 to t_2 for four given vertices. Li *et al.* [24] showed it is NP-hard to find two vertex-disjoint (arc-disjoint) paths from s to t while minimizing the length of the longer one.

Theorem 3.9. *Algorithm 3.1 runs in $O(nm)$ time.*

Proof. The dominators of a flowgraph can be found in linear time [1,3]. In lines 2–7 of Procedure 3.1, the construction of $G'(v) = (V', E', v)$ takes linear time because the graph G' has $|V'| = n + d_{\text{out}}(v) < 2n$ vertices and $|E'| = m + d_{\text{out}}(v) < m + n$ edges. Moreover, lines 9–11 of Procedure 3.1 take $O(n)$ time since the number of direct successors of v in the dominator tree $DT'(v)$ is at most $2(n - 1)$. Since the number of iterations of the for loop in lines 1–11 of Procedure 3.1 is n , the running time of Procedure 3.1 is $O(nm)$. One can test whether a directed graph is 2-vertex-connected in linear time using the algorithm of Italiano *et al.* [20]. The initialization of matrix A requires $O(n^2)$ time. The undirected graph $G^* = (V, E^*)$ can also be constructed in $O(n^2)$ time. Furthermore, the blocks of an undirected graph can be computed in linear time using Tarjan’s algorithm [28]. The total cost is therefore $O(nm + n^2) = O(nm)$. \square

Let $G = (V, E)$ be a strongly connected graph. By definition, the 2-directed blocks of G are the maximal cliques of the auxiliary graph G^* which is constructed in lines 4–12 of Algorithm 3.1. By Lemma 3.4, the auxiliary graph G^* is chordal. In line 13 of Algorithm 3.1, one can compute the maximal cliques of the auxiliary graph G^* instead of blocks since the maximal cliques of a chordal graph can be calculated in linear time [11,27].

4. COMPUTING 2-STRONG BLOCKS

In this section we present two algorithms for computing the 2-strong blocks of directed graphs. The 2-strong blocks of a directed graph are the union of the

2-strong blocks of its SCCs. Let $G = (V, E)$ be a strongly connected graph. We define a relation $\overset{2s}{\rightsquigarrow}$ as follows. For any distinct vertices $x, y \in V$, we write $x \overset{2s}{\rightsquigarrow} y$ if for any vertex $z \in V \setminus \{x, y\}$, the vertices x, y lie in the same SCC of $G \setminus \{z\}$. By definition, the 2-strong blocks are maximal subsets of V of size at least 2 closed under $\overset{2s}{\rightsquigarrow}$. Let v, w be distinct vertices in V such that $(v, w) \in E$ and $w \overset{2}{\rightsquigarrow} v$. While v, w are in one 2-strong block, these vertices do not necessarily lie in the same 2-directed block of G .

Lemma 4.1. *Each 2-directed block in a strongly connected graph G is a subset of a 2-strong block in G .*

Proof. Immediate from Lemma 3.1. □

Lemma 4.2. *Let $G = (V, E)$ be a strongly connected graph. Let C_1^{2s}, C_2^{2s} be distinct 2-strong blocks in G . Then C_1^{2s} and C_2^{2s} have at most one vertex in common.*

Proof. Assume for a contradiction that $|C_1^{2s} \cap C_2^{2s}| > 1$. Then there exist at least two vertices $x \in C_1^{2s}, y \in C_2^{2s}$ with $x, y \notin C_1^{2s} \cap C_2^{2s}$ and a vertex $z \in V \setminus \{x, y\}$ such that the vertices x, y lie in different SCCs of $G \setminus \{z\}$. Let w be a vertex in $(C_1^{2s} \cap C_2^{2s}) \setminus \{z\}$. Since $x, w \in C_1^{2s}$, these vertices lie in the same SCC of $G \setminus \{z\}$, similarly for $w, y \in C_2^{2s}$. Hence x, y lie in the same SCC of $G \setminus \{z\}$, a contradiction. □

As with 2-directed blocks, there can not be cycles of 2-strong blocks.

Lemma 4.3. *Let $G = (V, E)$ be a strongly connected graph and let v_0, v_1, \dots, v_l be distinct vertices of G such that $v_0 \overset{2s}{\rightsquigarrow} v_l$ and $v_{i-1} \overset{2s}{\rightsquigarrow} v_i$ for $i \in \{1, 2, \dots, l\}$. Then all the vertices v_0, v_1, \dots, v_l lie in the same 2-strong block of G .*

Proof. Let v_r, v_q be two vertices such that $r, q \in \{0, 1, \dots, l\}$ and $r < q$. Let w be a vertex in $V \setminus \{v_r, v_q\}$. We consider two cases.

- (1) $w \notin \{v_{r+1}, v_{r+2}, \dots, v_{q-1}\}$. Then, for each $i \in \{r+1, r+2, \dots, q\}$, the vertices v_{i-1}, v_i lie in the same SCC of $G \setminus \{w\}$. Thus the vertices v_r, v_q lie in the same SCC of $G \setminus \{w\}$.
- (2) $w \in \{v_{r+1}, v_{r+2}, \dots, v_{q-1}\}$. Then the vertices v_{i-1}, v_i lie in the same SCC of $G \setminus \{w\}$ for each $i \in \{1, 2, \dots, r\} \cup \{q+1, q+2, \dots, l\}$. Furthermore, the vertices v_0, v_l lie in the same SCC of $G \setminus \{w\}$ since $v_0 \overset{2s}{\rightsquigarrow} v_l$. Thus the vertices v_r, v_q lie in the same SCC of $G \setminus \{w\}$.

Since the vertices v_r, v_q lie in the same SCC of $G \setminus \{w\}$ for any vertex $w \in V \setminus \{v_r, v_q\}$, the vertices v_r, v_q lie in the same 2-strong block of G . □

Algorithm 4.1 shows our first algorithm for computing the 2-strong blocks of a strongly connected graph $G = (V, E)$.

Algorithm 4.1.**Input:** A strongly connected graph $G = (V, E)$.**Output:** The 2-strong blocks of G .

```

1  if  $G$  is 2-vertex-connected then
2    Output  $V$ .
3  else
4    Let  $A$  be an  $n \times n$  matrix.
5    Initialize  $A$  with 0s.
6    for each vertex  $v \in V$  do
7      Compute  $DT(v)$ .
8      for each direct successor  $w$  of  $v$  in  $DT(v)$  do
9         $A[v, w] \leftarrow 1$ .
10   Construct undirected graph  $G^* = (V, E^*)$  as follows.
11   for each pair  $(v, w) \in V \times V$  do
12     if  $A[v, w] = 1$  and  $A[w, v] = 1$  then
13       Add the undirected edge  $(v, w)$  to  $E^*$ .
14   Compute the blocks of size  $> 1$  of  $G^* = (V, E^*)$  and output them.

```

Using arguments similar to those in the proof of Lemma 3.6, one can show that Algorithm 4.1 is correct.

Theorem 4.4. *Algorithm 4.1 runs in $O(nm)$ time.*

Proof. The dominators of a flowgraph can be found in linear time [1, 3]. Therefore, lines 6–9 take $O(nm)$ time. \square

Lemma 4.5. *Let $G = (V, E)$ be a strongly connected graph and let x, y be distinct vertices in G . Let S be the set of all the SAPs in G . Then for any vertex $z \in V \setminus (S \cup \{x, y\})$, the vertices x and y lie in the same SCC of $G \setminus \{z\}$.*

Proof. Immediate from the definition. \square

This simple lemma gives rise to an alternative algorithm (Algorithm 4.2) that might be helpful if the number of the SAPs is small.

Algorithm 4.2.**Input:** A strongly connected graph $G = (V, E)$.**Output:** The 2-strong blocks of G .

```

1  if  $G$  is 2-vertex-connected then
2    Output  $V$ .
3  else
4    Let  $A$  be an  $n \times n$  matrix.
5    Initialize  $A$  with 1s.
6    Compute the SAPs of  $G$ .
7    for each  $s \in \text{SAPs of } G$  do
8      Compute the SCCs of  $G \setminus \{s\}$ .

```

```

9      for each pair  $(v, w) \in (V \setminus \{s\}) \times (V \setminus \{s\})$  do
10      if  $v, w$  in different SCCs of  $G \setminus \{s\}$  then
11       $A[v, w] \leftarrow 0$ .
12       $E^* \leftarrow \emptyset$ .
13      for each pair  $(v, w) \in V \times V$  do
14      if  $A[v, w] = 1$  and  $A[w, v] = 1$  then
15      Add the undirected edge  $(v, w)$  to  $E^*$ .
16      Compute the blocks of size  $> 1$  of  $G^* = (V, E^*)$  and output them.

```

Lemma 4.6. *Let v, w be distinct vertices in a strongly connected graph G . Then $v \overset{2s}{\rightsquigarrow} w$ if and only if $A[v, w] = 1$ and $A[w, v] = 1$ (when line 14 is reached).*

Proof. “ \Leftarrow ” If $A[v, w] = 1$ and $A[w, v] = 1$, then the vertices v, w lie in the same SCC of $G \setminus \{s\}$ for any SAP $s \in V \setminus \{v, w\}$ (see lines 7–11). By Lemma 4.5, the vertices v, w lie in the same SCC of $G \setminus \{z\}$ for any vertex $z \in V \setminus \{v, w\}$.

“ \Rightarrow ” This follows from Lemma 4.5. \square

Theorem 4.7. *The running time of Algorithm 4.2 is $O(t_{sap}n^2)$.*

Proof. The SAPs of a directed graph can be computed in linear time using the algorithm of Italiano *et al.* [20]. Lines 7–11 take $O(t_{sap}n^2)$ time. \square

Corollary 4.8. *The 2-strong blocks of a directed graph $G = (V, E)$ can be computed in $O(\min\{m, t_{sap}n\}n)$ time.*

5. COMPUTING THE 2-EDGE BLOCKS

In this section we present two algorithms for computing the 2-edge blocks of directed graphs. The 2-edge blocks of a directed graph are the union of the 2-edge blocks of its SCCs. We define a relation $\overset{2e}{\rightsquigarrow}$ as follows. For any distinct vertices $x, y \in V$, we write $x \overset{2e}{\rightsquigarrow} y$ if there exist two edge-disjoint paths from x to y and two edge-disjoint paths from y to x in G . The 2-edge blocks are maximal subsets closed under $x \overset{2e}{\rightsquigarrow} y$.

Lemma 5.1. *Let $G = (V, E)$ be a strongly connected graph and let x and y be distinct vertices in G . Then $x \overset{2e}{\rightsquigarrow} y$ if and only if for each edge $(v, w) \in E$, the vertices x, y lie in the same SCC of $G \setminus \{(v, w)\}$.*

Proof. This is an immediate consequence of Menger’s Theorem for edge connectivity [2]. \square

Lemma 5.2. *Let $G = (V, E)$ be a strongly connected graph. The 2-edge blocks of G are disjoint.*

Proof. Let C_1^{2e}, C_2^{2e} be two distinct 2-edge blocks of G . Assume for a contradiction that $C_1^{2e} \cap C_2^{2e} \neq \emptyset$. Then by Lemma 5.1, there are two vertices $x \in C_1^{2e}, y \in C_2^{2e}$ with $x, y \notin C_1^{2e} \cap C_2^{2e}$ and an edge $(v, w) \in E$ such that the vertices x, y lie in distinct SCCs of $G \setminus \{(v, w)\}$. Let z be a vertex in $C_1^{2e} \cap C_2^{2e}$. By Lemma 5.1, the vertices x, z lie in the same SCC of $G \setminus \{(v, w)\}$ since C_1^{2e} is a 2-edge block and the vertices z, y lie in the same SCC of $G \setminus \{(v, w)\}$ since C_2^{2e} is a 2-edge block. Hence x, y lie in the same SCC of $G \setminus \{(v, w)\}$, a contradiction. \square

Algorithm 5.1 shows our first algorithm for computing the 2-edge blocks of a strongly connected graph G .

Algorithm 5.1.

Input: A strongly connected graph $G = (V, E)$.

Output: The 2-edge blocks of G .

```

1  if  $G$  is 2-edge-connected then
2    Output  $V$ .
3  else
4    Let  $A$  be an  $n \times n$  matrix.
5    Initialize  $A$  with 0s.
6    for each vertex  $v \in V$  do
7      Compute the edge dominators of  $G(v) = (V, E, v)$ .
8      for each vertex  $w \in V \setminus \{v\}$  do
9        If there is no edge dominator of  $w$  then
10          $A[v, w] \leftarrow 1$ .
11      $E^* \leftarrow \emptyset$ .
12    for each pair  $(v, w) \in V \times V$  do
13      if  $A[v, w] = 1$  and  $A[w, v] = 1$  then
14        Add the undirected edge  $(v, w)$  to  $E^*$ .
15    Compute the connected components of size  $> 1$  of the graph
16     $G^* = (V, E^*)$  and output them.

```

Algorithm 5.1 works as follows. First, line 1 tests whether G is 2-edge-connected, and if it is, line 2 outputs V , since every 2-edge connected directed graph is a 2-edge block. Otherwise, for each vertex v in G , the algorithm computes the edge dominators of the flowgraph $G(v) = (V, E, v)$, and for each vertex $w \in V \setminus \{v\}$, line 10 sets $A[v, w]$ to 1 if there is no edge dominator of w . Let v, w be distinct vertices in G . Then $v \overset{2e}{\rightsquigarrow} w$ if and only if $A[v, w] = 1$ and $A[w, v] = 1$ in line 13. Lines 11–14 constructs an undirected graph $G^* = (V, E^*)$ as follows. For each pair $(v, w) \in V \times V$, we add an undirected edge (v, w) to E^* if $A[v, w] = 1$ and $A[w, v] = 1$. Finally, the algorithm finds the connected components of size at least 2 of G^* . This is correct by Lemma 5.2.

In [20], Italiano *et al.* presented two algorithms for calculating the strong bridges of a strongly connected graph $G = (V, E)$. We use them to implement lines 8–10 of Algorithm 5.1 as follows. Consider a flowgraph $G(v) = (V, E, v)$. For each edge $e = (x, y) \in E$, we delete this edge from $G(v)$ and we add

two new edges $(x, \varphi(e)), (\varphi(e), y)$ to $G(v)$. We obtain a new flowgraph, denoted $G'(v) = (V', E', v)$. Then, we compute the dominator tree $DT'(v)$ of $G'(v)$. Obviously, an edge e is an edge dominator of vertex $w \in V \setminus \{v\}$ in $G(v)$ if and only if the corresponding vertex $\varphi(e)$ is a dominator of w in $G'(v)$. We mark the vertices of G that have edge dominators in $G(v)$ by depth first search in $DT'(v)$. Therefore, lines 8–10 can be implemented in linear time. In [20], Italiano *et al.* observed that the strong bridges of G are the SAPs of the directed graph $G' = (V', E')$ that correspond to edges in G . We will use these strong bridges in our second algorithm for computing the 2-edge blocks of G .

Theorem 5.3. *Algorithm 5.1 runs in $O(nm)$ time.*

Proof. One can test whether a directed graph is 2-edge-connected in linear time using the algorithm of Italiano *et al.* [20]. Furthermore, the edge dominators of a flowgraph $G(v) = (V, E, v)$ can be computed in linear time [7, 20]. Lines 6–10 take $O(nm)$ time. The connected components of G^* can be found in $O(n^2)$ time. \square

Lemma 5.4. *Let $G = (V, E)$ be a strongly connected graph and let x, y be distinct vertices in G . Let S_{sb} be the set of all the strong bridges of G . Then for any edge $e \in E \setminus S_{sb}$, the vertices x and y lie in the same SCC of $G \setminus \{e\}$.*

Proof. Immediate from the definition. \square

This simple lemma leads to another algorithm (Algorithm 5.2) which might be useful when t_{sb} is small.

Algorithm 5.2.

Input: A strongly connected graph $G = (V, E)$.

Output: The 2-edge blocks of G .

```

1  If  $G$  is 2-edge-connected then.
2      Output  $V$ .
3  else
4      Let  $A$  be an  $n \times n$  matrix.
5      Initialize  $A$  with 1s.
6      for each strong bridge  $e$  of  $G$  do
7          for each pair  $(v, w) \in V \times V$  do
8              if  $v, w$  in distinct SCCs of  $G \setminus \{e\}$  then
9                   $A[v, w] \leftarrow 0$ .
10      $E^* \leftarrow \emptyset$ .
11     for each pair  $(v, w) \in V \times V$  do
12         if  $A[v, w] = 1$  and  $A[w, v] = 1$  then
13             Add the undirected edge  $(v, w)$  to  $E^*$ .
14     Compute the connected components of size  $> 1$  of  $G^*$  and output them.
```

The correctness of this algorithm follows from the following lemma.

Lemma 5.5. *Let v, w be distinct vertices in a strongly connected graph G . Then $v \overset{2e}{\rightsquigarrow} w$ if and only if $A[v, w] = 1$ and $A[w, v] = 1$ (when line 10 is reached).*

Proof. Similar to the proof of Lemma 4.6 using Lemma 5.4. \square

Theorem 5.6. *Algorithm 5.2 runs in $O(t_{sb}n^2)$ time.*

Proof. The strong bridges of a directed graph can be found in linear time using the algorithm of Italiano *et al.* [20]. Lines 7–11 take $O(t_{sb}n^2)$ time. \square

Let G a directed graph. Italiano *et al.* [20] showed that $t_{sb} \leq (2n - 2)$.

Corollary 5.7. *The 2-edge blocks of a directed graph $G = (V, E)$ can be computed in $O(\min\{m, t_{sb}n\}n)$ time.*

Now we show that the 2-edge block that contains a certain vertex can be computed in linear time. Let $G = (V, E)$ be a strongly connected graph and let $v \in V$. By $U(v)$ we denote the set of vertices that do not have edge dominators in $G(v)$ and by $U^R(v)$ we denote the set of vertices that do not have edge dominators in $G^R(v)$.

Lemma 5.8. *Let $G = (V, E)$ be a strongly connected graph and let $v \in V$. Let C^{2e} be the 2-edge block of G that includes v . Then $w \in C^{2e}$ if and only if $w \in U(v) \cap U^R(v)$*

Proof. “ \Leftarrow ” Let $w \in (U(v) \cap U^R(v)) \setminus \{v\}$. w does not have any edge dominator in $G(v)$. Therefore, by Menger’s Theorem for edge connectivity, there exist two edge-disjoint paths from v to w in $G(v)$. Furthermore, there are two edge-disjoint paths from v to w in $G^R(v)$ since w does not have any edge dominator in $G^R(v)$. Thus, there are also two edge-disjoint paths from w to v in G .

“ \Rightarrow ” Immediate from definition. \square

We have seen that $U(v)$ can be computed in linear time. Therefore, $U(v) \cap U^R(v)$ can be computed in linear time.

6. THE RELATION BETWEEN 2-DIRECTED BLOCKS, 2-STRONG BLOCKS AND 2-EDGE BLOCKS

In this section we consider the relation between 2-directed blocks, 2-strong blocks and 2-edge blocks.

Lemma 6.1. *Let $G = (V, E)$ be a strongly connected graph and let x, y be distinct vertices in G . Then $x \overset{2}{\rightsquigarrow} y$ if and only if $x \overset{2s}{\rightsquigarrow} y$ and $x \overset{2e}{\rightsquigarrow} y$.*

Proof. “ \Leftarrow ”: By Lemma 5.1, for each edge $e \in E$ the vertices x, y lie in the same SCC of $G \setminus \{e\}$ since $x \overset{2e}{\rightsquigarrow} y$. Because the vertices x, y lie in the same SCC of $G \setminus \{(x, y)\}$, there exist a path from x to y in $G \setminus \{(x, y)\}$. There is also a path from y to x in $G \setminus \{(y, x)\}$ since x, y lie in the same SCC of $G \setminus \{(y, x)\}$. As a consequence, the vertices x, y lie in the same SCC of $G \setminus \{(x, y), (y, x)\}$. By definition, the vertices

x, y lie in the same SCC of $G \setminus \{w\}$ for any vertex $w \in V \setminus \{x, y\}$ since $x \overset{2s}{\rightsquigarrow} y$. Therefore, by Lemma 3.1, we have $x \overset{2}{\rightsquigarrow} y$.

“ \Rightarrow ”: this direction follows from Lemmas 3.2 and 4.1. \square

Now we describe our second algorithm for computing all the 2-directed blocks of a strongly connected graph G . First, we execute lines 1–11 of Algorithm 4.2. Next, we execute lines 6–9 of Algorithm 5.2. Finally, we execute lines 12–16 of Algorithm 4.2. The correctness of our algorithm follows from Lemma 6.1.

Theorem 6.2. *The 2-directed blocks of a directed graph G can be computed in $O((t_{sap} + t_{sb})n^2)$ time.*

Proof. This follows from Theorems 4.7 and 5.6. \square

Corollary 6.3. *The 2-directed blocks of a directed graph G can be computed in $O(\min\{m, (t_{sap} + t_{sb})n\}n)$ time.*

Theorem 6.4. *All algorithms in Sections 3, 4, 5 and 6 require $\Theta(n^2)$ space.*

Proof. Clearly, all these algorithms get by with $O(n^2)$ space and they need $\Omega(n^2)$ space to store the matrix A and the auxiliary graph G^* . \square

7. THE 2-DIRECTED BLOCKS THAT CONTAIN A CERTAIN VERTEX

Let $G = (V, E)$ be a strongly connected graph and let v be a vertex in G . In this section we present an algorithm for computing the 2-directed blocks of G that contain v in $O(t^*m)$ time, where t^* is the number of these blocks. This algorithm is based on Lemmas 3.3 and 3.4. It offers two advantages, first, it does not need to construct the auxiliary graph G^* . Second, it runs in linear time when v is contained in a constant number of 2-directed blocks. By $B(v)$ we denote the set of all vertices $w \in V \setminus \{v\}$ such that $v \overset{2}{\rightsquigarrow} w$. One can compute $B(v)$ by using Procedure 7.1 in linear time.

Procedure 7.1.

Input: A strongly connected graph $G = (V, E)$ and vertex $v \in V$.

Output: $B(v)$.

- 1 $B_1(v) \leftarrow \emptyset, B_2(v) \leftarrow \emptyset, B(v) \leftarrow \emptyset.$
- 2 $E' \leftarrow E.$
- 3 $V' \leftarrow V.$
- 4 **for** each edge $e = (v, w) \in E$ **do**
- 5 $E' \leftarrow E' \setminus \{(v, w)\}.$
- 6 $V' \leftarrow V' \cup \{u_e\}.$
- 7 $E' \leftarrow E' \cup \{(v, u_e), (u_e, w)\}.$
- 8 Compute the dominator tree $DT'(v)$ of the flowgraph $G'(v) = (V', E', v)$.
- 9 **for** each direct successor w of v in $DT'(v)$ **do**
- 10 **if** $w \in V$ **then**

```

11    $B_1(v) \leftarrow B_1(v) \cup \{w\}.$ 
12    $E_1 \leftarrow E^R.$ 
13    $V_1 \leftarrow V.$ 
14   for each edge  $e = (v, w) \in E^R$  do
15      $E_1 \leftarrow E_1 \setminus \{(v, w)\}.$ 
16      $V_1 \leftarrow V_1 \cup \{u_e\}.$ 
17      $E_1 \leftarrow E_1 \cup \{(v, u_e), (u_e, w)\}.$ 
18   Compute the dominator tree  $DT_1(v)$  of  $G_1(v) = (V_1, E_1, v)$ .
19   for each direct successor  $w$  of  $v$  in  $DT_1(v)$  do
20     if  $w \in V$  then
21        $B_2(v) \leftarrow B_2(v) \cup \{w\}.$ 
22    $B(v) \leftarrow B_1(v) \cap B_2(v).$ 

```

The correctness of Procedure 7.1 follows from Lemma 3.7 and the fact that $v \overset{2}{\rightsquigarrow} v$ in G if and only if $v \overset{2}{\rightsquigarrow} w$ in G^R .

Algorithm 7.1.

Input: A strongly connected graph $G = (V, E)$ and vertex $v \in V$.

Output: The 2-directed blocks of G that contain v .

```

1   if  $G$  is 2-vertex-connected then
2     Output  $V$ .
3   else
4      $R \leftarrow B(v)$ .
5     while  $R$  is not empty do
6       Choose arbitrarily a vertex  $w \in R$ .
7       output  $(R \cap B(w)) \cup \{v, w\}.$ 
8        $R \leftarrow R \setminus ((R \cap B(w)) \cup \{w\})$ .

```

Lemma 7.1. *Algorithm 7.1 calculates the 2-directed blocks that include v .*

Proof. Let $C_1^{2d}, C_2^{2d}, \dots, C_t^{2d}$ be the 2-directed blocks which contain v . By Lemma 3.3, these blocks include only the vertex v in common. Thus, $C_1^{2d} \setminus \{v\}, C_2^{2d} \setminus \{v\}, \dots, C_t^{2d} \setminus \{v\}$ are disjoint. Obviously, $\bigcup_{1 \leq i \leq t} (C_i^{2d} \setminus \{v\}) \subseteq B(v)$. Let w be a vertex in $B(v)$ and let C^{2d} be the 2-directed block of G such that $v, w \in C^{2d}$. It is sufficient to show that $C^{2d} = (B(w) \cap B(v)) \cup \{v, w\}$. Let x be a vertex in $B(w) \cap B(v)$. Since $v \overset{2}{\rightsquigarrow} w, w \overset{2}{\rightsquigarrow} x$ and $x \overset{2}{\rightsquigarrow} v$, by Lemma 3.4, the vertices x, v, w lie in the same 2-directed block of G . Conversely, let x be a vertex in $C^{2d} \setminus \{v, w\}$. Since $v \overset{2}{\rightsquigarrow} x$ and $w \overset{2}{\rightsquigarrow} x$, we have $x \in B(v)$ and $x \in B(w)$. \square

Theorem 7.2. *Algorithm 7.1 runs in $O(t^*m)$, where t^* is the number of the 2-directed blocks that contain v .*

Proof. We have seen that $B(v)$ can be computed in linear time. Furthermore, the number of iterations of the while-loop in lines 5–8 is t^* . Thus, the total time is $O(t^*m)$. \square

8. APPROXIMATION ALGORITHM FOR THE MS-SAPs PROBLEM

In this section we show that there is a $17/3$ approximation algorithm for the MS-SAPs problem. In [13], Georgiadis presented a linear time 3-approximation algorithm for the problem of finding a minimum-cardinality 2-vertex connected spanning subgraph (2VCSS) of 2-vertex-connected directed graphs. This algorithm is based on the works [12, 14, 20]. We slightly modify this algorithm and combine it with the algorithm of Zhao *et al.* [31] in order to obtain a $17/3$ approximation algorithm for the MS-SAPs problem. We first briefly describe Georgiadis algorithm [13]. Let $G = (V, E)$ be a 2-vertex-connected directed graph and let v be a vertex in G . Menger's Theorem for vertex connectivity [2] implies that the flow-graph $G(v)$ has no non-trivial dominators. In [14], Georgiadis and Tarjan proved that there exist two independent spanning trees of $G(v)$. Algorithm 8.1 shows the algorithm of Georgiadis [13].

Algorithm 8.1. (from [13])

Input: A 2-vertex-connected directed graph $G = (V, E)$.

Output: A 2-vertex-connected spanning subgraph G^* of G .

- 1 Choose arbitrarily a vertex $v \in V$.
- 2 Compute two independent spanning trees T_1, T_2 of $G(v)$.
- 3 Compute two independent spanning trees T_3, T_4 of $G^R(v)$.
- 4 Construct a strongly connected spanning subgraph (SCSS)
- 6 $G' = (V \setminus \{v\}, E')$ of $G \setminus \{v\}$ with $|E'| \leq 2(n - 2)$.
- 7 $E^* \leftarrow T_1 \cup T_2 \cup T_3^R \cup T_4^R \cup E'$.
- 8 Output $G^* = (V, E^*)$.

By ([13], Lem. 2), the flowgraphs $(V, T_1 \cup T_2, v)$ and $(V, T_3 \cup T_4, v)$ have only trivial dominators. Let w be a vertex in $G \setminus \{v\}$. As is well known, it is easy to calculate a SCSS $G' = (V \setminus \{v\}, E')$ of $G \setminus \{v\}$ with $|E'| \leq 2(n - 2)$. Just take the union of outgoing branching rooted at w and incoming branching rooted at w [9, 22]. Since $G^* \setminus \{v\}$ is strongly connected, the vertex v is not a SAP in G^* . Therefore, by ([20], Thm. 5.2) the directed graph G^* has no SAPs. By definition, G^* is 2-vertex-connected. Algorithm 8.1 has an approximation ratio of 3 and runs in linear time [13].

In [31], Zhao *et al.* gave a linear time $5/3$ approximation algorithm for the problem of finding a MSCSS of a strongly connected graph. We briefly describe this algorithm. The algorithm of Zhao *et al.* [31] is based on repeatedly contracting special cycles. The idea of contracting cycles was first introduced by Khuller *et al.* [22]. Let $G = (V, E)$ be a strongly connected graph and let $U \subseteq V$. By $\delta^+(U)$ we denote the set of edges leaving U , i.e., $\delta^+(U) = \{(u, v) \in E | u \in U \text{ and } v \notin U\}$. By G/U we denote the directed graph obtained from G by contracting the vertex set U . Let l be a cycle of G . By V_l and E_l we denote the set of vertices and the set of edges of the cycle l , respectively. The cycle l conceals U if $\delta^+(U)$ is not empty and the edges in $\delta^+(U)$ are deleted in G/V_l . The algorithm of Zhao

et al. [31] repeatedly contracts concealing cycles. Algorithm 8.2 shows a high-level description of this algorithm.

Algorithm 8.2. (from [31])

Input: A strongly connected graph $G = (V, E)$.

Output: A SCSS $G^* = (V, E^*)$ of G .

```

1   $E^* \leftarrow \emptyset$ .
2   $G_1 \leftarrow G$ .
3  while  $G_1$  has a cycle of length at least 3 do
4    Find a concealing cycle  $l$  in  $G_1$  with  $E_l \geq 3$ .
5     $E^* \leftarrow E^* \cup E_l$ .
6     $G_1 \leftarrow G_1/V_l$ .
7     $E_1 \leftarrow$  the set of edges of  $G_1$ .
8     $E^* \leftarrow E^* \cup E_1$ .
9  Output  $G^* = (V, E^*)$ .
```

Zhao *et al.* [31] proved that Algorithm 8.2 returns a feasible solution for the MSCSS problem and has an approximation factor of $5/3$. In [31], they also showed that Algorithm 8.2 can be implemented in linear time.

Now we modify Georgiadis's algorithm [13] and combine it with the algorithm of Zhao *et al.* [31] in order to obtain an approximation algorithm for the MS-SAPs problem. See Algorithm 8.3.

Algorithm 8.3.

Input: A strongly connected graph $G = (V, E)$.

Output: A SCSS $G^* = (V, E^*)$ of G with the same SAPs.

```

1  if  $G$  is 2-vertex-connected then
2    Run Algorithm 8.1 on  $G$ .
3  else
4    Compute the SAPs of  $G$ .
5    If all vertices in  $V$  are SAPs then
6      Compute a SCSS  $G^* = (V, E^*)$  of  $G$  using the Algorithm of
7      Zhao et al. [31] and output  $G^*$ .
8    else
9      Choose a vertex  $v \in V$  such that  $v$  is not a SAP of  $G$ .
10     Compute a SCSS  $G' = (V \setminus \{v\}, E')$  of  $G \setminus \{v\}$  using the Algorithm
11     of Zhao et al. [31].
12     Compute two independent spanning trees  $T_1, T_2$  of  $G(v)$ .
13     Compute two independent spanning trees  $T_3, T_4$  of  $G^R(v)$ .
14      $E^* \leftarrow E' \cup T_1 \cup T_2 \cup T_3^R \cup T_4^R$ .
15     Output  $G^* = (V, E^*)$ .
```

The following lemma shows that the output G^* of Algorithm 8.3 is a feasible solution for the MS-SAPs problem.

Lemma 8.1. *The output G^* is strongly connected, and the directed graphs G^* and G have the same SAPs.*

Proof. We need only to show that this lemma is correct when G is not 2-vertex-connected. Let G be a strongly connected graph which is not 2-vertex-connected. If all the vertices in V are SAPs in G , then, for any SCSS G' of G , the graphs G' and G have the same SAPs. Otherwise, G has at least one vertex v which is not a SAP of G (see lines 9–14). In this case, by ([20], Thm. 5.2), $D(v) \cup D^R(v)$ is the set of all SAPs of G . Georgiadis and Tarjan [14] showed that there exist two independent trees of $G(v)$ and the flowgraphs $G(v)$ and $(V, T_1 \cup T_2, v)$ have the same non-trivial dominators. Thus, the flowgraphs $G^R(v)$ and $(V, T_3 \cup T_4, v)$ have the same non-trivial dominators. Clearly, $(V, T_1 \cup T_3^R)$ is strongly connected. Therefore, G^* is strongly connected. Let $D_1(v)$ be the set of all non-trivial dominators of $G^*(v)$ and let $D_1^R(v)$ be the set of all non-trivial dominators of $G^{*R}(v)$, where G^{*R} is the reversal graph of G^* . Since $T_1 \cup T_2 \subseteq E^*$, we have $D_1(v) \subseteq D(v)$. Moreover, $D_1^R(v) \subseteq D^R(v)$ because $T_3^R \cup T_4^R \subseteq E^*$. As a consequence, $D_1(v) \cup D_1^R(v) \subseteq D(v) \cup D^R(v)$. Assume for a contradiction that $D_1(v) \cup D_1^R(v) \neq D(v) \cup D^R(v)$. Then there is at least vertex $x \in D(v) \cup D^R(v)$ such that $x \notin D_1(v) \cup D_1^R(v)$. By ([20], Thm. 5.2), the vertex x is not a SAP in G^* but x is a SAP in G . Thus, $G^* \setminus \{x\}$ is strongly connected. Because $G^* \setminus \{x\}$ is a spanning subgraph of $G \setminus \{x\}$, the directed graph $G \setminus \{x\}$ is strongly connected, which contradicts that x is a SAP in G . Since v is not a SAP of G^* and $D(v) \cup D^R(v) = D_1(v) \cup D_1^R(v)$, by ([20], Thm. 5.2) the set of all SAPs of G^* is $D(v) \cup D^R(v)$. \square

Theorem 8.2. *Algorithm 8.3 achieves an approximation ratio of 17/3.*

Proof. The algorithm of Zhao *et al.* [31] has an approximation factor of 5/3. Thus, it is enough to consider the case when G is not 2-vertex-connected and includes at least one vertex which is not a SAP. Let E_{opt} be an optimal solution for the MS-SAPs problem. Let E'_{opt} be any optimal solution for the problem of finding a MSCSS of $G \setminus \{v\}$. Since the vertex v is not a SAP in $G[E_{\text{opt}}]$, the graph $G[E_{\text{opt}}] \setminus \{v\}$ is strongly connected. Thus, we have $|E_{\text{opt}}| \geq |E'_{\text{opt}}|$. Consequently, by ([31], Thm. 3), we have $|E'|/|E_{\text{opt}}| \leq |E'|/|E'_{\text{opt}}| \leq 5/3$. Since $|T_1 \cup T_2 \cup T_3^R \cup T_4^R| \leq 4(n - 1)$ and $|E_{\text{opt}}| \geq n$, Algorithm 8.3 has approximation ratio $|E^*|/|E_{\text{opt}}| \leq 4 - 4/n + 5/3 = 17/3 - 4/n$. \square

Theorem 8.3. *Algorithm 8.3 runs in linear time.*

Proof. The SAPs of G can be calculated in linear time using the algorithm of Italiano *et al.* [20]. Two independent spanning trees of $G(v)$ can also be constructed in linear time [15]. Furthermore, the algorithm of Zhao *et al.* [31] can be implemented in linear time. \square

9. APPROXIMATION ALGORITHM FOR THE MS-2SBS PROBLEM

In this section we present an approximation algorithm for the MS-2SBS problem. Let $G = (V, E)$ be a strongly connected graph such that G is not 2-vertex-connected. Our algorithm consists of two main steps. The first step finds a SCSS

G^* of G such that G^* and G have the same SAPs. The following lemma explains the purpose of this step.

Lemma 9.1. *Let $G = (V, E)$ be a strongly connected graph and let S be the set of all SAPs of G . Let $G^* = (V, E^*)$ be a feasible solution for the MS-SAPs problem and let x, y be distinct vertices in V . Then for any vertex $z \in V \setminus (S \cup \{x, y\})$, the vertices x, y lie in the same SCC of $G^* \setminus \{z\}$.*

Proof. Immediate from the definition of SAPs, since G and G^* have the same SAPs. \square

Let x, y be two vertices in G such that $x \overset{2s}{\rightsquigarrow} y$ and let z be a vertex in $V \setminus \{x, y\}$ such that z is not SAP in G . The first step ensures that there exist at least one path from x to y and from y to x in $G^* \setminus \{z\}$. The second step computes, for each SAP v of G , strongly connected spanning subgraphs of the subgraphs induced by the SCCs of $G \setminus \{v\}$.

Algorithm 9.1.

Input: A strongly connected graph $G = (V, E)$.

Output: A SCSS of G with the same 2-strong blocks.

```

1  if  $G$  is 2-vertex-connected then
2    Run algorithm of Cheriyan and Thurimella [5] for minimum
3    cardinality 2-VCSS problem, improved in [13].
4  else
5    lines 4–14 of Algorithm 8.3, giving  $G^* = (V, E^*)$ .
6  for each SAP  $v$  of  $G$  do
7    Compute the SCCs of  $G \setminus \{v\}$ .
8    for each SCC  $C$  of  $G \setminus \{v\}$  do
9      if  $G^*[C]$  is not strongly connected then
10       Find a SCSS  $(C, E')$  of  $G[C]$  with  $|E'| \leq 2(|C| - 1)$ .
11        $E^* \leftarrow E^* \cup E'$ .
12  Output  $G^* = (V, E^*)$ .
```

In the following lemma we show that Algorithm 9.1 returns a feasible solution for the MS-2SBs problem.

Lemma 9.2. *Let G^* be the output of Algorithm 9.1. Then G^*, G have the same 2-strong blocks and G^* is strongly connected.*

Proof. Since each 2-vertex-connected graph is a 2-strong block, the algorithm of Cheriyan and Thurimella [5] returns a feasible solution when G is 2-vertex-connected. Let $G = (V, E)$ be a strongly connected graph such that G is not 2-vertex-connected. By Lemma 8.1, the directed graph G^* which is calculated in line 5 and the directed graph G have the same SAPs, and G^* is strongly connected. Therefore, the output G^* of Algorithm 9.1 and G have the same SAPs. Obviously, it is sufficient to show the following. Let $x, y \in V$ be distinct vertices such that

$x \overset{2s}{\rightsquigarrow} y$ in G . We must show that $x \overset{2s}{\rightsquigarrow} y$ in the output G^* of Algorithm 9.1. Let $v \in V \setminus \{x, y\}$ be some vertex. By Lemma 9.1, we may assume that v is a SAP. Then x, y lie in the same SCC of $G \setminus \{v\}$. The execution of the loop in lines 6–11 for v enforces that x, y are also in the same SCC of $G^* \setminus \{v\}$. \square

Theorem 9.3. *Algorithm 9.1 has an approximation factor of $(2t_{sap} + 17/3)$.*

Proof. If G is 2-vertex-connected, the algorithm of Cheriyan and Thurimella [5] for the minimum cardinality 2-VCSS problem achieves an approximation ratio of $3/2$. We consider now the case when G is not 2-vertex-connected. Let E_{opt} be an optimal solution for the MS-2SBs problem. The output G^* of Algorithm 9.1 consists of two edge sets E_1, E_2 , where the edge set E_1 is computed in line 5 and the edge set E_2 is computed in lines 6–11. By Theorem 8.2, $|E_1|/|E_{opt}| \leq 17/3$. The number of iterations of the for-loop in lines 6–11 is t_{sap} . Because the SCCs of a directed graph are disjoint, we have $|E_2| < 2t_{sap}n$. Since $|E_{opt}| \geq n$, we have $|E_2|/|E_{opt}| < 2t_{sap}$. \square

Theorem 9.4. *Algorithm 9.1 runs in $O(m(\sqrt{n} + t_{sap}) + n^2)$ time.*

Proof. The algorithm of Cheriyan and Thurimella [5] for the minimum cardinality 2-VCSS problem has running time $O(m^2)$. In 2011, Georgiadis [13] improved it to $O(m\sqrt{n} + n^2)$. By Theorem 8.3, line 5 takes $O(n + m)$ time. The SCCs of a directed graph can be found in linear time using Tarjan’s algorithm [28]. Thus, lines 6–11 take $O(t_{sap}m)$ time. \square

Notice that in lines 9–11 of Algorithm 9.1, every SCC C which does not contain any vertex of the 2-strong blocks of G can be safely disregarded.

10. APPROXIMATION ALGORITHM FOR THE MS-2EBs PROBLEM

In this section we present an approximation algorithm for the MS-2EBs problem. The idea of this algorithm (Algorithm 10.1) is similar to the idea of Algorithm 9.1.

Algorithm 10.1.

Input: A strongly connected graph $G = (V, E)$.

Output: A SCSS of G with the same 2-edge blocks.

- 1 Choose a vertex v of G .
- 2 Compute two spanning trees T_1, T_2 of $G(v)$ (rooted at v) such
- 3 that T_1, T_2 have only the edge dominators of $G(v)$ in common.
- 4 Compute two spanning trees T_3, T_4 of $G^R(v)$ (rooted at v) such
- 5 that T_3, T_4 have only the edge dominators of $G^R(v)$ in common.
- 6 $E^* \leftarrow T_1 \cup T_2 \cup T_3^R \cup T_4^R$.
- 7 Find all the strong bridges in G .
- 8 **for** each strong bridge e of G **do**
- 9 Compute the SCCs of $G \setminus \{e\}$.

```

10   for each SCC  $C$  of  $G \setminus \{e\}$  do
11     if  $G^*[C]$  is not strongly connected then
12       Find a SCSS  $(C, E')$  of  $G[C]$  with  $|E'| \leq 2(|C| - 1)$ .
13        $E^* \leftarrow E^* \cup E'$ .
14   Output  $G^* = (V, E^*)$ .

```

Lemma 10.1. *Let G^* be the output of Algorithm 10.1. Then G^* is strongly connected and the directed graphs G^*, G have the same strong bridges.*

Proof. Since $T_1 \cup T_3^R \subseteq E^*$, the graph G^* is strongly connected. Tarjan [29] proved that there exist two spanning trees (rooted at v) of $G(v)$ that have only the edge dominators of $G(v)$ in common and he gave algorithms for computing them. Italiano *et al.* [20] showed that edge $e \in E$ is strong bridge if and only if e is an edge dominator in $G(v)$ or in $G^R(v)$. Therefore, the directed graphs $G, (V, T_1 \cup T_2 \cup T_3^R \cup T_4^R)$ have the same strong bridges. Since $(V, T_1 \cup T_2 \cup T_3^R \cup T_4^R)$ is a subgraph of G^* and G^* is a subgraph of G , the directed graphs G^*, G have the same strong bridges. \square

Next we show that Algorithm 10.1 outputs a feasible solution for the MS-2EBs problem.

Lemma 10.2. *Let $G^* = (V, E^*)$ be the output of Algorithm 10.1. Then G^*, G have the same 2-edge blocks.*

Proof. Let x, y be distinct vertices such that $x \overset{2e}{\rightsquigarrow} y$ in G . We must show that $x \overset{2e}{\rightsquigarrow} y$ in G^* . By Lemma 5.1, we need to show that x, y lie in the same SCC of $G^* \setminus \{e\}$ for any edge $e \in E^*$. Let e be an edge in G^* . We consider two cases.

1. e is not a strong bridge in G^* . By Lemma 10.1, G^* is strongly connected. Hence, by definition of strong bridges, the vertices x, y lie in the same SCC of $G^* \setminus \{e\}$.
2. e is a strong bridge in G^* . By Lemma 10.1, G, G^* have the same strong bridges. Since x, y lie in the same SCC of $G \setminus \{e\}$, the execution of the loop in lines 8–13 enforces that the vertices x, y are also in the same SCC of $G^* \setminus \{e\}$. \square

Theorem 10.3. *Let $G^* = (V, E^*)$ be the output of Algorithm 10.1. Then $|E^*| < (4 + 2t_{sb})n$.*

Proof. Let E_1 be the edge set which is computed in lines 8–13. Notice that the SCCs of a directed graph are disjoint and we add at most $2(|C| - 1)$ edges for each SCC C in lines 12–13. Since the number of iterations of the for-loop in lines 8–13 is t_{sb} , we have $|E_1| \leq 2t_{sb}(n-1)$. Therefore, we have $|E^*| = |T_1 \cup T_2 \cup T_3^R \cup T_4^R| + |E_1| \leq 4(n-1) + 2t_{sb}(n-1) < (4 + 2t_{sb})n$. \square

Let G be a strongly connected graph. Since each SCSS of G has at least n edges, Algorithm 10.1 achieves an approximation ratio of $(4 + 2t_{sb})$.

Theorem 10.4. *The running time of Algorithm 10.1 is $O((t_{sb} + \alpha(n, m))m)$.*

Proof. Two spanning trees T_1, T_2 of $G(v)$ (rooted at v) such that T_1, T_2 have only the edge dominators of $G(v)$ in common can be computed in $O(m\alpha(n, m))$ time by using Tarjan’s algorithm [29], where $\alpha(n, m)$ is a very slowly function related to a functional inverse of Ackermann’s function. The strong bridges of a strongly connected graph can be computed in linear time using the algorithm of Italiano *et al.* [20]. Moreover, the number of iterations of the for-loop in lines 6–13 is t_{sb} . The total time for Algorithm 10.1 is therefore $O((t_{sb} + \alpha(n, m))m)$. \square

Notice that by Lemma 6.1, we can obtain a $(2(t_{sap} + t_{sb}) + 29/3)$ approximation algorithm for the MS-2DBs problem by combining Algorithm 9.1 and Algorithm 10.1. This algorithm whose running time is $O((t_{sap} + t_{sb} + \sqrt{n} + \alpha(n, m))m + n^2)$ might be useful when $t_{sap} + t_{sb}$ is small.

11. OPEN PROBLEMS

The k -strong blocks of directed graphs, which are natural generalization of 2-strong blocks, are similar to the k -blocks of undirected graphs [4]. Let $G = (V, E)$ be a directed graph. We define a relation $\overset{ks}{\rightsquigarrow}$ as follows. For any pair of distinct vertices $x, y \in V$, we write $x \overset{ks}{\rightsquigarrow} y$ if for each subset $X \subseteq V \setminus \{x, y\}$ with $|X| < k$, the vertices x and y lie in the same SCC of $G \setminus X$. A k -strong block in a directed graph G is a maximal vertex set $C^{ks} \subseteq V$ with $|C^{ks}| \geq k$ such that for each pair of distinct vertices $x, y \in C^{ks}$, we have $x \overset{ks}{\rightsquigarrow} y$. One can show that any two k -strong blocks have at most $k - 1$ vertices in common. A simple algorithm was given in [4] to find the k -blocks of an undirected graph. We noticed that this algorithm is also able to compute the k -strong blocks of a directed graph in $O(\min\{k, \sqrt{n}\}n^4)$ -time. We just need to modify the pre-processing step and the definition of separation. Let $G = (V, E)$ be a directed graph. An ordered pair (C, D) such that $C, D \subseteq V$ and $C \cup D = V$ is a *separation* of G if there is no edge from $C \setminus D$ to $D \setminus C$ or there is no edge from $D \setminus C$ to $C \setminus D$. In the pre-processing step we construct a new undirected graph $H_k = (V, E_k)$ as follows. For each pair of distinct vertices x, y of G , if $x \overset{ks}{\rightsquigarrow} y$ in G , we add an undirected edge (x, y) to E_k (see Appendix A). Furthermore, for every pair (x, y) of H_k with $(x, y) \notin E_k$, we label it with some separation (C, D) such that $|C \cap D| < k$ and $x \in C, y \in D$.

We leave as an open problem whether there exists any approximation algorithm for the problem of finding MSCSS with same k -strong blocks of a strongly connected graph for $k > 2$. Another open question is whether there is an approximation algorithm for the problem of finding MSCSS with the same k -directed blocks when $k > 2$.

It is also possible to generalize the MS-2EBs problem. Let $G = (V, E)$ be a strongly connected graph. We define a relation $\overset{ke}{\rightsquigarrow}$ as follows. For any pair of distinct vertices $x, y \in V$, we write $x \overset{ke}{\rightsquigarrow} y$ if for each edge subset $Y \subseteq E$ with $|Y| < k$, the vertices x, y lie in the same SCC of $G \setminus Y$. The k -edge blocks of G are maximal subsets of V of size $\geq k$ closed under $\overset{ke}{\rightsquigarrow}$.

Lemma 11.1. *The k -edge blocks of a strongly connected graph are disjoint.*

Proof. The proof is similar to our proof of Lemma 5.2. \square

The k -edge blocks of a directed graph can be found in $O(n^3m)$ time using maximum flow algorithms [19, 26] since the relation $\overset{ke}{\rightsquigarrow}$ is symmetric and transitive.

A. LEMMAS RELATED TO k -STRONG BLOCKS

Lemma A.1. *Let $G = (V, E)$ be a directed graph and let x, y be distinct vertices of G . Then $x \overset{ks}{\rightsquigarrow} y$ if and only if x, y satisfy one of the following conditions.*

1. $\{(x, y), (y, x)\} \subseteq E$.
2. $(x, y) \in E, (y, x) \notin E$ and there are k -vertex-disjoint paths from y to x in G .
3. $(y, x) \in E, (x, y) \notin E$ and there are k -vertex-disjoint paths from x to y in G .
4. $\{(x, y), (y, x)\} \cap E = \emptyset$ and there exist k -vertex-disjoint paths from x to y and from y to x in G .

Proof. This follows immediately from Menger's Theorem for vertex connectivity. \square

Lemma A.2. *Let (C, D) be a separation of a directed graph $G = (V, E)$ such that $|C \cap D| < k$. Then $C \setminus D$ and $D \setminus C$ lie in different k -strong blocks of G .*

Proof. Let x be any vertex in $C \setminus D$ and let y be any vertex in $D \setminus C$. By the definition of separation, there is either no path from x to y or no path from y to x in $G \setminus (C \cap D)$. Thus, x, y do not lie in the same SCC of $G \setminus (C \cap D)$. \square

Acknowledgements. The author would like to thank Martin Dietzfelbinger for helpful comments and interesting discussions. He also would like to thank the anonymous reviewers for their many helpful comments and suggestions, which helped in improving the exposition.

REFERENCES

- [1] S. Alstrup, D. Harel, P.W. Lauridsen and M. Thorup, Dominators in linear time. *SIAM J. Comput.* **28** (1999) 2117–2132.
- [2] R. Balakrishnan and K. Ranganathan, A Textbook of graph theory, 2nd edn. Springer (2012) 66.
- [3] A.L. Buchsbaum, L. Georgiadis, H. Kaplan, A. Rogers, R.E. Tarjan and J.R. Westbrook, Linear-time algorithms for dominators and other path-evaluation problems. *SIAM J. Comput.* **38** (2008) 1533–1573.
- [4] J. Carmesin, R. Diestel, M. Hamann and F. Hundertmark, k -Blocks, a connectivity invariant for graphs (2013). Preprint [ArXiv:1305.4557](https://arxiv.org/abs/1305.4557).
- [5] J. Cheriyan and R. Thurimella, Approximating minimum-size k -connected spanning subgraphs via matching. *SIAM J. Comput.* **30** (2000) 528–560.
- [6] Y.M. Erusalimskii and G.G. Svetlov, Bijoin points, bibridges, and biblocks of directed graphs. *Cybernet. Systems Anal.* **16** (1980) 41–44.

- [7] D. Firmani, G.F. Italiano, L. Laura, A. Orlandi and F. Santaroni, Computing strong articulation points and strong bridges in large scale graphs, SEA. *Lect. Notes Comput. Sci.* **7276** (2012) 195–207.
- [8] S. Fortune, J.E. Hopcroft and J. Wyllie, The Directed Subgraph Homeomorphism Problem. *Theoret. Comput. Sci.* **10** (1980) 111–121.
- [9] G.N. Frederickson, J. JáJá, Approximation Algorithms for Several Graph Augmentation Problems. *SIAM J. Comput.* **10** (1981) 270–283.
- [10] M.R. Garey and D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, W.H. Freeman, San Francisco (1979).
- [11] F. Gavril, Algorithms for Minimum Coloring, Maximum Clique, Minimum Covering by Cliques, and Maximum Independent Set of a Chordal Graph. *SIAM J. Comput.* **1** (1972) 180–187.
- [12] L. Georgiadis, Testing 2-vertex connectivity and computing pairs of vertex-disjoint s-t paths in digraphs, In vol. 6189 of *Proc. of 37th ICALP*, Part I. *Lect. Notes Comput. Sci.* (2010) 738–749.
- [13] L. Georgiadis, Approximating the smallest 2-vertex connected spanning subgraph of a directed graph, In *Proc. of 19th European Symposium on Algorithms* (2011) 13–24.
- [14] L. Georgiadis and R.E. Tarjan, Dominator tree verification and vertex- disjoint paths, In *Proc. of the 16th ACM-SIAM Symposium on Discrete Algorithms* (2005) 433–442.
- [15] L. Georgiadis and R.E. Tarjan, Dominators, Directed Bipolar Orders, and Independent Spanning Trees. *ICALP* (2012) 375–386.
- [16] L. Georgiadis, G.F. Italiano, L. Laura and N. Parotsidis, 2-Edge Connectivity in Directed Graphs, CoRR abs/1407.3041 (2014).
- [17] L. Georgiadis, G.F. Italiano, L. Laura and N. Parotsidis, 2-Vertex Connectivity in Directed Graphs, CoRR abs/1409.6277 (2014).
- [18] L. Georgiadis, G.F. Italiano, L. Laura and N. Parotsidis, 2-Edge Connectivity in Directed Graphs, *SODA* (2015) 1988–2005.
- [19] A.V. Goldberg and R.E. Tarjan, Efficient maximum flow algorithms. *Commun. ACM* **57** (2014) 82–89.
- [20] G.F. Italiano, L. Laura and F. Santaroni, Finding strong bridges and strong articulation points in linear time. *Theoret. Comput. Sci.* **447** (2012) 74–84.
- [21] R. Jaber, On Computing the 2-vertex-connected components of directed graphs, CoRR abs/1401.6000 (2014).
- [22] S. Khuller, B. Raghavachari and N.E. Young, Approximating the Minimum Equivalent Digraph. *SODA* (1994) 177–186.
- [23] T. Lengauer and R.E. Tarjan, A fast algorithm for finding dominators in a flowgraph. *ACM Trans. Program. Lang. Syst.* **1** (1979) 121–141.
- [24] Chung-Lun Li, S. Thomas McCormick and D. Simchi-Levi, The complexity of finding two disjoint paths with min-max objective function. *Discrete. Appl. Math.* **26** (1990) 105–115.
- [25] E.S. Lowry and C.W. Medlock, Object code optimization. *Commun. ACM* **12** (1969) 13–22.
- [26] J.B. Orlin, Max Flows in $O(nm)$ time, or better. In *Proc. of the Annual ACM Symposium on Theory of Computing*. ACM Press, New York (2011) 765–774.
- [27] D.J. Rose and R.E. Tarjan, Algorithmic Aspects of Vertex Elimination. *Proc. of 7e Annual ACM Symposium on Theory of Computing*. ACM Press, New York (1975) 245–254.
- [28] R.E. Tarjan, Depth-first search and linear graph algorithms. *SIAM J. Comput.* **1** (1972.) 146–160
- [29] R.E. Tarjan, Edge-disjoint spanning trees and depth-first search. *Acta Inf.* **6** (1976) 171–185.
- [30] S. Vempala and A. Vetta, Factor 4/3 approximations for minimum 2-connected subgraphs. *Proc. of APPROX* (2000) 262–273.
- [31] L. Zhao, H. Nagamochi and T. Ibaraki, A linear time 5/3-approximation for the minimum strongly-connected spanning subgraph problem. *Inf. Process. Lett.* **86** (2003) 63–70.

Communicated by C. de Figueiredo.

Received September 5, 2014. Accepted February 19, 2015.