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Energy estimates and symmetry breaking in attractive Bose–Einstein condensates with ring-shaped potentials

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Abstract

This paper is concerned with the properties of L^2-normalized minimizers of the Gross–Pitaevskii (GP) functional for a two-dimensional Bose–Einstein condensate with attractive interaction and ring-shaped potential. By establishing some delicate estimates on the least energy of the GP functional, we prove that symmetry breaking occurs for the minimizers of the GP functional as the interaction strength $a > 0$ approaches a critical value a^*, each minimizer of the GP functional concentrates to a point on the circular bottom of the potential well and then is non-radially symmetric as $a \nearrow a^*$. However, when $a > 0$ is suitably small we prove that the minimizers of the GP functional are unique, and this unique minimizer is radially symmetric.

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1. Introduction

Since the remarkable experiments on Bose–Einstein condensates (BEC) in dilute gases of alkali atoms in 1995 [1,6], much attention has been attracted to the experimental studies on BEC over the last two decades, and many new phenomena of BEC have been observed in experiments [6]. These new experimental progresses also inspired the theoretical research in BEC, especially, the theory of Gross–Pitaevskii (GP) equations proposed by Gross and Pitaevskii [11,12,29]. There has been a growing interest in the mathematical theories and numerical methods of GP equations [2]. Several rigorous mathematical verifications of GP theory were established, see e.g. [8,22–25]. It is known that the classical trapping potential used in the study of BEC is the harmonic potential. With the advance of experimental techniques for BEC, some different trapping potentials have been used in the experiments [4,14,16,31,32]. Theoretically, it is also interesting to discuss mathematically how the shapes of trapping potentials affect the behavior

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of BEC. Very recently, Guo and Seiringer [13] studied the BEC with attractive interactions in $\mathbb{R}^2$ described by the following GP functional

$$E_a(u) := \int_{\mathbb{R}^2} \left(|\nabla u(x)|^2 + V(x)|u(x)|^2\right)dx - \frac{a}{2}\int_{\mathbb{R}^2} |u(x)|^4 dx, \quad u \in \mathcal{H}, \tag{1.1}$$

where $a > 0$ describes the strength of the attractive interactions, and $\mathcal{H}$ is a real-valued function space defined by

$$\mathcal{H} := \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} V(x)|u(x)|^2 dx < \infty \right\}, \tag{1.2}$$

with a trapping potential of the form

$$V(x) = h(x)\prod_{i=1}^{n} |x - x_i|^{p_i}, \quad p_i > 0 \text{ and } C < h(x) < 1/C \tag{1.3}$$

for some $C > 0$ and all $x \in \mathbb{R}^2$. The authors in [2,13] proved that there exists $a^* > 0$ such that the constrained minimization problem

$$e(a) := \inf\left\{ E_a(u) : u \in \mathcal{H} \text{ and } \int_{\mathbb{R}^2} u^2 dx = 1 \right\} \tag{1.4}$$

has at least one minimizer if and only if $a \in [0, a^*)$. Moreover,

$$a^* = \int_{\mathbb{R}^2} |Q(x)|^2 dx, \tag{1.5}$$

and $Q(x)$ is the unique positive solution (up to translations) of the scalar field equation

$$-\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^2, \ u \in H^1(\mathbb{R}^2). \tag{1.6}$$

The existence of $Q(x)$ is well known and $Q(x)$ is actually radially symmetric, see e.g. [9,19,20,27].

In what follows, we call $e(a)$ the GP energy, which is also the least energy of a BEC system. As mentioned in [13], the parameter a in (1.1) has to be interpreted as the particle number times the interaction strength, the existence of the threshold value a^* described above shows that there exists a critical particle number for collapse of the BEC [6]. Theorem 1 of [13] implies that the shape of trapping potential does not affect the critical particle number. However, the behavior of the minimizers for (1.4) as $a \nearrow a^*$ does depend on the shape of potentials. In fact, for the trapping potential (1.3), a detailed description of the behavior of the minimizers for (1.4) is given in Theorem 2 of [13], which shows that minimizers of (1.4) must concentrate at one of the flattest minima x_{i_0} $(1 \le i \le n)$ of $V(x)$ as $a \nearrow a^*$. This also implies the presence of symmetry breaking of the minimizer. Note that the method of [13] depends heavily on the potential $V(x)$ of (1.3) having a finite number of minima $\{x_i \in \mathbb{R}^2, \ i = 1, \cdots, n\}$.

It is natural to ask what would happen if $V(x)$ has infinitely many minima. Hence, in this paper we are mainly interested in studying the GP functional with a trapping potential $V(x)$ with infinitely many minima and analyzing the detailed behavior of its minimizers as $a \nearrow a^*$. For this purpose, we focus on the following ring-shaped trapping potential:

$$V(x) = (|x| - A)^2, \quad \text{where } A > 0, \ x \in \mathbb{R}^2, \tag{1.7}$$

which is essentially an important potential used in BEC experiments, see e.g. [15,16,31]. Clearly, all points in the set $\{x \in \mathbb{R}^2 : |x| = A\}$ are minima of the potential given by (1.7). Concerning the existence of minimizers of problem (1.4), much more general potentials $V(x)$ than (1.7) are allowed, see [13, Theorem 1]. But to demonstrate clearly that symmetry breaking does occur in the minimizers of problem (1.4), the uniqueness of the minimizers of problem (1.4) is used in our Corollary 1.4. So, we first give the theorem as follows.

Theorem 1.1. *Let a^* be given by (1.5), and let $V(x)$ be such that*

$$0 \le V(x) \in L^\infty_{\text{loc}}(\mathbb{R}^2), \quad \lim_{|x|\to\infty} V(x) = \infty \quad \text{and} \quad \inf_{x\in\mathbb{R}^2} V(x) = 0.$$

Then

- **(i)** *For all $a \in [0, a^*)$ Eq. (1.4) has at least one minimizer, and there is no minimizer for (1.4) if $a \ge a^*$. Moreover, $e(a) > 0$ if $a < a^*$, $\lim_{a\nearrow a^*} e(a) = e(a^*) = 0$ and $e(a) = -\infty$ if $a > a^*$.*
- **(ii)** *When $a \in [0, a^*)$ is suitably small, Eq. (1.4) has a unique non-negative minimizer in $\mathcal{H}$.*

Part (i) of the above theorem is just Theorem 1 of [13]. For part (ii), a proof based on an implicit function theorem is given in Appendix A.

To analyze the detailed behavior of the minimizers for problem (1.4), a delicate estimate on the GP functional is required. As far as we know, it is usually not easy to derive directly the optimal energy estimates for the GP functional (1.1) under general trapping potentials. Although the authors in [13] developed an approach to establish this kind of energy estimates for the potential (1.3), it does not work well for our potential (1.7). In fact, by following the method of [13] we are only able to get the following type of estimates

$$C_1(a^* - a)^{\frac{2}{3}} \le e(a) \le C_2(a^* - a)^{\frac{1}{2}} \quad \text{as } a \nearrow a^*, \tag{1.8}$$

see Lemma 2.1 in Section 2. Therefore, one of the aims of the paper is to provide some new ways to estimate precisely the GP energy under the potential (1.7), which may be used effectively to handle some general type potentials. Based on the estimates, we may improve the power $\frac{2}{3}$ at the left of (1.8) to be the same as that at the right, namely $\frac{1}{2}$, see our Theorem 2.1 for the details. Then, we may continue to analyze in detail the behavior of the minimizers of (1.4), and we finally have the following theorem.

Theorem 1.2. *Let $V(x)$ be given by (1.7) and let u_a be a non-negative minimizer of (1.4) for $a < a^*$. For any given sequence $\{a_k\}$ with $a_k \nearrow a^*$ as $k \to \infty$, there exists a subsequence, still denoted by $\{a_k\}$, such that each u_{a_k} has a unique maximum point x_k and $x_k \to y_0$ as $k \to \infty$ for some $y_0 \in \mathbb{R}^2$ satisfying $|y_0| = A > 0$. Moreover,*

$$\lim_{k\to\infty} \frac{|x_k| - A}{(a^* - a_k)^{\frac{1}{4}}} = 0, \tag{1.9}$$

and

$$(a^* - a_k)^{\frac{1}{4}} u_{a_k}\big(x_k + (a^* - a_k)^{\frac{1}{4}}\, x\big) \xrightarrow{k} \frac{\lambda_0 Q(\lambda_0 x)}{\|Q\|_2} \quad \text{strongly in } H^1(\mathbb{R}^2), \tag{1.10}$$

where $\lambda_0 > 0$ satisfies

$$\lambda_0 = \left(\frac{1}{2}\int_{\mathbb{R}^2} |x|^2 Q^2(x)dx\right)^{\frac{1}{4}}. \tag{1.11}$$

As we mentioned above, our Theorem 2.1 gives the optimal power of the estimates of the GP energy $e(a)$ as $a \nearrow a^*$. Can we determine precisely the coefficients of the estimates in Theorem 2.1? Our following theorem answers the question.

Theorem 1.3. *Let $V(x)$ be given by (1.7), then the GP energy $e(a)$ satisfies*

$$\lim_{a\nearrow a^*} \frac{e(a)}{(a^* - a)^{\frac{1}{2}}} = \frac{2\lambda_0^2}{\|Q\|_2^2} \tag{1.12}$$

where λ_0 is given by (1.11).

Since the potential $V(x)$ in (1.7) has infinitely many global minima, the method used in [13] cannot be applied directly in our case. For this reason, we have to introduce some new tricks to prove Theorem 1.2. Moreover, there are also some new difficulties to be overcome in finding the exact value of λ_0 in Theorem 1.2. Noting that the trapping potential $V(x)$ of (1.7) is radially symmetric, it then follows from Theorem 1.1(ii) that $e(a)$ has a unique non-negative minimizer which is also radially symmetric for small $a > 0$. On the other hand, Theorem 1.2 shows that any non-negative minimizer of $e(a)$ concentrates at a point on the ring $\{x \in \mathbb{R}^2 : |x| = A\}$ as $a \nearrow a^*$, and thus it cannot be radially symmetric. This implies that, as the strength of the interaction a increases from 0 to a^*, symmetry breaking occurs in the minimizers of $e(a)$. Therefore, the above arguments yield immediately the following corollary.

Corollary 1.4. *Let $V(x)$ be given by (1.7). Then there exist $a_* > 0$ and $a_{**} > 0$ satisfying $a_{**} \le a_* < a^*$ such that*

(i) *$e(a)$ has a unique non-negative minimizer which is radially symmetric about the origin if $a \in [0, a_{**})$.*
(ii) *$e(a)$ has infinitely many different non-negative minimizers, which are not radially symmetric if $a \in [a_*, a^*)$.*

We end this section by recalling some useful information related to the unique positive solution $Q = Q(|x|)$ of (1.6). Taking $N = 2$ in (I.2) of [35], we then have the following Gagliardo–Nirenberg inequality

$$\int_{\mathbb{R}^2} |u(x)|^4 dx \le \frac{2}{\|Q\|_2^2} \left(\int_{\mathbb{R}^2} |\nabla u(x)|^2 dx \right) \int_{\mathbb{R}^2} |u(x)|^2 dx, \quad u \in H^1(\mathbb{R}^2), \tag{1.13}$$

which can be an equality when $u(x) = Q(|x|)$. Since Q is a solution of (1.6), it is easy to see that

$$\int_{\mathbb{R}^2} |\nabla Q|^2 dx = \int_{\mathbb{R}^2} |Q|^2 dx = \frac{1}{2} \int_{\mathbb{R}^2} |Q|^4 dx, \tag{1.14}$$

see also [3, Lemma 8.1.2] for the details. Furthermore, by the results of [9, Proposition 4.1], we know that

$$Q(x),\ |\nabla Q(x)| = O\left(|x|^{-\frac{1}{2}} e^{-|x|}\right) \quad \text{as } |x| \to \infty. \tag{1.15}$$

Throughout the paper, we denote the norm of $L^p(\mathbb{R}^2)$ by $\|\cdot\|_p$ for $p \in (1, +\infty)$, and define the norms of the real-valued function spaces $\mathcal{H}$ and $H^1(\mathbb{R}^2)$ by

$$\|u\|_{\mathcal{H}}^2 = \int_{\mathbb{R}^2} \left(|\nabla u|^2 + V(x)|u|^2\right) dx \quad \text{for } u \in \mathcal{H},$$

and

$$\|u\|^2 = \int_{\mathbb{R}^2} \left(|\nabla u|^2 + |u|^2\right) dx \quad \text{for } u \in H^1(\mathbb{R}^2), \text{ respectively.}$$

Also, the scalar product of $\mathcal{H}$ is given by

$$\langle u, v\rangle_{\mathcal{H},\mathcal{H}} = \int_{\mathbb{R}^2} \left(\nabla u \nabla v + V(x) uv\right) dx \quad \text{for any real-valued functions } u, v \in \mathcal{H}.$$

This paper is organized as follows: in Section 2 we first establish some preparatory energy estimates and then prove our Theorem 2.1 which gives the refined estimates of the energy $e(a)$. Theorems 1.2 and 1.3 are showed in Section 3, where the phenomena of concentration and symmetry breaking of the minimizers of (1.4) are also discussed. Finally, by using an implicit function theorem, Theorem 1.1(ii) is proved in Appendix A.

2. Estimates in the energy $e(a)$ as $a \nearrow a^*$

In this section, we mainly establish the following estimates on the energy $e(a)$.

Theorem 2.1. *Let $V(x)$ be given by (1.7). Then, there exist two positive constants C_1 and C_2, independent of a, such that*

$$C_1\big(a^*-a\big)^{\frac{1}{2}} \le e(a) \le C_2\big(a^*-a\big)^{\frac{1}{2}} \quad \text{as } a \nearrow a^*. \tag{2.1}$$

In order to prove Theorem 2.1, we first give a rough estimates as (1.8) for the energy $e(a)$ of (1.4) by using some ideas of [13]. Based on these estimates, some detailed properties of the minimizers of $e(a)$ can be obtained. Finally, we can complete the proof of Theorem 2.1.

Lemma 2.1. *Let $V(x)$ be given by (1.7). Then, there exist two positive constants C_1 and C_2, independent of a, such that*

$$C_1\big(a^*-a\big)^{\frac{2}{3}} \le e(a) \le C_2\big(a^*-a\big)^{\frac{1}{2}} \quad \text{as } a \nearrow a^*. \tag{2.2}$$

Proof. For any $\lambda > 0$ and $u \in \mathcal{H}$ with $\|u\|_2^2 = 1$, using (1.13),

$$\begin{aligned}
E_a(u) &\ge \int_{\mathbb{R}^2} (|x|-A)^2 \big|u(x)\big|^2 dx + \frac{a^*-a}{2} \int_{\mathbb{R}^2} \big|u(x)\big|^4 dx \\
&= \lambda + \int_{\mathbb{R}^2} \big[(|x|-A)^2 - \lambda\big] \big|u(x)\big|^2 dx + \frac{a^*-a}{2} \int_{\mathbb{R}^2} \big|u(x)\big|^4 dx \\
&\ge \lambda - \frac{1}{2(a^*-a)} \int_{\mathbb{R}^2} \big[\lambda - (|x|-A)^2\big]_+^2 dx,
\end{aligned} \tag{2.3}$$

where $A > 0$ and $[\cdot]_+ = \max\{0, \cdot\}$ denotes the positive part. For $\lambda > 0$ small enough, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} \big[\lambda - (|x|-A)^2\big]_+^2 dx &= 2\pi \int_{A-\sqrt{\lambda}}^{A+\sqrt{\lambda}} \big[\lambda - (r-A)^2\big]^2 r\,dr \\
&= 2\pi\lambda^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\beta (A + \sqrt{\lambda}\sin\beta)\sqrt{\lambda}\cos\beta\, d\beta \le C\lambda^{\frac{5}{2}},
\end{aligned}$$

where we change the variable $r = A + \sqrt{\lambda}\sin\beta$ with $-\frac{\pi}{2} \le \beta \le \frac{\pi}{2}$ in the second identity. The lower estimate of (2.2) therefore follows from the above estimate and (2.3) by taking $\lambda = [4(a^*-a)/(5C)]^{2/3}$ and $a \nearrow a^*$.

We next prove the upper estimate of (2.2) as follows. For this purpose, we let

$$u(x) = \frac{\tau}{\|Q\|_2} Q\big(\tau(x-x_0)\big), \quad \text{for any } \tau > 0. \tag{2.4}$$

Then $\int_{\mathbb{R}^2} u^2(x)dx = 1$, and it follows from (1.14) that

$$\begin{aligned}
\int_{\mathbb{R}^2} \big|\nabla u(x)\big|^2 dx - \frac{a}{2}\int_{\mathbb{R}^2} u^4(x)dx &= \frac{\tau^2}{\|Q\|_2^2}\Big[\int_{\mathbb{R}^2} \big|\nabla Q(x)\big|^2 dx - \frac{a}{2\|Q\|_2^2}\int_{\mathbb{R}^2} Q^4(x)dx\Big] \\
&= \frac{\tau^2}{2\|Q\|_2^2}\Big[\Big(1 - \frac{a}{\|Q\|_2^2}\Big)\int_{\mathbb{R}^2} Q^4(x)dx\Big].
\end{aligned} \tag{2.5}$$

Moreover, by the exponential decay of (1.15), we have

$$\int_{\mathbb{R}^2} (|x|-A)^2|u|^2dx = \frac{1}{\|Q\|_2^2}\int_{\mathbb{R}^2}\left(\left|\frac{x}{\tau}+x_0\right|-|x_0|\right)^2 Q^2(x)dx$$
$$\le \frac{1}{\|Q\|_2^2}\int_{\mathbb{R}^2}\left|\frac{x}{\tau}\right|^2 Q^2(x)dx = \frac{C}{\tau^2}. \tag{2.6}$$

It then follows from (2.5) and (2.6) that

$$e(a) \le C\left(a^*-a\right)\tau^2 + \frac{C}{\tau^2}.$$

By taking $\tau = (a^*-a)^{-\frac{1}{4}}$, the above inequality implies the desired upper estimate of (2.2). □

Motivated by [13, Lemma 4], we have the following lemma.

Lemma 2.2. *Let $V(x)$ be given by (1.7) and suppose u_a is a non-negative minimizer of (1.4), then there exists a positive constant K, independent of a, such that*

$$0 < K\left(a^*-a\right)^{-\frac{1}{4}} \le \int_{\mathbb{R}^2} |u_a|^4 dx \le \frac{1}{K}\left(a^*-a\right)^{-\frac{1}{2}} \quad \text{as } a \nearrow a^*. \tag{2.7}$$

Proof. Noting from (2.3) that

$$e(a) = E_a(u_a) \ge \frac{a^*-a}{2}\int_{\mathbb{R}^2}\left|u_a(x)\right|^4 dx,$$

and the upper bound of (2.7) then follows from Lemma 2.1.

To prove the lower bound of (2.7), we choose $0 < b < a < a^*$ so that

$$e(b) \le E_b(u_a) = e(a) + \frac{a-b}{2}\int_{\mathbb{R}^2}\left|u_a(x)\right|^4 dx.$$

It then follows from Lemma 2.1 that

$$\frac{1}{2}\int_{\mathbb{R}^2}\left|u_a(x)\right|^4 dx \ge \frac{e(b)-e(a)}{a-b} \ge \frac{C_1(a^*-b)^{\frac{2}{3}} - C_2(a^*-a)^{\frac{1}{2}}}{a-b}.$$

Taking $b = a - C_0(a^*-a)^{\frac{3}{4}}$, where $C_0 > 0$ is large enough such that $C_1C_0^{\frac{2}{3}} > 2C_2$, we then obtain from the above inequality that

$$\int_{\mathbb{R}^2} |u_a|^4 dx \ge C\left(a^*-a\right)^{-\frac{1}{4}},$$

which therefore implies the lower bound of (2.7). □

Remark 2.1. Once our Theorem 2.1 is proved, then we can improve the power $-1/4$ to $-1/2$ in estimate of left hand side of (2.7). In fact, by applying Theorem 2.1, instead of using Lemma 2.1 in the proof of Lemma 2.2, and taking $b = a - C_0(a^*-a)$, we then have

$$\int_{\mathbb{R}^2} |u_a|^4 dx \ge C\left(a^*-a\right)^{-\frac{1}{2}}.$$

This will be used in Section 3.

Lemma 2.3. *For $V(x)$ satisfying (1.7), let u_a be a non-negative minimizer of (1.4), and set*

$$\epsilon_a^{-2} := \int_{\mathbb{R}^2} \left|\nabla u_a(x)\right|^2 dx. \tag{2.8}$$

Then

(i) $\epsilon_a \to 0$ *as* $a \nearrow a^*$.

(ii) *There exist a sequence* $\{y_{\epsilon_a}\} \subset \mathbb{R}^2$ *and positive constants* R_0, η *such that the sequence*

$$w_a(x) := \epsilon_a u_a(\epsilon_a x + \epsilon_a y_{\epsilon_a}) \tag{2.9}$$

satisfies

$$\liminf_{a \nearrow a^*} \int_{B_{R_0}(0)} |w_a|^2 dx \geq \eta > 0. \tag{2.10}$$

(iii) *The sequence* $\{\epsilon_a y_{\epsilon_a}\}$ *is bounded uniformly for* $\epsilon_a \to 0$. *Moreover, for any sequence* $\{a_k\}$ *with* $a_k \nearrow a^*$, *there exists a convergent subsequence, still denoted by* $\{a_k\}$, *such that*

$$\bar{x}_k := \epsilon_{a_k} y_{\epsilon_{a_k}} \to x_0 \quad \text{as } a_k \nearrow a^* \tag{2.11}$$

for some $x_0 \in \mathbb{R}^2$ *being a global minimum point of* $V(x)$, *i.e.,* $|x_0| = A > 0$. *Furthermore, we also have*

$$w_{a_k} \xrightarrow{k} \frac{\beta_1}{\|Q\|_2} Q(\beta_1|x - \bar{y}_0|) \quad \text{in } H^1(\mathbb{R}^2) \text{ for some } \bar{y}_0 \in \mathbb{R}^2 \text{ and } \beta_1 > 0. \tag{2.12}$$

Proof. (i): Applying (1.13), it follows from Lemma 2.1 that

$$\int_{\mathbb{R}^2} V(x)\left|u_a(x)\right|^2 dx \leq e(a) \leq C_1\left(a^* - a\right)^{\frac{1}{2}} \quad \text{as } a \nearrow a^*, \tag{2.13}$$

and

$$0 \leq \int_{\mathbb{R}^2} \left|\nabla u_a(x)\right|^2 dx - \frac{a}{2}\int_{\mathbb{R}^2} \left|u_a(x)\right|^4 dx = \epsilon_a^{-2} - \frac{a}{2}\int_{\mathbb{R}^2} \left|u_a(x)\right|^4 dx \leq e(a) \xrightarrow{a \nearrow a^*} 0.$$

By Lemma 2.2,

$$\int_{\mathbb{R}^2} \left|u_a(x)\right|^4 dx \to +\infty \quad \text{as } a \nearrow a^*,$$

then we see that

$$0 \leq \frac{\epsilon_a^{-2}}{\int_{\mathbb{R}^2} |u_a(x)|^4 dx} - \frac{a}{2} \leq \frac{e(a)}{\int_{\mathbb{R}^2} |u_a(x)|^4 dx} \xrightarrow{a \nearrow a^*} 0,$$

i.e.,

$$\frac{\epsilon_a^{-2}}{\int_{\mathbb{R}^2} |u_a(x)|^4 dx} \to \frac{a^*}{2} \quad \text{as } a \nearrow a^*.$$

So, by taking $m = \max\{\frac{4}{a^*}, \frac{3a^*}{4}\}$ we have

$$0 < \frac{1}{m}\epsilon_a^{-2} \leq \int_{\mathbb{R}^2} \left|u_a(x)\right|^4 dx \leq m\epsilon_a^{-2} \quad \text{as } a \nearrow a^*, \tag{2.14}$$

this and (2.7) imply that there exist $C_2 > 0$ and $C_3 > 0$ such that

$$C_2\big(a^*-a\big)^{-\frac{1}{4}} \le \int_{\mathbb{R}^2} \big|\nabla u_a(x)\big|^2 dx \le C_3\big(a^*-a\big)^{-\frac{1}{2}} \quad \text{as } a \nearrow a^*. \tag{2.15}$$

Hence, $\epsilon_a \to 0$ as $a \nearrow a^*$, and part **(i)** is proved.

(ii): Let

$$\tilde{w}_a(x) := \epsilon_a u_a(\epsilon_a x). \tag{2.16}$$

From (2.8) and (2.14), we see that

$$\int_{\mathbb{R}^2} |\nabla \tilde{w}_a|^2 dx = \int_{\mathbb{R}^2} |\tilde{w}_a|^2 dx = 1, \qquad \frac{1}{m} \le \int_{\mathbb{R}^2} |\tilde{w}_a|^4 dx \le m. \tag{2.17}$$

We claim that there exist a sequence $\{y_{\epsilon_a}\} \subset \mathbb{R}^2$ and $R_0 > 0$, $\eta > 0$ such that

$$\liminf_{\epsilon_a \to 0} \int_{B_{R_0}(y_{\epsilon_a})} |\tilde{w}_a|^2 dx \ge \eta > 0. \tag{2.18}$$

We argue this by contradiction. If (2.18) is not true, then, for any $R > 0$, there exists a sequence $\{\tilde{w}_{a_k}\}$ with $a_k \nearrow a^*$ such that

$$\lim_{k\to\infty} \sup_{y \in \mathbb{R}^2} \int_{B_R(y)} |\tilde{w}_{a_k}|^2 dx = 0.$$

By Lemma I.1 in [26] (or, Theorem 8.10 in [21]), we know that $\tilde{w}_{a_k} \overset{k}{\to} 0$ in $L^p(\mathbb{R}^2)$ for all $p \in (2, +\infty)$, which however contradicts (2.17) if we take $p = 4$. Thus, Eq. (2.18) holds. By applying (2.16) and (2.18), we therefore conclude (2.10), which gives part **(ii)**.

(iii): By (2.13), we see that

$$\int_{\mathbb{R}^2} V(x)\big|u_a(x)\big|^2 dx = \int_{\mathbb{R}^2} V(\epsilon_a x + \epsilon_a y_{\epsilon_a})\big|w_a(x)\big|^2 dx \to 0 \quad \text{as } a \nearrow a^*. \tag{2.19}$$

We first *claim* that

$$\lim_{\epsilon_a \to 0} |\epsilon_a y_{\epsilon_a}| = A.$$

Indeed, if this is false, then there exist a constant $\alpha > 0$ and a subsequence $\{a_n\}$ with $a_n \nearrow a^*$ as $n \to \infty$, such that

$$\epsilon_n := \epsilon_{a_n} \to 0 \quad \text{and} \quad \big||\epsilon_n y_{\epsilon_n}| - A\big| \ge \alpha > 0 \quad \text{as } n \to \infty.$$

Hence,

$$V(\epsilon_n y_{\epsilon_n}) = (|\epsilon_n y_{\epsilon_n}| - A)^2 \ge \alpha^2 > 0 \quad \text{as } n \to \infty.$$

Therefore, it follows from (2.10) and Fatou's Lemma that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} V(\epsilon_n x + \epsilon_n y_{\epsilon_n})\big|w_{a_n}(x)\big|^2 dx \ge \int_{\mathbb{R}^2} \lim_{n\to\infty} V(\epsilon_n x + \epsilon_n y_{\epsilon_n})\big|w_{a_n}(x)\big|^2 dx \ge \frac{\alpha^2}{2}\eta > 0,$$

which contradicts (2.19). So, the above claim is proved. This claim implies that $\{\epsilon_a y_{\epsilon_a}\}$ is bounded uniformly as $\epsilon_a \to 0$, and (2.11) follows from these conclusions.

We now turn to proving (2.12). Since u_a is a non-negative minimizer of (1.4), it satisfies the Euler–Lagrange equation

$$-\Delta u_a(x) + V(x)u_a(x) = \mu_a u_a(x) + a u_a^3(x) \quad \text{in } \mathbb{R}^2, \tag{2.20}$$

where $\mu_a \in \mathbb{R}$ is a Lagrange multiplier, and

$$\mu_a = e(a) - \frac{a}{2} \int_{\mathbb{R}^2} |u_a|^4 dx.$$

It then follows from Lemma 2.1 and (2.14) that there exist two positive constants C_1 and C_2, independent of a, such that

$$-C_2 < \epsilon_a^2 \mu_a < -C_1 < 0 \quad \text{as } a \nearrow a^*.$$

Using (2.20), we know that $w_a(x)$ defined in (2.9) satisfies the elliptic equation

$$-\Delta w_a(x) + \epsilon_a^2 V(\epsilon_a x + \epsilon_a y_{\epsilon_a}) w_a(x) = \epsilon_a^2 \mu_a w_a(x) + a w_a^3(x) \quad \text{in } \mathbb{R}^2. \tag{2.21}$$

Therefore, for the convergent subsequence $\{a_k\}$ obtained in (2.11), we may assume that $\epsilon_k^2 \mu_{a_k} \overset{k}{\to} -\beta_1^2 < 0$ for some $\beta_1 > 0$, and $w_{a_k} \overset{k}{\rightharpoonup} w_0 \geq 0$ weakly in $H^1(\mathbb{R}^2)$ for some $w_0 \in H^1(\mathbb{R}^2)$. Since $\{\epsilon_a y_{\epsilon_a}\}$ is bounded uniformly in ϵ_a, by passing to the weak limit of (2.21), we see that $w_0 \geq 0$ satisfies

$$-\Delta w_0(x) = -\beta_1^2 w_0(x) + a^* w_0^3(x) \quad \text{in } \mathbb{R}^2. \tag{2.22}$$

Furthermore, it follows from (2.10) that $w_0 \not\equiv 0$, and therefore we have $w_0 > 0$ by the strong maximum principle. By a simple rescaling, the uniqueness (up to translations) of positive solutions for the nonlinear scalar field equation (1.6) implies that

$$w_0(x) = \frac{\beta_1}{\|Q\|_2} Q(\beta_1 |x - \bar{y}_0|) \quad \text{for some } \bar{y}_0 \in \mathbb{R}^2, \tag{2.23}$$

where $\|w_0\|_2^2 = 1$. By the norm preservation we further conclude that w_{a_k} converges to w_0 strongly in $L^2(\mathbb{R}^2)$ and in fact, strongly in $L^p(\mathbb{R}^2)$ for any $2 \leq p < \infty$ because of $H^1(\mathbb{R}^2)$ boundedness. Also, since w_{a_k} and w_0 satisfy (2.21) and (2.22), respectively, a simple analysis shows that w_{a_k} converges to w_0 strongly in $H^1(\mathbb{R}^2)$, and thus (2.12) holds. □

Lemma 2.4. *Under the assumptions of Lemma 2.3, let $\{a_k\}$ be the convergent subsequence given by Lemma 2.3(iii). Then, for any $R > 0$, there exists $C_0(R) > 0$, independent of a_k, such that*

$$\lim_{\epsilon_{a_k} \to 0} \frac{1}{\epsilon_{a_k}^2} \int_{B_R(0)} V(\epsilon_{a_k} x + \epsilon_{a_k} y_{\epsilon_{a_k}}) \left| w_{a_k}(x) \right|^2 dx \geq C_0(R). \tag{2.24}$$

Proof. Since $V(x) = (|x| - A)^2$ with $A > 0$, we have

$$V(\epsilon_a x + \epsilon_a y_{\epsilon_a}) = \epsilon_a^2 \left(|x + y_{\epsilon_a}| - \frac{A}{\epsilon_a} \right)^2, \tag{2.25}$$

where the term $|x + y_{\epsilon_a}|$ can be rewritten as

$$|x + y_{\epsilon_a}| = \sqrt{|x|^2 + |y_{\epsilon_a}|^2} \sqrt{1 + \frac{2x \cdot y_{\epsilon_a}}{|x|^2 + |y_{\epsilon_a}|^2}}. \tag{2.26}$$

For the convergent sequence $\{a_k\}$ given by Lemma 2.3(iii), since $\epsilon_k y_{\epsilon_k} \overset{k}{\to} x_0$ with $|x_0| = A > 0$, we have $|y_{\epsilon_k}| \overset{k}{\to} \infty$, and hence $\frac{2x \cdot y_{\epsilon_k}}{|x|^2 + |y_{\epsilon_k}|^2} \overset{k}{\to} 0$ uniformly for $x \in B_R(0)$. Using the Taylor expansion we obtain that

$$\sqrt{1 + \frac{2x \cdot y_{\epsilon_k}}{|x|^2 + |y_{\epsilon_k}|^2}} = 1 + \frac{x \cdot y_{\epsilon_k}}{|x|^2 + |y_{\epsilon_k}|^2} + O\left(\frac{1}{|y_{\epsilon_k}|^2} \right) \quad \text{for all } x \in B_R(0),$$

which, together with (2.25) and (2.26), then implies that

$$\frac{1}{\epsilon_k^2} V(\epsilon_k x + \epsilon_k y_{\epsilon_k}) = \left| \sqrt{|x|^2 + |y_{\epsilon_k}|^2} + \frac{x \cdot y_{\epsilon_k}}{\sqrt{|x|^2 + |y_{\epsilon_k}|^2}} - \frac{A}{\epsilon_k} + O\left(\frac{1}{|y_{\epsilon_k}|} \right) \right|^2. \tag{2.27}$$

For any $x \in \mathbb{R}^2$, let $\arg x$ be the angle between x and the positive x-axis, and $\langle x, y\rangle$ be the angle between the vectors x and y. Without loss of generality, we may assume that $x_0 = (A, 0)$, and it then follows that $\arg y_{\epsilon_k} \xrightarrow{k} 0$. Thus, we can choose $0 < \delta < \frac{\pi}{16}$ small enough such that

$$-\delta < \arg y_{\epsilon_k} < \delta \quad \text{as } \epsilon_k \to 0. \tag{2.28}$$

Denote

$$\begin{aligned}\Omega^1_{\epsilon_k} &= \left\{x \in B_R(0) : \sqrt{|x|^2 + |y_{\epsilon_k}|^2} \le \frac{A}{\epsilon_k}\right\} \\ &= \left\{x \in B_R(0) : |x|^2 \le \left(\frac{A}{\epsilon_k}\right)^2 - |y_{\epsilon_k}|^2\right\},\end{aligned} \tag{2.29}$$

and

$$\begin{aligned}\Omega^2_{\epsilon_k} &= \left\{x \in B_R(0) : \sqrt{|x|^2 + |y_{\epsilon_k}|^2} > \frac{A}{\epsilon_k}\right\} \\ &= \left\{x \in B_R(0) : \left(\frac{A}{\epsilon_k}\right)^2 - |y_{\epsilon_k}|^2 < |x|^2 < R^2\right\},\end{aligned} \tag{2.30}$$

so that $B_R(0) = \Omega^1_{\epsilon_k} \cup \Omega^2_{\epsilon_k}$ and $\Omega^1_{\epsilon_k} \cap \Omega^2_{\epsilon_k} = \emptyset$. Since

$$\left|\Omega^1_{\epsilon_k}\right| + \left|\Omega^2_{\epsilon_k}\right| = \left|B_R(0)\right| = \pi R^2,$$

there exists a subsequence, still denoted by $\{\epsilon_k\}$, of $\{\epsilon_k\}$ such that

$$\text{either} \quad \left|\Omega^1_{\epsilon_k}\right| \ge \frac{\pi R^2}{2} \quad \text{or} \quad \left|\Omega^2_{\epsilon_k}\right| \ge \frac{\pi R^2}{2}.$$

We finish the proof by considering the following two cases:

Case 1: $|\Omega^1_{\epsilon_k}| \ge \frac{\pi R^2}{2}$. In this case, we have $B_{\frac{R}{\sqrt{2}}}(0) \subset \Omega^1_{\epsilon_k}$, and set

$$\Omega_1 := \left(B_{\frac{R}{\sqrt{2}}}(0) \setminus B_{\frac{R}{2}}(0)\right) \cap \left\{x : \frac{\pi}{2} + 2\delta < \arg x < \frac{3\pi}{2} - 2\delta\right\} \subset \Omega^1_{\epsilon_k}.$$

Then

$$|\Omega_1| = \frac{(\pi - 4\delta)}{8} R^2. \tag{2.31}$$

By (2.28), one can easily check that for any $x \in \Omega_1$,

$$x \cdot y_{\epsilon_k} = |x||y_{\epsilon_k}|\cos\langle x, y_{\epsilon_k}\rangle < 0 \quad \text{and} \quad \left|\cos\langle x, y_{\epsilon_k}\rangle\right| > -\cos\left(\frac{\pi}{2} + \delta\right) > 0. \tag{2.32}$$

We thus derive from (2.27) that

$$\begin{aligned}&\sqrt{|x|^2 + |y_{\epsilon_k}|^2} + \frac{x \cdot y_{\epsilon_k}}{\sqrt{|x|^2 + |y_{\epsilon_k}|^2}} - \frac{A}{\epsilon_k} + O\left(\frac{1}{|y_{\epsilon_k}|}\right) \\ &\quad \le \frac{x \cdot y_{\epsilon_k}}{\sqrt{|x|^2 + |y_{\epsilon_k}|^2}} + O\left(\frac{1}{|y_{\epsilon_k}|}\right) \le \frac{x \cdot y_{\epsilon_k}}{2\sqrt{|x|^2 + |y_{\epsilon_k}|^2}} \le \frac{|x||y_{\epsilon_k}|\cos(\frac{\pi}{2} + \delta)}{2\sqrt{|x|^2 + |y_{\epsilon_k}|^2}} < 0 \quad \text{for } x \in \Omega_1.\end{aligned}$$

Noting that $\lim_{\epsilon_k \to 0}|y_{\epsilon_k}| = \infty$, we thus have

$$\frac{1}{\epsilon_k^2} V(\epsilon_k x + \epsilon y_{\epsilon_k}) \ge \frac{\cos^2(\frac{\pi}{2} + \delta)|x|^2}{8} \quad \text{for } x \in \Omega_1. \tag{2.33}$$

Taking $\delta = \frac{\pi}{20}$, the above estimate implies that

$$\lim_{\epsilon_k \to 0} \frac{1}{\epsilon_k^2} \int\limits_{B_R} V(\epsilon_k x + \epsilon_k y_{\epsilon_k}) |w_a(x)|^2 dx \ge \lim_{\epsilon_k \to 0} \frac{1}{\epsilon_k^2} \int\limits_{\Omega_1} V(\epsilon_k x + \epsilon_k y_{\epsilon_k}) |w_a(x)|^2 dx$$
$$\ge \frac{\cos^2 \frac{11\pi}{20}}{8} \int\limits_{\Omega_1} |x|^2 |w_0(x)|^2 dx := C(R) > 0, \tag{2.34}$$

and then (2.24) is proved.

Case 2: $|\Omega^2_{\epsilon_k}| \ge \frac{\pi R^2}{2}$. In this case, we deduce that the annular region $D_R := B_R \setminus B_{\frac{R}{\sqrt{2}}} \subset \Omega^2_{\epsilon_k}$. Set $\Omega_2 := D_R \cap \{x; -\frac{\pi}{2} + 2\delta < \arg x < \frac{\pi}{2} - 2\delta\} \subset \Omega^2_{\epsilon_k}$. Then,

$$|\Omega_2| = \frac{(\pi - 4\delta)}{4} R^2. \tag{2.35}$$

One can check that for any $x \in \Omega_2$,

$$x \cdot y_{\epsilon_k} = |x||y_{\epsilon_k}| \cos\langle x, y_{\epsilon_k}\rangle > 0 \quad \text{and} \quad \cos\langle x, y_{\epsilon_k}\rangle > \cos\left(\frac{\pi}{2} - \delta\right) > 0. \tag{2.36}$$

It then follows from (2.27) and (2.36) that

$$\sqrt{|x|^2 + |y_{\epsilon_k}|^2} + \frac{x \cdot y_{\epsilon_k}}{\sqrt{|x|^2 + |y_{\epsilon_k}|^2}} - \frac{A}{\epsilon_k} + O\left(\frac{1}{|y_{\epsilon_k}|}\right)$$
$$\ge \frac{x \cdot y_{\epsilon_k}}{\sqrt{|x|^2 + |y_{\epsilon_k}|^2}} + O\left(\frac{1}{|y_{\epsilon_k}|}\right) \ge \frac{x \cdot y_{\epsilon_k}}{2\sqrt{|x|^2 + |y_{\epsilon_k}|^2}} \ge \frac{|x||y_{\epsilon_k}| \cos(\frac{\pi}{2} - \delta)}{2\sqrt{|x|^2 + |y_{\epsilon_k}|^2}} > 0 \quad \text{for } x \in \Omega_2.$$

Hence

$$\frac{1}{\epsilon_k^2} V(\epsilon_k x + \epsilon_k y_{\epsilon_k}) \ge \frac{\cos^2(\frac{\pi}{2} - \delta)|x|^2}{8} \quad \text{for } x \in \Omega_2.$$

Thus, by taking $\delta = \frac{\pi}{20}$, the above estimate gives that

$$\lim_{\epsilon_k \to 0} \frac{1}{\epsilon_k^2} \int\limits_{B_R} V(\epsilon_k x + \epsilon_k y_{\epsilon_k}) |w_a(x)|^2 dx \ge \lim_{\epsilon_k \to 0} \frac{1}{\epsilon_k^2} \int\limits_{\Omega_2} V(\epsilon_k x + \epsilon_k y_{\epsilon_k}) |w_a(x)|^2 dx$$
$$\ge \frac{\cos^2 \frac{9\pi}{20}}{8} \int\limits_{\Omega_2} |x|^2 |w_0(x)|^2 dx := C_0(R) > 0. \tag{2.37}$$

Therefore, Eq. (2.24) also follows from (2.34) and (2.37) in this case. □

We end this section by proving Theorem 2.1, which gives the refined estimates for $e(a)$.

Proof of Theorem 2.1. By Lemma 2.1, it suffices to prove that there exists a positive $C > 0$, independent of a, such that

$$e(a) \ge C(a^* - a)^{\frac{1}{2}} \quad \text{as } a \nearrow a^*. \tag{2.38}$$

In fact, by the proof of Lemma 2.3(iii), we see that for any sequence $\{a_k\}$ with $a_k \nearrow a^*$, there exists a convergent subsequence, still denoted by $\{a_k\}$, such that $w_{a_k} \to w_0 > 0$ strongly in $L^4(\mathbb{R}^2)$, where w_0 satisfies (2.23). This implies that there exists a constant $M_1 > 0$, independent of a_k, such that

$$\int\limits_{\mathbb{R}^2} |w_{a_k}(x)|^4 dx \ge M_1 \quad \text{as } a_k \nearrow a^*.$$

Moreover, applying (2.24) with $R = 1$ yields that there exists a constant $M_2 > 0$, independent of a_k, such that

$$\int_{B_1(0)} V(\epsilon_k x+\epsilon_k y_{\epsilon_k})\big|w_{a_k}(x)\big|^2dx \ge M_2\epsilon_k^2 \quad \text{as } a_k \nearrow a^*.$$

Thus,

$$\begin{aligned} e(a_k)=E_{a_k}(u_{a_k}) &= \frac{1}{\epsilon_k^2}\Big[\int_{\mathbb{R}^2}\big|\nabla w_{a_k}(x)\big|^2dx-\frac{a^*}{2}\int_{\mathbb{R}^2}\big|w_{a_k}(x)\big|^4dx\Big] \\ &\quad +\frac{a^*-a_k}{2\epsilon_k^2}\int_{\mathbb{R}^2}\big|w_{a_k}(x)\big|^4dx+\int_{\mathbb{R}^2}V(\epsilon_k x+\epsilon_k y_{\epsilon_k})\big|w_{a_k}(x)\big|^2dx \\ &\ge \frac{a^*-a_k}{2\epsilon_k^2}M_1+M_2\epsilon_k^2 \ge \sqrt{2M_1M_2}\big(a^*-a_k\big)^{\frac12} \quad \text{as } a_k\nearrow a^*, \end{aligned} \tag{2.39}$$

and (2.38) therefore holds for the subsequence $\{a_k\}$.

Actually, the above argument can be carried out for any subsequence $\{a_k\}$ satisfying $a_k \nearrow a^*$, which then implies that (2.38) holds for all $a \nearrow a^*$. This completes the proof of Theorem 2.1. □

3. Mass concentration and symmetry breaking

In this section we complete the proof of Theorems 1.2 and 1.3 under the ring-shaped potential $V(x)=(|x|-A)^2$ with $A>0$, which addresses the mass concentration and symmetry breaking of minimizers as $a\nearrow a^*$. Let u_a be a non-negative minimizer of (1.4). By Remark 2.1, it is easy to see that there exists a constant $M>0$, independent of a, such that

$$0<M\big(a^*-a\big)^{-\frac12} \le \int_{\mathbb{R}^2}|u_a|^4dx \le \frac{1}{M}\big(a^*-a\big)^{-\frac12} \quad \text{as } a\nearrow a^*. \tag{3.1}$$

Stimulated by above estimates, we define

$$\varepsilon_a := \big(a^*-a\big)^{\frac14}>0. \tag{3.2}$$

From (1.13) we conclude that

$$e(a)\ge \Big(1-\frac{a}{a^*}\Big)\int_{\mathbb{R}^2}\big|\nabla u_a(x)\big|^2dx+\int_{\mathbb{R}^2}(|x|-A)^2u_a^2(x)dx,$$

and it hence follows from Theorem 2.1 that

$$\int_{\mathbb{R}^2}\big|\nabla u_a(x)\big|^2dx \le C\varepsilon_a^{-2} \quad \text{and} \quad \int_{\mathbb{R}^2}(|x|-A)^2u_a^2(x)dx\le C\varepsilon_a^2. \tag{3.3}$$

Similar to Lemma 2.3(ii), for ε_a given by (3.2), we know that there exist a sequence $\{y_{\varepsilon_a}\}\subset\mathbb{R}^2$ and positive constants R_0 and η such that

$$\liminf_{a\nearrow a^*}\int_{B_{R_0}(0)}|w_a|^2dx\ge\eta>0, \tag{3.4}$$

where we define the $L^2(\mathbb{R}^2)$-normalized function

$$w_a(x)=\varepsilon_a u_a(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}). \tag{3.5}$$

Note from (3.1) and (3.3) that

$$\int_{\mathbb{R}^2}|\nabla w_a|^2dx\le C, \qquad M\le\int_{\mathbb{R}^2}|w_a|^4dx\le\frac{1}{M}, \tag{3.6}$$

where the positive constants C and M are independent of a.

Lemma 3.1. *For any given sequence* $\{a_k\}$ *with* $a_k \nearrow a^*$, *let* $\varepsilon_k := \varepsilon_{a_k} = (a^* - a_k)^{\frac{1}{4}} > 0$, $u_k(x) := u_{a_k}(x)$ *be a non-negative minimizer of (1.4), and* $w_k := w_{a_k} \geq 0$ *be defined by (3.5). Then, there is a subsequence, still denoted by* $\{a_k\}$, *such that*

$$z_k := \varepsilon_k y_{\varepsilon_k} \xrightarrow{k} y_0 \quad \text{for some } y_0 \in \mathbb{R}^2 \text{ and } |y_0| = A. \tag{3.7}$$

Moreover, for any $\delta > 0$ *small enough, we have*

$$u_k(x) = \frac{1}{\varepsilon_k} w_k\Big(\frac{x - z_k}{\varepsilon_k}\Big) \xrightarrow{k} 0, \quad \forall x \in B_\delta^c(y_0). \tag{3.8}$$

Proof. By (2.20) and (3.5), we see that w_k satisfies

$$-\Delta w_k(x) + \varepsilon_k^2(|\varepsilon_k x + \varepsilon_k y_{\varepsilon_k}| - A)^2 w_k(x) = \mu_k \varepsilon_k^2 w_k(x) + a_k w_k^3(x) \quad \text{in } \mathbb{R}^2, \tag{3.9}$$

where $\mu_k \in \mathbb{R}^2$ is a Lagrange multiplier. Similar to the proof of Lemma 2.3(iii), we can prove that there exists a subsequence of $\{w_k\}$, still denoted by $\{w_k\}$, such that (3.7) holds and $w_k \xrightarrow{k} w_0$ strongly in $H^1(\mathbb{R}^2)$ for some positive function w_0 satisfying

$$-\Delta w_0(x) = -\beta^2 w_0(x) + a^* w_0^3(x) \quad \text{in } \mathbb{R}^2, \tag{3.10}$$

where $\beta > 0$ is a positive constant. Hence, for any $\alpha > 2$,

$$\int\limits_{|x|\geq R} |w_k|^\alpha dx \to 0 \quad \text{as } R \to \infty \text{ uniformly for large } k. \tag{3.11}$$

Note from (3.9) that $-\Delta w_k - c(x) w_k \leq 0$, where $c(x) = a_k w_k^2(x)$. By applying De Giorgi–Nash–Moser theory, see e.g. [17, Theorem 4.1], we have

$$\max_{B_1(\xi)} w_k \leq C\Big(\int\limits_{B_2(\xi)} |w_k|^\alpha dx\Big)^{\frac{1}{\alpha}},$$

where ξ is an arbitrary point in $\mathbb{R}^2$, and C is a constant depending only on the bound of $\|w_k\|_{L^\alpha(B_2(\xi))}$. We hence deduce from (3.11) that

$$w_k(x) \to 0 \quad \text{as } |x| \to \infty \text{ uniformly in } k. \tag{3.12}$$

Since w_k satisfies (3.9), one can use the comparison principle as in [18] to compare w_k with $Ce^{-\frac{\beta}{2}|x|}$, which then shows that there exists a large constant $R > 0$, independent of k, such that

$$w_k(x) \leq Ce^{-\frac{\beta}{2}|x|} \quad \text{for } |x| > R \text{ as } k \to \infty. \tag{3.13}$$

For any $x \in B_\delta^c(y_0)$, it then follows from (3.7) that

$$\frac{|x - z_k|}{\varepsilon_k} \geq \frac{1}{2}\frac{|x - y_0|}{\varepsilon_k} \geq \frac{\delta}{2\varepsilon_k} \xrightarrow{k} +\infty,$$

which, together with (3.13), yields that

$$u_k(x) = \frac{1}{\varepsilon_k} w_k\Big(\frac{x - z_k}{\varepsilon_k}\Big) \leq \frac{1}{\varepsilon_k} e^{-\frac{\beta\delta}{2\varepsilon_k}} \xrightarrow{k} 0, \quad \forall x \in B_\delta^c(y_0),$$

i.e., Eq. (3.8) holds. □

Remark 3.1. The estimate (3.8) shows that, for any $x \in \mathbb{R}^2$ and out of any small ball centered at y_0, $u_k(x)$ vanishes as $k \to \infty$. That is, Eq. (3.8) implies that, for any given sequence $\{x_k\} \subset \mathbb{R}^2$, if there exists $\alpha > 0$ such that $u_k(x_k) \geq \alpha > 0$, then, $x_k \to y_0$ as $k \to \infty$. This fact is used to prove (3.15) below.

We are now ready to prove Theorem 1.2, which is partially motivated by [13,34].

Proof of Theorem 1.2. We still set $\varepsilon_k = (a^* - a_k)^{\frac{1}{4}} > 0$, where $a_k \nearrow a^*$, and $u_k(x) := u_{a_k}(x)$ is a non-negative minimizer of (1.4). We start the proof by establishing first the detailed concentration behavior of u_k.

Let $\bar{z}_k$ be any local maximum point of u_k. It then follows from (2.20) that

$$u_k(\bar{z}_k) \geq \left(\frac{-\mu_k}{a_k}\right)^{\frac{1}{2}} \geq C\varepsilon_k^{-1}. \tag{3.14}$$

This estimate and (3.8) (see Remark 3.1) imply that, by passing to a subsequence,

$$\bar{z}_k \overset{k}{\to} y_0 \in \mathbb{R}^2 \quad \text{with } |y_0| = A. \tag{3.15}$$

Set

$$\bar{w}_k = \varepsilon_k u_k(\varepsilon_k x + \bar{z}_k). \tag{3.16}$$

It then follows from (3.9) that

$$-\Delta \bar{w}_k(x) + \varepsilon_k^2(|\varepsilon_k x + \bar{z}_k| - A)^2 \bar{w}_k(x) = \mu_k \varepsilon_k^2 \bar{w}_k(x) + a_k \bar{w}_k^3(x) \quad \text{in } \mathbb{R}^2. \tag{3.17}$$

We claim that $\bar{w}_k$ satisfies (3.4) for some positive constants R_0 and η. For this purpose, we first show that $\{\frac{\bar{z}_k - z_k}{\varepsilon_k}\} \subset \mathbb{R}^2$ is bounded uniformly in k. Otherwise, if $|\frac{\bar{z}_k - z_k}{\varepsilon_k}| \to \infty$ as $k \to \infty$, it then follows from the exponential decay (3.13) that

$$u_k(\bar{z}_k) = \frac{1}{\varepsilon_k} w_k\left(\frac{\bar{z}_k - z_k}{\varepsilon_k}\right) \leq \frac{C}{\varepsilon_k} e^{-\frac{\beta}{2}|\frac{\bar{z}_k - z_k}{\varepsilon_k}|} = o\left(\varepsilon_k^{-1}\right) \quad \text{as } k \to \infty,$$

which however contradicts (3.14). Therefore, there exists a constant $R_1 > 0$, independent of k, such that $|\frac{\bar{z}_k - z_k}{\varepsilon_k}| < \frac{R_1}{2}$. Note from (3.16) and (3.5) that

$$\bar{w}_k(x) = w_k\left(x + \frac{\bar{z}_k - z_k}{\varepsilon_k}\right).$$

Since w_k satisfies (3.4), we know that

$$\lim_{k\to\infty} \int_{B_{R_0+R_1}(0)} |\bar{w}_k|^2 dx = \lim_{k\to\infty} \int_{B_{R_0+R_1}(\frac{\bar{z}_k - z_k}{\varepsilon_k})} |w_k|^2 dx \geq \int_{B_{R_0}(0)} |w_k|^2 dx \geq \eta > 0, \tag{3.18}$$

and the claim is proved, that is, Eq. (3.4) holds also for $\bar{w}_k$.

As in the proof of Lemma 3.1 (see also Lemma 2.3(iii)), one can further derive that there exists a subsequence, still denoted by $\{\bar{w}_k\}$, of $\{\bar{w}_k\}$ such that

$$\bar{w}_k \overset{k}{\to} \bar{w}_0 \quad \text{strongly in } H^1(\mathbb{R}^2) \quad \text{and} \quad \mu_k \varepsilon_k^2 \overset{k}{\to} -\beta^2, \tag{3.19}$$

for some $0 \leq \bar{w}_0 \in H^1(\mathbb{R}^2)$ and some $\beta > 0$, where $\bar{w}_0$ satisfies (3.10). Note from (3.18) that $\bar{w}_0 \not\equiv 0$. Thus, the strong maximum principle yields that $\bar{w}_0(x) > 0$ in $\mathbb{R}^2$. Since the origin is a critical point of $\bar{w}_k$ for all $k > 0$, it is also a critical point of $\bar{w}_0$. We therefore conclude from the uniqueness (up to translations) of positive radial solutions for (1.6) that $\bar{w}_0$ is spherically symmetric about the origin, and for the above $\beta > 0$,

$$\bar{w}_0 = \frac{\beta}{\|Q\|_2} Q(\beta|x|). \tag{3.20}$$

Using (3.17) and (3.19), we know that $\bar{w}_k \geq (\frac{\beta^2}{2a^*})^{\frac{1}{2}}$ at each local maximum point. Since $\bar{w}_k$ decays to zero uniformly in k as $|x| \to \infty$, all local maximum points of $\bar{w}_k$ stay in a finite ball in $\mathbb{R}^2$. *We claim that* $\bar{w}_k \to \bar{w}_0$ *in* $C^2_{loc}(\mathbb{R}^2)$ *as* $k \to \infty$. In fact, by (3.12) and the definition of $\bar{w}_k(x)$ we see that $\{\bar{w}_k\}$ is bounded in $L^\infty(\mathbb{R}^2)$, uniformly in k. Applying L^p-theory (see e.g., Theorem 9.11 of [10]), it follows from (3.17) that $\{\bar{w}_k\}$ is bounded uniformly in $W^{2,q}_{loc}(\mathbb{R}^2)$ for any $q > 2$. Thus, the standard Sobolev embedding theorem implies that $\{\bar{w}_k\}$ is bounded uniformly in

$C^{1,\alpha}_{\text{loc}}(\mathbb{R}^2)$ for some $\alpha \in (0,1)$. Furthermore, since $\bar{V}_k(x) := \varepsilon_k^2(|\varepsilon_k x + \bar{z}_k| - A)^2$ is locally Lipschitz continuous in $\mathbb{R}^2$, it follows from (3.17) and Theorem 6.2 in [10] that $\{\bar{w}_k\}$ is bounded uniformly in $C^{2,\alpha}_{\text{loc}}(\mathbb{R}^2)$. Therefore, there exists $\tilde{w}_0 \in C^{2,\alpha}_{\text{loc}}(\mathbb{R}^2)$ such that

$$\bar{w}_k \to \tilde{w}_0 \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^2) \text{ as } k \to \infty.$$

Further, by using (3.19) we conclude that $\tilde{w}_0 = \bar{w}_0$, and the claim is therefore established.

Note that the origin is the only critical point of $\bar{w}_0$, then the above claim shows that all local maximum points of $\{\bar{w}_k\}$ must approach the origin and hence stay in a small ball $B_\epsilon(0)$ as $k \to \infty$. One can take ϵ small enough such that $\bar{w}_0''(r) < 0$ for $0 \le r \le \epsilon$. It then follows from Lemma 4.2 in [28] that for large k, each $\bar{w}_k$ has no critical points other than the origin. This gives the uniqueness of local maximum points for each $\bar{w}_k(x)$, which therefore implies that there exists a subsequence of $\{u_k\}$ concentrating at a *unique* global minimum point of potential $V(x) = (|x| - A)^2$.

To complete the proof of Theorem 1.2, we need to determine the exact value of β in (3.20). From (3.16), we have

$$\begin{aligned} e(a_k) = E_{a_k}(u_k) &= \frac{1}{\varepsilon_k^2}\Bigg[\int_{\mathbb{R}^2} |\nabla \bar{w}_k(x)|^2 dx - \frac{a^*}{2}\int_{\mathbb{R}^2} \bar{w}_k^4(x)dx\Bigg] \\ &\quad + \frac{\varepsilon_k^2}{2}\int_{\mathbb{R}^2} \bar{w}_k^4(x)dx + \int_{\mathbb{R}^2} (|\varepsilon_k x + \bar{z}_k| - |y_0|)^2 \bar{w}_k^2(x)dx, \end{aligned} \tag{3.21}$$

where $\bar{z}_k$ is the unique global maximum point of u_k, and $\bar{z}_k \to y_0 \in \mathbb{R}^2$ as $k \to \infty$ for some $|y_0| = A > 0$. The term in square brackets is non-negative which can be ignored for the lower bound of $e(a_k)$. The $L^4(\mathbb{R}^2)$ norm of $\bar{w}_k$ converges to that of $\bar{w}_0$ as $k \to \infty$.

To estimate the last term of (3.21), we claim that $\{\frac{|\bar{z}_k| - |y_0|}{\varepsilon_k}\} \subset \mathbb{R}$ is bounded uniformly for $k \to \infty$. Otherwise, there must exist a subsequence of $\{a_k\}$, still denoted by $\{a_k\}$, such that $|\frac{|\bar{z}_k| - |y_0|}{\varepsilon_k}| \to \infty$ as $k \to \infty$, then, for any constant $C > 0$, using (3.18) we see that

$$\lim_{k\to\infty} \varepsilon_k^{-2} \int_{\mathbb{R}^2} (|\varepsilon_k x + \bar{z}_k| - |y_0|)^2 \bar{w}_k^2(x)dx = \lim_{k\to\infty} \int_{\mathbb{R}^2} \left(\left|x + \frac{\bar{z}_k}{\varepsilon_k}\right| - \frac{|y_0|}{\varepsilon_k}\right)^2 \bar{w}_k^2(x)dx \ge C.$$

This estimate and (3.21) then imply that

$$e(a_k) \ge C\varepsilon_k^2 = C\left(a^* - a_k\right)^{\frac{1}{2}}$$

holds for any constant $C > 0$, which however contradicts Theorem 2.1, and the claim is proved.

So, by our above claim we know that there exists a subsequence still denoted by $\{a_k\}$ such that

$$\frac{|\bar{z}_k| - |y_0|}{\varepsilon_k} \to C_0 \quad \text{as } k \to \infty \tag{3.22}$$

for some constant C_0. Since Q is a radially symmetric function and decays exponentially as $|x| \to \infty$, we then deduce from (3.20) that

$$\begin{aligned} \lim_{k\to\infty} \frac{1}{\varepsilon_k^2} \int_{\mathbb{R}^2} (|\varepsilon_k x + \bar{z}_k| - |y_0|)^2 \bar{w}_k^2(x)dx &= \lim_{k\to\infty} \int_{\mathbb{R}^2} \left(\frac{|\varepsilon_k x + \bar{z}_k|}{\varepsilon_k} - \frac{|y_0|}{\varepsilon_k}\right)^2 \bar{w}_k^2(x)dx \\ &= \lim_{k\to\infty} \int_{\mathbb{R}^2} \left(\frac{|\varepsilon_k x + \bar{z}_k| - |\bar{z}_k|}{\varepsilon_k} + \frac{|\bar{z}_k| - |y_0|}{\varepsilon_k}\right)^2 \bar{w}_k^2(x)dx \\ &= \int_{\mathbb{R}^2} \left(\frac{y_0 \cdot x}{|y_0|} + C_0\right)^2 \bar{w}_0^2(x)dx \ge \int_{\mathbb{R}^2} \frac{|y_0 \cdot x|^2}{A^2} \bar{w}_0^2(x)dx, \end{aligned} \tag{3.23}$$

where the equality holds if and only if $C_0 = 0$. We hence infer from (3.21) and (3.23) that

$$\lim_{k\to\infty}\frac{e(a_k)}{(a^*-a_k)^{1/2}}\geq\frac{1}{2}\|\bar{w}_0\|_4^4+\frac{1}{A^2}\int_{\mathbb{R}^2}|y_0\cdot x|^2\bar{w}_0^2(x)dx$$
$$=\frac{1}{a^*}\left(\beta^2+\frac{1}{A^2\beta^2}\int_{\mathbb{R}^2}|y_0\cdot x|^2Q^2(x)dx\right), \tag{3.24}$$

where (1.14) is used in the equality. So, for any $\beta>0$, we have

$$\lim_{k\to\infty}\frac{e(a_k)}{(a^*-a_k)^{1/2}}\geq\frac{2}{a^*}\left(\frac{\int_{\mathbb{R}^2}|y_0\cdot x|^2Q^2(x)dx}{A^2}\right)^{\frac{1}{2}}, \tag{3.25}$$

where the equality is achieved at

$$\beta=\lambda_0:=\left(\frac{\int_{\mathbb{R}^2}|y_0\cdot x|^2Q^2(x)dx}{A^2}\right)^{\frac{1}{4}}=\left(\frac{1}{2}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx\right)^{\frac{1}{4}},$$

here a suitable rotation and $Q(x)=Q(|x|)$ in $\mathbb{R}^2$ are used in getting the last equality.

We finally note that the limit in (3.25) actually exists, and it is equal to the right hand side of (3.25). To see this, one simply takes

$$u(x)=\frac{\beta}{\varepsilon\|Q\|_2}Q\left(\frac{\beta|x-y_0|}{\varepsilon}\right)$$

as a trial function for $E_a(\cdot)$ and minimizes over $\beta>0$. By applying (3.25), this leads to

$$\lim_{a_k\nearrow a^*}\frac{e(a_k)}{(a^*-a_k)^{1/2}}=\frac{2}{a^*}\left(\frac{\int_{\mathbb{R}^2}|y_0\cdot x|^2Q^2(x)dx}{A^2}\right)^{\frac{1}{2}}. \tag{3.26}$$

The equality (3.26) leads to two conclusions. Firstly, β is unique, which is independent of the choice of the subsequence, and takes the value of λ_0 as above. Secondly, Eq. (3.23) is indeed an equality, and thus $C_0=0$, i.e., Eq. (1.9) holds. Moreover, combining (3.15), (3.19) and (3.20), we see that

$$\bar{w}_k(x)=\varepsilon_k u(\varepsilon_k x+\bar{z}_k)\overset{k}{\to}\frac{\lambda_0}{\|Q\|_2}Q(\lambda_0|x|)\quad\text{strongly in } H^1(\mathbb{R}^2),$$

where $\bar{z}_k$ is the unique maximum point of u_k and $\bar{z}_k\overset{k}{\to}y_0$ for some $y_0\in\mathbb{R}^2$ satisfying $|y_0|=A>0$. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. By (3.26), it is clear that (1.12) holds for the subsequence $\{a_k\}$. In fact, by the proof of Theorem 1.2, we know that (3.26) is essentially true for any subsequence $\{a_k\}$ with $a_k\nearrow a^*$, this implies that (1.12) holds also for all $a\nearrow a^*$. □

Remark 3.2. Theorem 1.2 gives a detailed description on the concentration behavior of the minimizers of $e(a)$ when a is close to a^*, upon which the phenomena of symmetry breaking of the minimizers of $e(a)$ can be demonstrated by Corollary 1.4. That is, when a increases from 0 to a^*, the minimizers of GP energy $e(a)$ have essentially different properties: the GP energy $e(a)$ has a unique non-negative minimizer which is radially symmetric if $a>0$ is small, but $e(a)$ has infinity many minimizers which are non-radially symmetric if a approaches a^*.

Conflict of interest statement

We declare that we have no conflict of interest.

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Appendix A. Proof of Theorem 1.1(ii)

This appendix is devoted to the proof of Theorem 1.1(ii), for which we always assume that

$$0 \le V(x) \in L^\infty_{\text{loc}}(\mathbb{R}^2), \quad \lim_{|x|\to\infty} V(x) = \infty \quad \text{and} \quad \inf_{x\in\mathbb{R}^2} V(x) = 0. \tag{A.1}$$

The properties of the Schrödinger operator $-\Delta + V(x)$ with $V(x) \in L^\infty(\mathbb{R}^2)$ are well known, see e.g. [33], but we could not find a reference for that of $V(x)$ satisfying (A.1) although we guess it should exist somewhere. For the sake of completeness, we begin this appendix by giving some properties of the Schrödinger operator $-\Delta + V(x)$ under conditions (A.1), which are required in proving the uniqueness of non-negative minimizers. Before going to the properties of $-\Delta + V(x)$, we recall the following embedding lemma, which can be found in [30, Theorem XIII.67], [7, Lemma 2.1] or [2, Lemma 2.1], etc.

Lemma A.1. *Suppose $V(x)$ satisfies (A.1). Then, the embedding $\mathcal{H} \hookrightarrow L^q(\mathbb{R}^2)$ is compact for all $q \in [2,\infty)$.* □

Define

$$\mu_1 = \inf\Big\{\int_{\mathbb{R}^2} |\nabla u|^2 + V(x)u^2 dx : u \in \mathcal{H} \text{ and } \int_{\mathbb{R}^2} u^2 dx = 1\Big\}. \tag{A.2}$$

By Lemma A.1, it is not difficult to know that μ_1 is simple and can be attained by a positive function $\phi_1 \in \mathcal{H}$. We now define

$$\mu_2 = \inf\Big\{\int_{\mathbb{R}^2} |\nabla u|^2 + V(x)u^2 dx : u \in Z \text{ and } \int_{\mathbb{R}^2} u^2 dx = 1\Big\}, \tag{A.3}$$

where

$$Z = \text{span}\{\phi_1\}^\perp = \Big\{u : u \in \mathcal{H}, \int_{\mathbb{R}^2} u\phi_1 dx = 0\Big\}.$$

It is known that $\mu_2 > \mu_1$ and

$$\mathcal{H} = \text{span}\{\phi_1\} \oplus Z. \tag{A.4}$$

Then, we have the following lemma, its proof is somehow standard, we omit it here.

Lemma A.2. *Under the assumption of (A.1), we have*

(i) $\ker(-\Delta + V(x) - \mu_1) = \text{span}\{\phi_1\}$;
(ii) $\phi_1 \notin (-\Delta + V(x) - \mu_1)Z$;
(iii) $\text{Im}(-\Delta + V(x) - \mu_1) = (-\Delta + V(x) - \mu_1)Z$ *is closed in* $\mathcal{H}^*$;
(iv) $\text{codim}\,\text{Im}(-\Delta + V(x) - \mu_1) = 1$,

where $\mathcal{H}^*$ denotes the dual space of $\mathcal{H}$. □

Motivated by Theorem 3.2 in [5], we have the following lemma. For the sake of completeness, we give a short proof here.

Lemma A.3. *Define the following C^1 functional $F : \mathcal{H} \times \mathbb{R}^2 \mapsto \mathcal{H}^*$*

$$F(u,\mu,a) = \big(-\Delta + V(x) - \mu\big)u - au^3. \tag{A.5}$$

Then, there exist $\delta > 0$ and a unique function $(u(a),\mu(a)) \in C^1(B_\delta(0); B_\delta(\mu_1,\phi_1))$ such that

$$\begin{cases} \mu(0) = \mu_1, \qquad u(0) = \phi_1; \\ F(u(a),\mu(a),a) = 0; \\ \|u(a)\|_2^2 = 1. \end{cases} \tag{A.6}$$

Proof. Let $g : Z \times \mathbb{R}^3 \mapsto \mathcal{H}^*$ be defined by

$$g(z,\tau,s,a) := F\big((1+s)\phi_1 + z, \mu_1 + \tau, a\big).$$

Then $g \in C^1(Z \times \mathbb{R}^3, \mathcal{H}^*)$ and

$$g(0,0,0,0) = F(\phi_1,\mu_1,0) = 0 \quad \text{and} \quad g_s(0,0,0,0) = F_u(\phi_1,\mu_1,0)\phi_1 = \big(-\Delta + V(x) - \mu_1\big)\phi_1 = 0. \tag{A.7}$$

Moreover, for any $(\hat{z},\hat{\tau}) \in Z \times \mathbb{R}$, we have

$$g_{(z,\tau)}(0,0,0,0)(\hat{z},\hat{\tau}) = F_u(\phi_1,\mu_1,0)\hat{z} + F_\mu(\phi_1,\mu_1,0)\hat{\tau} = \big(-\Delta + V(x) - \mu_1\big)\hat{z} - \hat{\tau}\phi_1. \tag{A.8}$$

Then, by Lemma A.2, $g_{(z,\tau)}(0,0,0,0) : Z \times \mathbb{R} \mapsto \mathcal{H}^*$ is an isomorphism. Therefore, by the implicit function theorem, there exist $\delta_1 > 0$ and a unique function $(z(s,a),\tau(s,a)) \in C^1(B_{\delta_1}(0,0); B_{\delta_1}(0,0))$ such that

$$\begin{cases} g(z(s,a),\tau(s,a),s,a) = F((1+s)\phi_1 + z(s,a), \qquad \mu_1 + \tau(s,a), a) = 0, \\ z(0,0) = 0, \qquad \tau(0,0) = 0, \\ z_s(0,0) = -g_{(z,\tau)}^{-1}(0,0,0,0) \cdot g_s(0,0,0,0) = 0. \end{cases} \tag{A.9}$$

Now, let

$$u(s,a) = (1+s)\phi_1 + z(s,a), \quad (s,a) \in B_{\delta_1}(0,0),$$

and define

$$f(s,a) = \|u(s,a)\|_2^2 = (1+s)^2 + \int_{\mathbb{R}^2} z(s,a)^2 dx, \quad (s,a) \in B_{\delta_1}(0,0).$$

It follows from (A.9) that

$$f(0,0) = 1, \qquad f_s(0,0) = 2 + 2\int_{\mathbb{R}^2} z_s(0,0)z(0,0)dx = 2.$$

Then, by applying implicit function theorem again, there exist $\delta \in (0,\delta_1)$ and a unique function $s = s(a) \in C^1(B_\delta(0); B_\delta(0))$ such that

$$f\big(s(a),a\big) = \big\|u\big(s(a),a\big)\big\|_2^2 = f(0,0) = 1, \quad a \in B_\delta(0).$$

This and (A.9) show that, for $a \in B_\delta(0)$, there exists a unique function:

$$\big(u(a) := u\big(s(a),a\big), \mu(a) := \mu_1 + \tau\big(s(a),a\big)\big) \in C^1\big(B_\delta(0); B_\delta(\phi_1,\mu_1)\big)$$

such that (A.6) holds, and the proof is therefore complete. □

Proof of Theorem 1.1(ii). Let $u_a(x) > 0$ be a minimizer of $e(a)$ with $a \in [0,a^*)$. It is easy to see that

$$e(0) = \mu_1 \quad \text{and} \quad e(a) \le e(0) = \mu_1, \tag{A.10}$$

where μ_1 is defined by (A.2). Moreover, $e(a)$ is a concave function of a and then

$$e(a) \in C\big([0, a^*), \mathbb{R}^+\big). \tag{A.11}$$

For any $a_0 \in [0, a^*)$, it follows from (1.13) that

$$\int_{\mathbb{R}^2} u_a^4 dx \le \frac{2e(a)}{a^* - a} \le \frac{4\mu_1}{a^*} \quad \text{for } 0 \le a \le \frac{a^*}{2}. \tag{A.12}$$

Since u_a is a minimizer of (1.4), it satisfies the following Euler–Lagrange equation

$$-\Delta u_a(x) + V(x)u_a(x) - \mu_a u_a(x) - a u_a^3(x) = 0 \quad \text{in } \mathbb{R}^2,$$

where $\mu_a \in \mathbb{R}$ is a suitable Lagrange multiplier, i.e.,

$$F(u_a, \mu_a, a) = 0, \quad \text{where } F(\cdot) \text{ is defined by (A.5)}. \tag{A.13}$$

Since

$$\mu_a = e(a) - \frac{a}{2} \int_{\mathbb{R}^2} |u_a|^4 dx,$$

it then follows from (A.10)–(A.12) that there exists $a_1 > 0$ small such that

$$|\mu_a - \mu_1| \le \big|e(a) - \mu_1\big| + \frac{a}{2} \int_{\mathbb{R}^2} |u_a|^4 dx \le \delta \quad \text{for } 0 \le a < a_1, \tag{A.14}$$

where $\delta > 0$ is as in Lemma A.3. On the other hand, since

$$E_0(u_a) = e(a) + \frac{a}{2} \int_{\mathbb{R}^2} |u_a|^4 dx \to e(0) = \mu_1 \quad \text{as } a \searrow 0,$$

i.e., $\{u_a \ge 0\}$ is a minimizing sequence of $e(0) = \mu_1$ as $a \searrow 0$. Noting that μ_1 is simple, we can easily deduce from Lemma A.1 that

$$u_a \to \phi_1 \quad \text{in } \mathcal{H} \text{ for all } a \searrow 0.$$

This implies that there exists $a_2 > 0$ such that

$$\|u_a - \phi_1\|_{\mathcal{H}} < \delta \quad \text{for } 0 \le a < a_2. \tag{A.15}$$

Then using (A.13)–(A.15) and Lemma A.3, we obtain that

$$\mu_a = \mu(a); \quad u_a = u(a) \quad \text{for } 0 \le a < \min\{a_1, a_2\},$$

i.e., $e(a)$ has a unique non-negative minimizer $u(a)$ if $a > 0$ is small. □

References

[1] M.H. Anderson, J.R. Ensher, M.R. Matthews, C.E. Wieman, E.A. Cornell, Observation of Bose–Einstein condensation in a dilute atomic vapor, Science 269 (1995) 198–201.

[2] W.Z. Bao, Y.Y. Cai, Mathematical theory and numerical methods for Bose–Einstein condensation, Kinet. Relat. Models 6 (2013) 1–135.

[3] T. Cazenave, Semilinear Schrödinger Equations, Courant Lecture Notes in Math., vol. 10, Courant Institute of Mathematical Science/AMS, New York, 2003.

[4] D.I. Choi, Q. Niu, Bose Einstein condensation in an optical lattice, Phys. Rev. Lett. 82 (1999) 2022–2025.

[5] M.G. Crandall, P.H. Rabinowitz, Bifurcation, perturbation of simple eigenvalues and linearized stability, Arch. Ration. Mech. Anal. 52 (1973) 161–180.

[6] F. Dalfovo, S. Giorgini, L.P. Pitaevskii, S. Stringari, Theory of Bose–Einstein condensation in trapped gases, Rev. Mod. Phys. 71 (1999) 463–512.

[7] Y.H. Ding, S.J. Li, Homoclinic orbits for first order Hamiltonian systems, J. Math. Anal. Appl. 189 (1995) 585–601.

[8] L. Erdős, B. Schlein, H.T. Yau, Derivation of the Gross–Pitaevskii equation for the dynamics of Bose–Einstein condensate, Ann. Math. 172 (2010) 291–370.

[9] B. Gidas, W.M. Ni, L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$, in: Mathematical Analysis and Applications Part A, in: Adv. in Math. Suppl. Stud., vol. 7, Academic Press, New York, 1981, pp. 369–402.
[10] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, 1997.
[11] E.P. Gross, Structure of a quantized vortex in boson systems, Nuovo Cimento 20 (1961) 454–466.
[12] E.P. Gross, Hydrodynamics of a superfluid condensate, J. Math. Phys. 4 (1963) 195–207.
[13] Y.J. Guo, R. Seiringer, On the mass concentration for Bose–Einstein condensates with attractive interactions, Lett. Math. Phys. 104 (2014) 141–156.
[14] S. Gupta, K.W. Murch, K.L. Moore, T.P. Purdy, D.M. Stamper-Kurn, Bose–Einstein condensation in a circular waveguide, Phys. Rev. Lett. 95 (2005) 143201.
[15] P.L. Halkyard, Dynamics in cold atomic gases: resonant behaviour of the quantum delta-kicked accelerator and Bose–Einstein condensates in ring traps, Ph.D. Thesis, Durham University, 2010.
[16] P.L. Halkyard, M.P.A. Jones, S.A. Gardiner, Rotational response of two-component Bose–Einstein condensates in ring traps, Phys. Rev. A 81 (2010) 061602(R).
[17] Q. Han, F.H. Lin, Elliptic Partial Differential Equations, second edition, Courant Lect. Notes Math., vol. 1, Courant Institute of Mathematical Science/AMS, New York, 2011.
[18] O. Kavian, F.B. Weissler, Self-similar solutions of the pseudo-conformally invariant nonlinear Schrödinger equation, Mich. Math. J. 41 (1) (1994) 151–173.
[19] M.K. Kwong, Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^N$, Arch. Ration. Mech. Anal. 105 (1989) 243–266.
[20] Y. Li, W.M. Ni, Radial symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$, Commun. Partial Differ. Equ. 18 (1993) 1043–1054.
[21] E.H. Lieb, M. Loss, Analysis, second edition, Grad. Stud. in Math., vol. 14, Amer. Math. Soc., Providence, RI, 2001.
[22] E.H. Lieb, R. Seiringer, Proof of Bose–Einstein condensation for dilute trapped gases, Phys. Rev. Lett. 88 (2002) 170409-1–170409-4.
[23] E.H. Lieb, R. Seiringer, J.P. Solovej, J. Yngvason, The Mathematics of the Bose Gas and Its Condensation, Oberwolfach Seminars, vol. 34, Birkhäuser Verlag, Basel, 2005.
[24] E.H. Lieb, R. Seiringer, J. Yngvason, Bosons in a trap: a rigorous derivation of the Gross–Pitaevskii energy functional, Phys. Rev. A 61 (2000) 043602-1–043602-13.
[25] E.H. Lieb, R. Seiringer, J. Yngvason, A rigorous derivation of the Gross–Pitaevskii energy functional for a two-dimensional Bose gas, Commun. Math. Phys. 224 (2001) 17–31.
[26] P.L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case II, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 1 (1984) 223–283.
[27] K. McLeod, J. Serrin, Uniqueness of positive radial solutions of $\Delta u + f(u) = 0$ in $\mathbb{R}^n$, Arch. Ration. Mech. Anal. 99 (1987) 115–145.
[28] W.-M. Ni, I. Takagi, On the shape of least-energy solutions to a semilinear Neumann problem, Commun. Pure Appl. Math. 44 (1991) 819–851.
[29] L.P. Pitaevskii, Vortex lines in an imperfect Bose gas, Sov. Phys. JETP 13 (1961) 451–454.
[30] M. Reed, B. Simon, Methods of Modern Mathematical Physics. IV. Analysis of Operators, Academic Press, New York–London, 1978.
[31] C. Ryu, M.F. Andersen, P. Cladé, Vasant Natarajan, K. Helmerson, W.D. Phillips, Observation of persistent flow of a Bose–Einstein condensate in a toroidal trap, Phys. Rev. Lett. 99 (2007) 260401.
[32] J. Smyrnakis, S. Bargi, G.M. Kavoulakis, M. Magiropoulos, K. Karkkainen, S.M. Reimann, Mixtures of Bose gases confined in a ring potential, Phys. Rev. Lett. 103 (2009) 100404.
[33] C.A. Stuart, An introduction to elliptic equations on $\mathbb{R}^N$, in: A. Ambrosetti, K.-C. Chang, I. Ekeland (Eds.), Nonlinear Functional Analysis and Applications to Differential Equations, World Scientific, Singapore, 1998.
[34] X.F. Wang, On concentration of positive bound states of nonlinear Schrödinger equations, Commun. Math. Phys. 153 (1993) 229–244.
[35] M.I. Weinstein, Nonlinear Schrödinger equations and sharp interpolations estimates, Commun. Math. Phys. 87 (1983) 567–576.