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Corresponding Author: Dr Jonathan Elmer, PhD

Corresponding Author's Institution: RWTH Aachen

First Author: Jonathan Elmer, PhD

Order of Authors: Jonathan Elmer, PhD

Depth and Detection in Modular Invariant Theory

Jonathan Elmer
Lehrstuhl A für Mathematik
RWTH-Aachen
Aachen 52062
Germany

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Abstract

Let G be a finite group acting linearly on a vector space V over a field of characteristic p dividing the group order, and let $R := S(V^*)$. We study the R^G modules $H^i(G, R)$, for $i \geq 0$ with R^G itself as a special case. There are lower bounds for $\operatorname{depth}_{R^G}(H^i(G, R))$ and for $\operatorname{depth}(R^G)$. We show that a certain sufficient condition for their attainment (due to Fleischmann, Kemper and Shank [14]) may be modified to give a condition which is both necessary and sufficient. We apply our main result to classify the representations of the Klein four-group for which $\operatorname{depth}(R^G)$ attains its lower bound, a process begun in [10]. We also use our new condition to show that the if $G = P \times Q$, with P a p-group and Q an abelian p'-group, then the depth of R^G attains its lower bound if and only if the depth of R^P does so.

1 Introduction

Let G be a finite group acting on a polynomial ring $R := k[x_1, x_2, \ldots, x_n]$. Then the set of fixed points under this action form a ring R^G called the ring of invariants. Suppose in addition that k is a field of characteristic p which divides the group order. Then R^G is the central object of study in modular invariant theory. Since R^G can be regarded as the zeroth cohomology $H^0(G, R)$, it is often worthwhile studying it in conjunction with the higher cohomology modules $H^i(G, R)$, which become R^G-modules via the cup product.

In this paper we will concentrate on the standard situation in which $R := \operatorname{Sym}(V^*)$ (the symmetric algebra of the dual) for some finite dimensional left kG-module V, and the (right) action of G on R is by graded algebra automorphisms. If k is algebraically closed we can regard R as the space $k[V]$ of polynomial functions on V, and R^G as the set of G-invariant polynomial functions on V, or the ring of coordinate functions on the quotient space V/G. With this identification, we may define, for any ideal I of R^G, the variety

$$\mathcal{V}(I) := \{v \in V : f(v) = 0 \ \forall \ f \in I\} \tag{1}$$

and for any subset of points $U \subseteq V$ the ideal

$$\mathcal{I}(U) := \{f \in R^G : f(u) = 0 \ \forall \ u \in U\}. \tag{2}$$

Note that in order to apply Hilbert's Nullstellensatz directly we should instead take varieties in the categorical quotient V/G. However, a version of the Nullstellensatz exists which allows us to use standard techniques on the objects defined above:

Lemma 1.1 (See Kemper [18], Lemma 3.3)**.** *For any ideal J of R^G, and with $\mathcal{I}$ and $\mathcal{V}$ defined as above, we have*

$$\mathcal{I}(\mathcal{V}(J)) = \sqrt{J}.$$

1.1 Depth

Let $A = \oplus_{i \in \mathbb{N}_0} A_i$ be a graded connected k - algebra, by which we mean that A is a graded k-algebra with $A_0 = k$, and define $A_+ := \oplus_{i>0} A_i$; furthermore let $J \subseteq A_+$ be a homogeneous ideal and M be a graded A-module. A sequence of homogeneous elements $(a_1, \cdots, a_k)$ with $a_i \in J$ is called M-**regular**, if for every $i = 1, \cdots, k$ the multiplication by a_i induces an injective map on the quotient ring $M/(a_1, \cdots, a_{i-1})M$. It is known that all maximal M-regular sequences in J have the same length $\text{grade}(J, M)$, called the grade of J on M, and one now defines

$$\text{depth}(M) := \text{grade}(A^+, M).$$

It is clear from the construction that $\text{depth}(A) \leq \dim(A)$. For other standard results concerning depth and grade, the reader is referred to [4]. We will be concerned with calculating the depth of the invariant ring R^G, or more generally the R^G-modules $H^i(G, R)$. It is well known (see [6]) that in the non-modular case, all (standard) invariant rings are Cohen-Macaulay, that is, $\text{depth}(R^G) = \dim(R^G)$, so from this point onwards we will consider only the modular situation. Little is known in general about the depth of modular invariant rings and cohomology modules. One significant result, due to Ellingsrud and Skjelbred [7], gives us a lower bound for the depth of modular invariant rings. Their result was strengthened in [14] to the following:

Theorem 1.2. *Let G be a finite group and k a field of characteristic p. Let P be a Sylow-p-subgroup and let V be a left kG-module. Let R denote the symmetric algebra $S(V^*)$ which has a natural right-module structure. Then*

$$\text{depth}(R^G) \geq \min\{\dim(V), \dim(V^P) + cc_G(R) + 1\}$$

where V^P denotes the fixed point space of P on V and $cc_G(R)$ is called the cohomological connectivity *and defined as* $\min\{i > 0 : H^i(G, R) \neq 0\}$.

In this article, we shall say that an invariant ring R^G has *minimal depth* if the above is an equality. Ellingsrud and Skjelbred proved that R^G has minimal depth when G is cyclic of prime order, and that $H^i(G, R)$ is Cohen-Macaulay for all $i > 0$ in this situation. In [14] it was shown that R^G has minimal depth for every p-nilpotent group with cyclic Sylow-p-subgroups, and in [15] the same authors were able to calculate $\mathrm{depth}_{R^G}(H^i(G, R))$ for any i when G is a cyclic p-group. In [10], similar techniques were used to show that R^G has minimal depth whenever $G = C_2 \times C_2$ and V is indecomposable and not projective. The authors also showed that R^G has minimal depth for many decomposable representations of $C_2 \times C_2$, but were unable to classify completely those representations of G with this property. We return to this question in section 3.

The starting point for all our depth calculations is the following result due to Kemper ([19], Theorem 1.5):

Theorem 1.3. *Let G be a finite group acting linearly on $R := S(V^*)$. Let $U \leq V$ be a kG-submodule for which the kernel of the action of G on U has index in G not divisible by p - we will call this a non-modular submodule. Let $M := H^i(G, R)$ for some $i \geq 0$. Then we have*

$$\mathrm{depth}_{R^G}(M) = \mathrm{grade}(\mathcal{I}(U), M) + \dim(U)$$

Let P denote a fixed Sylow-p-subgroup of G. Then we observe that, for any group G and kG-module V, the fixed point set V^P is a non-modular submodule. Further, by [12], Theorem 5.9, $\mathcal{I}(V^P) = \sqrt{I^G_{\mathcal{X}(P)}}$ where $I^G_{\mathcal{X}(P)}$ is a relative transfer ideal as defined in the next section. Although it may seem that Theorem 1.3 merely changes the question of calculating $\mathrm{depth}_{R^G}(H^i(G, R))$ to one of calculating $\mathrm{grade}(\mathcal{I}(U), H^i(G, R))$, it is nonetheless an extremely useful result, since when $i > 0$ the latter quantity is often zero, and these occasions are not so difficult to spot. It follows easily from this result that

$$\mathrm{depth}_{R^G}(H^i(G, R)) \geq \dim(V^P)$$

for any kG-module V. Accordingly, we shall say that $H^i(G, R)$ $(i > 0)$ has minimal depth when $\mathrm{depth}_{R^G}(H^i(G, R)) = \dim(V^P)$. (Note that when $i = 0$, we do not recover the original definition of minimal depth).

2 Relative Transfer Ideals

If $X < G$, we may define a mapping $\mathrm{Tr}^G_X : R^X \to R^G$ as follows: let S be a set of right coset representatives of X in G. Then we define

$$\mathrm{Tr}^G_X(x) := \sum_{g \in S} xg. \tag{3}$$

This mapping is often called the *relative transfer*, and induces mappings Tr^G_X : $H^i(X, R) \to H^i(G, R)$ also called the relative transfer. Both are surjective when the index of X in G is coprime to p. The image of the transfer map $\mathrm{Tr}_X^G(R^X)$ is an ideal in R^G called the *relative transfer ideal* which we denote by I_X^G. We may generalise this definition and define

$$I_\chi^G := \sum_{X \in \chi} I_X^G$$

for any set χ of subgroups of G.

In this section we will study I_χ^G for various choices of χ, which is always assumed to be a set of p-subgroups of G and closed under taking subgroups. We will assume throughout that k is algebraically closed. For convenience, let $\mathfrak{i}_\chi$ denote $\sqrt{I_\chi^G}$. Then we have the following result: due to Fleischmann ([12], Theorem 5.9):

$$\mathcal{V}(I_\chi^G) = \{v \in V : p \mid [G_v : G_v \cap X] \ \forall\ X \in \chi\} = \bigcup_{Q \in \chi'} V^Q \tag{4}$$

where χ' denotes the set of p-subgroups of G not conjugate to any subgroup in χ.

Lemma 2.1 ([14], Corollary 3.2)**.** *Suppose χ_1 and χ_2 are sets of p-subgroups of G, closed under conjugation and taking subgroups. Then $\mathcal{V}(I_{\chi_1}^G) \subseteq \mathcal{V}(I_{\chi_2}^G)$ if and only if, for every $Q \in \chi_2 \setminus \chi_1$ there exists $Q' \in \chi_2'$ such that Q is a proper subgroup of Q', but $V^Q = V^{Q'}$.*

Proof. Suppose $\mathcal{V}(I_{\chi_1}^G) \subseteq \mathcal{V}(I_{\chi_2}^G)$ and let $Q \in \chi_2 \backslash \chi_1$. Then (4) shows that $V^Q \subsetneqq \mathcal{V}(I_{\chi_1}^G) \subseteq \mathcal{V}(I_{\chi_2}^G)$. In particular, $V^Q \subseteq \mathcal{V}(I_{\chi_2}^G)$ and so again by (4), $p \mid [G_v : Q]$ for every $v \in V^Q$. This means there is a Sylow-p-subgroup P_v of G_v which satisfies $Q < P_v$, and $V^{P_v} \subseteq V^Q$ while $V^Q \leq \bigcup_{v \in V^Q} V^{P_v}$. Since we're assuming k is algebraically closed and therefore infinite, we must have $V^Q \leq V^{P_w}$ for some $w \in V^Q$, and so $V^Q = V^{P_w}$ as required.

Conversely suppose for each $Q \in \chi_2 \backslash \chi_1$ there exists $Q' \in \chi_2'$ such that Q is a proper subgroup of Q', but $V^Q = V^{Q'}$. This means for each $Q \in \chi_2 \cap \chi_1'$, $V^Q = V^{Q'}$ for some $Q' \in \chi_2'$. Therefore for any $Q \in \chi_1'$, $V^Q \subseteq \bigcup_{Q' \in \chi_2'} V_{Q'}$ and the result follows from (4). ☐

Corollary 2.2. *Let $\chi(P)$ denote the set of all proper subgroups of a fixed Sylow-p-subgroup P of G, and let $\psi(P) := \{Q < P : V^P \subsetneqq V^Q\}$. Also define $\chi := \{Q < G : Q \text{ is a } p-\text{group and } p \mid [G:Q]\}$ and $\psi := \{Q < G : Q \text{ is a } p-\text{group and } V^P \subsetneqq V^Q \text{ for every Sylow}-p-\text{subgroup } P \geq Q\}$. Then $\mathfrak{i}_\chi = \mathfrak{i}_{\chi(P)} = \mathfrak{i}_{\psi(P)} = \mathfrak{i}_\psi$.*

Proof. The formula $\mathrm{Tr}_Q^G(x) = \mathrm{Tr}_{Q^g}^G(xg)$ shows that $I_\chi^G = I_{\chi(P)}^G$ and $I_\psi^G = I_{\psi(P)}^G$, from which the first and third equalities follow immediately. Note that ψ and χ are closed under conjugation. By Lemma 1.1 it suffices to show $\mathcal{V}(I_\chi^G) = \mathcal{V}(I_\psi^G)$. Since $\psi \subseteq \chi$ it is clear that $\mathcal{V}(I_\chi^G) \subseteq \mathcal{V}(I_\psi^G)$ and so we need prove only the converse. Note that if $Q \in \chi \setminus \psi$, then $V^P = V^Q$ for every Sylow-p-subgroup P containing Q as a subgroup. The result now follows from Lemma 2.1. ☐

As well as the transfer maps, we shall consider the restriction maps $\mathrm{res}^G_Q : H^m(G,R) \to H^m(Q,R)$, which are the maps on cohomology induced by the inclusion $Q \subset G$. We shall say that $H^m(G,R)$ is *detected* on a set $\mathcal{Q}$ of subgroups of G if the product of maps

$$\prod_{Q\in\mathcal{Q}} \mathrm{res}^G_Q : H^m(G,R) \to \prod_{Q\in\mathcal{Q}} H^m(Q,R)$$

is an injection. Many of the results of [14] and [10] depend on the observation that $\mathrm{grade}(\mathfrak{i}, H^m(G,R)) = 0$ if $H^m(G,R)$ is not detected on the set $\chi(P)$. Here we are able to prove a variation on this result which is both necessary and sufficient:

Proposition 2.3. *Let G be a finite group acting linearly on a k-vector space V, where k is an algebraically closed field of characteristic p dividing the group order. Let P be a Sylow-p-subgroup of G and let $R := \mathrm{Sym}(V^*)$. Let $\mathfrak{i} := \mathfrak{i}_\chi = \mathfrak{i}_\psi$ as above, and let m be a strictly positive integer. Then the following are equivalent:*

1. $\mathrm{depth}_{R^G}((H^m(G,R)) = \dim(V^P)$
2. $\mathrm{grade}(\mathfrak{i}, H^m(G,R) = 0$
3. *$\mathfrak{i}$ is an associated prime of $H^m(G,R)$*
4. *$H^m(G,R)$ is not detected on $\psi(P)$.*

Remark: The equivalence of (1), (2) and (3) was noted in [14], and the fact that (4) implies any of these is shown using only the results of [14]. The implication (3) $\Rightarrow$ (4) is new, and requires the following explicit description of the associated primes of $H^m(G,R)$ found in [8]:

Theorem 2.4 ([8], Theorem 1.1)**.** *Let $\mathfrak{p}$ be an associated prime ideal of the R^G-module $\oplus_{i>0} H^i(G,R)$. Then $\mathfrak{p} = \sqrt{I^G_\chi}$ for some set χ of subgroups of G.*

We now prove Proposition 2.3.

Proof. (1) $\Leftrightarrow$ (2) follows immediately from Theorem 1.3. If $\mathrm{grade}(\mathfrak{i}, H^m(G,R)) = 0$, then $\mathfrak{i}$ consists of zero-divisors and consequently is contained in an associated prime; see for example [4], Theorem 1.2.1. Now suppose $\mathfrak{i}$ is an associated prime of the R^G-module $H^m(G,R)$. Then by [8], Proposition 2.5, we have for some $\alpha \in H^m(G,R)$

$$\mathfrak{i} = \sqrt{\mathrm{Ann}_{R^G}(\alpha)} = \mathfrak{i}_\upsilon = \mathfrak{i}_{\upsilon(P)}$$

where $\upsilon = \{Q < G : Q \text{ is a } p-\text{group and } \mathrm{res}^G_Q(\alpha) = 0\}$ (which is closed under conjugation) and $\upsilon(P) = \{Q < P : \mathrm{res}^G_Q(\alpha) = 0\}$. Note that υ cannot contain any Sylow-p-subgroup of G, since res^G_P is injective for every Sylow-p-subgroup P of G. Clearly $H^m(G,R)$ is not detected on $\upsilon(P)$. We will show that $\psi \subset \upsilon$, and hence that $\psi(P) \subset \upsilon(P)$. For, suppose that $Q < G$ satisfies $\mathrm{res}^G_Q \neq 0$. Applying Lemma 2.1 with $\chi_1 = \upsilon$ and $\chi_2 = \chi$, we see that there exists a Sylow-p-subgroup $P \geq Q$ with $V^P = V^Q$, since χ' is the set of Sylow-p-subgroups of G. Consequently, $Q \notin \psi$ as required.

Finally to show that (4) $\Rightarrow$ (2), we notice by [20], Lemma 1.3, that if $\mathrm{res}^G_Q(\alpha) = 0$ for every $Q \in \psi$, then $\mathrm{Ann}_{R^G}(\alpha) \supseteq I^G_\psi$. Consequently we have

$$0 = \mathrm{grade}(I^G_\psi, H^m(G,R)) = \mathrm{grade}(\sqrt{I^G_\psi}, H^m(G,R)) = \mathrm{grade}(\mathfrak{i}, H^m(G,R))$$

where the first equality follows from, for example, [4], Proposition 1.2.10(b). □

Now let $m := cc_G(R)$. Note that by [2], Theorem 4.1, there exists $i > 0$ such that $H^i(G,k) \neq 0$, so we know that m is finite. Note that if $m + 1 \leq \mathrm{codim}(V^P)$, then R^G has minimal depth - this follows immediately from Theorem 1.2. Assuming the opposite, a spectral sequence argument given in [14], section 7, shows that

$$\mathrm{grade}(\mathfrak{i}, R^G) = m + 1 \Leftrightarrow \mathrm{grade}(\mathfrak{i}, H^m(G,R)) = 0. \tag{5}$$

So we can also use our detection condition to say definitively whether or not the depth of R^G is minimal.

2.1 Vector Invariants

One notable consequence of Proposition 2.3 is in the study of vector invariants. Suppose V is a kG-module and consider the direct sum $W = V^{\oplus r}$ for some $r \geq 1$. Then invariants in $\mathrm{Sym}(W^*)^G$ are often called vector invariants. We have the following result:

Proposition 2.5. *Let G, V be as above with $W := V^{\oplus r}$, $W' := V^{\oplus s}$. Suppose that $H^1(G, \mathrm{Sym}(W^*)) \neq 0$, $r > 0$ is sufficiently large that $\mathrm{codim}(W^P) > 2$ and that $s > r$. Then if $\mathrm{Sym}(W^*)^G$ has minimal depth, so does $\mathrm{Sym}(W'^*)^G$.*

Proof. Let $P \in Syl_p(G)$ and $Q < P$. Clearly

$$V^P \subsetneq V^Q \Leftrightarrow W^P \subsetneq W^Q \Leftrightarrow W'^P \subsetneq W'^Q,$$

and so $\psi(P)$ is the same for both W and W'. Let. Since $\mathrm{codim}(W^P) > 2$ and $\mathrm{Sym}(W^*)^G$ has minimal depth, Proposition 2.3 tells us that $H^1(G, \mathrm{Sym}(W^*))$ is not detected on $\psi(P)$. Since $H^1(G, \mathrm{Sym}(W^*))$ is a kP-direct summand of $H^1(G, \mathrm{Sym}(W'^*))$, the latter is not detected on $\psi(P)$ either. Then by Proposition 2.3 once more, the depth of $\mathrm{Sym}(W'^*)^G$ is minimal. □

We can give the following example where this result is used. Suppose $k := \mathbb{F}_p$ and let G be the group of 3×3 unipotent upper triangular matrices (which is sometimes denoted U_3. This is a p-group, so we have $cc_G(R) = 1$ for any representation of G over k. Let V be the natural 3 dimensional kG-module and let $W := V^{\oplus r}$. Furthermore, define $R := \mathrm{Sym}(W^*)$. The invariant rings

R^G for $r = 2$ were studied by Shank and Wehlau [22]; they found that R^G is Cohen-Macaulay if and only if $p = 2$. Let p be an odd prime. When $r = 2$, we have $\dim(W^P) + 2 = 4$ and $\dim(W) = 6$, so the depth of R^G is either 4 or 5. A direct calculation in MAGMA tells us that $\text{depth}(R^G) = 4$ when $p = 3$. [1] Combining this with Proposition 2.5 we obtain:

Corollary 2.6. *Let $G := U_3(\mathbb{F}_3)$ as above and let $W := V^{\oplus r}$. Let $R :=$ $\text{Sym}(W^*)$. Then*

$$\text{depth}(R^G) = r + 2$$

3 The Klein Four-Group

In this section we specialize to the case $G = P = C_2 \times C_2$. Let X and Y be generators of this group. Note that $cc_G(R) = 1$ for any representation of P (in fact this is true of any p-group). We recall from [10] the following theorem (in slightly different language):

Theorem 3.1. *Let V be a (faithful) representation of P with $R := \text{Sym}(V^*)$. Then if $H^1(P, R)$ is detected on the set of proper subgroups of P, V must be isomorphic to some direct sum of modules in the following set*[2]

$$\mathcal{S} := \{V_1, V_{2,0}, V_{2,1}, V_{2,\infty}, V_3, V_{-3}, \overline{V_4}\}$$

where the action of P on each of these modules is given by the following matrices:

- *V_1 denotes the one-dimensional trivial module*
- *For $V_{2,0}$ we have*

$$X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$Y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

- *For $V_{2,1}$ we have*

$$X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$Y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

- *For $V_{2,\infty}$*[3] *we have*

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

[1] This calculation took approximately 90 minutes.

[2] The notation for these modules is adapted from [1]

[3] Of course, none of these two dimensional modules are faithful. But we may form faithful modules by taking direct sums of them.

$$Y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

- *For V_3 we have*

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Y = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- *For V_{-3} we have*

$$X = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Y = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- *$\overline{V}_4$ is the unique projective indecomposable or kP-module - the regular kP-module.*

Consequently, unless V is isomorphic to a direct sum of modules as above, we have $\operatorname{depth}(R^G) = \dim(V^P) + 2$ *and* $\operatorname{depth}_{R^G}(H^1(G, R)) = \dim(V^P)$. *Furthermore, if V contains a direct summand isomorphic to $V_3 \oplus V_3$ or $V_{-3} \oplus V_{-3}$ then $H^1(G, R)$ is not detected on the set of subgroups of P, and we have the same conclusion.*

The proof of ([10], Corollary 3) uses the classification of kP-modules for $C_2 \times C_2$ ([1], Theorem 4.3.3). On the other hand, no classification of such modules for the groups $C_p \times C_p$ exists, so the Klein four-group is very much a special case. We are now able, using Proposition 2.3, to classify completely the representations of P for which R^G has minimal depth. We will need to calculate $\psi(P)$ for each V which is a direct summand of modules isomorphic to those in $\mathcal{S}$. Since we're fixing P throughout, we abuse our notation from section 2 by defining

$$\psi(V) = \{Q < P : V^P \subsetneqq V^Q\}.$$

It is then easy to see that

$$\psi(V \oplus W) = \psi(V) \cup \psi(W). \tag{6}$$

We will need the following lemmata:

Lemma 3.2. *Let V be a faithful representation of P which is a direct summand of modules isomorphic to those in $\mathcal{S}$. Then $\psi(V)$ consists of the three maximal subgroups of P unless $V \cong V_3^{\oplus a} \oplus W^{\oplus b} \oplus W'^{\oplus c} \oplus V_1^{\oplus d}$ where a, b, c and d are any integers ≥ 0 and W and W' are two of the three two dimensional modules in $\mathcal{S}$.*

Proof. We calculate $\psi(V_1) = \emptyset$, $\psi(V_{2,0}) = \{1, \langle Y \rangle\}$, $\psi(V_{2,1}) = \{1, \langle XY \rangle\}$, $\psi(V_{2,\infty}) = \{1, \langle X \rangle\}$, $\psi(V_3) = 1$, while both $\psi(V_{-3})$ and $\psi(V_4)$ consists of all proper subgroups of P. The result now follows immediately from (6). □

Lemma 3.3. *Let W be a right kP-module and let Q, Q' be a pair of distinct maximal subgroups of P. Then $H^1(P, W)$ is detected on the pair Q, Q' if and only if*

$$\mathrm{Tr}_Q^P(W^Q) = \mathrm{Tr}_1^{Q'}(W) \cap W^P$$

Proof. It is clear that LHS $\subseteq$ RHS with no assumptions on $H^1(P, W)$. Consider the composition

$$H^1(P/Q, W^Q) \to H^1(P, W) \to H^1(Q', W)^{P/Q'} \tag{7}$$

where the first map is the inflation inf_Q^P, which is the map induced on cohomology by the canonical quotient map $P \to P/Q$ and the module inclusion $W^Q \to W$, and the second is the restriction $\mathrm{res}_{Q'}^P$. For more details on these maps, we point the reader towards [11], chapter seven. In particular, by [11], Corollary 7.2.3, inf_Q^P is an injective map with image equal to the kernel of the restriction res_Q^P. It follows that $H^1(P, W)$ is detected on the pair Q, Q' if and only if the composition of maps (7) is injective. Now since P/Q is a cyclic group of order two, we have

$$H^1(P/Q, W^Q) \cong W^P / \mathrm{Tr}_Q^P(W^Q),$$

and, by [14] Lemma 6.2, if $u \in W^P$ represents a non-zero element of $H^1(P/Q, W^Q)$ then its image under the composition (7) is zero if and only if u represents zero in

$$H^1(Q'/(Q \cap Q'), W^{Q \cap Q'}) = H^1(Q', W) = W^{Q'} / \mathrm{Tr}_1^{Q'}(W)$$

from which the desired conclusion follows. □

Proposition 3.4. *Let V be a faithful representation of P and let $R := \mathrm{Sym}(V^*)$. Then $H^1(P, R)$ has minimal depth if and only if one of the following holds:*

1. *V contains a direct summand not isomorphic to any of the seven modules in $\mathcal{S}$*
2. *V contains a direct summand isomorphic to $V_3 \oplus V_3$*
3. *V contains a direct summand isomorphic to $V_{-3} \oplus V_{-3}$*
4. *V is isomorphic to $V_3 \oplus W^{\oplus a} \oplus V_1^{\oplus b}$ where a and b are any integers ≥ 0 and W is one of the three two dimensional modules in $\mathcal{S}$.*

R^P *has minimal depth if and only if either* $\operatorname{codim}(V^P) \leq 2$ *or one of the above holds.*

Proof. That the first three statements imply minimal depth for $H^1(P,R)$ is already covered by Theorem 3.1. If none of these three statements hold, then $H^1(G,R)$ is detected on the set of proper subgroups of P (see [10], Theorem 9 and its proof). If additionally statement (4) does not hold, then by Lemma 3.2, either $\psi(V)$ consists of all proper subgroups of P, in which case Proposition 2.3 tells us that $\operatorname{depth}_{R^P}(H^1(P,R)) > \dim(V^P)$, or else $V \cong V_3^{\oplus j} \oplus W^{\oplus a} \oplus W'^{\oplus b} \oplus V_1^{\oplus c}$, where j is zero or one, a and b are positive integers and c is a positive integer or zero, with $\psi(V)$ consisting of the two proper subgroups Q and Q' of P which act trivially on the summands W and W' respectively.

Suppose $V \cong V_3^{\oplus j} \oplus W^{\oplus a} \oplus W'^{\oplus b} \oplus V_1^{\oplus c}$, and there exists a nonzero cohomology class $\alpha \in H^1(P,R)$ which restricts to zero on Q and Q'. Since R decomposes as $\bigoplus_{i\geq 0} S^i(V^*)$, and in each degree $S^i(V^*)$ decomposes further into a direct sum of kP-modules, we may assume there exists a degree d and direct summand M of $S^d(V^*)$ such that $\alpha \in H^1(P,M)$. We may also evaluate

$$\operatorname{Sym}(V^*) \cong \operatorname{Sym}(V_3^*)^{\otimes j} \otimes \operatorname{Sym}(W^*)^{\otimes a} \otimes \operatorname{Sym}(W'^*)^{\otimes b}.$$

[4] Now every direct summand of $\operatorname{Sym}(W^*)$ is either trivial or isomorphic to W^*. Similarly, every direct summand of $\operatorname{Sym}(W'^*)$ is either trivial or isomorphic to W'^*, and we know from [10] (in the proof of Theorem 9) that every direct summand of $\operatorname{Sym}(V_3^*)$ is either trivial, isomorphic to V_3^* or isomorphic to $\overline{V}_4^*$. The following table (also in [10]; constructed using the 'meat-axe' function in MAGMA) tells us how tensor products of these modules decompose:

Table 1: Decomposing Tensor Products of kP-modules

$\otimes$	$V_{2,1}$	$V_{2,\infty}$	$V_{2,0}$	V_3	$\overline{V}_4$
$V_{2,1}$	$V_{2,1}^{\oplus 2}$	$\overline{V}_4$	$\overline{V}_4$	$V_{2,1} \oplus \overline{V}_4$	$\overline{V}_4^{\oplus 2}$
$V_{2,\infty}$		$V_{2,\infty}^{\oplus 2}$	$\overline{V}_4$	$V_{2,\infty} \oplus \overline{V}_4$	$\overline{V}_4^{\oplus 2}$
$V_{2,0}$			$V_{2,0}^{\oplus 2}$	$V_{2,0} \oplus \overline{V}_4$	$\overline{V}_4^{\oplus 2}$
V_3				$\overline{V}_4 \oplus V_5$	$\overline{V}_4^{\oplus 3}$
$\overline{V}_4$					$\overline{V}_4^{\oplus 4}$

We conclude, then, that M is isomorphic to one of the following:

$$\{V_1^*, W^*, W'^*, V_3^*, \overline{V}_4^*\}.$$

Since $\overline{V}_4^*$ is projective, $H^1(P,\overline{V}_4^*) = 0$ and so M cannot be isomorphic to $\overline{V}_4^*$. Suppose that $M \cong W'^*$. Then (remembering that Q' acts trivially on M)

[4] It is easy to see that W and W' are self dual, but we keep the asterisks to show we're thinking of right modules. Note also that $A \otimes V_1 \cong A$ for any kP-module A.

we have

$$0 = \mathrm{Tr}_Q^P(W^Q) = \mathrm{Tr}_1^{Q'}(W) \cap W^P$$

and it follows from Lemma 3.3 that $H^1(P, M)$ is detected on the pair Q, Q'. The corresponding result for W^* also follows similarly. If $M \cong V_1^*$ then both Q' and Q act trivially and again using Lemma 3.3, $H^1(P, M)$ is detected on the pair Q, Q'. So we may assume $M \cong V_3^*$. Let $\{y_1, y_2, y_3\}$ be a basis for M. Then we have, for any pair Q, Q' by direct calculation

$$\mathrm{Tr}_1^{Q'}(M) = \mathrm{Tr}_Q^P(M^Q) = M^P = \langle y_3 \rangle$$

so by Lemma 3.3, $H^1(P, M)$ is detected on the pair Q, Q' as required. So there cannot be a nonzero $\alpha \in H^1(P, R)$ which restricts to zero on a pair of proper subgroups Q, Q' in this case.

Finally we must show that statement (4) implies minimal depth for $H^1(P, R)$. In this case, $\psi(V)$ consists of the trivial group and one proper subgroup of P. Therefore, using Proposition 2.3 it is enough to show that $H^1(P, R)$ is not detected on any proper subgroup Q of P, that is to say, $\ker(\mathrm{res}_Q^P) \neq 0$. But this is in fact true for any faithful representation of a p-group (provided R is not projective, which is clear in this case) - see [11], Corollary 7.2.3. □

It was noted in [10] (on the final page) that when $V \cong V_3 \oplus V_{2,1} \oplus V_{2,1}$, R^P has minimal depth, but $H^1(P, R)$ is detected on the set of subgroups of P. The above proof explains why this is the case - it is an example of a representation where statement (4) applies.

4 Direct Products

As in section 1, let G be a finite group acting on a vector space V over a field k of characteristic p, with $R := \mathrm{Sym}(V^*)$ and P a Sylow-p-subgroup of G. It can be shown (see [17]) that

$$\mathrm{depth}(R^P) \leq \mathrm{depth}(R^G). \tag{8}$$

Consequently, R^G is Cohen-Macaulay if and only if R^P is so. This statement may be interpreted as saying that R^G has maximal depth if and only if R^P has maximal depth. In this section we ask to what extent this statement holds if maximal is replaced by minimal. Note that the above inequality already implies that the depth of R^P is minimal whenever the depth of R^G is minimal, so we need only to find when the reverse implication holds.

Let $G := P \times Q$ and L be any kG - module. Then the Lyndon-Hochschild-Serre spectral sequence (see, e.g. [11], chapter 7) gives two short exact sequences:

$$0 \to H^1(Q, L^P) = H^1(G/P, L^P) \to H^1(G, L) \to H^1(P, L)^{G/P} \tag{9}$$

$$0 \to H^1(P, L^Q) = H^1(G/Q, L^Q) \to H^1(G, L) \to H^1(Q, L)^{G/Q}. \tag{10}$$

Note that in order for these sequences to arise it is not necessary that P be a p-group and Q a p'-group. When they do take this form, the first short exact sequence specializes to the inclusion given by restriction

$$H^1(G, L) \hookrightarrow H^1(P, L),$$

whose image consists of G - stable cohomology, hence we get $H^1(G, L) \cong H^1(P, L)^Q$.
The second short exact sequence specializes to $H^1(P, L^Q) \cong H^1(G, L)$, since $H^1(Q, L) = 0$. So we have

$$H^1(P, L^Q) \cong H^1(G, L) \cong H^1(P, L)^Q.$$

In particular the cohomological connectivities of V and of $V_{|P}$ coincide. Since P is a p-group, we have $cc_G(R) = cc_P(R) = 1$.

Proposition 4.3 is the key to proving the main result of this section. We will need the following lemma describing the structure of the symmetric algebra:

Lemma 4.1. *Let V be a kP module for a p-group P and field k of characteristic p. Then the symmetric algebra $R := \mathrm{Sym}(V^*)$ splits as*

$$R = uR \oplus B$$

where the homogeneous invariant u and kP-submodule B are described below.

It is not entirely clear to whom this lemma should be attributed. An argument similar to the one below is used in [16], Lemma 2.9 but this result seems to have been known for some time. For lack of a good reference, we include a proof.

Proof. Since P is a p-group, the only irreducible kP-module is trivial. Consequently, P has an upper triangular representation on V and $R = k[x_1, \dots, x_n]$ may be viewed as $k[x_2, \dots, x_n][x_1]$, since $k[x_2, \dots, x_n]$ is a kP-module. So we view polynomials in R as polynomials in the single variable x_1 with coefficients in $k[x_2, \dots, x_n]$. If $r \in R$ we define $\deg(r)$ to be the degree of r when viewed as a polynomial in x_1. Then if $r \in R$ and $p \in P$, $\deg(r \cdot p) \leq \deg(r)$. Let B be the kP-submodule of R consisting of polynomials whose degree (as polynomials in x_1) is less than $|P|$. Consider the invariant $u := \prod_{g \in P} x_1.g$. In $k[x_2, \dots, x_n][x_1]$, this is a monic polynomial of degree $|P|$. Given any $r \in R$ with $\deg(r) \geq |P|$ we may perform successive long division by u, giving us a unique expression $r = qu^a + b$ for some $q \in R$, $b \in B$, $a \in \mathbb{N}$. Clearly $uR \cap B = 0$ and therefore $R = uR \oplus B$. □

Corollary 4.2. *If W is kP-module and W' is a direct summand of the lth symmetric power $S^l(W)$ then uW' is an isomorphic direct summand of $S^{l+|P|}(W)$.*

We say that the invariant u *propagates direct summands* of R.

Proposition 4.3. *Let G be as above and let ψ be a set of subgroups of P. Suppose in addition that k is algebraically closed. Then $H^1(G,R)$ is detected on ψ if and only if $H^1(P,R)$ is detected on ψ.*

Proof. Since $H^1(G,R)$ is a direct summand of $H^1(P,R)$, the "if" part is immediate. Suppose $H^1(P,R)$ is not detected on ψ, so we can find $0 \neq \alpha \in H^1(P,R)$ be such that $\mathrm{res}^P_N(\alpha) = 0$ for every $N \in \psi$. Then since k is algebraically closed, $V^* = \oplus_{j=1}^k W_j$ with $W_j \cong U_j \otimes k_{\epsilon_j}$, where each U_j is an indecomposable kP - module, k_{ϵ_j} is a one - dimensional Q - module with character $\epsilon_j \in \mathrm{Hom}(Q, k^\times)$ and diagonal action $(u \otimes \lambda)(p,q) := up \otimes \lambda\epsilon_j(q)$. Hence we have a decomposition of the G - module

$$R = \otimes_j \mathrm{Sym}(W_j) = \oplus_{s=0}^\infty \oplus_{\substack{\underline{\ell} \in \mathbb{N}^k, \\ |\underline{\ell}|=s}} X_{\underline{\ell}}$$

with $X_{\underline{\ell}} := X_{\ell_1} \otimes \cdots \otimes X_{\ell_k}$ and $X_{\ell_i} := S^{\ell_i}(W_i)$. Note that Q acts on each $S^{\ell_i}(W_i)$ by the linear character $\epsilon_i^{\ell_i}$ and on $X_{\underline{\ell}}$ by the character $\prod_{j=1}^k \epsilon_j^{\ell_j}$. For every $N \in \psi$ we have decompositions of k - spaces

$$H^1(P,R) = \oplus_{\underline{\ell}} H^1(P, X_{\underline{\ell}})$$

$$H^1(N,R) = \oplus_{\underline{\ell}} H^1(N, X_{\underline{\ell}})$$

which are preserved by the corresponding restriction map.
It follows that $\alpha = \oplus_{\underline{\ell}} \alpha_{\underline{\ell}}$ with $\alpha_{\underline{\ell}} \in H^1(P, X_{\underline{\ell}})$ and $\mathrm{res}^P_N(\alpha_{\underline{\ell}}) = 0$ for every $N \in \psi$ and $\underline{\ell}$. Hence we can assume that $0 \neq \alpha \in H^1(P, X_{\underline{\ell}})$. For every $j = 1, \cdots, k$, the space X_{ℓ_j} is a kG - direct summand of $\mathrm{Sym}(W_j)$ and by Lemma 4.1 there is a suitable homogeneous invariant $u_j \in \mathrm{Sym}(W_j)^P$ of degree $|P|$, propagating direct summands of $\mathrm{Sym}(W_j)$. Choosing $a_j, b_j \in \mathbb{N}$ such that $b_j|Q| - a_j|P| = \ell_j$, which we can do because $|P|$ and $|Q|$ are coprime, we see that $u_j^{a_j} \cdot X_{\ell_j}$ is a direct summand of $\mathrm{Sym}(W_j)$ and also a submodule of

$$S^{\ell_j + |P|a_j}(W_j) = S^{b_j|Q|}(W_j) \leq \mathrm{Sym}(W_j)^Q$$

Let $u := \otimes_{j=1}^k u_j^{a_j} \in R^P$, then $X_{\underline{\ell}}$ and $u \cdot X_{\underline{\ell}} = \otimes_{j=1}^k u_j^{a_j} \cdot X_{\ell_j}$ are isomorphic kP - modules and the latter one is a direct summand of R^Q. It follows that there is

$$0 \neq \tilde{\alpha} \in H^1(P, uX_{\underline{\ell}}) \mid H^1(P, R^Q)$$

satisfying $\mathrm{res}^P_N(\tilde{\alpha}) = 0$ for all $N \in \psi$. Therefore $\tilde{\alpha} = \mathrm{res}^G_P(\beta)$ with $\beta \in H^1(G,R)$ and

$$\mathrm{res}^G_N(\beta) = \mathrm{res}^P_N(\mathrm{res}^G_P(\beta)) = \mathrm{res}^P_N(\tilde{\alpha}) = 0$$

for all ψ, that is, $H^1(G,R)$ is not detected on ψ as required. ☐

Corollary 4.4. *Let G be of the form $P \times Q$ with P a p-group and Q an abelian p'-group. Then the depth of R^P is minimal if and only if the depth of R^G is minimal.*

Proof. Assume first that $\text{codim}(V^P) \leq 2$. Then by Theorem 1.2 we have $\text{depth}(R^G) = \text{depth}(R^P) = \dim(V)$ so the proposition is true in this case. Assuming the contrary, we must show that $\text{depth}(R^P)$ minimal implies $\text{depth}(R^G)$ minimal, the converse having been dealt with by (8).

Note that if $\overline{k}$ denotes the algebraic closure of k, then the extension $k \to \overline{k}$ is flat. Consequently for any R^G-module M, we have $\text{depth}(M) = \text{depth}(M \otimes \overline{k})$ (see [4], Proposition 1.2.16) and we may assume k is algebraically closed. If $\text{depth}(R^P)$ is minimal, then by Proposition 2.3, $H^1(P, R)$ is not detected on the set $\psi(P) := \{N < P : V^P \subsetneqq V^N\}$. Applying Proposition 4.3 with $\psi := \psi(P)$, we see that $H^1(G, R)$ is not detected on $\psi(P)$, and by Proposition 2.3 once more, the depth of R^G is minimal. $\square$

So for a group G of the form $P \times Q$ as above, the depth of R^G is minimal if and only if the depth of R^P is minimal, and the depth of R^G is maximal if and only if the depth of R^G is maximal. It is interesting to ask whether in fact $\text{depth}(R^P) = \text{depth}(R^G)$ for all groups of this form, and we can find no evidence to the contrary. We can however, show that we cannot conclude $\text{depth}(R^G)$ minimal if and only if $\text{depth}(R^P)$ is minimal when P is merely a normal Sylow-p-subgroup of G. The following example is used (for a different purpose) in [19], Example 4.6. Let p be any prime ≥ 5 and consider the following subgroup of $SL_2(p)$:

$$G := \left\{ \begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} : \ a \in \mathbb{F}_p^\times, \ x \in \mathbb{F}_p \right\}.$$

Then G is a semidirect product of the form $\mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$, and has a normal Sylow-p-subgroup P of order p consisting of those matrices above in which $a = 1$. The action of G on P by conjugation is given by the formula

$$\begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & a^2 y \\ 0 & 1 \end{pmatrix} \tag{11}$$

which is easily checked. Let $V := S^{p-1}(\mathbb{F}_p^2)$ be the $(p-1)$th symmetric power of the natural module. Note that the centre Z of G now acts trivially on V, so we regard V as a module for the group $H := G/Z$ which is a semidirect product of the form $\mathbb{Z}_p \rtimes \mathbb{Z}_{(p-1)/2}$. Now as a kP-module, V is clearly projective and indecomposable, hence isomorphic to the regular module. Moreover since $V^H \neq 0$, V is the unique projective indecomposable kH-module containing the trivial module. Therefore V is also a permutation module, there is a natural extension of the action of H on V to the symmetric group S_p, and since the action of H on P by conjugation is isomorphic to the action of $\mathbb{Z}_{(p-1)/2}$ on the additive group $\mathbb{Z}_p$ given by multiplication by squares, H is the normaliser of P in the alternating group A_p. Now A_p is a trivial intersection group (i.e. for each $g \in G$, $P \cap g^{-1}Pg$ is either P or the trivial group) and so the normaliser of P in A_p is strongly p-embedded into A_p (see [18], Corollary 1.2) and we conclude that

$$\text{depth}(R^H) = \text{depth}(R^{A_p})$$

It is well known that the invariant ring R^{A_p} for the natural action of A_p on a polynomial ring in p variables is a hypersurface (see e.g. [23], Corollary 1.3.2). In particular R^{A_p} is Cohen-Macaulay so we conclude that $\operatorname{depth}(R^H) = p$. On the other hand P is cyclic of prime order, so by [7], we have $\operatorname{depth}(R^P) = \dim(V^P)+2 = 3$ which is minimal. We must show that $\operatorname{depth}(R^H)$ is not minimal, which means that $\dim(V^P)+1+cc_H(R) < p$, or more simply $cc_H(R) < p-2$. In fact we will show that $cc_H(R) \leq (p-1)/2$ which is strictly less than $p-2$ for all $p \geq 5$. Now R contains a trivial direct summand in degree zero, and

$$H^*(H,k) = H^*(P,k)^H.$$

$H^*(P,k)$ is a polynomial ring over k in one variable z, and the action of H on $H^*(P,k)$ is given by the formula

$$\begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} \cdot z = a^{-2}z.$$

It follows that

$$\begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} \cdot z^{(p-1)/2} = (a^{-2}z)^{(p-1)/2} = a^{-(p-1)}z^{(p-1)/2} = z^{(p-1)/2}$$

and so $H^{(p-1)/2}(H,k) \neq 0$. Consequently we have $H^{(p-1)/2}(H,R) \neq 0$ which shows that $cc_H(R) \leq (p-1)/2$ as required.

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Depth and Detection in Modular Invariant Theory

Jonathan Elmer
Lehrstuhl A für Mathematik
RWTH-Aachen
Aachen 52062
Germany

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Abstract

Let G be a finite group acting linearly on a vector space V over a field of characteristic p dividing the group order, and let $R := S(V^*)$. We study the R^G modules $H^i(G,R)$, for $i \geq 0$ with R^G itself as a special case. There are lower bounds for $\mathrm{depth}_{R^G}(H^i(G,R))$ and for $\mathrm{depth}(R^G)$. We show that a certain sufficient condition for their attainment (due to Fleischmann, Kemper and Shank [14]) may be modified to give a condition which is both necessary and sufficient. We apply our main result to classify the representations of the Klein four-group for which $\mathrm{depth}(R^G)$ attains its lower bound, a process begun in [10]. We also use our new condition to show that the if $G = P \times Q$, with P a p-group and Q an abelian p'-group, then the depth of R^G attains its lower bound if and only if the depth of R^P does so.

1 Introduction

Let G be a finite group acting on a polynomial ring $R := k[x_1, x_2, \ldots, x_n]$. Then the set of fixed points under this action form a ring R^G called the ring of invariants. Suppose in addition that k is a field of characteristic p which divides the group order. Then R^G is the central object of study in modular invariant theory. Since R^G can be regarded as the zeroth cohomology $H^0(G,R)$, it is often worthwhile studying it in conjunction with the higher cohomology modules $H^i(G,R)$, which become R^G-modules via the cup product.

In this paper we will concentrate on the standard situation in which $R := \mathrm{Sym}(V^*)$ (the symmetric algebra of the dual) for some finite dimensional left kG-module V, and the (right) action of G on R is by graded algebra automorphisms. If k is algebraically closed we can regard R as the space $k[V]$ of polynomial functions on V, and R^G as the set of G-invariant polynomial functions on V, or the ring of coordinate functions on the quotient space V/G. With this identification, we may define, for any ideal I of R^G, the variety

$$\mathcal{V}(I) := \{v \in V : f(v) = 0 \ \forall \ f \in I\} \tag{1}$$

and for any subset of points $U \subseteq V$ the ideal

$$\mathcal{I}(U) := \{f \in R^G : f(u) = 0 \ \forall \ u \in U\}. \tag{2}$$

Note that in order to apply Hilbert's Nullstellensatz directly we should instead take varieties in the categorical quotient V/G. However, a version of the Nullstellensatz exists which allows us to use standard techniques on the objects defined above:

Lemma 1.1 (See Kemper [18], Lemma 3.3)**.** *For any ideal J of R^G, and with $\mathcal{I}$ and $\mathcal{V}$ defined as above, we have*

$$\mathcal{I}(\mathcal{V}(J)) = \sqrt{J}.$$

1.1 Depth

Let $A = \oplus_{i \in \mathbb{N}_0} A_i$ be a graded connected k - algebra, by which we mean that A is a graded k-algebra with $A_0 = k$, and define $A_+ := \oplus_{i>0} A_i$; furthermore let $J \subseteq A_+$ be a homogeneous ideal and M be a graded A-module. A sequence of homogeneous elements $(a_1, \cdots, a_k)$ with $a_i \in J$ is called M**-regular**, if for every $i = 1, \cdots, k$ the multiplication by a_i induces an injective map on the quotient ring $M/(a_1, \cdots, a_{i-1})M$. It is known that all maximal M-regular sequences in J have the same length grade(J, M), called the grade of J on M, and one now defines

$$\operatorname{depth}(M) := \operatorname{grade}(A^+, M).$$

It is clear from the construction that $\operatorname{depth}(A) \leq \dim(A)$. For other standard results concerning depth and grade, the reader is referred to [4]. We will be concerned with calculating the depth of the invariant ring R^G, or more generally the R^G-modules $H^i(G, R)$. It is well known (see [6]) that in the non-modular case, all (standard) invariant rings are Cohen-Macaulay, that is, $\operatorname{depth}(R^G) = \dim(R^G)$, so from this point onwards we will consider only the modular situation. Little is known in general about the depth of modular invariant rings and cohomology modules. One significant result, due to Ellingsrud and Skjelbred [7], gives us a lower bound for the depth of modular invariant rings. Their result was strengthened in [14] to the following:

Theorem 1.2. *Let G be a finite group and k a field of characteristic p. Let P be a Sylow-p-subgroup and let V be a left kG-module. Let R denote the symmetric algebra $S(V^*)$ which has a natural right-module structure. Then*

$$\operatorname{depth}(R^G) \geq \min\{\dim(V), \dim(V^P) + cc_G(R) + 1\}$$

where V^P denotes the fixed point space of P on V and $cc_G(R)$ is called the cohomological connectivity *and defined as* $\min\{i > 0 : H^i(G, R) \neq 0\}$.

In this article, we shall say that an invariant ring R^G has *minimal depth* if the above is an equality. Ellingsrud and Skjelbred proved that R^G has minimal depth when G is cyclic of prime order, and that $H^i(G, R)$ is Cohen-Macaulay for all $i > 0$ in this situation. In [14] it was shown that R^G has minimal depth for every p-nilpotent group with cyclic Sylow-p-subgroups, and in [15] the same authors were able to calculate $\mathrm{depth}_{R^G}(H^i(G, R))$ for any i when G is a cyclic p-group. In [10], similar techniques were used to show that R^G has minimal depth whenever $G = C_2 \times C_2$ and V is indecomposable and not projective. The authors also showed that R^G has minimal depth for many decomposable representations of $C_2 \times C_2$, but were unable to classify completely those representations of G with this property. We return to this question in section 3.

The starting point for all our depth calculations is the following result due to Kemper ([19], Theorem 1.5):

Theorem 1.3. *Let G be a finite group acting linearly on $R := S(V^*)$. Let $U \leq V$ be a kG-submodule for which the kernel of the action of G on U has index in G not divisible by p - we will call this a non-modular submodule. Let $M := H^i(G, R)$ for some $i \geq 0$. Then we have*

$$\mathrm{depth}_{R^G}(M) = \mathrm{grade}(\mathcal{I}(U), M) + \dim(U)$$

Let P denote a fixed Sylow-p-subgroup of G. Then we observe that, for any group G and kG-module V, the fixed point set V^P is a non-modular submodule. Further, by [12], Theorem 5.9, $\mathcal{I}(V^P) = \sqrt{I^G_{\mathcal{X}(P)}}$ where $I^G_{\mathcal{X}(P)}$ is a relative transfer ideal as defined in the next section. Although it may seem that Theorem 1.3 merely changes the question of calculating $\mathrm{depth}_{R^G}(H^i(G, R))$ to one of calculating $\mathrm{grade}(\mathcal{I}(U), H^i(G, R))$, it is nonetheless an extremely useful result, since when $i > 0$ the latter quantity is often zero, and these occasions are not so difficult to spot. It follows easily from this result that

$$\mathrm{depth}_{R^G}(H^i(G, R)) \geq \dim(V^P)$$

for any kG-module V. Accordingly, we shall say that $H^i(G, R)$ $(i > 0)$ has minimal depth when $\mathrm{depth}_{R^G}(H^i(G, R)) = \dim(V^P)$. (Note that when $i = 0$, we do not recover the original definition of minimal depth).

2 Relative Transfer Ideals

If $X < G$, we may define a mapping $\mathrm{Tr}^G_X : R^X \to R^G$ as follows: let S be a set of right coset representatives of X in G. Then we define

$$\mathrm{Tr}^G_X(x) := \sum_{g \in S} xg. \tag{3}$$

This mapping is often called the *relative transfer*, and induces mappings Tr^G_X : $H^i(X, R) \to H^i(G, R)$ also called the relative transfer. Both are surjective when the index of X in G is coprime to p. The image of the transfer map $\mathrm{Tr}_X^G(R^X)$ is an ideal in R^G called the *relative transfer ideal* which we denote by I_X^G. We may generalise this definition and define

$$I_\chi^G := \sum_{X \in \chi} I_X^G$$

for any set χ of subgroups of G.

In this section we will study I_χ^G for various choices of χ, which is always assumed to be a set of p-subgroups of G and closed under taking subgroups. We will assume throughout that k is algebraically closed. For convenience, let $\mathfrak{i}_\chi$ denote $\sqrt{I_\chi^G}$. Then we have the following result: due to Fleischmann ([12], Theorem 5.9):

$$\mathcal{V}(I_\chi^G) = \{v \in V : p \mid [G_v : G_v \cap X] \ \forall \ X \in \chi\} = \bigcup_{Q \in \chi'} V^Q \tag{4}$$

where χ' denotes the set of p-subgroups of G not conjugate to any subgroup in χ.

Lemma 2.1 ([14], Corollary 3.2)**.** *Suppose χ_1 and χ_2 are sets of p-subgroups of G, closed under conjugation and taking subgroups. Then $\mathcal{V}(I_{\chi_1}^G) \subseteq \mathcal{V}(I_{\chi_2}^G)$ if and only if, for every $Q \in \chi_2 \setminus \chi_1$ there exists $Q' \in \chi_2'$ such that Q is a proper subgroup of Q', but $V^Q = V^{Q'}$.*

Proof. Suppose $\mathcal{V}(I_{\chi_1}^G) \subseteq \mathcal{V}(I_{\chi_2}^G)$ and let $Q \in \chi_2 \backslash \chi_1$. Then (4) shows that $V^Q \subsetneqq \mathcal{V}(I_{\chi_1}^G) \subseteq \mathcal{V}(I_{\chi_2}^G)$. In particular, $V^Q \subseteq \mathcal{V}(I_{\chi_2}^G)$ and so again by (4), $p \mid [G_v : Q]$ for every $v \in V^Q$. This means there is a Sylow-p-subgroup P_v of G_v which satisfies $Q < P_v$, and $V^{P_v} \subseteq V^Q$ while $V^Q \leq \bigcup_{v \in V^Q} V^{P_v}$. Since we're assuming k is algebraically closed and therefore infinite, we must have $V^Q \leq V^{P_w}$ for some $w \in V^Q$, and so $V^Q = V^{P_w}$ as required.

Conversely suppose for each $Q \in \chi_2 \backslash \chi_1$ there exists $Q' \in \chi_2'$ such that Q is a proper subgroup of Q', but $V^Q = V^{Q'}$. This means for each $Q \in \chi_2 \cap \chi_1'$, $V^Q = V^{Q'}$ for some $Q' \in \chi_2'$. Therefore for any $Q \in \chi_1'$, $V^Q \subseteq \bigcup_{Q' \in \chi_2'} V_{Q'}$ and the result follows from (4). □

Corollary 2.2. *Let $\chi(P)$ denote the set of all proper subgroups of a fixed Sylow-p-subgroup P of G, and let $\psi(P) := \{Q < P : V^P \subsetneqq V^Q\}$. Also define $\chi := \{Q < G : Q \text{ is a } p-\text{group and } p \mid [G : Q]\}$ and $\psi := \{Q < G : Q \text{ is a } p-\text{group and } V^P \subsetneqq V^Q \text{ for every Sylow}-p-\text{subgroup } P \geq Q\}$. Then $\mathfrak{i}_\chi = \mathfrak{i}_{\chi(P)} = \mathfrak{i}_{\psi(P)} = \mathfrak{i}_\psi$.*

Proof. The formula $\mathrm{Tr}_Q^G(x) = \mathrm{Tr}_{Q^g}^G(xg)$ shows that $I_\chi^G = I_{\chi(P)}^G$ and $I_\psi^G = I_{\psi(P)}^G$, from which the first and third equalities follow immediately. Note that ψ and χ are closed under conjugation. By Lemma 1.1 it suffices to show $\mathcal{V}(I_\chi^G) = \mathcal{V}(I_\psi^G)$. Since $\psi \subseteq \chi$ it is clear that $\mathcal{V}(I_\chi^G) \subseteq \mathcal{V}(I_\psi^G)$ and so we need prove only the converse. Note that if $Q \in \chi \setminus \psi$, then $V^P = V^Q$ for every Sylow-p-subgroup P containing Q as a subgroup. The result now follows from Lemma 2.1. □

As well as the transfer maps, we shall consider the restriction maps $\mathrm{res}^G_Q : H^m(G,R) \to H^m(Q,R)$, which are the maps on cohomology induced by the inclusion $Q \subset G$. We shall say that $H^m(G,R)$ is *detected* on a set $\mathcal{Q}$ of subgroups of G if the product of maps

$$\prod_{Q\in\mathcal{Q}} \mathrm{res}^G_Q : H^m(G,R) \to \prod_{Q\in\mathcal{Q}} H^m(Q,R)$$

is an injection. Many of the results of [14] and [10] depend on the observation that $\mathrm{grade}(\mathfrak{i}, H^m(G,R)) = 0$ if $H^m(G,R)$ is not detected on the set $\chi(P)$. Here we are able to prove a variation on this result which is both necessary and sufficient:

Proposition 2.3. *Let G be a finite group acting linearly on a k-vector space V, where k is an algebraically closed field of characteristic p dividing the group order. Let P be a Sylow-p-subgroup of G and let $R := \mathrm{Sym}(V^*)$. Let $\mathfrak{i} := \mathfrak{i}_\chi = \mathfrak{i}_\psi$ as above, and let m be a strictly positive integer. Then the following are equivalent:*

1. $\mathrm{depth}_{R^G}((H^m(G,R)) = \dim(V^P)$
2. $\mathrm{grade}(\mathfrak{i}, H^m(G,R) = 0$
3. $\mathfrak{i}$ *is an associated prime of* $H^m(G,R)$
4. $H^m(G,R)$ *is not detected on* $\psi(P)$.

Remark: The equivalence of (1), (2) and (3) was noted in [14], and the fact that (4) implies any of these is shown using only the results of [14]. The implication (3) $\Rightarrow$ (4) is new, and requires the following explicit description of the associated primes of $H^m(G,R)$ found in [8]:

Theorem 2.4 ([8], Theorem 1.1)**.** *Let $\mathfrak{p}$ be an associated prime ideal of the R^G-module $\oplus_{i>0}H^i(G,R)$. Then $\mathfrak{p} = \sqrt{I^G_\chi}$ for some set χ of subgroups of G.*

We now prove Proposition 2.3.

Proof. (1) $\Leftrightarrow$ (2) follows immediately from Theorem 1.3. If $\mathrm{grade}(\mathfrak{i}, H^m(G,R)) = 0$, then $\mathfrak{i}$ consists of zero-divisors and consequently is contained in an associated prime; see for example [4], Theorem 1.2.1. Now suppose $\mathfrak{i}$ is an associated prime of the R^G-module $H^m(G,R)$. Then by [8], Proposition 2.5, we have for some $\alpha \in H^m(G,R)$

$$\mathfrak{i} = \sqrt{\mathrm{Ann}_{R^G}(\alpha)} = \mathfrak{i}_\upsilon = \mathfrak{i}_{\upsilon(P)}$$

where $\upsilon = \{Q < G : Q \text{ is a } p-\text{group and } \mathrm{res}^G_Q(\alpha) = 0\}$ (which is closed under conjugation) and $\upsilon(P) = \{Q < P : \mathrm{res}^G_Q(\alpha) = 0\}$. Note that υ cannot contain any Sylow-p-subgroup of G, since res^G_P is injective for every Sylow-p-subgroup P of G. Clearly $H^m(G,R)$ is not detected on $\upsilon(P)$. We will show that $\psi \subset \upsilon$, and hence that $\psi(P) \subset \upsilon(P)$. For, suppose that $Q < G$ satisfies $\mathrm{res}^G_Q \neq 0$. Applying Lemma 2.1 with $\chi_1 = \upsilon$ and $\chi_2 = \chi$, we see that there exists a Sylow-p-subgroup $P \geq Q$ with $V^P = V^Q$, since χ' is the set of Sylow-p-subgroups of G. Consequently, $Q \notin \psi$ as required.

Finally to show that (4) $\Rightarrow$ (2), we notice by [20], Lemma 1.3, that if $\mathrm{res}^G_Q(\alpha) = 0$ for every $Q \in \psi$, then $\mathrm{Ann}_{R^G}(\alpha) \supseteq I^G_\psi$. Consequently we have

$$0 = \mathrm{grade}(I^G_\psi, H^m(G,R)) = \mathrm{grade}(\sqrt{I^G_\psi}, H^m(G,R)) = \mathrm{grade}(\mathfrak{i}, H^m(G,R))$$

where the first equality follows from, for example, [4], Proposition 1.2.10(b). $\square$

Now let $m := cc_G(R)$. Note that by [2], Theorem 4.1, there exists $i > 0$ such that $H^i(G,k) \neq 0$, so we know that m is finite. Note that if $m + 1 \leq \mathrm{codim}(V^P)$, then R^G has minimal depth - this follows immediately from Theorem 1.2. Assuming the opposite, a spectral sequence argument given in [14], section 7, shows that

$$\mathrm{grade}(\mathfrak{i}, R^G) = m + 1 \Leftrightarrow \mathrm{grade}(\mathfrak{i}, H^m(G,R)) = 0. \tag{5}$$

So we can also use our detection condition to say definitively whether or not the depth of R^G is minimal.

2.1 Vector Invariants

One notable consequence of Proposition 2.3 is in the study of vector invariants. Suppose V is a kG-module and consider the direct sum $W = V^{\oplus r}$ for some $r \geq 1$. Then invariants in $\mathrm{Sym}(W^*)^G$ are often called vector invariants. We have the following result:

Proposition 2.5. *Let G, V be as above with $W := V^{\oplus r}$, $W' := V^{\oplus s}$. Suppose that $H^1(G, \mathrm{Sym}(W^*)) \neq 0$, $r > 0$ is sufficiently large that $\mathrm{codim}(W^P) > 2$ and that $s > r$. Then if $\mathrm{Sym}(W^*)^G$ has minimal depth, so does $\mathrm{Sym}(W'^*)^G$.*

Proof. Let $P \in Syl_p(G)$ and $Q < P$. Clearly

$$V^P \subsetneq V^Q \Leftrightarrow W^P \subsetneq W^Q \Leftrightarrow W'^P \subsetneq W'^Q,$$

and so $\psi(P)$ is the same for both W and W'. Let. Since $\mathrm{codim}(W^P) > 2$ and $\mathrm{Sym}(W^*)^G$ has minimal depth, Proposition 2.3 tells us that $H^1(G, \mathrm{Sym}(W^*))$ is not detected on $\psi(P)$. Since $H^1(G, \mathrm{Sym}(W^*))$ is a kP-direct summand of $H^1(G, \mathrm{Sym}(W'^*))$, the latter is not detected on $\psi(P)$ either. Then by Proposition 2.3 once more, the depth of $\mathrm{Sym}(W'^*)^G$ is minimal. $\square$

We can give the following example where this result is used. Suppose $k := \mathbb{F}_p$ and let G be the group of 3×3 unipotent upper triangular matrices (which is sometimes denoted U_3. This is a p-group, so we have $cc_G(R) = 1$ for any representation of G over k. Let V be the natural 3 dimensional kG-module and let $W := V^{\oplus r}$. Furthermore, define $R := \mathrm{Sym}(W^*)$. The invariant rings

R^G for $r = 2$ were studied by Shank and Wehlau [22]; they found that R^G is Cohen-Macaulay if and only if $p = 2$. Let p be an odd prime. When $r = 2$, we have $\dim(W^P) + 2 = 4$ and $\dim(W) = 6$, so the depth of R^G is either 4 or 5. A direct calculation in MAGMA tells us that $\text{depth}(R^G) = 4$ when $p = 3$. [1] Combining this with Proposition 2.5 we obtain:

Corollary 2.6. *Let $G := U_3(\mathbb{F}_3)$ as above and let $W := V^{\oplus r}$. Let $R := \text{Sym}(W^*)$. Then*

$$\text{depth}(R^G) = r + 2$$

3 The Klein Four-Group

In this section we specialize to the case $G = P = C_2 \times C_2$. Let X and Y be generators of this group. Note that $cc_G(R) = 1$ for any representation of P (in fact this is true of any p-group). We recall from [10] the following theorem (in slightly different language):

Theorem 3.1. *Let V be a (faithful) representation of P with $R := \text{Sym}(V^*)$. Then if $H^1(P, R)$ is detected on the set of proper subgroups of P, V must be isomorphic to some direct sum of modules in the following set*[2]

$$\mathcal{S} := \{V_1, V_{2,0}, V_{2,1}, V_{2,\infty}, V_3, V_{-3}, \overline{V_4}\}$$

where the action of P on each of these modules is given by the following matrices:

- *V_1 denotes the one-dimensional trivial module*
- *For $V_{2,0}$ we have*

$$X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$Y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

- *For $V_{2,1}$ we have*

$$X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$Y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

- *For $V_{2,\infty}$*[3] *we have*

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

[1] This calculation took approximately 90 minutes.

[2] The notation for these modules is adapted from [1]

[3] Of course, none of these two dimensional modules are faithful. But we may form faithful modules by taking direct sums of them.

$$Y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

- *For V_3 we have*

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Y = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- *For V_{-3} we have*

$$X = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Y = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- *$\overline{V}_4$ is the unique projective indecomposable or kP-module - the regular kP-module.*

Consequently, unless V is isomorphic to a direct sum of modules as above, we have $\mathrm{depth}(R^G) = \dim(V^P) + 2$ *and* $\mathrm{depth}_{R^G}(H^1(G, R)) = \dim(V^P)$. *Furthermore, if V contains a direct summand isomorphic to $V_3 \oplus V_3$ or $V_{-3} \oplus V_{-3}$ then $H^1(G, R)$ is not detected on the set of subgroups of P, and we have the same conclusion.*

The proof of ([10], Corollary 3) uses the classification of kP-modules for $C_2 \times C_2$ ([1], Theorem 4.3.3). On the other hand, no classification of such modules for the groups $C_p \times C_p$ exists, so the Klein four-group is very much a special case. We are now able, using Proposition 2.3, to classify completely the representations of P for which R^G has minimal depth. We will need to calculate $\psi(P)$ for each V which is a direct summand of modules isomorphic to those in $\mathcal{S}$. Since we're fixing P throughout, we abuse our notation from section 2 by defining

$$\psi(V) = \{Q < P : V^P \subsetneqq V^Q\}.$$

It is then easy to see that

$$\psi(V \oplus W) = \psi(V) \cup \psi(W). \tag{6}$$

We will need the following lemmata:

Lemma 3.2. *Let V be a faithful representation of P which is a direct summand of modules isomorphic to those in $\mathcal{S}$. Then $\psi(V)$ consists of the three maximal subgroups of P unless $V \cong V_3^{\oplus a} \oplus W^{\oplus b} \oplus W'^{\oplus c} \oplus V_1^{\oplus d}$ where a, b, c and d are any integers ≥ 0 and W and W' are two of the three two dimensional modules in $\mathcal{S}$.*

Proof. We calculate $\psi(V_1) = \emptyset$, $\psi(V_{2,0}) = \{1, \langle Y \rangle\}$, $\psi(V_{2,1}) = \{1, \langle XY \rangle\}$, $\psi(V_{2,\infty}) = \{1, \langle X \rangle\}$, $\psi(V_3) = 1$, while both $\psi(V_{-3})$ and $\psi(\overline{V}_4)$ consists of all proper subgroups of P. The result now follows immediately from (6). □

Lemma 3.3. *Let W be a right kP-module and let Q, Q' be a pair of distinct maximal subgroups of P. Then $H^1(P, W)$ is detected on the pair Q, Q' if and only if*

$$\mathrm{Tr}_Q^P(W^Q) = \mathrm{Tr}_1^{Q'}(W) \cap W^P$$

Proof. It is clear that LHS $\subseteq$ RHS with no assumptions on $H^1(P, W)$. Consider the composition

$$H^1(P/Q, W^Q) \to H^1(P, W) \to H^1(Q', W)^{P/Q'} \tag{7}$$

where the first map is the inflation $\inf_Q^P$, which is the map induced on cohomology by the canonical quotient map $P \to P/Q$ and the module inclusion $W^Q \to W$, and the second is the restriction $\mathrm{res}_{Q'}^P$. For more details on these maps, we point the reader towards [11], chapter seven. In particular, by [11], Corollary 7.2.3, $\inf_Q^P$ is an injective map with image equal to the kernel of the restriction res_Q^P. It follows that $H^1(P, W)$ is detected on the pair Q, Q' if and only if the composition of maps (7) is injective. Now since P/Q is a cyclic group of order two, we have

$$H^1(P/Q, W^Q) \cong W^P / \mathrm{Tr}_Q^P(W^Q),$$

and, by [14] Lemma 6.2, if $u \in W^P$ represents a non-zero element of $H^1(P/Q, W^Q)$ then its image under the composition (7) is zero if and only if u represents zero in

$$H^1(Q'/(Q \cap Q'), W^{Q \cap Q'}) = H^1(Q', W) = W^{Q'} / \mathrm{Tr}_1^{Q'}(W)$$

from which the desired conclusion follows. □

Proposition 3.4. *Let V be a faithful representation of P and let $R := \mathrm{Sym}(V^*)$. Then $H^1(P, R)$ has minimal depth if and only if one of the following holds:*

1. *V contains a direct summand not isomorphic to any of the seven modules in $\mathcal{S}$*
2. *V contains a direct summand isomorphic to $V_3 \oplus V_3$*
3. *V contains a direct summand isomorphic to $V_{-3} \oplus V_{-3}$*
4. *V is isomorphic to $V_3 \oplus W^{\oplus a} \oplus V_1^{\oplus b}$ where a and b are any integers ≥ 0 and W is one of the three two dimensional modules in $\mathcal{S}$.*

R^P *has minimal depth if and only if either* $\operatorname{codim}(V^P) \le 2$ *or one of the above holds.*

Proof. That the first three statements imply minimal depth for $H^1(P,R)$ is already covered by Theorem 3.1. If none of these three statements hold, then $H^1(G,R)$ is detected on the set of proper subgroups of P (see [10], Theorem 9 and its proof). If additionally statement (4) does not hold, then by Lemma 3.2, either $\psi(V)$ consists of all proper subgroups of P, in which case Proposition 2.3 tells us that $\operatorname{depth}_{R^P}(H^1(P,R)) > \dim(V^P)$, or else $V \cong V_3^{\oplus j} \oplus W^{\oplus a} \oplus W'^{\oplus b} \oplus V_1^{\oplus c}$, where j is zero or one, a and b are positive integers and c is a positive integer or zero, with $\psi(V)$ consisting of the two proper subgroups Q and Q' of P which act trivially on the summands W and W' respectively.

Suppose $V \cong V_3^{\oplus j} \oplus W^{\oplus a} \oplus W'^{\oplus b} \oplus V_1^{\oplus c}$, and there exists a nonzero cohomology class $\alpha \in H^1(P,R)$ which restricts to zero on Q and Q'. Since R decomposes as $\bigoplus_{i\ge 0} S^i(V^*)$, and in each degree $S^i(V^*)$ decomposes further into a direct sum of kP-modules, we may assume there exists a degree d and direct summand M of $S^d(V^*)$ such that $\alpha \in H^1(P,M)$. We may also evaluate

$$\operatorname{Sym}(V^*) \cong \operatorname{Sym}(V_3^*)^{\otimes j} \otimes \operatorname{Sym}(W^*)^{\otimes a} \otimes \operatorname{Sym}(W'^*)^{\otimes b}.$$

[4] Now every direct summand of $\operatorname{Sym}(W^*)$ is either trivial or isomorphic to W^*. Similarly, every direct summand of $\operatorname{Sym}(W'^*)$ is either trivial or isomorphic to W'^*, and we know from [10] (in the proof of Theorem 9) that every direct summand of $\operatorname{Sym}(V_3^*)$ is either trivial, isomorphic to V_3^* or isomorphic to $\overline{V}_4^*$. The following table (also in [10]; constructed using the 'meat-axe' function in MAGMA) tells us how tensor products of these modules decompose:

Table 1: Decomposing Tensor Products of kP-modules

$\otimes$	$V_{2,1}$	$V_{2,\infty}$	$V_{2,0}$	V_3	$\overline{V}_4$
$V_{2,1}$	$V_{2,1}^{\oplus 2}$	$\overline{V}_4$	$\overline{V}_4$	$V_{2,1} \oplus \overline{V}_4$	$\overline{V}_4^{\oplus 2}$
$V_{2,\infty}$		$V_{2,\infty}^{\oplus 2}$	$\overline{V}_4$	$V_{2,\infty} \oplus \overline{V}_4$	$\overline{V}_4^{\oplus 2}$
$V_{2,0}$			$V_{2,0}^{\oplus 2}$	$V_{2,0} \oplus \overline{V}_4$	$\overline{V}_4^{\oplus 2}$
V_3				$\overline{V}_4 \oplus V_5$	$\overline{V}_4^{\oplus 3}$
$\overline{V}_4$					$\overline{V}_4^{\oplus 4}$

We conclude, then, that M is isomorphic to one of the following:

$$\{V_1^*, W^*, W'^*, V_3^*, \overline{V}_4^*\}.$$

Since $\overline{V}_4^*$ is projective, $H^1(P,\overline{V}_4^*) = 0$ and so M cannot be isomorphic to $\overline{V}_4^*$. Suppose that $M \cong W'^*$. Then (remembering that Q' acts trivially on M)

[4] It is easy to see that W and W' are self dual, but we keep the asterisks to show we're thinking of right modules. Note also that $A \otimes V_1 \cong A$ for any kP-module A.

we have

$$0 = \mathrm{Tr}_Q^P(W^Q) = \mathrm{Tr}_1^{Q'}(W) \cap W^P$$

and it follows from Lemma 3.3 that $H^1(P, M)$ is detected on the pair Q, Q'. The corresponding result for W^* also follows similarly. If $M \cong V_1^*$ then both Q' and Q act trivially and again using Lemma 3.3, $H^1(P, M)$ is detected on the pair Q, Q'. So we may assume $M \cong V_3^*$. Let $\{y_1, y_2, y_3\}$ be a basis for M. Then we have, for any pair Q, Q' by direct calculation

$$\mathrm{Tr}_1^{Q'}(M) = \mathrm{Tr}_Q^P(M^Q) = M^P = \langle y_3 \rangle$$

so by Lemma 3.3, $H^1(P, M)$ is detected on the pair Q, Q' as required. So there cannot be a nonzero $\alpha \in H^1(P, R)$ which restricts to zero on a pair of proper subgroups Q, Q' in this case.

Finally we must show that statement (4) implies minimal depth for $H^1(P, R)$. In this case, $\psi(V)$ consists of the trivial group and one proper subgroup of P. Therefore, using Proposition 2.3 it is enough to show that $H^1(P, R)$ is not detected on any proper subgroup Q of P, that is to say, $\ker(\mathrm{res}_Q^P) \neq 0$. But this is in fact true for any faithful representation of a p-group (provided R is not projective, which is clear in this case) - see [11], Corollary 7.2.3. □

It was noted in [10] (on the final page) that when $V \cong V_3 \oplus V_{2,1} \oplus V_{2,1}$, R^P has minimal depth, but $H^1(P, R)$ is detected on the set of subgroups of P. The above proof explains why this is the case - it is an example of a representation where statement (4) applies.

4 Direct Products

As in section 1, let G be a finite group acting on a vector space V over a field k of characteristic p, with $R := \mathrm{Sym}(V^*)$ and P a Sylow-p-subgroup of G. It can be shown (see [17]) that

$$\mathrm{depth}(R^P) \leq \mathrm{depth}(R^G). \tag{8}$$

Consequently, R^G is Cohen-Macaulay if and only if R^P is so. This statement may be interpreted as saying that R^G has maximal depth if and only if R^P has maximal depth. In this section we ask to what extent this statement holds if maximal is replaced by minimal. Note that the above inequality already implies that the depth of R^P is minimal whenever the depth of R^G is minimal, so we need only to find when the reverse implication holds.

Let $G := P \times Q$ and L be any kG - module. Then the Lyndon-Hochschild-Serre spectral sequence (see, e.g. [11], chapter 7) gives two short exact sequences:

$$0 \to H^1(Q, L^P) = H^1(G/P, L^P) \to H^1(G, L) \to H^1(P, L)^{G/P} \tag{9}$$

$$0 \to H^1(P, L^Q) = H^1(G/Q, L^Q) \to H^1(G, L) \to H^1(Q, L)^{G/Q}. \tag{10}$$

Note that in order for these sequences to arise it is not necessary that P be a p-group and Q a p'-group. When they do take this form, the first short exact sequence specializes to the inclusion given by restriction

$$H^1(G, L) \hookrightarrow H^1(P, L),$$

whose image consists of G - stable cohomology, hence we get $H^1(G, L) \cong H^1(P, L)^Q$.
The second short exact sequence specializes to $H^1(P, L^Q) \cong H^1(G, L)$, since $H^1(Q, L) = 0$. So we have

$$H^1(P, L^Q) \cong H^1(G, L) \cong H^1(P, L)^Q.$$

In particular the cohomological connectivities of V and of $V_{|P}$ coincide. Since P is a p-group, we have $cc_G(R) = cc_P(R) = 1$.

Proposition 4.3 is the key to proving the main result of this section. We will need the following lemma describing the structure of the symmetric algebra:

Lemma 4.1. *Let V be a kP module for a p-group P and field k of characteristic p. Then the symmetric algebra $R := \mathrm{Sym}(V^*)$ splits as*

$$R = uR \oplus B$$

where the homogeneous invariant u and kP-submodule B are described below.

It is not entirely clear to whom this lemma should be attributed. An argument similar to the one below is used in [16], Lemma 2.9 but this result seems to have been known for some time. For lack of a good reference, we include a proof.

Proof. Since P is a p-group, the only irreducible kP-module is trivial. Consequently, P has an upper triangular representation on V and $R = k[x_1, \dots, x_n]$ may be viewed as $k[x_2, \dots, x_n][x_1]$, since $k[x_2, \dots, x_n]$ is a kP-module. So we view polynomials in R as polynomials in the single variable x_1 with coefficients in $k[x_2, \dots, x_n]$. If $r \in R$ we define $\deg(r)$ to be the degree of r when viewed as a polynomial in x_1. Then if $r \in R$ and $p \in P$, $\deg(r \cdot p) \leq \deg(r)$. Let B be the kP-submodule of R consisting of polynomials whose degree (as polynomials in x_1) is less than $|P|$. Consider the invariant $u := \prod_{g \in P} x_1.g$. In $k[x_2, \dots, x_n][x_1]$, this is a monic polynomial of degree $|P|$. Given any $r \in R$ with $\deg(r) \geq |P|$ we may perform successive long division by u, giving us a unique expression $r = qu^a + b$ for some $q \in R$, $b \in B$, $a \in \mathbb{N}$. Clearly $uR \cap B = 0$ and therefore $R = uR \oplus B$. □

Corollary 4.2. *If W is kP-module and W' is a direct summand of the lth symmetric power $S^l(W)$ then uW' is an isomorphic direct summand of $S^{l+|P|}(W)$.*

We say that the invariant u *propagates direct summands* of R.

Proposition 4.3. *Let G be as above and let ψ be a set of subgroups of P. Suppose in addition that k is algebraically closed. Then $H^1(G,R)$ is detected on ψ if and only if $H^1(P,R)$ is detected on ψ.*

Proof. Since $H^1(G,R)$ is a direct summand of $H^1(P,R)$, the "if" part is immediate. Suppose $H^1(P,R)$ is not detected on ψ, so we can find $0 \neq \alpha \in H^1(P,R)$ be such that $\mathrm{res}^P_N(\alpha) = 0$ for every $N \in \psi$. Then since k is algebraically closed, $V^* = \oplus_{j=1}^k W_j$ with $W_j \cong U_j \otimes k_{\epsilon_j}$, where each U_j is an indecomposable kP - module, k_{ϵ_j} is a one - dimensional Q - module with character $\epsilon_j \in \mathrm{Hom}(Q, k^\times)$ and diagonal action $(u \otimes \lambda)(p,q) := up \otimes \lambda\epsilon_j(q)$. Hence we have a decomposition of the G - module

$$R = \otimes_j \mathrm{Sym}(W_j) = \oplus_{s=0}^{\infty} \oplus_{\substack{\underline{\ell}\in\mathbb{N}^k,\\ |\underline{\ell}|=s}} X_{\underline{\ell}}$$

with $X_{\underline{\ell}} := X_{\ell_1} \otimes \cdots \otimes X_{\ell_k}$ and $X_{\ell_i} := S^{\ell_i}(W_i)$. Note that Q acts on each $S^{\ell_i}(W_i)$ by the linear character $\epsilon_i^{\ell_i}$ and on $X_{\underline{\ell}}$ by the character $\prod_{j=1}^k \epsilon_j^{\ell_j}$. For every $N \in \psi$ we have decompositions of k - spaces

$$H^1(P,R) = \oplus_{\underline{\ell}} H^1(P, X_{\underline{\ell}})$$

$$H^1(N,R) = \oplus_{\underline{\ell}} H^1(N, X_{\underline{\ell}})$$

which are preserved by the corresponding restriction map.
It follows that $\alpha = \oplus_{\underline{\ell}} \alpha_{\underline{\ell}}$ with $\alpha_{\underline{\ell}} \in H^1(P, X_{\underline{\ell}})$ and $\mathrm{res}^P_N(\alpha_{\underline{\ell}}) = 0$ for every $N \in \psi$ and $\underline{\ell}$. Hence we can assume that $0 \neq \alpha \in H^1(P, X_{\underline{\ell}})$. For every $j = 1, \cdots, k$, the space X_{ℓ_j} is a kG - direct summand of $\mathrm{Sym}(W_j)$ and by Lemma 4.1 there is a suitable homogeneous invariant $u_j \in \mathrm{Sym}(W_j)^P$ of degree $|P|$, propagating direct summands of $\mathrm{Sym}(W_j)$. Choosing $a_j, b_j \in \mathbb{N}$ such that $b_j|Q| - a_j|P| = \ell_j$, which we can do because $|P|$ and $|Q|$ are coprime, we see that $u_j^{a_j} \cdot X_{\ell_j}$ is a direct summand of $\mathrm{Sym}(W_j)$ and also a submodule of

$$S^{\ell_j + |P|a_j}(W_j) = S^{b_j|Q|}(W_j) \leq \mathrm{Sym}(W_j)^Q$$

Let $u := \otimes_{j=1}^k u_j^{a_j} \in R^P$, then $X_{\underline{\ell}}$ and $u \cdot X_{\underline{\ell}} = \otimes_{j=1}^k u_j^{a_j} \cdot X_{\ell_j}$ are isomorphic kP - modules and the latter one is a direct summand of R^Q. It follows that there is

$$0 \neq \tilde{\alpha} \in H^1(P, uX_{\underline{\ell}}) \mid H^1(P, R^Q)$$

satisfying $\mathrm{res}^P_N(\tilde{\alpha}) = 0$ for all $N \in \psi$. Therefore $\tilde{\alpha} = \mathrm{res}^G_P(\beta)$ with $\beta \in H^1(G,R)$ and

$$\mathrm{res}^G_N(\beta) = \mathrm{res}^P_N(\mathrm{res}^G_P(\beta)) = \mathrm{res}^P_N(\tilde{\alpha}) = 0$$

for all ψ, that is, $H^1(G,R)$ is not detected on ψ as required. □

Corollary 4.4. *Let G be of the form $P \times Q$ with P a p-group and Q an abelian p'-group. Then the depth of R^P is minimal if and only if the depth of R^G is minimal.*

Proof. Assume first that $\mathrm{codim}(V^P) \leq 2$. Then by Theorem 1.2 we have $\mathrm{depth}(R^G) = \mathrm{depth}(R^P) = \dim(V)$ so the proposition is true in this case. Assuming the contrary, we must show that $\mathrm{depth}(R^P)$ minimal implies $\mathrm{depth}(R^G)$ minimal, the converse having been dealt with by (8).

Note that if $\overline{k}$ denotes the algebraic closure of k, then the extension $k \to \overline{k}$ is flat. Consequently for any R^G-module M, we have $\mathrm{depth}(M) = \mathrm{depth}(M \otimes \overline{k})$ (see [4], Proposition 1.2.16) and we may assume k is algebraically closed. If $\mathrm{depth}(R^P)$ is minimal, then by Proposition 2.3, $H^1(P, R)$ is not detected on the set $\psi(P) := \{N < P : V^P \subsetneqq V^N\}$. Applying Proposition 4.3 with $\psi := \psi(P)$, we see that $H^1(G, R)$ is not detected on $\psi(P)$, and by Proposition 2.3 once more, the depth of R^G is minimal. □

So for a group G of the form $P \times Q$ as above, the depth of R^G is minimal if and only if the depth of R^P is minimal, and the depth of R^G is maximal if and only if the depth of R^G is maximal. It is interesting to ask whether in fact $\mathrm{depth}(R^P) = \mathrm{depth}(R^G)$ for all groups of this form, and we can find no evidence to the contrary. We can however, show that we cannot conclude $\mathrm{depth}(R^G)$ minimal if and only if $\mathrm{depth}(R^P)$ is minimal when P is merely a normal Sylow-p-subgroup of G. The following example is used (for a different purpose) in [19], Example 4.6. Let p be any prime ≥ 5 and consider the following subgroup of $SL_2(p)$:

$$G := \left\{ \begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} : \ a \in \mathbb{F}_p^{\times}, \ x \in \mathbb{F}_p \right\}.$$

Then G is a semidirect product of the form $\mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$, and has a normal Sylow-p-subgroup P of order p consisting of those matrices above in which $a = 1$. The action of G on P by conjugation is given by the formula

$$\begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & a^2 y \\ 0 & 1 \end{pmatrix} \tag{11}$$

which is easily checked. Let $V := S^{p-1}(\mathbb{F}_p^2)$ be the $(p-1)$th symmetric power of the natural module. Note that the centre Z of G now acts trivially on V, so we regard V as a module for the group $H := G/Z$ which is a semidirect product of the form $\mathbb{Z}_p \rtimes \mathbb{Z}_{(p-1)/2}$. Now as a kP-module, V is clearly projective and indecomposable, hence isomorphic to the regular module. Moreover since $V^H \neq 0$, V is the unique projective indecomposable kH-module containing the trivial module. Therefore V is also a permutation module, there is a natural extension of the action of H on V to the symmetric group S_p, and since the action of H on P by conjugation is isomorphic to the action of $\mathbb{Z}_{(p-1)/2}$ on the additive group $\mathbb{Z}_p$ given by multiplication by squares, H is the normaliser of P in the alternating group A_p. Now A_p is a trivial intersection group (i.e. for each $g \in G$, $P \cap g^{-1}Pg$ is either P or the trivial group) and so the normaliser of P in A_p is strongly p-embedded into A_p (see [18], Corollary 1.2) and we conclude that

$$\mathrm{depth}(R^H) = \mathrm{depth}(R^{A_p})$$

It is well known that the invariant ring R^{A_p} for the natural action of A_p on a polynomial ring in p variables is a hypersurface (see e.g. [23], Corollary 1.3.2). In particular R^{A_p} is Cohen-Macaulay so we conclude that $\operatorname{depth}(R^H) = p$. On the other hand P is cyclic of prime order, so by [7], we have $\operatorname{depth}(R^P) = \dim(V^P)+2 = 3$ which is minimal. We must show that $\operatorname{depth}(R^H)$ is not minimal, which means that $\dim(V^P)+1+cc_H(R) < p$, or more simply $cc_H(R) < p-2$. In fact we will show that $cc_H(R) \leq (p-1)/2$ which is strictly less than $p-2$ for all $p \geq 5$. Now R contains a trivial direct summand in degree zero, and

$$H^*(H,k) = H^*(P,k)^H.$$

$H^*(P,k)$ is a polynomial ring over k in one variable z, and the action of H on $H^*(P,k)$ is given by the formula

$$\begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} \cdot z = a^{-2}z.$$

It follows that

$$\begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} \cdot z^{(p-1)/2} = (a^{-2}z)^{(p-1)/2} = a^{-(p-1)}z^{(p-1)/2} = z^{(p-1)/2}$$

and so $H^{(p-1)/2}(H,k) \neq 0$. Consequently we have $H^{(p-1)/2}(H,R) \neq 0$ which shows that $cc_H(R) \leq (p-1)/2$ as required.

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