ON THE DEPTH OF SEPARATING ALGEBRAS FOR FINITE GROUPS

JONATHAN ELMER

Abstract. Consider a finite group G acting on a vector space V over a field $\mathbb{k}$ of characteristic $p > 0$. A separating algebra is a subalgebra A of the ring of invariants $\mathbb{k}[V]^G$ with the same point separation properties. In this article we compare the depth of an arbitrary separating algebra with that of the corresponding ring of invariants. We show that, in some special cases, the depth of A is bounded above by the depth of $\mathbb{k}[V]^G$.

1. Introduction

Let G be a group acting on a vector space V over a field $\mathbb{k}$ of characteristic $p > 0$. We denote by $\mathbb{k}[V]$ the ring of polynomial functions $V \to \mathbb{k}$, which carries a natural $\mathbb{k}G$-module structure defined by $\sigma \cdot f(v) := f(\sigma^{-1}v)$, where $v \in V$, $f \in \mathbb{k}[V]$, and $\sigma \in G$. The set of fixed points under this action forms a subring of $\mathbb{k}[V]$ which we denote by $\mathbb{k}[V]^G$, and call the ring of invariants.

If v and w are points of V in the same G-orbit, then it follows that $f(v) = f(w)$ for each $f \in \mathbb{k}[V]^G$. If there exists some $f \in \mathbb{k}[V]^G$ such that $f(v) \neq f(w)$, then v and w cannot be in the same orbit. In this situation we say that f *separates* the points v and w. This leads naturally to the following definition, due to Derksen and Kemper [3, Definition 2.3.8]:

Definition 1.1. A set $S \subset \mathbb{k}[V]^G$ is called a separating set if the following holds: given $v, w \in V$, if there exists $f \in \mathbb{k}[V]^G$ separating v and w, then $s(v) \neq s(w)$ for some $s \in S$.

In other words, a separating set is a subset of $\mathbb{k}[V]^G$ which separates the same points. In this article, all groups considered will be finite, and so in particular reductive. Thus, the categorical quotient map $V \to V//G$ will be surjective, and the ring of invariants will separate all points in distinct G-orbits. Therefore, we can take as our definition of a separating set that it separates the G-orbits. It is clear that if S is a separating set, then so is $\mathbb{k}[S]$, the k-algebra generated by S. A subalgebra of $\mathbb{k}[V]^G$ which separates orbits is called a *separating algebra*.

In this article, all separating algebras under consideration are *geometric* separating algebras. Roughly speaking, a geometric separating algebra A separates the orbits of the extended module $\overline{V} := V \otimes_{\Bbbk} \overline{\Bbbk}$ (see [4] for definition). There is no loss in assuming A to be geometric if $\Bbbk$ is algebraically closed. We also assume throughout that our subalgebras A are graded, that is, generated by homogeneous polynomials in $\Bbbk[V]$.

Subsequently, we may apply the following propositions taken from [5]:

Proposition 1.2. *If $A \subset \Bbbk[V]^G$ is a graded geometric separating algebra, then $A \subset \Bbbk[V]^G$ is an integral extension, and A is a finitely generated $\Bbbk$-algebra.*

Proposition 1.3. *Suppose $p > 0$. If $A \subset \Bbbk[V]^G$ is a graded subalgebra, then A is a geometric separating algebra if and only if $\Bbbk[V]^G$ is the purely inseparable closure of A in $\Bbbk[V]$, that is,*

$$\Bbbk[V]^G = \{f \in \Bbbk[V] \mid \quad \text{for some } m,\ f^{p^m} \in A\}.$$

Using the above, we deduce that if I is an ideal of $\Bbbk[V]^G$, and $A \subset \Bbbk[V]^G$ is a geometric separating algebra, then $\mathrm{ht}_{\Bbbk[V]^G}(I) = \mathrm{ht}_A(I \cap A)$; indeed, it is clear that the left hand side is greater than or equal to the right. If $a_1, a_2, \ldots a_h$ is a hsop for I, then, for some large enough p-power q, $a_1^q, a_2^q, \ldots a_h^q$ is an hsop for $I \cap A$.

An A-module M is called Cohen-Macaulay if one has $\mathrm{depth}(M) = \dim(M)$. It is known that, if $|G|$ is not divisible by p, then the ring of invariants $\Bbbk[V]^G$ is a Cohen-Macaualay ring (i.e. Cohen-Macaulay as a module over itself). The problem of calculating depth for invariant rings in the modular case has been widely explored, and solved in only a few simple cases, for instance, when G is cyclic [6], p-nilpotent with cyclic Sylow-p-subgroup [11], and when G is the Klein four group [9], [8]. The corresponding question of depth for separating algebras has not, to the best of our knowledge, been studied at all. In [5] it was shown that many of the criteria which imply a ring of invariants is not Cohen-Macaulay actually preclude the existence of a Cohen-Macaulay geometric separating algebra. This article is in many ways a sequel to [5]. In the former we obtained lower bounds for the Cohen-Macaulay defect of separating algebras. Here we obtain stronger lower bounds (viewed instead as upper bounds for depth) in situations where we can be more precise about the height of given annihilators in cohomology. Frequently these bounds coincide with the known value of the depth of the corresponding ring of invariants. Notably we prove:

Theorem 1.4. *Suppose $A \subset \Bbbk[V]^G$ is a separating algebra and that one of the following holds:*

1. *G is a cyclic p-group*
2. *G is p-nilpotent with a cyclic Sylow-p-subgroup*
3. *G is a shallow group (see section 3 for definition)*

Then $\mathrm{depth}(A) \leq \mathrm{depth}(\Bbbk[V]^G) = \min\{\dim(V^P) + 2, \dim(V)\}$.

Note that in the above, (1) is a special case of (2), included for emphasis.

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2. Depth and Cohomology

Let $A \subseteq \Bbbk[V]$ be a finitely generated graded subalgebra, and let A_+ denote its maximal homogeneous ideal. Homogeneous elements $a_1, \ldots, a_k$ in A_+ form a *partial homogeneous system of parameters (phsop)* if they generate an ideal of height k in A. If in addition $k = \dim A$, then they form a *homogeneous system of parameters (hsop)*. Noether's normalization theorem guarantees that an A always contains an hsop. An element $a \in A$ is called regular if it is not a zero-divisor on A. If, for $i = 1, \ldots, k$, the element a_i is regular on $A/(a_1, \ldots, a_{i-1})A$, then the elements $a_1, \ldots, a_k$ are said to form a *regular sequence*. Every regular sequence is a phsop. We say A is *Cohen-Macaulay* when every phsop is a regular sequence. The depth of a homogeneous ideal $I \subseteq A_+$, written $\operatorname{depth}_A(I)$, is the maximal length of a regular sequence in I. Note that the height of I, $\operatorname{ht}(I)$, is equal to the maximal length of a phsop in I. We write $\operatorname{depth}(A) := \operatorname{depth}_A(A_+)$, and define the *Cohen-Macaulay defect* of A to be $\operatorname{cmdef}(A) := \dim A - \operatorname{depth}(A)$. Thus, A is Cohen-Macaulay precisely when $\operatorname{depth}(A) = \dim(A)$, that is, when $\operatorname{cmdef} A = 0$. We also define the Cohen-Macaulay defect of an arbitrary ideal as $\operatorname{cmdef}(I) := \operatorname{ht}(I) - \operatorname{depth}(I)$.

We want to use the following result to find upper bounds for depth:

Proposition 2.1. *Suppose A is a finitely generated graded connected $\Bbbk$-algebra, and I is an ideal of A. Then* $\operatorname{cmdef}(A) \geq \operatorname{cmdef}(I)$.

Proof. The proof is a simple adaption to the graded case of [1, Exercise 1.2.23]. □

It follows from the above that if I is an ideal of A, then we have $\operatorname{depth}(A) \leq \dim(A) + \operatorname{depth}(I) - \operatorname{ht}(I)$. In particular A is not Cohen-Macaulay if there exists an ideal I of A for which $\operatorname{depth}(I) < \operatorname{ht}(I)$. Recall that the cohomology modules $H^i(G, \Bbbk[V])$, $i > 0$ are modules over the ring of invariants $\Bbbk[V]^G$ via the cup product, and can therefore by regarded as A-modules for any subalgebra A of R^G. We define the annihilator ideal, $\operatorname{Ann}_A(\alpha)$ of a cohomology class $\alpha \in H^i(G, \Bbbk[V])$ to be $\{a \in A : a \cdot \alpha = 0\}$. These annihilators provide a good source of ideals whose depth and height can be calculated.

Lemma 2.2. *Let* $\alpha \in H^i(G, \Bbbk[V])$ *for some* $i > 0$*, and let* $A \leq \Bbbk[V]^G$ *be a graded geometric separating algebra. Then* $\mathrm{ht}(\mathrm{Ann}_A(\alpha)) = \min\{\mathrm{codim}(V^X) : X \leq G, \mathrm{res}^G_X(\alpha) \neq 0\}$*.*

Proof. By [7, Lemma 2.4, Proposition 2.5] we have

$$\sqrt{\mathrm{Ann}_{R^G}(\alpha)} = \bigcap_{X \leq G : \mathrm{res}^G_X(\alpha) \neq 0} \mathcal{I}(V^X) \cap \Bbbk[V]^G$$

where for a subset $W \leq V$, $\mathcal{I}(W)$ denotes the set of functions in $\Bbbk[V]$ vanishing on W. Note that $\mathrm{ht}(\mathcal{I}(V^X)) = \mathrm{codim}(V^X)$ and that $\mathrm{ht}(\mathcal{I}(V^X)) = \mathrm{ht}(\mathcal{I}(V^X) \cap \Bbbk[V]^G)$ by the "going up" and "going down" theorems. Thus $\mathrm{ht}(\mathrm{Ann}_{R^G}(\alpha)) = \min\{\mathrm{codim}(V^X) : X \leq G, \mathrm{res}^G_X(\alpha) \neq 0\}$. By the remarks following Proposition 1.3, we get the required result. □

If $p > 0$, the m-fold Frobenius homomorphism $\Bbbk[V] \to \Bbbk[V]$, $f \mapsto f^{p^m}$, induces a map $H^n(G, \Bbbk[V]) \to H^n(G, \Bbbk[V])$. We write α^{p^m} for the image of an element $\alpha \in H^n(G, \Bbbk[V])$ under this map.

Lemma 2.3.
(1) *Let* $n \geq 1$ *be the smallest integer such that there exists a nonzero homogeneous* $\alpha \in H^n(G, \Bbbk[V])$*. Then* $I := \mathrm{Ann}_{\Bbbk[V]^G}(\alpha)$*, has depth equal to* $\min(n+1, \dim(V))$*.*
(2) *Let* $n' \geq 1$ *be the smallest integer such that there exists a homogeneous* $\alpha \in H^{n'}(G, \Bbbk[V])$ *such that* α^{p^m} *is nonzero for every* $m \geq 0$*. If* A *is a graded geometric separating algebra in* $\Bbbk[V]^G$*, then* $I := \mathrm{Ann}_A(\alpha)$*, has depth at most* $n' + 1$*.*

Proof. For (1), see [12, Corollary 1.6]. For (2), see [5, Theorem 3.1]. □

Corollary 2.4. *Let* n' *be as above, and suppose there exists a cohomology class* $\alpha \in H^{n'}(G, R)$ *with the following properties:*
(1) $\alpha^{p^m} \neq 0$ *for every* $m \geq 0$
(2) $\mathrm{res}^G_Q(\alpha) = 0$ *for every subgroup* Q *of some Sylow-p-subsgroup* P *for which* $\dim(V^Q) > \dim(V^P)$

Then for every graded geometric separating algebra $A \leq \Bbbk[V]^G$*, one has* $\mathrm{depth}(A) \leq \dim(V^P) + n' + 1$*.*

Proof. Let $I := \mathrm{Ann}_A(\alpha)$. Then by Lemma 2.3 we have $\mathrm{depth}(I) \leq n' + 1$ and by Lemma 2.2 we have $\mathrm{ht}(I) = \mathrm{codim}(V^P)$. Therefore $\mathrm{depth}(A) \leq \dim(V) - \mathrm{codim}(V^P) + n' + 1 = \dim(V^P) + n' + 1$ as required. □

3. Proof of main results

Proposition 3.1. *Suppose* $A \subset \Bbbk[V]^G$ *is a graded geometric separating algebra and* G *is a finite group which is p-nilpotent with a cyclic Sylow-p-subgroup (this includes the case where* G *is a cyclic group of p-power order). Then* $\mathrm{depth}(A) \leq \min\{\dim(V^P) + 2, \dim(V)\} = \mathrm{depth}(\Bbbk[V]^G)$*.*

Proof. By the proof of [11, Theorem 5.3], there exists a nonzero cohomology class α in the direct summand $H^1(G, \Bbbk)$ of $H^1(G, \Bbbk[V])$ which restricts to zero on the unique maximal subgroup of the Sylow-p-subgroup P. By [5, Lemma 2.2], we have $\alpha^{p^m} \neq 0$ for all m. From Corollary 2.4 we now obtain that $\operatorname{depth}(A) \leq \min\{\dim(V^P)+2\}$. The second statement is [11, Theorem 5.3]. □

Let G be a finite group and P a Sylow-p-subgroup, and denote by G'_p the subgroup of G generated by P and the commutator subgroup G' of G. We say G is a *shallow group* if there exists a normal subgroup N of index p in G such that the set

$$\{\sigma \in G : V^\sigma > V^{G'_p}\}$$

is contained in N. For examples of shallow groups, see [2].

Proposition 3.2. *Suppose $A \leq \Bbbk[V]^G$ is a graded geometric separating algebra and G is a shallow group. Then* $\operatorname{depth}(A) \leq \min\{\dim(V^P) + 2, \dim(V)\} = \operatorname{depth}(\Bbbk[V]^G)$.

Proof. By [2, Lemma 8], we have $\dim(V^P) = \dim(V^{G'_p})$. Therefore, for every subgroup $Q < P$ such that $V^Q > V^P$, we have $Q \leq N$. Since the index of N in G is p, G/N is a cyclic group of order p and we can find a nonzero cohomology class $\beta \in H^1(G/N, \Bbbk)$. Consider the inflation $\alpha := \operatorname{inf}_N^G(\beta)$ (see [10, Chapter 7] for definition; in particular, the inflation map inf_N^G is injective and its image equals the kernel of the restriction res_N^G). Then $0 \neq \alpha \in H^1(G, \Bbbk)$ satisfies $\operatorname{res}_N^G(\alpha) = 0$, and further $\operatorname{res}_Q^G(\alpha) = 0$ for every subgroup $Q < P$ such that $V^Q > V^P$. Moreover, by [5, Lemma 2.2], we have $\alpha^{p^m} \neq 0$ for each m, and so by Corollary 2.4 we obtain $\operatorname{depth}(A) \leq \dim(V^P) + 2$. The second statement is a combination of [2, Theorem 1, Theorem 2]. □

This completes the proof of Theorem 1.1. We can also prove:

Proposition 3.3. *Suppose G is a finite group such that $|G|$ is divisible by p but not by p^2. Then for any permutation module V of G we have*

$$\operatorname{depth}(A) \leq \operatorname{depth}(\Bbbk[V]^G)$$

for any graded geometric separating algebra $A \subset \Bbbk[V]^G$.

Proof. As usual, let P be a Sylow-p-subgroup of G. Since $|G|$ is divisible by p but not by p^2, it follows that P is cyclic of prime order. In particular, P has no non-trivial proper subgroups. By [13, Theorem 3.1], we have $\operatorname{depth}(\Bbbk[V]^G) = \min\{\dim(V^P)+n+1, \dim(V)\}$ where $n = \min\{i : H^i(G, \Bbbk[V]) \neq 0\}$, and by Corollary 2.4 we have, for any graded geometric separating algebra A, $\operatorname{depth}(A) \leq \min\{\dim(V^P) + n' + 1, \dim(V)\}$ where $n' = \min\{\exists\ \alpha \in H^i(G, \Bbbk[V])$ such that $\alpha^{p^m} \neq 0\ \forall\ m\}$. We must prove that $n = n'$; indeed, by [5, Theorem 2.11], we have $\alpha^{p^m} \neq 0$ for all nonzero cohomology classes α. The result follows. □

Remark 3.4. The exact value of n, and hence of $\mathrm{depth}(\Bbbk[V]^G)$, has in this case been described by Kemper [13, Theorem 3.3].

As an important special case, we obtain the following:

Corollary 3.5. *(Compare with* [13, Corollary 3.6]*). Suppose G is the symmetric group S_l on l letters, $V := W^{\oplus r}$ is isomorphic to r copies of the natural module, and $p \leq l \leq 2p$. Then for any graded geometric separating algebra A in $\Bbbk[V]^G$ we have* $\mathrm{depth}(A) \leq rl - (p-1)(r-2)$. *In particular, there cannot exist a Cohen-Macaulay graded geometric separating algebra unless $r \leq 2$.*

Proof. The restriction on the value of l ensures that $l! =: |G|$ is divisible by p but not by p^2. By [13, Corollary 3.6], we have $\mathrm{depth}(\Bbbk[V]^G) \leq rl - (p-1)(r-2)$. The result now follows by Proposition 3.3 □

4. The group $C_2 \times C_2$

The group $P := C_2 \times C_2$ is, in some sense, the next simplest group after cyclic groups. If $\Bbbk$ is a field of characteristic two and V is indecomposable, then the depth of $\Bbbk[V]^G$ is known by [9, Theorem 7]. In this section we will explain why, for many choices of V, the depth of any graded geometric separating algebra is bounded above by the depth of the ring of invariants, but that this is not true in general. Throughout we assume that V is an indecomposable representation of P over a field $\Bbbk$ of characteristic two.

The following lemma provides an upper bound for the depth of a graded geometric separating algebra for this group:

Lemma 4.1. *Let P be the group $C_2 \times C_2$, V a $\Bbbk P$-module and $A \subset \Bbbk[V]^G$ a graded geometric separating algebra. Let Q_1, Q_2, Q_3 be the three maximal proper subgroups of P, ordered so that* $\dim(V^{Q_1}) \leq \dim(V^{Q_2}) \leq \dim(V^{Q_3})$. *Then* $\mathrm{depth}(A) \leq \dim(V^{Q_2}) + 2$.

Proof. Let $0 \neq \alpha \in H^1(P/Q_3, \Bbbk) \subset H^1(P/Q_3, \Bbbk[V])$ and consider $\beta := \mathrm{inf}^P_{Q_3}(\alpha)$. This is a nonzero cohomology class in $H^1(P, \Bbbk)$ satisfying $\mathrm{res}^P_{Q_3} = 0$, however $\mathrm{res}^P_{Q_2}(\beta)$ and $\mathrm{res}^P_{Q_1}(\beta)$ cannot be zero. By Lemma 2.2 we have $\mathrm{ht}(\mathrm{Ann}_A(\beta)) = \mathrm{codim}(V^{Q_2})$. On the other hand, by [5, Lemma 2.2], we have $\beta^{p^m} \neq 0$ for all m since β is contained in the direct summand $H^1(P, \Bbbk)$ of $H^1(P, \Bbbk[V])$, and so by Lemma 2.3 we obtain $\mathrm{depth}(\mathrm{Ann}_A(\beta)) \leq 2$. Therefore $\mathrm{depth}(A) \leq \dim(V) - \mathrm{codim}(V^{Q_2}) + 2 = \dim(V^{Q_2}) + 2$ as required. □

Suppose first that the dimension of V is at most 4. By [9, Theorem 7], the ring of invariants $\Bbbk[V]^G$ is Cohen-Macaulay, and so if A is a graded geometric separating algebra, we have trivially $\mathrm{depth}(A) \leq \mathrm{depth}(\Bbbk[V]^G)$. Now suppose that V is indecomposable of even dimension $2n > 4$; then it follows from the classification of indecomposable

$\Bbbk P$-modules that, with respect to some basis, the generators of P act on V (from the left) via the matrices

$$\begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}$$

$$\begin{pmatrix} I_n & J_\lambda \\ 0 & I_n \end{pmatrix}.$$

where J_λ is a Jordan block of size n with eigenvalue λ and $\lambda \in \Bbbk$. In this case we have $\dim(V^P) = n$, and if $\lambda \in \{0,1\}$ then $\dim(V^{Q_1}) = n$, $\dim(V^{Q_2}) = n$, $\dim(V^{Q_3}) = n+1$, while if $\lambda \notin \{0,1\}$, then $\dim(V^{Q_1}) = \dim(V^{Q_2}) = \dim(V^{Q_3}) = n$. . The depth of $\Bbbk[V]^G$ is equal to $\dim(V^P) + 2 = n+2$ by [9, Theorem 7], while Lemma 4.1 tells us that if $A \subset \Bbbk[V]^G$ is a graded geometric separating algebra, then $\operatorname{depth}(A) \leq n+2$. So in this case, the depth of A is bounded above by $\operatorname{depth}(\Bbbk[V]^G)$.

Now suppose that V is indecomposable of odd dimension $2n+1 > 4$. In each dimension there are two possible isomorphism classes for V. With respect to some basis, the action of P on V is given either by the matrices

$$\begin{pmatrix} & 0 & \\ I_n & \vdots & I_n \\ & 0 & \\ & & \\ 0 & & I_{n+1} \end{pmatrix}$$

$$\begin{pmatrix} & & 0 \\ I_n & I_n & \vdots \\ & & 0 \\ & & \\ 0 & I_{n+1} & \end{pmatrix}$$

or by the matrices

$$\begin{pmatrix} I_{n+1} & 0\ldots0 \\ & I_n \\ & \\ 0 & I_n \end{pmatrix}$$

$$\begin{pmatrix} I_{n+1} & I_n \\ & 0\ldots0 \\ & \\ 0 & I_n \end{pmatrix}$$

We shall say that $V \cong V_{-(2n+1)}$ in the first instance, and $V \cong V_{2n+1}$ in the second. If $V \cong V_{2n+1}$, then $\dim(V^P) = n+1 = \dim(V^{Q_1}) = \dim(V^{Q_2}) = \dim(V^{Q_3})$. By [9, Theorem 7], the depth of $\Bbbk[V]^G$ is $\dim(V^P) + 2 = n+3$ and by Lemma 4.1, if $A \subset \Bbbk[V]^G$ is a graded geometric separating algebra then $\operatorname{depth}(A) \leq n+3$. So the depth of A

is bounded above by the depth of $\Bbbk[V]^G$ in this case. Finally we consider the case $V \cong V_{2n-1}$. In this case $\dim(V^P) = n$ and $\dim(V^{Q_1}) = \dim(V^{Q_2}) = \dim(V^{Q_3}) = n+1$. By [9, Theorem 7], the depth of $\Bbbk[V]^G$ is $\dim(V^P)+2 = n+2$ and by Lemma 4.1, if $A \subset \Bbbk[V]^G$ is a geometric separating algebra then $\operatorname{depth}(A) \leq n+3$. So the upper bound for the depth of a graded geometric separating algebra exceeds the depth of $\Bbbk[V]^G$ by one in this case.

We have therefore proved the following;

Proposition 4.2. *Let V be an indecomposable represenation of P over a field of characteristic two, and let $A \subset \Bbbk[V]^G$ be a graded geometric separating algebra. Then either* $\operatorname{depth}(A) \leq \operatorname{depth}(\Bbbk[V]^G)$ *or the dimension of V is an odd number greater than* 4 *and* $\operatorname{depth}(A) \leq \operatorname{depth}(\Bbbk[V]^G) + 1$.

Note in particular that the representation V_{-5} is the only indecomposable modular representation of P with a non-Cohen-Macaulay ring of invariants which can possibly contain a Cohen-Macaulay graded geometric separating algebra. An example of such a separating algebra for this example can be found in [5].

5. Lower bounds - an example

It is known that the integer

$$\min\{\dim(V^P) + n + 1, \dim(V)\},$$

which appears in our work as an upper bound for the depth of certain separating algebras and rings of invariants, is in fact also a lower bound for the depth of any ring of invariants in general [11, Theorem 1.2]. It is not, however, a general lower bound for the depth of a graded geometric separating algebra, as the following example (due to Martin Kohls) shows:

Example 5.1. Let $G := \langle \sigma \rangle$ be the cyclic group of order two, acting on $\Bbbk[V] := \Bbbk[x_1, x_2]$ by permuting the variables, where $\Bbbk$ is a field of characteristic two. Then

$$\{a_1 := (x_1+x_2)^2 + x_1x_2, a_2 := (x_1x_2)^2, a_3 := (x_1+x_2)^3, a_4 := (x_1x_2)^3\}$$

is a geometric separating set. To see this, observe that $b_1 := x_1 + x_2$ and $b_2 := x_1x_2$ generate the ring of invariants, and that we have

$$b_1^4 = a_1^2 + a_2, \ b_2^2 = a_2$$

which shows that $\Bbbk[V]^G$ is contained in the purely inseparable closure of the algebra $A := \Bbbk[a_1, a_2, a_3, a_4]$. Therefore A is a geometric separating algebra by Proposition 1.3. We claim that A is not Cohen-Macaulay. It's clear that $\{a_3^2, a_4\}$ is an hsop. The relation

$$a_1^4a_4 + a_2^2a_3^2 + a_1a_3^2a_4 + a_2^2a_4 = 0$$

shows that $a_2^2 a_3^2 = 0 \mod a_4$, and since $a_2^2 \notin Aa_4$, we have shown that a_4, a_3^2 is not a regular sequence. Therefore, $\mathrm{depth}(A) = 1$. Now since G is a p-group, we have $H^1(G, \Bbbk) \neq 0$, and so $n = 1 = n'$ (where [5, Remark 2.2] is used in the calculation of n'), while $\min(\dim(V^P) + 2, \dim(V)) = 2$. Therefore, we have a counterexample as claimed. ◁

The above example also shows that graded geometric separating algebras do not, in general, satisfy Serre's condition S_2, and, unlike rings of invariants, are not necessarily normal domains.

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School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London E1 4NS

E-mail address: j.elmer@qmul.ac.uk