ZERO-SEPARATING INVARIANTS
FOR LINEAR ALGEBRAIC GROUPS

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ABSTRACT. Let G be linear algebraic group over an algebraically closed field $\Bbbk$ acting rationally on a G-module V, and $\mathcal{N}_{G,V}$ its nullcone. Let $\delta(G,V)$ and $\sigma(G,V)$ denote the minimal number d, such that for any $v \in V^G \setminus \mathcal{N}_{G,V}$ and $v \in V \setminus \mathcal{N}_{G,V}$ respectively, there exists a homogeneous invariant f of positive degree at most d such that $f(v) \neq 0$. Then $\delta(G)$ and $\sigma(G)$ denote the supremum of these numbers taken over all G-modules V. For positive characteristics, we show that $\delta(G) = \infty$ for any subgroup G of $\mathrm{GL}_2(\Bbbk)$ which contains an infinite unipotent group, and $\sigma(G)$ is finite if and only if G is finite. In characteristic zero, $\delta(G) = 1$ for any group G, and we show that if $\sigma(G)$ is finite, then G^0 is unipotent. Our results also lead to a more elementary proof that $\beta_{\mathrm{sep}}(G)$ is finite if and only if G is finite.

1. INTRODUCTION

In invariant theory, the notion of geometric reductivity is of great importance. It implies finite generation of the invariants, the separability of disjoint orbit closures by invariants, and in characteristic zero even algebraic properties like the Cohen-Macaulayness of the invariant ring. It is defined to be the property that any non-zero fixed point of a finite dimensional rational representation can be separated from zero by a homogeneous invariant of positive degree. Similarly, by definition any point outside the nullcone can be separated from zero by a homogeneous positive degree invariant. It is a natural question to ask what is the maximum degree needed for a given representation. While in our recent paper [5] we gave some (partial) answers to these questions for the case of finite groups, the current paper concentrates on the case of infinite groups. Before we go into more details, we fix our setup.

Let G be a linear algebraic group over an algebraically closed field $\Bbbk$, V a finite dimensional rational representation of G (which we will call a G-module), and denote by $\Bbbk[V] \cong S(V^*)$ the ring of polynomial functions $V \to \Bbbk$. The action of G on V induces an action of G on $\Bbbk[V]$ via $(g \cdot f)(v) := f(g^{-1}v)$ for $g \in G$, $f \in \Bbbk[V]$ and $v \in V$. The set of G-invariant polynomial functions under this action is denoted by $\Bbbk[V]^G$, and inherits a natural grading from $\Bbbk[V]$, since the given action is degree-preserving. We denote by $\Bbbk[V]^G_d$ the set of polynomial invariants of degree d and the zero-polynomial, and by $\Bbbk[V]^G_{\leq d}$ the set of polynomial invariants of degree at most d. For any subset S of $\Bbbk[V]$, we define S_+ as the set of elements in S with constant term zero. Then $\mathcal{N}_{G,V} := \mathcal{V}(\Bbbk[V]^G_+)$ denotes the nullcone of V. A linear algebraic group is said to be geometrically reductive, if for any G-module V, we have $V^G \cap \mathcal{N}_{G,V} = \{0\}$, i.e. for all nonzero $v \in V^G$, there exists $f \in \Bbbk[V]^G_+$ such that $f(v) \neq 0$. This inspires the definition of a δ-set: for a linear algebraic group G, let us say a subset $S \subseteq \Bbbk[V]^G$ is a *δ-set* if, for all $v \in V^G \setminus \mathcal{N}_{G,V}$, there exists

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an $f \in S_+$ such that $f(v) \neq 0$. We shall call a subalgebra of $\Bbbk[V]^G$ a *δ-subalgebra* if it is a δ-set. The quantity $\delta(G,V)$ is then defined as

$$\delta(G,V) = \min\{d \geq 0 | \quad \Bbbk[V]^G_{\leq d} \text{ is a } \delta\text{-set}\}.$$

Define further

$$\delta(G) := \sup\{\delta(G,V) | \quad V \text{ a } G\text{-module}\},$$

where we take the supremum of an unbounded set to be infinity. A reductive group is called linearly reductive if $\delta(G) = 1$. Over a field of characteristic zero, Nagata and Miyata [11] have shown that reductive groups are linearly reductive. In fact their proof shows that in characteristic zero, for any linear algebraic group G and any G-module V, $\delta(G,V)$ equals 1 or 0 (the latter being the case when $V^G \subseteq \mathcal{N}_{G,V}$), see Proposition 2.1. A natural, but seemingly neglected question, is: for which geometrically reductive groups G is $\delta(G)$ strictly greater than 1, but still finite? For finite groups, we gave the following answer in [5]:

Theorem 1.1 ([5, Theorem 1.1])**.** *Let G be a finite group, $\Bbbk$ an algebraically closed field of characteristic p, and P a Sylow-p-subgroup of G. Then $\delta(G) = |P|$.*

Thus, $\delta(G)$ is finite for all finite groups, and strictly greater than 1 if and only if $|G|$ is divisible by p. In this article we investigate $\delta(G)$ for infinite groups. In particular, we make and investigate the following conjecture:

Conjecture 1.2. *Suppose G is an infinite reductive group over a field of positive characteristic whose connected component G^0 is not a torus. Then $\delta(G) = \infty$.*

The assumption on G^0 cannot be dropped, as the following result, a generalisation of Theorem 1.1 shows:

Theorem 1.3. *Let G be a reductive linear algebraic group over a field of characteristic $p > 0$ such that G^0 is a torus or trivial. Let P be a Sylow-p-subgroup of the (finite) group G/G^0. Then $\delta(G) = |P|$.*

While we do not succeed in proving our conjecture, we are able to establish:

Theorem 1.4. *Let $\Bbbk$ be a field of characteristic $p > 0$. Suppose G is a closed subgroup of $\mathrm{GL}_2(\Bbbk)$ containing an infinite unipotent subgroup. Then $\delta(G) = \infty$.*

In particular, $\delta(\mathrm{SL}_2(\Bbbk)) = \delta(\mathrm{GL}_2(\Bbbk)) = \delta(\mathbb{G}_a) = \infty$ (where $\mathbb{G}_a = (\Bbbk, +)$ is the additive group of the ground field) in positive characteristics, supporting the conjecture.

In addition to $\delta(G)$, we study the closely related quantity $\sigma(G)$. We shall say a subset $S \subseteq \Bbbk[V]^G$ is a *σ-set* if, for all $v \in V \setminus \mathcal{N}_{G,V}$, there exists an $f \in S_+$ such that $f(v) \neq 0$. We shall call a subalgebra of $\Bbbk[V]^G$ a *σ-subalgebra* if it is a σ-set. Then the quantities $\sigma(G,V)$ and $\sigma(G)$ are defined along the same lines as $\delta(G,V)$ and $\delta(G)$. For a motivation of the importance of this number we content ourselves here by saying that at least for linearly reductive groups in characteristic zero, the knowledge of $\sigma(G,V)$ gives upper bounds for the maximal degrees of generating sets (for example in Derksen's famous bound [2]), and refer the reader to [1] and [5] for more details and some elementary properties of this number.

In the latter paper, the authors investigated $\sigma(G)$ for finite groups G, mainly for positive characteristic. In sections 4 and 5 of the present article we investigate $\sigma(G)$ for infinite linear algebraic groups. Our main results are as follows:

Theorem 1.5. *Let G be a linear algebraic group over a field of characteristic $p > 0$. Then $\sigma(G)$ is finite if and only if $|G|$ is finite.*

Theorem 1.6. *Let G be a linear algebraic group over a field of characteristic 0. Then if $\sigma(G)$ is finite, G^0 is unipotent, i.e. either G is finite or G^0 is infinite unipotent.*

As reductive groups do not contain a non-trivial connected unipotent normal subgroup, we get as an immediate corollary:

Corollary 1.7. *Let G be a reductive group over a field of arbitrary characteristic. Then $\sigma(G)$ is finite if and only if G is finite.*

Somewhat surprisingly, for the (infinite) additive group $\mathbb{G}_a = (\Bbbk, +)$ of a field $\Bbbk$ of characteristic zero, we will see $\sigma(\mathbb{G}_a) = 2$. We do not know whether $\sigma(G)$ is finite for all unipotent groups in characteristic zero.

Another quantity associated with $\delta(G, V)$ and $\sigma(G, V)$, which has attracted some attention in recent years, is $\beta_{\text{sep}}(G, V)$. It is defined as follows: a subset $S \subseteq \Bbbk[V]^G$ is called a *separating set* if, for any pair $v, w \in V$ such that there exists $f \in \Bbbk[V]^G$ with $f(v) \neq f(w)$, there exists $s \in S$ with $s(v) \neq s(w)$. Now again, $\beta_{\text{sep}}(G, V)$ and $\beta_{\text{sep}}(G)$ are defined along the same lines as $\sigma(G, V)$ and $\sigma(G)$. Our point of view is that δ- and σ-sets are "zero-separating" sets. This leads to the inequalities [5, Proposition 1.4]

$$\delta(G, V) \leq \sigma(G, V) \leq \beta_{\text{sep}}(G, V) \leq \beta(G, V)$$

for any linear algebraic group G and G-module V, hence

$$\delta(G) \leq \sigma(G) \leq \beta_{\text{sep}}(G) \leq \beta(G),$$

where $\beta(G, V)$ and $\beta(G)$ are the classical local and global Noether number. The second author and Kraft have shown [9] that $\beta_{\text{sep}}(G)$ is finite if and only if G is finite (indepedently of the characteristic of $\Bbbk$). Some parts of the proof of this result required some deep results from geometric invariant theory. The results of our current paper allows one to replace these parts of the proof by more elementary arguments, see section 4.

2. General results on the $\delta-$ number

In this section, we prove various general results on $\delta(G)$. For the convenience of the reader, we present the proof of the following result of Nagata and Miyata in language consistent with this article.

Proposition 2.1 (Nagata and Miyata [11, Proof of Theorem 1])**.** *Let G be a linear algebraic group over a field $\Bbbk$, and V a G-module. Suppose $v \in V^G$ and $f \in \Bbbk[V]^G_+$ is homogeneous such that $f(v) \neq 0$. If the characteristic of $\Bbbk$ does not divide the degree of f, then there exists a homogeneous invariant $\tilde{f} \in \Bbbk[V]^G_1$ of degree one satisfying $\tilde{f}(v) \neq 0$.*

Proof. Write $d := \deg(f)$. Choose a basis $\{v =: v_0, v_1, \ldots, v_n\}$ of V and let $\{x_0, x_1, \ldots, x_n\}$ be the corresponding dual basis. Since $f(v) \neq 0$, we can write $f = \sum_{i=0}^{d} x_0^{d-i} c_i$ with $c_i \in \Bbbk[x_1, x_2, \ldots, x_n]_i$ for each $i = 0, \ldots, d$ and $c_0 \in \Bbbk \setminus \{0\}$. Without loss we assume $c_0 = 1$. Further, since $v \in V^G$, note that $\langle x_1, x_2, \ldots, x_n \rangle$ is a G-invariant space, and we can write $g \cdot x_0 = x_0 + y(g)$ with $y(g) \in \langle x_1, x_2, \ldots, x_n \rangle$ for each $g \in G$. For any $g \in G$, we have

$$\begin{aligned} g \cdot f &= (g \cdot x_0)^d + (g \cdot c_1)(g \cdot x_0)^{d-1} + (\text{terms of } x_0\text{-degree} \leq d-2) \\ &= (x_0 + y(g))^d + (g \cdot c_1)(x_0 + y(g))^{d-1} + (\text{terms of } x_0\text{-degree} \leq d-2) \\ &= x_0^d + (dy(g) + (g \cdot c_1))x_0^{d-1} + (\text{terms of } x_0\text{-degree} \leq d-2) = f, \end{aligned}$$

since f is invariant. Comparing coefficients of x_0^{d-1} tells us that for any $g \in G$ we have $c_1 = dy(g) + (g \cdot c_1)$. By assumption, the degree d is invertible in $\Bbbk$, and we now set $\tilde{f} := x_0 + d^{-1}c_1$. Notice that $\deg(\tilde{f}) = 1$, and for any $g \in G$ we have

$$g \cdot \tilde{f} = g \cdot x_0 + d^{-1}(g \cdot c_1) = x_0 + y(g) + d^{-1}(c_1 - dy(g)) = \tilde{f},$$

so $\tilde{f} \in \Bbbk[V]_1^G$. Furthermore $\tilde{f}(v) = x_0(v) + d^{-1}c_1(v) = x_0(v) = 1 \neq 0$, completing the proof. □

Corollary 2.2. *Let G be a linear algebraic group and V a G-module. Then $\delta(G, V)$ equals either 0 or 1 or is divisible by the characteristic of $\Bbbk$. In particular, if $\Bbbk$ is a field of characteristic zero, then $\delta(G) = 1$.*

Proof. Note firstly that if $V^G \subseteq \mathcal{N}_{G,V}$, then $\delta(G, V) = 0$. Otherwise, $\delta(G, V) \geq 1$. Applying the above proposition shows that for any δ-set S consisting of homogeneous invariants, the set $\Bbbk[V]_1^G \cup \{f \in S \mid \deg(f)$ divisible by the characteristic$\}$ is also a δ-set. Finally, since $\delta(G, V) = 1$ when $V = \Bbbk$ is the trival module, we must have $\delta(G) \geq 1$ for any linear algebraic group G. □

The proof of the following result is a slight adaption of Nagata [10, Lemma 3.1], where it is shown that if N is a closed normal subgroup of G such that N and G/N are reductive, then G is reductive.

Proposition 2.3. *Let N be a closed normal subgroup of G such that G/N is reductive. Then for any G-module V, we have*

$$\delta(G, V) \leq \delta(N, V)\delta(G/N) \leq \delta(N)\delta(G/N),$$

so in particular we have $\delta(G) \leq \delta(N)\delta(G/N)$.

Proof. Take a point $v \in V^G \setminus \mathcal{N}_{G,V}$. As a G-invariant separating v from zero is clearly also an N-invariant, we see that $v \in V^N \setminus \mathcal{N}_{N,V}$. Therefore there is a homogeneous $f_0 \in \Bbbk[V]^N$ of positive degree $d \leq \delta(N, V)$ satisfying $f_0(v) \neq 0$. Without loss, we assume $f_0(v) = 1$. Note that as N is a normal subgroup of G, we have that $U := \Bbbk[V]_d^N$ is a G-module on which N acts trivially, so it can be considered as a G/N-module. Further we define $U_0 := \{f \in U \mid f(v) = 0\}$. Note that U_0 is a G-invariant subspace of U, since $v \in V^G$. As $f_0 \notin U_0$, we have $U_0 \neq U$. For any $f \in U$, we have $f = (f - f(v)f_0) + f(v)f_0$ with $f - f(v)f_0 \in U_0$, hence $U = U_0 \oplus \Bbbk f_0$ as a vector space. We can therefore define $\varphi \in U^*$ by $\varphi(u_0 + \lambda f_0) := \lambda$ for $u_0 \in U_0$ and $\lambda \in \Bbbk$. It is easily seen that φ is G-invariant. As mentioned, we can consider U as a G/N-module, and then we have $\varphi \in (U^*)^{G/N} \setminus \{0\}$. By assumption, G/N is a reductive group, so there exists a homogeneous $F \in \Bbbk[U^*]_{d'}^{G/N} = S^{d'}(U)^{G/N}$ of some positive degree $d' \leq \delta(G/N)$ such that $F(\varphi) \neq 0$. Let $\{f_1, \ldots, f_r\}$ denote a basis of U_0. Since $\varphi|_{U_0} = 0$, the fact that $F \in S^{d'}(\langle f_0, f_1, \ldots, f_r\rangle)^{G/N}$ such that $F(\varphi) \neq 0$ implies $F = c \cdot f_0^{d'} + \tilde{F}$, where $c \in \Bbbk \setminus \{0\}$ and $\tilde{F}$ is an element of the ideal $(f_1, \ldots, f_r)S(U)$. Note that as $U = \Bbbk[V]_d^N$, there is a canonical map $S^{d'}(U)^{G/N} \to \Bbbk[V]_{dd'}^G$, so we can take F as an element of $\Bbbk[V]_{dd'}^G$. Clearly, $F(v) = cf_0(v)^{d'} \neq 0$ as $f_i(v) = 0$ for $i = 1, \ldots, r$ by the definition of U_0, showing that $\delta(G, V) \leq \delta(N, V)\delta(G/N)$. □

Corollary 2.4. *Let G be a linear algebraic group and let G^0 denote the connected component of the identity. Then we have*

$$\delta(G) \leq \delta(G/G^0)\delta(G^0).$$

In particular, $\delta(G)$ is finite if $\delta(G^0)$ is finite.

Remark 2.5. If N is a normal subgroup of G, then $\delta(G/N) \leq \delta(G)$, since any G/N-module becomes a G-module via the map $G \to G/N$.

Proof of the Theorem 1.3. As tori are linearly reductive, $\delta(G^0) = 1$. Hence we get $\delta(G/G^0) \leq \delta(G) \leq \delta(G^0)\delta(G/G^0) = \delta(G/G^0)$, so $\delta(G) = \delta(G/G^0)$. As G/G^0 is a finite group, the value of $\delta(G/G^0)$ is the size of a Sylow-p-subgroup by Theorem 1.1. □

Theorem 1.3 shows that there are many examples of infinite groups G with finite $\delta(G) > 1$; simply define $G = P \times T$ where P is a finite p-group and T a nontrivial torus, then $\delta(G) = |P|$. For a more interesting example, consider $G = \mathrm{O}_2(\Bbbk)$ with $\Bbbk$ an algebraically closed field of characteristic 2. It is well known that $G \cong \Bbbk^* \rtimes Z_2$, where Z_2 denotes the cyclic group of order 2. Therefore $G^0 \cong \Bbbk^*$ is a torus, and $G/G^0 \cong Z_2$. By Theorem 1.3, $\delta(\mathrm{O}_2(\Bbbk)) = 2$.

3. The δ-number for subgroups of $\mathrm{GL}_2(\Bbbk)$

The goal of this section is to prove Theorem 1.4. Throughout we assume $\Bbbk$ is a field of characteristic $p > 0$. We begin by introducing another number associated to a representation of a group, which is useful for finding lower bounds for both the δ-number and σ-number. Let G be a linear algebraic group and V a G-module. Let $v \in V$. Then we set

$$\epsilon(G, v) := \inf\{d \in \mathbb{N}_{>0} \mid \text{ there exists } f \in \Bbbk[V]^G_d \text{ such that } f(v) \neq 0\},$$

where the infimum of an empty set is infinity. Notice that if $V^G \setminus \mathcal{N}_{G,V} \neq \emptyset$, then

$$\delta(G, V) = \sup\{\epsilon(G, v) \mid v \in V^G \setminus \mathcal{N}_{G,V}\},$$

and if $V \setminus \mathcal{N}_{G,V} \neq \emptyset$, then

$$\sigma(G, V) = \sup\{\epsilon(G, v) \mid v \in V \setminus \mathcal{N}_{G,V}\}.$$

For a submodule $W \subseteq V$ we define

$$\epsilon(G, W, V) := \inf\{\epsilon(G, v) \mid v \in W \setminus \mathcal{N}_{G,V}\},$$

and we set

$$\epsilon(G, V) := \epsilon(G, V^G, V) \quad \text{ and } \quad \tau(G, V) := \epsilon(G, V, V).$$

It is immdiately clear that for any linear algebraic group G we have $\delta(G, V) \geq \epsilon(G, V)$ if $V^G \setminus \mathcal{N}_{G,V} \neq \emptyset$ and $\sigma(G, V) \geq \tau(G, V)$ if $V \setminus \mathcal{N}_{G,V} \neq \emptyset$. In fact, we have the following slightly stronger result, which we mainly use for H a *finite* subgroup of G (the second inequality is not used and only stated for completeness):

Lemma 3.1. *Let G be a linear algebraic group, V a G-module and H a subgroup of G. Then $\delta(G, V) \geq \epsilon(H, V)$ if $V^G \setminus \mathcal{N}_{G,V} \neq \emptyset$ and $\sigma(G, V) \geq \tau(H, V)$ if $V \setminus \mathcal{N}_{G,V} \neq \emptyset$.*

Proof. Choose a $v \in V^G \setminus \mathcal{N}_{G,V}$ such that $\delta(G, V) = \epsilon(G, v)$. Clearly $v \in V^H \setminus \mathcal{N}_{H,V}$, hence $\delta(G, V) = \epsilon(G, v) \geq \epsilon(H, v) \geq \epsilon(H, V^H, V) = \epsilon(H, V)$. For the second inequality, choose $v \in V \setminus \mathcal{N}_{G,V}$ such that $\sigma(G, V) = \epsilon(G, v)$. As also $v \in V \setminus \mathcal{N}_{H,V}$, $\sigma(G, V) = \epsilon(G, v) \geq \epsilon(H, v) \geq \epsilon(H, V, V) = \tau(H, V)$. □

We believe a thorough investigation of the numbers $\epsilon(G, V)$ when G is a finite group may hold the key to proving Conjecture 1.2. In order to prove Proposition 1.4, we require only the corollary of the following lemma, whose proof is very similar to the proof of [5, Proposition 2.5], but the point of view is different. For any finite group G, let $V_{\mathrm{reg},G} := \Bbbk G$ denote its regular representation.

Lemma 3.2. *Suppose G is a finite group and P a Sylow-p-subgroup of G. If $V = V^n_{\mathrm{reg},G}$ is a free G-module over $\Bbbk$, then $\epsilon(G, v) = |P|$ for any $v \in V^G \setminus \{0\}$.*

Proof. For each $i = 1, \dots, n$ choose a permutation basis $\{v_{g,i} \mid g \in G\}$ of the ith summand (which is isomorphic to $V_{\mathrm{reg},G}$), so that $\{v_{g,i} \mid g \in G,\ i = 1, \dots, n\}$ is a basis of V. Let $\{x_{g,i} \mid g \in G,\ i = 1, \dots, n\}$ be the basis dual to our chosen basis of V, so that $\Bbbk[V] = \Bbbk[x_{g,i} : g \in G,\ i = 1, \dots, n]$. The fixed point space of the ith summand is spanned by $v_i := \sum_{g \in G} v_{g,i}$, therefore $V^G = \langle v_1, \dots, v_n \rangle$. For a point $v = \sum_{i=1}^n \lambda_i v_i \in V^G \setminus \{0\}$ with scalars $\lambda_i \in \Bbbk$ (not all of them zero), we will show

$\epsilon(G,v) = |P|$. We show first $\epsilon(G,v) \geq |P|$, i.e. $\deg(f) \geq |P|$ for any homogeneous $f \in \Bbbk[V]^G_+$ such that $f(v) \neq 0$. Since V is a permutation module, such an f is a linear combination of orbit sums of monomials

$$O_G(m) := \sum_{m' \in G \cdot m} m',$$

where m is a monomial in $\Bbbk[V]_+$. It follows that there exists a monomial $m \in \Bbbk[V]_+$, whose degree is the same as $\deg(f)$, such that $O_G(m)(v) \neq 0$. Now if $m' \in G \cdot m$ then $m' = g \cdot m$ for some $g \in G$, and $m'(v) = (g \cdot m)(v) = m(g^{-1}v) = m(v)$ since $v \in V^G$. Therefore

$$O_G(m)(v) = |G \cdot m| m(v) = (G : \mathrm{Stab}_G(m)) m(v) \neq 0.$$

This implies that $\mathrm{Stab}_G(m)$ contains a Sylow-p-subgroup of G, which without loss we can assume to be P. Therefore, if $x_{g,i}$ is any variable dividing m, then m is also divisible by $x_{g'g,i}$ for every $g' \in P$. In particular, since m is not constant, we obtain $\deg(f) = \deg(m) \geq |P|$ as required. Secondly, choose an i such that $\lambda_i \neq 0$ and define $m := \prod_{g \in P} x_{g,i}$. Then $O_G(m)$ is an invariant of degree $|P|$ satisfying

$$O_G(m)(v) = (G : \mathrm{Stab}_G(m)) m(v) = (G : P) \lambda_i^{|P|} \neq 0,$$

showing $\epsilon(G,v) \leq |P|$. □

Corollary 3.3. *Suppose G is a finite group and P a Sylow-p-subgroup of G. If U is a projective G-module over $\Bbbk$, then $\epsilon(G,u) = |P|$ for any $u \in U^G \setminus \{0\}$. In particular, $\epsilon(G,U) = |P|$ if $U^G \neq \{0\}$.*

Proof. By definition, U is a direct summand of a free module, i.e. there exists (up to isomorphism) a decomposition $U \oplus W = V = V^n_{\mathrm{reg},G}$ of some free module V into U and a G-module complement W. Take $u \in U^G \setminus \{0\}$. As $u \in V^G \setminus \{0\}$, by the previous lemma there is an $f \in \Bbbk[V]^G_{|P|}$ satisfying $f(u) \neq 0$. As $f|_U \in \Bbbk[U]^G_{|P|}$ satisfies $f|_U(u) = f(u) \neq 0$, we have $\epsilon(G,u) \leq |P|$. On the other hand, as we have an algebra-inclusion $\Bbbk[U] \subseteq \Bbbk[V]$, any homogeneous $f \in \Bbbk[U]^G_+$ satisfying $f(u) \neq 0$ can be considered as an element of $\Bbbk[V]^G$, hence $\deg(f) \geq |P|$ by the same lemma, so $\epsilon(G,u) \geq |P|$. □

It is worth recalling that for a p-group, "projective" and "free" means the same for a module. It is now clear how our proof should proceed - we need to find a sequence of finite (p-)subgroups H of $G = \mathrm{GL}_2(\Bbbk)$ and G-modules V which become projective (free) on restriction to H. The following result provides a good source of such modules.

Proposition 3.4. *Let $p > 0$ be a prime and let $\Bbbk$ be an algebraically closed field of characteristic p. Let $G_n = (\mathbb{F}_{p^n}, +)$ be the additive group of the finite subfield $\mathbb{F}_{p^n}$ of $\Bbbk$. Let V be the G_n-module spanned by vectors X and Y such that the action $*$ of G_n on V is given by*

$$\begin{array}{rrcll} & t * X & = & X & \\ \text{and} & t * Y & = & Y + tX & \text{for all } t \in G_n. \end{array}$$

Then $S^{p^n-1}(V)$ is isomorphic to the regular representation of G_n.

Proof. We will show that $S := \{t * Y^{p^n-1} \mid t \in G_n\}$ is a basis of $S^{p^n-1}(V)$, which clearly implies $S^{p^n-1}(V) \cong V_{\mathrm{reg},G_n}$. As $|G_n| = p^n$ equals the dimension of $S^{p^n-1}(V)$, it is enough to show that the $p^n \times p^n$ matrix A with columns formed by the coordinate vectors of the elements $t * Y^{p^n-1}$, $t \in G_n$ with respect to the standard basis $\{Y^{p^n-1-i}X^i \mid i \in \{0,\dots,p^n-1\}\}$ of $S^{p^n-1}(V)$ has a nonzero determinant. Using the binomial theorem and Lemma 3.5 we compute

$$\begin{aligned} t * Y^{p^n-1} &= (Y+tX)^{p^n-1} = \sum_{i=0}^{p^n-1} \binom{p^n-1}{i} Y^{p^n-1-i}(tX)^i \\ &\overset{\text{L. 3.5}}{=} \sum_{i=0}^{p^n-1} (-1)^i Y^{p^n-1-i}(tX)^i = \sum_{i=0}^{p^n-1} (-t)^i Y^{p^n-1-i} X^i. \end{aligned}$$

Thus, $A = ((-t)^i)_{i\in\{0,\dots,p^n-1\},t\in G_n} \in \Bbbk^{p^n\times p^n}$, where we enumerated the p^n columns of A by the set G_n – which is harmless as the order of the columns only affects the sign of the determinant of A. Note that A is the $p^n \times p^n$ Vandermonde matrix of the p^n *different* elements of $-G_n(= G_n)$, hence $\det(A) \neq 0$, which proves the claim. □

In the preceding proof we used the following number-theoretic lemma, of which we provide a proof for the convenience of the reader.

Lemma 3.5. *Let p be a prime number and $0 \le k \le p^n - 1$. Then*

$$\binom{p^n-1}{k} \equiv (-1)^k \mod p.$$

Proof. We have $\binom{p^n-1}{k} = \prod_{m=1}^{k} \frac{p^n-m}{m}$. We show that the reduced fraction of each factor has a denominator coprime to p, and equals $-1+p\mathbb{Z}$ if computed in the field $\mathbb{Z}/p\mathbb{Z}$. For this sake, for $1 \le m \le k \le p^n-1$, write $m = p^r s$ where s and p are coprime. Then $r < n$, and $\frac{p^n-m}{m} = \frac{p^n-p^r s}{p^r s} = \frac{p^{n-r}-s}{s}$. In the field $\mathbb{Z}/p\mathbb{Z}$, the last fraction equals -1. □

Having set up all the necessary machinery, we are now in a position to prove Theorem 1.4. Let G be a subgroup of $\mathrm{GL}_2(\Bbbk)$ containing an infinite unipotent subgroup U. As U is conjugate in $\mathrm{GL}_2(\Bbbk)$ to the subgroup of unipotent upper triangular 2×2 matrices (see [8, Corollary 17.5]), we can replace G by a conjugate subgroup and assume $U = \{u_t \mid t \in \Bbbk\}$, where $u_t = \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} \in G$. Note that U is ismorphic to the additive group of the ground field $\mathbb{G}_a = (\Bbbk, +)$. Let V denote the restriction of the natural 2-dimensional $\mathrm{GL}_2(\Bbbk)$-module to G. We may choose a basis $\{X, Y\}$ of V such that

$$u_t * X = X \quad \text{and} \quad u_t * Y = Y + tX \quad \text{for all } t \in \mathbb{G}_a.$$

Theorem 1.4 follows immediately from the following:

Proposition 3.6. *For any integer n set $V_n := \mathrm{Hom}_\Bbbk(S^{p^n-1}(V), S^{p^n-1}(V))$. Then $\delta(G, V_n) \ge p^n$.*

Proof. First note that $V_n^G \setminus \mathcal{N}_{G,V_n} \neq \emptyset$: to see this consider the identity homomorphism $\mathrm{id} : S^{p^n-1}(V) \to S^{p^n-1}(V)$, which is an element of V_n^G. The determinant map $\det : V_n \to \Bbbk$ is an element of $\Bbbk[V_n]^G$, and $\det(\mathrm{id}) = 1 \neq 0$, so $\mathrm{id} \in V_n^G \setminus \mathcal{N}_{G,V_n}$. Therefore we can apply Lemma 3.1 to G and its finite subgroup $U_n := \{u_t \mid t \in \mathbb{F}_{p^n}\}$, hence $\delta(G, V_n) \ge \epsilon(U_n, V_n)$. Note that $U_n \cong (\mathbb{F}_{p^n}, +)$. By Proposition 3.4, $S^{p^n-1}(V)$ is a free U_n-module. Recall that tensoring a free/projective module with any other module yields again a free/projective module, hence $V_n = \mathrm{Hom}_\Bbbk(S^{p^n-1}(V), S^{p^n-1}(V)) \cong S^{p^n-1}(V) \otimes (S^{p^n-1}(V))^*$ is also a free U_n-module. Using Corollary 3.3 we obtain

$$\delta(G, V_n) \ge \epsilon(U_n, V_n) = |U_n| = p^n$$

as required. □

We record the following observation for later use:

Corollary 3.7. *Let G be an infinite connected unipotent algebraic group over an algebraically closed field of positive characteristic. Then $\delta(G) = \infty$.*

Proof. It is well-known that such a group G contains a closed normal subgroup N such that $G/N \cong \mathbb{G}_a$. We can embed $\mathbb{G}_a$ in $\mathrm{GL}_2(\Bbbk)$ as above. Now using Remark 2.5 and Theorem 1.4, we have $\delta(G) \geq \delta(G/N) = \delta(\mathbb{G}_a) = \infty$. □

Combining Theorem 1.4 and Proposition 2.3 leads to more examples of groups with infinite δ-value: Whenever $\delta(G) = \infty$ and N is a closed normal subgroup of G such that G/N is reductive, either $\delta(N) = \infty$ or $\delta(G/N) = \infty$.

Example 3.8. Take $G = \mathrm{GL}_2(\Bbbk)$ and consider its centre $Z(G) = \{aI_2 \mid a \in \Bbbk \setminus \{0\}\}$. As a torus, $Z(G)$ is linearly reductive, hence $\delta(Z(G)) = 1$. Therefore, $\delta(\mathrm{PGL}_2(\Bbbk)) = \delta(G/Z(G)) = \infty$. Note also that $\delta(\mathrm{PSL}_2(\Bbbk)) = \infty$, because over an algebraically closed field, we have $\mathrm{PSL}_2(\Bbbk) \cong \mathrm{PGL}_2(\Bbbk)$.

4. The σ-number of infinite groups

In this section we prove Theorems 1.5 and 1.6. Some of the groundwork was done in [5]. In particular, we recall the following result:

Proposition 4.1. [5, Corollary 3.13] *Let G be a linear algebraic group, with G^0 the connected component of G containing the identity. We have the inequalities*

$$\sigma(G^0) \leq \sigma(G) \leq (G : G^0)\sigma(G^0).$$

In particular, $\sigma(G)$ and $\sigma(G^0)$ are either both finite or infinite.

The following proposition is key to the proofs:

Proposition 4.2. *Let G be a linear algebraic group over a field $\Bbbk$ of arbitrary characteristic. Suppose G contains a non-trivial torus. Then $\sigma(G) = \infty$.*

Proof. We exhibit a sequence of G-modules $\{U_m \mid m \in \mathbb{N}\}$ such that $\sigma(G, U_m) \geq m + 1$ for all $m \in \mathbb{N}$. By assumption, G contains a subgroup $T \cong \Bbbk^*$, so there is an isomorphism $\Bbbk^* \to T$, $t \mapsto a_t$. As a linear algebraic group, G can be considered as a closed subgroup of some $\mathrm{GL}_{n+1}(\Bbbk)$, and then $V = \Bbbk^{n+1}$ becomes a faithful G-module. We can choose a basis $\{v_0, v_1, \ldots, v_n\}$ of V on which T acts diagonally, and as T acts faithfully, it acts non-trivially on at least one basis vector, say v_0. Therefore, for some $r \in \mathbb{Z} \setminus \{0\}$, we have $a_t * v_0 = t^r v_0$ for all $t \in \Bbbk^*$. Write $\{y_0, y_1, \ldots, y_n\}$ for the basis of V^* dual to $\{v_0, v_1, \ldots, v_n\}$. A basis for $S^m(V^*)$ is then given by the set of monomials

$$\left\{ \mathbf{y}^{\mathbf{e}} := \prod_{i=0}^{n} y_i^{e_i} \in S^m(V^*) \;\middle|\; \mathbf{e} \in \mathbb{N}_0^{n+1},\ |\mathbf{e}| := \sum_{i=0}^{n} e_i = m \right\}$$

of degree m in this basis. Let further

$$\{Z_{\mathbf{e}} \in S^m(V^*)^* \mid \mathbf{e} \in \mathbb{N}_0^{n+1},\ |\mathbf{e}| = m\}$$

denote the corresponding dual basis of $S^m(V^*)^*$ i.e. $Z_{\mathbf{e}}(y^{\mathbf{e}'}) = \delta_{\mathbf{e},\mathbf{e}'}$ (the Kronecker-delta). Now we set $U_m := V \oplus S^m(V^*)$. We may identify $\Bbbk[U_m]$ with

$$S(U_m^*) = S(V^* \oplus S^m(V^*)^*) = \Bbbk[y_0, y_1, \ldots, y_n]\left[Z_{\mathbf{e}} : \mathbf{e} \in \mathbb{N}_0^{n+1},\ |\mathbf{e}| = m\right].$$

Consider the point $v := v_0 + y_0^m \in U_m$. We claim that $v \notin \mathcal{N}_{G,U_m}$, and we will show that $\epsilon(G, v) = m + 1$. As a consequence, $\sigma(G, U_m) \geq \epsilon(G, v) = m + 1$, finishing the proof. To see this, we define the polynomial

$$f := \sum_{\mathbf{e} \in \mathbb{N}_0^{n+1},\, |\mathbf{e}| = m} \mathbf{y}^{\mathbf{e}} Z_{\mathbf{e}} \in \Bbbk[U_m],$$

which can be interpreted as the identity map id : $S^m(V^*) \to S^m(V^*)$, and is hence an invariant, i.e. $f \in \Bbbk[U_m]^G$. Note that here we used the isomorphism

$$\mathrm{Hom}_{\Bbbk}(S^m(V^*), S^m(V^*)) \cong S^m(V^*) \otimes S^m(V^*)^*$$

and that

$$\Bbbk[U_m] \cong S(V^* \oplus S^m(V^*)^*) \cong S(V^*) \otimes S(S^m(V^*)^*)$$

contains a direct summand isomorphic to $S^m(V^*)\otimes S^m(V^*)^*$. Clearly $f(v) = 1 \neq 0$, which shows that $v \notin \mathcal{N}_{G,U_m}$. Furthermore we have $\deg(f) = m + 1$, so we have $\epsilon(G, v) \leq m+1$. It remains to show that $\epsilon(G, v) \geq m+1$. Suppose a homogeneous $f' \in \Bbbk[U_m]^G_+$ also satisfies $f'(v) \neq 0$; we will show that $\deg(f') \geq m + 1$. Observe that a fortiori we have $f' \in \Bbbk[U_m]^T_+$. Therefore f' can be written as a sum of T invariant monomials, so in particular there exists a T-invariant monomial h (of the same degree as f') satisfying $h(v) \neq 0$. As $v = v_0 + y_0^m$, the only variables that can appear in h are those dual to v_0 and y_0^m, i.e. the variables y_0 and $Z_{\mathbf{e}_0}$ with $\mathbf{e}_0 := (m, 0, 0, \dots, 0)$. We thus have $h = y_0^k Z_{\mathbf{e}_0}^l$ with $k, l \in \mathbb{N}_0$, and $\deg(h) = k + l > 0$. On the other hand, since $h \in \Bbbk[U_m]^T$ we have

$$\begin{aligned} y_0^k Z_{\mathbf{e}_0}^l &= h = a_t * h = (a_t * y_0)^k (a_t * Z_{\mathbf{e}_0})^l \\ &= (t^{-r} y_0)^k (t^{mr} Z_{\mathbf{e}_0})^l = t^{mrl-kr} \cdot y_0^k Z_{\mathbf{e}_0}^l \quad \text{for all } t \in \Bbbk^*, \end{aligned}$$

i.e. $r(ml - k) = 0$. Since $r \neq 0$ and $k + l > 0$ it must be the case that $k = ml \geq m$ and $l \geq 1$. Therefore $\deg(f') = \deg(h) = ml + l \geq m + 1$ as required. □

Corollary 4.3. *Suppose G is a linear algebraic group such that $\sigma(G)$ is finite. Then G^0 is unipotent, i.e. either G is finite or G^0 is infinite unipotent.*

Proof. If $\sigma(G)$ is finite, $\sigma(G^0)$ is finite by Proposition 4.1. It follows from Proposition 4.2 that G^0 does not contain any non-trivial torus, i.e. the rank of the connected group G^0 (the dimension of a maximal torus) is zero, hence G^0 is unipotent by [8, Exercise 21.4.1]. □

Specialising to the case of $\Bbbk$ a field of characteristic zero, this completes the proof of Theorem 1.6. To finish the proof of Theorem 1.5, it remains to show that over a field of positive characterstic, if G^0 is infinite unipotent, we have $\sigma(G) = \infty$. This follows from $\sigma(G^0) \leq \sigma(G)$ (Proposition 4.1), the inequality $\delta(G^0) \leq \sigma(G^0)$ and from $\delta(G^0) = \infty$ (Corollary 3.7). The following proposition, which provides some examples of their own interest, gives a more direct proof that $\delta(\mathbb{G}_a) = \sigma(\mathbb{G}_a) = \infty$ for a field of positive characteristic. Additionally, it gives another proof of $\beta_{\mathrm{sep}}(\mathbb{G}_a) = \infty$ for such a field, which is also shown in [9, Proposition 4], see also the following remark for more details. As before, it follows $\delta(G) = \sigma(G) = \infty$ for any infinite unipotent connected group, via a normal subgroup N such that $G/N \cong \mathbb{G}_a$. We want to mention that $\mathbb{G}_a$-modules of the type as in the proposition are also investigated in [6, 12]. The generators of the considered invariant ring would also follow from the latter paper, but we give a self-contained argument.

Proposition 4.4. *Assume $\Bbbk$ is a field of characteristic $p > 0$, and let $V_n = \Bbbk^3$ ($n \geq 1$) be the $\mathbb{G}_a = (\Bbbk, +)$-module given by the representation*

$$\mathbb{G}_a \mapsto \mathrm{GL}_3(\Bbbk), \quad t \mapsto \begin{pmatrix} 1 & 0 & 0 \\ -t & 1 & 0 \\ -t^{p^n} & 0 & 1 \end{pmatrix}.$$

If we write $\Bbbk[V_n] = \Bbbk[x_0, x_1, x_2]$, then we have

$$\Bbbk[V_n]^{\mathbb{G}_a} = \Bbbk[x_0, x_2 x_0^{p^n-1} - x_1^{p^n}] \quad \text{and} \quad \delta(\mathbb{G}_a, V_n) = \sigma(\mathbb{G}_a, V_n) = p^n.$$

Consequently, $\delta(\mathbb{G}_a) = \sigma(\mathbb{G}_a) = \infty$.

Proof. The action $*$ of $\mathbb{G}_a$ on $\Bbbk[V_n]$ is given by

$$t * f(x_0, x_1, x_2) = f(x_0, x_1 + tx_0, x_2 + t^{p^n} x_0) \quad \text{for } t \in \mathbb{G}_a,\ \ f(x_0, x_1, x_2) \in \Bbbk[V_n].$$

If f is an invariant, the equation $t * f = f$ for all $t \in \mathbb{G}_a$ implies that for an additional independent variable t, the equation

$$f(x_0, x_1, x_2) = f(x_0, x_1 + tx_0, x_2 + t^{p^n} x_0)$$

holds in the polynomial ring $\Bbbk[V_n][t]$. Substituting $t := -\frac{x_1}{x_0}$ leads to

$$(1) \quad f(x_0, x_1, x_2) = f\left(x_0, 0, x_2 - \frac{x_1^{p^n}}{x_0^{p^n}} x_0\right) = f\left(x_0, 0, \frac{x_2 x_0^{p^n-1} - x_1^{p^n}}{x_0^{p^n-1}}\right).$$

We have to show that $\Bbbk[V_n]^{\mathbb{G}_a} \subseteq \Bbbk[x_0, x_2 x_0^{p^n-1} - x_1^{p^n}]$, as the reverse inclusion is checked immediately. For an $f \in \Bbbk[V_n]^{\mathbb{G}_a}$, write $f = \sum_{k=0}^m a_k(x_0, x_1) x_2^k$ with polynomials $a_k \in \Bbbk[x_0, x_1]$. Equation (1) implies

$$(2)\ f = \sum_{k=0}^m a_k(x_0, 0) \left(\frac{x_2 x_0^{p^n-1} - x_1^{p^n}}{x_0^{p^n-1}}\right)^k = \sum_{k=0}^m \frac{b_k(x_0)}{(x_0^{p^n-1})^k} (x_2 x_0^{p^n-1} - x_1^{p^n})^k,$$

with polynomials $b_k(x_0) := a_k(x_0, 0) \in \Bbbk[x_0]$. Substituting $x_2 := 0$ leads to

$$f(x_0, x_1, 0) = \sum_{k=0}^m \frac{b_k(x_0)}{(x_0^{p^n-1})^k} (-x_1^{p^n})^k \in \Bbbk[x_0, x_1],$$

which implies that $c_k(x_0) := \frac{b_k(x_0)}{(x_0^{p^n-1})^k}$ is actually a polynomial, i.e. an element of $\Bbbk[x_0]$. Resubstituting in (2) implies

$$f = \sum_{k=0}^m c_k(x_0)(x_2 x_0^{p^n-1} - x_1^{p^n})^k \in \Bbbk[x_0, x_2 x_0^{p^n-1} - x_1^{p^n}],$$

as desired. It follows that $\sigma(\mathbb{G}_a, V_n) \leq p^n$ and $\mathcal{N}_{\mathbb{G}_a, V_n} = \{(0, 0, a_2) \in V_n \mid a_2 \in \Bbbk\}$, and clearly we have $V_n^{\mathbb{G}_a} = \{(0, a_1, a_2) \in V_n \mid a_1, a_2 \in \Bbbk\}$. Now the point $v := (0, 1, 0) \in V_n^{\mathbb{G}_a} \setminus \mathcal{N}_{\mathbb{G}_a, V_n}$ satisfies $x_0(v) = 0$ and $(x_2 x_0^{p^n-1} - x_1^{p^n})(v) = -1$, which shows $\delta(\mathbb{G}_a, V_n) = \sigma(\mathbb{G}_a, V_n) = p^n$. □

Remark 4.5. Theorems 1.5 and 1.6 were proved by "elementary" means, in the sense that we did not use any geometric invariant theory. We can use these results to give an elementary proof of [9, Theorem A], which states that $\beta_{\mathrm{sep}}(G)$ is finite if and only if G is finite. That $\beta_{\mathrm{sep}}(G)$ is finite for a finite group G is well known (see [3, Corollary 3.9.14]) so it remains to prove the converse. Suppose $\beta_{\mathrm{sep}}(G)$ is finite. The inequality $\sigma(G) \leq \beta_{\mathrm{sep}}(G)$ implies in particular $\sigma(G)$ is finite, so if $\Bbbk$ has characteristic $p > 0$ we are done by Theorem 1.5. Otherwise we conclude that G^0 is unipotent from Theorem 1.6. Now the results $\beta_{\mathrm{sep}}(\mathbb{G}_a) = \infty$ and $\beta_{\mathrm{sep}}(G^0) \leq \beta_{\mathrm{sep}}(G)$, which are both proven elementarily in [9, Proposition 5 and Theorem B], imply that $\beta_{\mathrm{sep}}(G) = \infty$ when G^0 is an infinite unipotent group. Hence, if $\beta_{\mathrm{sep}}(G) < \infty$, G^0 and G are finite.

We do not know very much about $\sigma(G)$ when G is an infinite unipotent group over a field of characteristic zero. Unlike $\beta_{\mathrm{sep}}(G)$, it is not always infinite, as the following surprising result shows:

Proposition 4.6. *Assume* $\Bbbk$ *is a field of characteristic* 0*. Then* $\sigma(\mathbb{G}_a) = 2$.

Proof. In [4, Section 3], we give for any $\mathbb{G}_a$-module V an explicit set of invariants of degree at most 2 that cuts out the nullcone. It follows that $\sigma(\mathbb{G}_a) = 2$. □

We conclude with an example which shows that $\sigma(\mathbb{G}_a \times \mathbb{G}_a) \geq 3$.

Example 4.7. Let $\Bbbk$ be an algebraically closed field of characteristic zero and $V = \Bbbk^4$. Consider an action of $G := \mathbb{G}_a \times \mathbb{G}_a$ defined as follows: $(s,t) \in \Bbbk \times \Bbbk$ acts on V as multiplication by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -s & 1 & 0 & 0 \\ \frac{1}{2}s^2 - t & -s & 1 & 0 \\ -\frac{1}{6}s^3 + st & \frac{1}{2}s^2 - t & -s & 1 \end{pmatrix}.$$

Let $\{x_0, x_1, x_2, x_3\}$ denote the basis of V^* dual to the standard basis of V. Then we claim that the ring of invariants $\Bbbk[V]^G$ is generated by the invariants x_0 and $f := x_1^3 - 3x_0x_1x_2 + 3x_0^2x_3$. Under this assumption we have that the point $v = (0,1,0,0) \in V$ is not contained in the nullcone, since $f(v) = 1 \neq 0$, and is not separated from zero by any invariant of degree less than 3, which shows that $\sigma(G, V) = 3$ and hence $\sigma(G) \geq 3$.

To prove the claim, consider the subgroup $H := \{(0,t) \in G \mid t \in \Bbbk\}$ of G. The action of H on $\Bbbk[V]$ is given by

$$\begin{aligned} (0,t) * x_0 &= x_0 \\ (0,t) * x_1 &= x_1 \\ (0,t) * x_2 &= x_2 + tx_0 \\ (0,t) * x_3 &= x_3 + tx_1 \qquad \text{for all } t \in \Bbbk. \end{aligned}$$

This $\mathbb{G}_a$-action corresponds to the direct sum of two copies of the natural representation of $\mathbb{G}_a$, and the invariant ring is well known to be given by $\Bbbk[V]^H = \Bbbk[x_0, x_1, x_0x_3 - x_2x_1]$. Crucially, this is a polynomial ring in three variables. Now $\Bbbk[V]^G = \Bbbk[x_0, x_1, x_0x_3 - x_2x_1]^{G/H}$ is isomorphic to the ring of invariants of a non-linear action of $\mathbb{G}_a$ on a polynomial ring in three variables; by a theorem of Miyanishi (see [7, Theorem 5.1]) this ring of invariants is again polynomial, with two generators. Therefore, $\Bbbk[V]^G$ is a graded polynomial ring with two generators. One may readily check that x_0 is the only invariant of degree one, and as f is an invariant of smallest possible degree not contained in $\Bbbk[x_0]$, we see that $\Bbbk[V]^G = \Bbbk[x_0, f]$ as claimed.

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