Topological Semantics for da Costa Paraconsistent Logics C_ω and C_ω^*

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Abstract

In this work, we consider a well-known and well-studied system of paraconsistent logic which is due to Newton da Costa, and present a topological semantics for it.

Keywords Paraconsistency, Da Costa logics, topological semantics.

1 Introduction

Paraconsistency is an umbrella term for the logical systems where the *explosion principle* fails. Namely, in paraconsistent systems there are some formula φ, ψ such that $\{\varphi, \neg\varphi\} \nvdash \psi$ for a logical consequence relation $\vdash$. In this work, we will focus on a well-known and well-studied paraconsistent logic which is due to Newton da Costa, and present a topological semantics for it.

Da Costa's hierarchical systems C_n and C_n^* are one of the earliest systems of paraconsistent logic (da Costa, 1974). Da Costa systems C_n where $n < \omega$ are consistent and finitely trivializable. Yet, for the limit ordinal ω, one can obtain logic C_ω which is not finitely trivializable (da Costa & Alves, 1977). In this work, we focus on the paraconsistent system C_ω and its first-order cousin C_ω^*.

Conceptually, Da Costa systems are not unfamiliar. As Priest remarked, the logic C_ω can be thought of the positive intuitionistic logic with dualized negation to give truth value gluts (Priest, 2011). We define C_ω with the following postulates:

1. $\varphi \to (\psi \to \varphi)$
2. $(\varphi \to \psi) \to ((\varphi \to (\psi \to \chi)) \to (\varphi \to \chi))$
3. $\varphi \wedge \psi \to \varphi$
4. $\varphi \wedge \psi \to \psi$
5. $\varphi \to (\psi \to \varphi \wedge \psi)$
6. $\varphi \to \varphi \vee \psi$

7. $\psi \to \varphi \vee \psi$
8. $(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$
9. $\varphi \vee \neg\varphi$
10. $\neg\neg\varphi \to \varphi$

The rule of inference that we need is modus ponens: $\varphi, \varphi \to \psi \therefore \psi$.

Based on this axiomatization, Baaz gave a Kripke-type semantics which uses Gentzen style proof theory (Baaz, 1986). Baaz's C_ω-Kripke model is a tuple $M = \langle W, \leq, V, T \rangle$ where W is a non-empty set, $\leq$ is a partial order, V is a valuation that returns a subset of W for every propositional variable in the language. The additional component T is a function defined from possible worlds (in W) to the sets of negated propositional forms. The imposed condition on T is the monotonicity: $w \leq v$ implies $T(w) \subseteq T(v)$. Monotonicity condition, as it will be clear later, resembles the hereditary condition of intuitionistic logic. The valuation respects the monotonicity and is assumed to return upsets.

Also note that the relation $\leq$ is a partial order rendering the frame $\langle W, \leq \rangle$ an **S4**-frame. The fact that the frame of Baaz's model is **S4** will be central in our topological investigations later.

One of the most interesting properties of da Costa systems is the principle of non-substitution for negated formulas. For instance, even if p and $p \wedge p$ are logically equivalent, i.e. $p \equiv p \wedge p$, we do not necessarily have that $\neg p \equiv \neg(p \wedge p)$ in da Costa systems, where $\equiv$ denotes logical equivalence. In Baaz's construction, the function T is the operator that keeps track of the negated formulas with respect to possible worlds. Therefore, at a possible world, the function T returns a set of formulas which are negated at that possible world. Moreover, for a possible world w, $T(w)$ need not be closed under logical equivalence. Namely, at a state w, we can have $\neg p \in T(w)$, but this does *not* mean that $\neg(p \wedge p) \in T(w)$. Thus, the function T returns a set which is not closed under logical consequence relation, in other words, which is not necessarily a theory. Monotonicity of T, on the other hand, reflects the *intuitionistic* side of da Costa systems. In other words, in the partially ordered Kripkean frame for C_ω, children nodes have the same formulas as their parents and possibly more under T.

Baaz gave a Kripkean semantics for C_ω as follows (Baaz, 1986). First, let $\neg^0\varphi \equiv \varphi$ and $\neg^{n+1}\varphi \equiv \neg(\neg^n\varphi)$ for a φ which does not include a negation sign in the front.

$w \models p$	iff	for all v such that $w \leq v$, $v \models p$ for atomic p
$w \models \varphi \wedge \psi$	iff	$w \models \varphi$ and $w \models \psi$
$w \models \varphi \vee \psi$	iff	$w \models \varphi$ or $w \models \psi$
$w \models \varphi \supset \psi$	iff	for all v such that $w \leq v$, $v \models \varphi$ implies $v \models \psi$
$w \models \neg^1\varphi$	iff	$\neg^1\varphi \in T(w)$ or $\exists v. v \leq w$ and $v \not\models \varphi$
$w \models \neg^{n+2}\varphi$	iff	$\neg^{n+2}\varphi \in T(w)$ and $w \models \neg^n\varphi$, or $\exists v. v \leq w$ and $v \not\models \neg^{n+1}\varphi$
$w \models \varphi_1, \ldots, \varphi_n \to \psi_1, \ldots, \psi_n$	iff	$\forall v. w \leq v$, $v \models \varphi_1, \ldots, v \models \varphi_n$ imply $v \models \psi_1$ or … or $v \models \psi_n$

Let us now briefly comment on the semantics. First, notice the hereditary condition for propositional variables. In other words, propositional variables remain true in the accessible states from the actual state once they are true at the actual state. And, notice that the hereditary condition is given only for positive propositional atoms. Second, as it can be observed, the original semantical definition that Baaz gave does not include the valuation for the propositional variables. In the topological semantics we introduce, we will address this issue. Third, as we already underlined the similarity, the implication in this system behaves like the intuitionistic implication. The crucial point, however, is the negation. A negated formula is true at a state w if it is in $T(w)$ or there is a predecessor state at which the formula does not hold. The auxiliary function T helps us to keep track of paraconsistent negations which are different than intuitionistic negation or its dual. Finally, for the last connective $\rightarrow$, we can have empty set as antecedent. Therefore, $\rightarrow p, q$ is a well-formed formula. However, we will take this connective as a shorthand.

By using proof theory of (propositional) intuitionistic logic and Gentzen style calculus, Baaz showed the soundness, completeness and decidability of this system (Baaz, 1986). We will henceforth denote this system as KC_ω.

2 Topological Models TC_ω

In this section, we give a topological semantics for da Costa's system C_ω, and call our formalism as TC_ω. First, notice that the topological semantics for (modal) logics have been presented in early 1920s preceding the well-known Kripke / Hintikka semantics (Goldblatt, 2006). The major developments in the field of topological semantics for modalities have been initiated by J. C. C. McKinsey and Alfred Tarski in 1940s in a series of papers (McKinsey & Tarski, 1944; McKinsey, 1945; McKinsey & Tarski, 1946).

In the classical modal case, the modal operator $\Box$ is associated with the topological interior operator (and, similarly $\Diamond$ with closure operator in a dual fashion). Moreover, a closer look reveals that the interior and closure operators behave as $\mathbf{S4}$ modalities (normal, reflexive and transitive). Therefore, the connection between topological semantics and $\mathbf{S4}$ modalities becomes obvious. The well-known McKinsey - Tarski result showed that $\mathbf{S4}$ is the modal logic of topological spaces, in fact, of any metric, separable, dense-in-itself space. Moreover, the topological semantics for intuitionistic and some paraconsistent logics are also rather straight forward by using open and closed complements (i.e. interior / closure of the complement) respectively (Goodman, 1981; Mints, 2000; Başkent, 2013).

2.1 Basics

The language of TC_ω is the language of propositional logic with the usual Boolean conjunction, disjunction and implication, and we will allow iterated negations. Let $\mathbf{P}$ be a countable set of propositional variables. We denote the closure of a set X by $\mathsf{Clo}(X)$. If a set $\{x\}$ is a singleton, we write $\mathsf{Clo}(x)$ instead of $\mathsf{Clo}(\{x\})$ provided no confusion arises. Also note that in this case $\mathsf{Clo}(x)$ is the intersection of all closed sets containing x.

Now, we start with defining TC_ω models.

Definition 2.1. A TC_ω model M is a tuple $M = \langle S, \sigma, V, N \rangle$ where S is a non-empty set, σ is an Alexandroff topology on S, $V : \mathbf{P} \to \wp(S)$ is a valuation function, and N is a (full) function which takes possible worlds $s \in S$ as inputs and returns sets of negated propositional forms (possibly empty) in such a way that $w \in \mathsf{Clo}(v)$ implies $N(w) \subseteq N(v)$.

Here, note that we resort to the standard method to obtain a topological model from a given (classical) Kripke model, and vice versa. In other words, given a topological space we put $w \leq v$, for $w \in \mathsf{Clo}(v)$ to obtain a partially ordered tree. And conversely, given a partially ordered tree, we consider the upward closed (or dually, downward closed) branches of the tree as opens to construct a topology, indeed an Alexandroff topology - which is closed under arbitrary intersections. In other words, since the Baaz's frames are already **S4**, the topology we obtain (after translating the given **S4** frame) is an Alexandroff topology. We refer the reader to (van Benthem & Bezhanishvili, 2007) for a detailed treatment of the subject for the classical case.

Interestingly, the fact that we obtain Alexandroff spaces in TC_ω raises the question of handling non-Alexandroff spaces in the topological models of C_ω. Granted, this is a very interesting question with a possibility of producing weaker logics than C_ω. In order not to digress from our current focus, we leave it to a future work.

Now, we give the semantics of TC_ω as follows. We abbreviate $\neg^0\varphi := \varphi$, and $\neg^{n+1}\varphi := \neg(\neg^n\varphi)$ for a φ which does not include a negation sign in the front. Similarly, we assume that the valuation function V returns closed sets (Başkent, 2013).

$w \models p$	iff	$\forall v.w \in \mathsf{Clo}(v)$, $v \models p$ for atomic p
	iff	$w \in V(p)$
$w \models \varphi \wedge \psi$	iff	$w \models \varphi$ and $w \models \psi$
$w \models \varphi \vee \psi$	iff	$w \models \varphi$ or $w \models \psi$
$w \models \varphi \supset \psi$	iff	$\forall v.w \in \mathsf{Clo}(v)$, $v \models \varphi$ implies $v \models \psi$
$w \models \neg^1\varphi$	iff	$\neg^1\varphi \in N(w)$ or $\exists v.v \in \mathsf{Clo}(w)$ and $v \not\models \varphi$
$w \models \neg^{n+2}\varphi$	iff	$\neg^{n+2}\varphi \in N(w)$ and $w \models \neg^n\varphi$ or $\exists v.v \in \mathsf{Clo}(w)$ and $v \not\models \neg^{n+1}\varphi$
$w \models \varphi_1, \ldots, \varphi_n \to \psi_1, \ldots, \psi_n$	iff	$\forall v.w \in \mathsf{Clo}(v)$, $v \models \varphi_1, \ldots, v \models \varphi_n$ imply $v \models \psi_1$ or ... or $v \models \psi_n$

Following the usual representation, we denote the extension of a formula φ in a model M by $[\varphi]^M$, and define $[\varphi]^M := \{w : M, w \models \varphi\}$.

Based on the given semantics, now we can discuss the satisfiability problem (SAT) and its complexity in logic KC_ω. First of all, note that the complexity of SAT for basic modal logic is known to be PSPACE-complete. In Kripkean frames, searching for a satisfying assignment may not be efficient timewise, but it uses the space efficiently yielding a PSPACE-complete complexity. Think of it as searching the branches of a Kripke model (which is a tree) starting from the root. Once you are done with one branch, you do not need to remember it, and thus you can reuse the same space. And, the extent of the tree you need the search, i.e. the depth, solely depends on the length of the formula. Therefore, the given formula determines the space you need to check. In KC_ω, the only issue is checking the satisfiability for negation. However, a careful examination shows that it has a rather immediate solution. The case for $\neg^1$ requires two operations: check whether a given $\neg^1$ is in the image set of T at the given state, and check if there exists a state that sees the current state with the desired condition. The latter part is clearly PSPACE considering the standard modal argument for SAT. The prior part is also polynomial - it is a sequential check for membership. Moreover, one can easily construct a polynomial transformation from modal SAT with topological semantics to KC_ω satisfiability yielding the fact that SAT for KC_ω is also PSPACE. Considering $\neg^n$ as a nested (intuitionistic) modality, one can come up with the obvious translation giving the PSPACE-completeness of the satisfiability problem for KC_ω. Now, based on the above mentioned (polynomial) transformations between topological spaces and Kripke frames, it is immediate to observe that SAT for TC_ω is also PSPACE-complete.

Theorem 2.2. *The satisfiability problem for both KC_ω and TC_ω is PSPACE-complete.*

Corollary 2.3. *Both KC_ω and TC_ω are decidable.*

In his work, Baaz mentioned several results (Baaz, 1986). Here, we observe that they hold in TC_ω as well. The following results make it clear how negation behaves in TC_ω, and our definition for the function N works as expected.

Proposition 2.4. $w \models \varphi$ *iff for all* v *such that* $w \in \mathsf{Clo}(v)$*, we have* $v \models \varphi$.

Proof. The proof is by induction on the length of the formula as usual. The only interesting case is the negation.Assume $\varphi \equiv \neg^1\psi$. Then, suppose $w \models \neg^1\psi$. By definition, either $\neg^1\psi \in N(w)$ or there exists a x such that $x \in \mathsf{Clo}(w)$ and $x \not\models \psi$.Now, let v be such that $w \in \mathsf{Clo}(v)$. Then, by the definition of N, we observe $N(w) \subseteq N(v)$. Thus, $\neg^1\psi \in N(v)$. On the other hand, $w \in \mathsf{Clo}(v)$ implies that $\mathsf{Clo}(w) \subseteq \mathsf{Clo}(v)$. Therefore, $x \in \mathsf{Clo}(w) \subseteq \mathsf{Clo}(v)$ with $v \not\models \psi$. Then, we have either $\neg^1\psi \in N(x)$ or there exists x such that $x \in \mathsf{Clo}(v)$ with $x \not\models \psi$. Thus, $v \models \neg^1\psi$.

The cases for $\neg^{n+1}$ are similar by using the induction hypothesis. ⊠

Proposition 2.5. $w \not\models \varphi$ *implies that there is no* $v \in \mathsf{Clo}(w)$ *such that* $v \not\models \neg\varphi$.

Proof. Let $w \not\models \varphi$. Towards a contradiction, assume that there is a $v \in \mathsf{Clo}(w)$ with $v \not\models \neg\varphi$.On the other hand, by Proposition 2.4, $v \not\models \neg\varphi$ means that for all w such that $v \in \mathsf{Clo}(w)$ we have $w \not\models \neg\varphi$.Thus, we conclude $w \not\models \varphi$ and $w \not\models \neg\varphi$. Contradiction. ⊠

Proposition 2.6. $w \models \neg\neg\varphi \to \varphi$.

Proof. We will show that $w \not\models \varphi$ implies $w \not\models \neg\neg\varphi$.Let $w \not\models \varphi$. Then, by Proposition 2.5, there is no $v \in \mathsf{Clo}(w)$ with $v \not\models \neg\varphi$. Then, by definition of $\neg^2$, we conclude that $w \not\models \neg\neg\varphi$. ⊠

Note that $\neg\varphi \leftrightarrow \neg(\varphi \wedge \varphi)$ is not valid in KC_ω. We next observe that it is not valid in TC_ω as well.

Proposition 2.7. $\neg\varphi \leftrightarrow \neg(\varphi \wedge \varphi)$ *is not valid.*

Proof. Take a state w such that $\mathsf{Clo}(w) \subseteq [\varphi]$ and $\neg\varphi \in N(w)$. Thus, $w \models \neg\varphi$. Stipulate further that $\neg(\varphi \wedge \varphi) \notin N(w)$ to get a counter-model. ⊠

Proposition 2.8. $w \models \to \varphi, \neg\varphi$.

Proof. Recall that $\to \varphi_1, \ldots, \varphi_n$ means that $\varphi_1 \vee \cdots \vee \varphi_n$ holds. Then, the result follows from the axiomatization of C_ω. ⊠

As we emphasized already, most theoretical of paraconsistency is proof theoretical and sometimes resorts to Gentzen style constructions. For the completeness of our arguments in this work, here, we present the (semantical) validity of cut elimination. The proof is straightforward, hence, we skip it.

Proposition 2.9. $w \models \Pi \to \Gamma, \varphi$ *and* $w \models \varphi, \Delta \to \Lambda$ *imply* $w \models \Pi, \Delta \to \Gamma, \Lambda$.

We now state the soundness theorem without a proof.

Theorem 2.10. $\vdash \varphi \to \psi$ *implies* $\models \varphi \to \psi$.

Baaz uses Gentzen style sequent calculus to show the completeness of his system. He concludes that if $\Pi \to \Gamma$ is not provable without cuts, there is a KC_ω-Kripke model $M = \langle W, \leq, v, T\rangle$ such that $0 \in W$ and $0 \not\models \Pi' \to \Gamma'$ where $\Pi' \equiv \Pi, \Delta$ and $\Gamma' \equiv \Gamma, \Psi$. Namely, $M \not\models \Pi \to \Gamma$. Here, 0 is the lowest top sequent in the reduction tree of $\Pi \to \Gamma$.

Now, to show the topological completeness of our system TC_ω, we again make use of the translation which we mentioned earlier. Given a KC_ω model $M = \langle W, \leq, v, T\rangle$, we can construct TC_ω model $M' = \langle S, \sigma, V, N\rangle$ as follows. Let $S := W$, and $V := v$. Now, we need to define the topology σ, and the open and closed sets in σ. Define closed sets as the upsets, and observe that $v \in \mathsf{Clo}(w)$ whenever $v \leq w$. For a tree model, it is easy to observe that the closed sets we defined produces an Alexandroff topology. This is the standard translation between the classical Kripke models and topological models (van Benthem & Bezhanishvili, 2007). As we mentioned earlier, as the Kripke model in our case is a **S4** frame, we obtain an Alexandroff topology.

Furthermore, put $N(w) := T(w)$. Then, the topology σ is defined as the collection upward closed sets in W with respect to the order $\leq$ in the standard fashion. Therefore, given a TC_ω model, we can effectively convert it to KC_ω which is known to be complete. This is the immediate method. Alternatively, we can start off with the topological TC_ω model, and give a topological completeness proof. For the completeness of our arguments here in this paper, we will sketch the completeness proof here.For the topological completeness of TC_ω, we use *maximal nontrivial sets of formulas*. First, note that we call a set X trivial if every formula in the language is deducible from X, otherwise we call it nontrivial. Then, a nontrivial set X is called a maximal nontrivial set of formulas if for all φ, if $\varphi \notin X$, then $X \cup \{\varphi\}$ is trivial. Now, we construct the maximal nontrivial consistent sets. First, observe the following.

Proposition 2.11. *If Γ is a maximal non-trivial set of formulas, then we have $\Gamma \vdash \varphi$ iff $\varphi \in \Gamma$.*

Using canonical sets, we construct the canonical TC_ω model $\langle S', \sigma', V', N' \rangle$. Let us first start with the canonical topological space. The canonical topological space is the pair $\langle S', \sigma' \rangle$ where S' is the set of all maximal non-trivial sets of formulas, and σ' is the set generated by the basis $B = \{\widehat{\neg\varphi} : \text{ any formula } \varphi\}$ where we define $\widehat{\varphi} := \{s' \in S' : \varphi \in s'\}$. Here, our construction is very similar to the classical case: instead of (classical) modal formula, we use negated formulas in the construction of the canonical model (and its topology). The reason for this choice is the fact that in TC_ω negated formulas resort to the closure operator - similar to the modal operators in the classical case.

In order to show that B is a basis for the topology σ', we need to show:

1. For any $U, U' \in B$ and any $x \in U \cap U'$, there is $U_x \in B$ such that $x \in U_x \subseteq U \cap U'$
2. For any $x \in S'$, there is $U \in B$ with $x \in U$

For the first item, observe that $\widehat{\neg\varphi \wedge \chi} = \widehat{\neg\varphi} \wedge \widehat{\neg\chi}$. Therefore, $U \cap U' \in B$ which argues for finite intersection.

For the second item, observe that $\neg\bot \in x$ for any maximal consistent set x in the canonical TC_ω. Therefore, for any $x \in S'$, there is a $\widehat{\neg\bot} \in B$ that includes x.

This argument shows that B is a basis for the topology of the canonical model.

Now, the valuation V' is defined in the standard way: $V'(p) := \{s' \in S' : p \in s'\}$. Similarly, define N' from S' to sets of formulas, and put, $N'(s') \subseteq N'(t')$ if $s' \in \mathsf{Clo}(t')$ for $s', t' \in S'$. Additionally, we impose that $N'(s') \subseteq s'$. In other words, we lift N to the level of maximal consistent sets, and impose a closure condition for the negated formulas with respect to the actual maximal consistent set. Another way of looking at it is to include $N'(s')$ formulas to the s' in the construction of the maximal non-trivial set s'. Therefore, we close not only under logical connectives but also under the N' function which amounts to adding negated formulas to the maximal (nontrivial) consistent sets. The significant difference of our logic here is the function N', and, we need to close the non-trivial sets under it. The truth of classical Booleans are defined as usual in the canonical models. For negation, we put the following.

$$s' \models \neg^1\varphi \text{ iff } \neg^1\varphi \in N'(s') \text{ or } \exists t' \in \mathsf{Clo}(s') \text{ such that } t' \not\models \varphi$$

For the truth lemma, we only need to observe that, $s' \models \varphi$ if and only if $\varphi \in s'$.

The standard Boolean cases are immediate. So, let us take $\varphi = \neg^1\psi$ for some ψ. For "truth to membership" direction, if $\neg^1\psi \in N'(s')$, then we are done as $N'(s') \subseteq s'$. Otherwise, we need to find a t' in $\mathsf{Clo}(s')$ which does not satisfy ψ. Since the topology σ is constructed by using a basis with opens, we can easily pick t' from the boundary points $\partial(s')$ which are not in the interior of the extension (by definition). For instance, if the space is discrete and the boundary is empty, then we can take any point from s' as each subset of the space is clopen (both closed and open) so that $\mathsf{Clo}(s') = s' = \mathsf{Int}(s')$. Therefore, let us here argue assuming that the boundary is not empty (if it is, we still know what to do as described above).

Take such a $t' \in \partial(s')$ such that $t' \not\models \psi$. Then, by the induction hypothesis, $\psi \notin t'$. The set t' is maximal and non-trivial, so $\neg^1\psi \in t'$. Recall that $t' \in \mathsf{Clo}(s')$, thus $\neg^1\psi \in \mathsf{Clo}(s')$.

This was the direction from "truth to membership". The direction from "membership to truth" is similar using some properties of closure operators, so we skip it. Similarly, we leave the case $\varphi = \neg^{n+2}\psi$ to the reader which only requires an inductive proof.

After establishing the truth lemma, we have the following completeness result.

Theorem 2.12. *For any set of formulas Σ in TC_ω, if $\Sigma \models \varphi$ then, $\Sigma \vdash \varphi$.*

Proof. We will show the contrapositive of the statement. Assume, $\Sigma \not\vdash \varphi$. Then, $\Sigma \cup \{\neg\varphi\}$ is non-trivial, and can be extended to a maximal non-trivial set Σ'. By the truth lemma, $\Sigma' \models \neg\varphi$ yielding $\Sigma' \not\models \varphi$. This is the counter-model we were looking for. ⊠

Note that so far, we have simply showed that Baaz's results in KC_ω can be carried over to TC_ω without much difficulty. This is achieved relatively easily because of the immediate and effective translation between KC_ω and TC_ω, and the similarity between the classical modalities and the da Costa negation operator. Such similarities between classical modalities and paraconsistent operators were also addressed in some other work (Béziau, 2005; Béziau, 2002).

2.2 Further Results

In this section, we reconsider TC_ω models in various topological spaces, and investigate how topological properties and TC_ω models interact. Here, we discuss separation axioms, regular spaces and connected spaces. The main motivation behind choosing these structures is the fact that the semantics of the negation operator in TC_ω deals with the closure (and then indirectly, with the boundary) of the sets. Thus, topological notions that are relevant to the boundary become our main focus. Moreover, many of our results heavily depend on the fact that Baaz's construction uses a partial-order which produces an **S4** frame.

Moreover, we remind the reader that our treatment is by no means complete. Various other topological, mereotopological and geometrical notions can further be investigated within the framework of da Costa logics or paraconsistent logics in general. Nevertheless, in this work, we confine ourselves to the aforementioned issues, and leave the rest to future work.

2.2.1 Separation Axioms

Let us first recall some of the well-known separation axioms for topological spaces. Two points are called *topologically indistinguishable* if both have the same neighborhoods. They are topologically distinguishable if they are not topologically indistinguishable. Indiscrete space (trivial topology) is perhaps the simplest example where any two points are topologically indistinguishable. Moreover, two points are *separated* if each of the points has a neighborhood which is distinct from the other's neighborhoods.

Separation axioms present an interesting take on paraconsistency. Traditionally, paraconsistent logics are known as the logics with truth value *gluts* as opposed to intuitionistic logics which have truth value *gaps*. Therefore, separating the points in the model with respect to paraconsistent negation is an intriguing direction to pursue.

Let us now define the separation axioms that we need. A topological space is called,

- $\mathbf{T_0}$ if any two distinct points in it are topologically distinguishable.
- $\mathbf{T_1}$ if any two distinct points in it are separated.
- $\mathbf{R_0}$ if any two topologically distinguishable points are separable.
- $\mathbf{T_2}$ if any two distinct points in it are separated by neighborhoods.
- $\mathbf{T_{2^{1/2}}}$ if any two distinct points in it are separated by closed neighborhoods.

Notice that while discussing the semantics of TC_ω above, we made use of the relation $w \in \mathsf{Clo}(v)$ quite often. This relation is called *the specialization order*: $w \leq v$ if and only if $w \in \mathsf{Clo}(v)$. It is a partial-order if and only if the space is $\mathbf{T_0}$. In this case, if the relation $\leq$ is symmetric, then the space we obtain is $\mathbf{R_0}$.

We do not force TC_ω models to be $\mathbf{T_1}$ models or even $\mathbf{R_0}$ models. Then the natural question is the following: Can we have TC_ω models which are not even $\mathbf{T_0}$ or $\mathbf{T_1}$?

Proposition 2.13. *Given a KC_ω model M. The TC_ω model M' obtained from M is $\mathbf{T_0}$, and not necessarily $\mathbf{T_1}$.*

Proof. The argument is quite straight forward. While generating M' from M, we use the partial order of the Kripke model to obtain the opens and closed sets of the topology, as we remarked earlier. In this case, the topology we obtain in M' is an Alexandroff topology as the specialization order of the Alexandroff topology is precisely the partial order that comes from the Kripke model. Therefore, since the specialization order is a pre-order, as we noted before, the space we obtain is $\mathbf{T_0}$, so is M'.

However, M' is not $\mathbf{T_1}$ as Alexandroff spaces are not necessarily $\mathbf{T_1}$. They are $\mathbf{T_1}$ if only if they are discrete - each s having a neighborhood of $\{s\}$ only (Arenas, 1999). ⊠

Now, we focus on $\mathbf{T_{2^{1/2}}}$ spaces as the closed sets and closure operator play a central role in paraconsistent semantics. Our main theorem is the following.

Theorem 2.14. *Let $M = \langle S, \sigma, V, N \rangle$ be a $\mathbf{T_{2^{1/2}}}$ TC_ω model which admits true contradictions, then N cannot be empty.*

Proof. In TC_ω (and similarly in KC_ω) models, N (or T) function tracks the negated formulas in an ad hoc way. In this fashion, nonemptiness of N means that the model cannot have truth value gluts. Intuitively, this is because of the imposition of the separation axiom. Let us now see the proof.Let $M = \langle S, \sigma, V, N \rangle$ be a $\mathbf{T_{2^{1/2}}}$ TC_ω model. Assume N is empty. Let w be a state where we have a true contradiction $\varphi \wedge \neg\varphi$ for some φ. Thus, $w \models \varphi$, and moreover, since N is empty, there is $v \in \mathsf{Clo}(w)$ such that $v \not\models \varphi$. Since we are in a $\mathbf{T_{2^{1/2}}}$ space, w and v must be separable. However, since $v \in \mathsf{Clo}(w)$, it means that v is in the intersection of all closed sets in σ containing w. Thus, they are not separable by closed neighborhoods. Contradiction. Thus, N cannot be empty, and such a point v cannot exists in a $\mathbf{T_{2^{1/2}}}$ space that admit true contradictions. ⊠

Corollary 2.15. *Let $M = \langle S, \sigma, V, N \rangle$ be a TC_ω space with true contradictions. If N is empty, then M cannot be $\mathbf{T_{2^{1/2}}}$.*

In order to see the correctness of the previous corollary in an example, first assume that N is empty; and design a model where for some formula φ and its negation $\neg\varphi$, at some points $w \in [\varphi]$ and $w' \in [\neg\varphi]$, the only closed sets around w and w' will be $[\varphi]$ and $[\neg\varphi]$ respectively. Let $S = \{1, 2, 3\}$, and $\sigma = \{\emptyset, S, \{1, 2\}, \{2, 3\}, \{2\}\}$. Let $[\varphi] = \{1, 2\}$, and $[\neg\varphi] = \{2, 3\}$. (Consider the formula $\varphi \wedge \neg\varphi$ at 2.) Then, observe that the points 1 and 3 are not separable by closed sets. Thus, this model cannot be $\mathbf{T_{2^{1/2}}}$. However, if N was not empty, in an ad-hoc way, we would have defined the truth of negated formula $\neg\varphi$ in a way to overcome this issue by letting $N(2) = \{\neg\varphi\}$.

Mortensen, in an earlier paper, investigated the connection between similar separation axioms and paraconsistent theories where he made several observations about discrete spaces, and $\mathbf{T_1}$ and $\mathbf{T_2}$ spaces (Mortensen, 2000).

Moreover, similar connections can be made between paraconsistency in general, and connectedness and continuity. In this work, we refer the reader to (Başkent, 2013) where such properties are studied.

2.2.2 Regular Spaces

Regular (open) sets are the sets which are equal to the interior of their closure. They play an important role not only in topology but also in mereotopology where the relationship between parts and the whole is investigated (Pratt-Hartman, 2007).

Even if we will not dwell on it further, it is worthwhile to mention that the algebra of closed sets and the topological models for paraconsistent logic do have the same algebraic structure, they both are co-Heyting algebras. Co-Heyting algebras are duals of Heyting algebras which were first proposed as the algebraic counterpart of intuitionistic logics. Some region based logics, on the other hand, utilize both Heyting and co-Heyting algebras (Mormann, 2012; Stell & Worboys, 1997). From an algebraic perspective, we observe that regular sets play an important role in paraconsistency. Now we will consider the matter from a model theoretical perspective with topological semantics, and focus on TC_ω. We start with definitions.

Definition 2.16. Let $\langle S, \sigma \rangle$ be a topology. A subset $X \subseteq S$ is called a *regular open* set if X is equal to the interior of its closure, namely if $X = \mathsf{Int}(\mathsf{Clo}(X))$. Similarly, a subset $Y \subseteq S$ is called a *regular closed* set if Y is equal to the closure of its interior, namely if $Y = \mathsf{Clo}(\mathsf{Int}(Y))$. We call a space *regular open (closed)* if all the opens (or dually closeds) are regular. We call a model *regular open (closed)* if its topological space is regular open (closed).

For example, regular open sets in the standard topology of $\mathbb{R}^2$ are the open sets with no "holes" or "cracks". Also note that the complement of a regular open is a regular closed and vice versa.

We then observe the following.

Proposition 2.17. *Let $M = \langle S, \sigma, V, N \rangle$ be a TC_ω model with discrete topology σ. If $N = \emptyset$, then we have $w \models \neg\varphi$ if and only if $w \not\models \varphi$ for all $w \in S$ and for all φ.*

Proof. It is a well-known fact that, in a discrete topology every subset is closed (or open dually). Now, let N be empty. Assume, for an arbitrary $w \in S$, an arbitrary formula φ, we have $w \models \neg\varphi$. Then, by definition, considering the discrete topology and the emptiness of N, we have $w \not\models \varphi$. Converse direction is also similar, and we leave it to the reader. ⊠

Clearly, the converse of the above statement is not necessarily true, as it is very much possible to add "redundant" elements to N to make it non-empty.

2.2.3 Connectedness

A topological space is called *connected* if it cannot be written as the disjoint union of two open sets. We define *connected component* as the maximal connected subset of a given space. Moreover, in a connected topological space $\langle S, \sigma \rangle$, the only subsets with empty boundary are S and $\emptyset$. This fact, together with the semantics of the negation, plays an important role in TC_ω.

Proposition 2.18. *Let $M = \langle S, \sigma, V, N \rangle$ be a TC_ω model that admits a true contradiction whose extension is in the topology. If the space is disconnected and $|M| > 1$, then N cannot be empty.*

Proof. Proof follows from the fact that in disconnected spaces, there are sets with empty boundary beyond the space itself and the empty set. So, we briefly mention the proof idea here.Let a contradiction $\varphi \wedge \neg\varphi$ satisfied in the model. Then, in this case, the positive φ and negative φ conjuncts of the contradiction will lie in the different connected components. However, if N is empty, it means that the extensions of each conjunct is connected via the boundary - which creates the contradiction as the space is assumed to be disconnected. ⊠

Corollary 2.19. *If N is empty, and M admits true contradictions whose extensions are in the topology, then M cannot be disconnected.*

Namely, if the extension of a contradiction is not connected, then the negation function N needs to be non-empty. Clearly, one can define a topology that does not include such true contradictions to get around the connectedness issue (or any other issue about any topological property). Recall that, the definition of a TC_ω model does not require that the extensions of formulas should be open sets. Therefore, in the above observations we focused only on the true contradictions $\varphi \wedge \neg\varphi$ where $[\varphi \wedge \neg\varphi] \in \sigma$.

3 Topological First-Order Models TC_ω^*

The logic C_ω can be extended to first-order level by introducing quantifiers, and the resulting first order da Costa logic is called C_ω^* (da Costa, 1974).

Note that in his work which we mentioned in the previous section, Baaz considered only the propositional case C_ω, and did not take the next step to introduce Kripke semantics for C_ω^*. Priest, later on presented a Kripke semantics and tableaux style completeness for first-order da Costa logic (Priest, 2011). Here, we introduce a topological semantics for C_ω^*, and call our system TC_ω^*.

First, let us set a piece of notation. For a formula φ, we abbreviate $\varphi^\circ := \neg(\varphi \wedge \neg\varphi)$. Moreover, we let, $\varphi^{(1)} := \varphi^\circ$, $\varphi^{(n)} := \varphi^{(n-1)} \wedge (\varphi^{(n-1)})^\circ$ for $2 \leq n \leq \omega$. We will often abuse the notation, and write φ^n instead of $\varphi^{(n)}$ for easy read.

Let us now start with introducing the axioms for C_ω^*. The axioms of C_ω^* are the axioms of C_ω together with the following additional axioms.

1. $\forall x F(x) \to F(y)$.
2. $F(y) \to \exists x F(x)$.
3. $\forall x (F(x))^{(n)} \to (\forall x F(x))^{(n)}$ for $n \leq \omega$.
4. $\forall x (F(x))^{(n)} \to (\exists x F(x))^{(n)}$ for $n \leq \omega$.

5. Given F and F', if either one is obtained from the other by replacing bound variables or by suppressing vacuous quantification (without confusion of variables), then $F \leftrightarrow F'$ is an axiom.

The rules of inference are modus ponens, $\varphi \to F(x) \therefore \varphi \to \forall x F(x)$ where x does not occur in F, and $F(x) \to \varphi \therefore \exists x F(x) \to \varphi$. Based on the given axiomatization, note that C_n^* is finitely trivializable for $n < \omega$ while C_ω^* is not. Also, observe that C_0^* is the classical first-order logic.

Our goal now is to give a topological semantics for C_ω^*, and show that the given axiomatization is sound with respect to the topological semantics we will propose. First, how can we give a topological semantics for C_ω^*? Some ideas which we have used in propositional case will also be useful for the predicate case. Nevertheless, we need to be more careful.

In the case of TC_ω^*, we will make use of *denotational semantics* akin to Awodey and Kishida's work on topological first-order classical modal logic. In their work, they used sheaves to express the quantification domain of predicated modal formulas together with various category theoretical tools in the proofs (Awodey & Kishida, 2008). Their semantics is elegant, and simply explain how we should read predication in a natural way in the case of topological modal models. The use of denotational semantics will be helpful for TC_ω^* as it is also a quite natural way to handle the non-truth functional behavior of the negation.

We start by introducing TC_ω^* models, and the related denotational interpretation function.

Definition 3.1. A first-order topological da Costa model TC_ω^* is given as the tuple $\langle S, D, |\cdot|, N^*, \sigma\rangle$ where S is a non-empty set with topology σ on it, $\emptyset \neq D \subseteq S$ is called the domain of individuals, $|\cdot|$ is a denotational interpretation function that assigns denotations in S to formulas, and N^* is the extension of the propositional negation function N to the first-order case defined over S.

A brief explanation of TC_ω^* models is in order here. The denotational interpretation function $|\cdot|$ takes formulas (with or without free variables), and returns individuals from S. Domain D, on the other hand, is given to precise the quantification. Similar to first-order classical modal logic, we use the domain set in the definition of the semantics of the quantifiers (Fitting & Mendelsohn, 1998). Here, we take D as a subset of S, so that we can make use of the topology σ defined on S for the objects in the domain. Alternatively, domain D and the topological space S can be taken as disjoint, and there can be defined a homeomorphic map from D to S (Awodey & Kishida, 2008). Nevertheless, for simplicity reasons, we avoid such complications here. Finally, the function N^* is similar to the propositional one N, and makes the semantics for negation non-truth functional, which we need in da Costa systems.

As we have remarked already, da Costa negation, in both propositional and first-order cases, is not truth functional. Note that there are, however, some paraconsistent logics with topological semantics where negation behaves truth functionally (Goodman, 1981). In such systems, the extension of each and every propositional variable is associated with a closed set while this condition is not a requirement in the topological semantics for classical modal logics. The reason for that is the fact that in classical modal logics, only modal formulas are forced to have open or closed extensions. Propositional formulas do not necessarily have such extensions in classical case. They may have, or they may not. Then, the negation in such (standard) paraconsistent topological systems is defined as the *closure of the compliment* (Goodman, 1981). The reason for this is quite immediate. While attempting to take the negation of a given formula, the usual way is to consider the set theoretical complement of the extension of the given formula. However, the complement of a closed set (which is the extension of the given formula) may not be closed, thus, may not be in the topology since the topology in question is a closed set topology. Therefore, in order to maintain the closed set topological structure, negation needs to be defined in that way to produce a closed set.

This idea, however, does not work in da Costa logics. For instance, assume that we endorse the aforementioned definition of negation for TC_ω^*. Namely, consider the following definition $|\overline{x}; \neg F| = \mathsf{Clo}(S^n - |\overline{x}; F|)$.

However, a closer inspection immediately reveals that the above semantics for negation is truth functional. In order to see the failure of this definition within the context of TC_ω^*, consider the logically equivalent formulas $\neg p$ and $\neg(p \wedge p)$. Based on the proposed semantics, the denotations of $\neg p$ and $\neg(p \wedge p)$ are necessarily the same. However, in da Costa systems, recall that the extensions of both $\neg p$ and $\neg(p \wedge p)$ are not necessarily identical. Therefore, the proposed (standard) paraconsistent topological semantics does not work for da Costa systems. This is another way of seeing why we need the N^* function (or N function in the propositional case) in da Costa systems. Here, we suggest a working topological semantics for C_ω^*. Let us now make it clear.

With $\overline{x}$ for variables $x_1, \ldots, x_n$ of appropriate arity n in the formula F, we represent the function that maps all free variables in F to some objects. We denote the denotational interpretation of F with $\overline{x}$ by $|\overline{x}; F|$, which is a tuple in S^n. For the formulas of different arity for free variables, we simply adjust the arity of the function $\overline{x}$ per each of its occurrence (or we can simply define the arity $\overline{x}$ as the largest arity of the predicates involved). Moreover, we denote the compliment of $|\overline{x}; F|$ by $|\overline{x}; F|^c$.

We denote the variable assignment by v. The function v assigns the variables in logical terms to the objects in the model, and this construction is a familiar one from first-order logic matching individual atoms with objects in the model. Moreover, we also define *terms* following the standard construction in first-order logic. Based on these specifications, here we spell out the semantics which we suggest for the logic TC_ω^*.

- $|\overline{x}; c| \in S$ for a constant c
- $|\overline{x}; F| \subseteq S^n$ for a n-place predicate FIn particular, take an atomic sentence $F(t_1, \ldots, t_n)$ with terms t_i for $1 \leq i \leq n$. If $d_1, \ldots, d_n$ are the evaluation of the terms $t_1, \ldots, t_n$ under the variable assignment v, then we have the following in S:$|\overline{t}; F(t_1, \ldots, t_n)| = v(F)(v(t_1), \ldots v(t_n))$

- $|\overline{x}; F \wedge G| = |\overline{x}; F| \cap |\overline{x}; G|$
- $|\overline{x}; F \vee G| = |\overline{x}; F| \cup |\overline{x}; G|$
- $|\overline{x}; \neg F| = |\overline{x}; N^*| \cup \mathsf{Clo}(|\overline{x}; F|^c)$
- $|\overline{x}; \exists y F| = \bigcup_{d \in D} |\overline{d}, d; F|$ where $\overline{d} \in D^n$
- $|\overline{x}; \forall y F| = \bigcap_{d \in D} |\overline{d}, d; F|$ where $\overline{d} \in D^n$

We can furthermore define the truth in a TC_ω^* model M. We say that a formula $F(\overline{x})$ is true in the denotational interpretation $|\cdot|$, if $|\overline{x}; F| = S$.

Let us now explicate the given semantics a bit further. The denotational semantics for the negation ensures that the negated denotation is among the formulas determined by N^* function. So, $|\overline{x}; N^*|$ can be thought of as the collection of the denotations of the formulas returned by N^*. The closure operator Clo in the definition functions as the *classical* (or standard) part of the definition. Similarly, the denotational semantics for the quantifier varies over the objects in the domain even though the denotation of the formula in question will eventually be in S.

Recall that in the (standard) topological semantics for paraconsistent logics, the negation is defined as the closure of the complement. Therefore, the inconsistencies occur at the boundary points which makes it quite easy and straightforward to keep track of inconsistencies. Yet, note that this is not necessarily the case in da Costa systems. The Clo operator in our case introduces the element of inconsistencies at boundaries, and the function N^* makes sure that negation behaves as it should in da Costa systems. The following example illustrates how such constructed negation behaves in TC_ω^*. Consider the denotational semantics of the formula $\exists y(\neg F \wedge F)$ with x.

$$
\begin{aligned}
|x; \exists y(\neg F \wedge F)| &= \bigcup_{d \in D} |d, d'; (\neg F \wedge F)| \\
&= \bigcup_{d \in D} \{|d, d'; \neg F| \cap |d, d'; F|\} \\
&= \bigcup_{d \in D} \{(|d, d'; N^*| \cup \mathsf{Clo}(|d, d'; F|^c)) \cap |d, d'; F|\} \\
&= \bigcup_{d \in D} \{(|d, d'; N^*| \cap |d, d'; F|) \cup \partial(|d, d'; F|)\}
\end{aligned}
$$

where $\partial(\cdot)$ is the topological boundary operator for sets. In this example, the individuals $d \in D$ which exist and satisfy the contradictory formula $F \wedge \neg F$ lie in the boundary of the denotation of F, or in the intersection of the denotation of F, and the denotation of the formulas returned by N^*.

Also, note that quantified De Morgan's laws are not valid in da Costa systems - even if the set theoretical De Morgan's laws hold (da Costa *et al.*, 2007).

As an illustration, consider the following classical first-order logical equality $\forall x Fx \leftrightarrow \neg\exists x \neg Fx$. Let us see the denotation of $\neg\exists x \neg Fx$.

$$\begin{aligned}
|x; \neg\exists x \neg Fx| &= |d; N^*| \cup \mathsf{Clo}(|d; \exists x \neg Fx|^c) \\
&= |d; N^*| \cup \mathsf{Clo}((\cup_{d \in D} |d; \neg F|)^c) \\
&= |d; N^*| \cup \mathsf{Clo}((\cup_{d \in D} (|d; N^*| \cup \mathsf{Clo}(|d; F|)^c))^c)
\end{aligned}$$

Therefore, if $|d; N^*|$ is not empty, we cannot generally obtain $\bigcap_{d \in D} |d; F|$ - which is the denotation of $\forall x Fx$. Other quantified De Morgan laws can be given similar arguments (Ferguson, 2012).

Soundness of the axioms of TC^*_ω with respect to the given semantics above is a straightforward symbolic manipulation. However, in order to illustrate our point, let us consider those axioms which are unique to da Costa systems, and show their soundness.

Now, as the first case, take the following formula as an instantiation of the axiom scheme (3) with $n = 1$.

$$\forall x F^1 x \to (\forall x Fx)^1$$

In order to have an idea what to expect, observe the following logical equalities.

$$\forall x F^1 x = \forall x F^\circ x = \forall x \neg(Fx \wedge \neg Fx)$$

and

$$(\forall x Fx)^1 = (\forall x Fx)^\circ = \neg(\forall x Fx \wedge \neg\forall x Fx).$$

So let us now assume, $\forall x F^1 x$, namely $\forall x . \neg(Fx \wedge \neg Fx)$. Then, we have the following.

$$
\begin{aligned}
|x;\forall x.F^1x| &= |x;\forall x.\neg(Fx \wedge \neg Fx)| \\
&= \bigcap_{d\in D} |d;\neg(Fx \wedge \neg Fx)| \\
&= \bigcap_{d\in D} \{|d;N^*| \cup \mathsf{Clo}(|d;Fx \wedge \neg Fx|^c)\} \\
&= \bigcap_{d\in D} \{|d;N^*| \cup \mathsf{Clo}((|d;Fx| \cap |d;\neg Fx|)^c)\} \\
&= \bigcap_{d\in D} \{|d;N^*| \cup \mathsf{Clo}((|d;Fx| \cap (|d;N^*| \cup \mathsf{Clo}(|d;Fx|^c)))^c)\} \\
&= \bigcap_{d\in D} \{|d;N^*| \cup \\
&\quad \mathsf{Clo}((|d;Fx| \cap |d;N^*|) \cup (|d;Fx| \cap \mathsf{Clo}(|d;Fx|^c)))^c)\} \\
&\text{(as intersection operation commutes with closure operator)} \\
&= \cap_{d\in D}|d;N^*| \cup \cap_{d\in D}\mathsf{Clo}(|d;Fx| \cap |d;N^*|)^c \cup (|d;Fx| \cap \mathsf{Clo}(|d;Fx|^c))^c)) \\
&= \cap_{d\in D}|d;N^*| \cup \mathsf{Clo}(\cap_{d\in D}|d;Fx| \cap |d;N^*|)^c \cup \cap_{d\in D}(|d;Fx| \cap \mathsf{Clo}(|d;Fx|^c))^c)) \\
&\text{(as the interior of a set is its subset)} \\
&\subseteq |d;N^*| \cup \mathsf{Clo}(\cap_{d\in D}|d;Fx|)^c \cup (\cap_{d\in D}(|d;Fx| \cap \mathsf{Clo}(|d;Fx|^c))^c)) \\
&\subseteq |d;N^*| \cup \mathsf{Clo}(\cap_{d\in D}|d;Fx|)^c \cup ((|d;Fx| \cap \mathsf{Clo}\cap_{d\in D}(|d;Fx|^c))^c)) \\
&\subseteq |d;N^*| \cup \mathsf{Clo}(\cap_{d\in D}|d;Fx| \cap ((|d;Fx| \cap \mathsf{Clo}\cap_{d\in D}(|d;Fx|^c))) \\
&\subseteq \neg(\forall xFx \wedge \neg\forall xFx) \\
&\subseteq (\forall xFx)^1
\end{aligned}
$$

Thus, we obtain $\forall xF^1x \to (\forall xFx)^1$.

As the second case, take the axiom scheme (4) instantiated with $n = 1$. Thus, we consider the following with term x.

$$\forall xF^1x \to (\exists xFx)^1$$

Now, we have the following.

$$
\begin{aligned}
|x;\forall x.F^1x| &= |x;\forall x.\neg(Fx\wedge\neg Fx)|\\
&= \bigcap_{d\in D}\{|d;N^*|\cup \mathsf{Clo}(|d;Fx\wedge\neg Fx|^c)\}\\
&= \bigcap_{d\in D}\{|d;N^*|\cup \mathsf{Clo}(|d;Fx|\cap|d;\neg Fx|)^c\}\\
&= \bigcap_{d\in D}\{|d;N^*|\cup \mathsf{Clo}(|d;Fx|^c\cup|d;\neg Fx|^c)\}\\
&= \bigcap_{d\in D}\{|d;N^*|\cup \mathsf{Clo}(|d;Fx|^c\cup(|d;N^*|\cup\mathsf{Clo}(|d;Fx|^c))^c))\}\\
&= \bigcap_{d\in D}\{|d;N^*|\cup \mathsf{Clo}(|d;Fx|^c\cup(|d;N^*|^c\cap\mathsf{Int}(|d;Fx|)))\}\\
&\subseteq \cap_{d\in D}|d;N^*|\cup\mathsf{Clo}(\cap_{d\in D}|d;Fx|^c\cup(\cap_{d\in D}|d;N^*|^c\cap\mathsf{Int}(\cap_{d\in D}|d;Fx|)))\\
&\subseteq |d;N^*|\cup\mathsf{Clo}(\cap_{d\in D}|d;Fx|^c\cup(|d;N^*|^c\cap\mathsf{Int}(\cup_{d\in D}|d;F|)))\\
&\text{by set theoretical De Morgan's Laws}\\
&\subseteq |d;N^*|\cup\mathsf{Clo}(\cup_{d\in D}|d;Fx|\cap(|d;N^*|\cup\mathsf{Clo}(\cup_{d\in D}|d;F|^c)^c))\\
&\subseteq |d;N^*|\cup\mathsf{Clo}((\exists xFx\wedge\neg\exists xFx)^c)\\
&\subseteq \neg(\exists xFx\wedge\neg\exists xFx)\\
&\subseteq (\exists xFx)^1
\end{aligned}
$$

Finally, we obtain $\forall xF^1x\to(\exists xFx)^1$.

A closer inspection reveals that soundness of the axioms we discussed relies on several simple topological facts. The remaining axioms can also be given rather straight forward argumentations for their soundness, thus we leave them to the reader.

This was soundness. However, we still do not have a completeness result (or lack thereof) for our system. We leave it to further work.

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