Social Capital, Communication Channels and Opinion Formation*

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Abstract

We study how different forms of social capital lead to different distributions of multidimensional opinions by affecting the channels through which individuals communicate. We develop a model to compare and contrast the evolution of opinions between societies whose members communicate through bonding associations (i.e., which bond similar people together) and societies where communication is through bridging associations (i.e., which bridge the gap among different people). Both processes converge towards opinion distributions where there are groups within which there is consensus in all issues. Bridging processes are more likely to lead to society-wide consensus and converge to distributions that have, on average, fewer opinion groups. The latter result holds even when the confidence bound that allows successful communication in the bridging process is much smaller than the respective bound in the bonding process.

JEL Classification: D71, D83, P16, Z1

Keywords: Social Capital, Opinion Formation, Bounded Confidence, Bonding versus Bridging Associations.

1 Introduction

People's opinions are not fixed. Once an individual has formed an opinion about an issue, she may revisit it in the future, once she gets more information about it. Arguably, one

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of the most important avenues through which information is exchanged is by individually meeting and discussing with others. Of course, one might not necessarily take another's opinion into account when updating her own, especially if this opinion is very different than hers. Oftentimes though, when two individuals who hold different opinions discuss, they see each others' points; this way their opinions on the subject they discuss become more similar.

The opportunities available to people to meet, interact and exchange opinions measure the stock of social capital of the community they live in. Depending on whether these opportunities result in individuals meeting other individuals similar to them, or individuals quite different to them, social capital can be classified as *bonding* or *bridging* respectively.

In this paper, we present a theoretical framework that incorporates the existence of communication channels with different features into a standard model of multidimensional opinion formation with bounded confidence (Deffuant et al., 2000; Lorenz, 2005). These differences are driven by the type of social capital of the community they live in (Putnam, 1995). Our goal is then to study how different types of social capital, bonding or bridging, affect the distribution of opinions within a society. Of particular interest to our analysis is the comparison between the two processes with respect to the likelihood of leading to society-wide consensus and the expected level of opinion fractionalization.

In order to do that, we formulate two dynamic processes of opinion formation with bounded confidence, based on Deffuant et al. (2000), that capture the features of communication through bonding and bridging associations respectively. In each of the two processes, citizens hold opinions over two issues, which they revise after meeting and discussing with other citizens of the society. A society in which citizens interact predominantly through bonding associations is modeled by a process in which two citizens may discuss and agree only as long as their current opinions are sufficiently close *in both issues*, henceforth called *bonding process*. On the contrary, a society in which citizens interact predominantly through bridging associations is modeled by a process in which two citizens may discuss and agree as long as their current opinions are sufficiently close *in at least one of the two issues* (and irrespectively of how far they might be in the other one), henceforth called *bridging process*.

We find that both processes lead societies to an "island"-type distribution of opinions, where groups of citizens reach consensus between them in both issues,[1] and we also show that these islands are sufficiently far from each other with probability one. Despite this similarity in the shape of the final distributions, we show that the two processes have distinct characteristics that determine how these groups are formed and how many of them survive in the long run. In particular, we show that it is tougher for a group of citizens to form a stable common neighborhood, i.e. a group of people who can communicate successfully only with each other and who will eventually converge towards a common opinion in all issues, in a bridging process than in a bonding process.

[1]This is in fact a recurrent result in this type of processes, see Deffuant et al. (2000) and Lorenz (2005).

We then consider a simplified setup of a small society of three citizens for which we obtain additional results. We show that for any confidence bound d of the bonding process, there exists a confidence bound k of the bridging process, such that for any bridging process with a confidence bound larger than k the society is more likely to reach society-wide consensus and has a lower expected number of islands compared to the bonding process with confidence bound d. Importantly, this lower bound is always strictly lower than d and in fact it is typically much lower than d, at a similar order to d^2. Given the limitations of our simplified setup, we also perform an extensive set of simulations on larger societies and we find that our results are not only robust to larger societies but are in fact strengthened.

These last results contain the main message of the paper. They suggest that *even minimally outward-looking bridging processes lead to more cohesive societies than modestly outward-looking bonding processes.* This means that enhancing social capital that bridges distinct social groups is essential for reducing opinion fractionalization within societies. Note that this might seem as an unquestionably positive feature, yet it is probably more accurate to consider it as an opinion coordination mechanism, as there is no guarantee that societies will converge towards opinions that are necessarily positive from a social point of view. For instance, Satyanath et al. (2017) find that in Weimar Germany participation in bridging associations was strongly related to higher enrollment in the Nazi party.

2 Literature Review

A large branch of the literature on opinion formation has focused on conditions that allow individuals to reach consensus (or learn some "correct" action) via communication, considering both Bayesian and non-Bayesian updating processes.[2] Probably closer to the spirit of our analysis is Golub and Jackson (2012) where the authors study the speed of convergence in a fixed network of individuals that update their opinions using average-based updating (DeGroot, 1974) and find homophily to speed down the process. Here, we consider a process that allows the network to coevolve with opinions (on that, see also Holme and Newman, 2006) and second, the connection between the updating process and the forms of social capital allows us to capture the impact of different forms of homophily on the network structure.

Social capital is a term that has been used to refer to a variety of things. According to Putnam (1995), social capital refers to *"features of social organization such as networks, norms, and social trust that facilitate coordination and cooperation for mutual benefit"*.[3] In this sense the stock of social capital is greater where individuals have many opportunities to

[2]See Bala and Goyal (1998); Gale and Kariv (2003); Banerjee and Fudenberg (2004); Acemoglu and Ozdaglar (2011); DeMarzo et al. (2003); Golub and Jackson (2010); Mueller-Frank (2013, 2015) and also Golub and Sadler (2017) for an excellent review of the different branches of literature on learning on networks.

[3]For other definitions of social capital and for a research survey on the topic see Durlauf and Fafchamps (2005). For an empirical decomposition of the concept see Bjørnskov (2006).

interact and cooperate within a community. This would seem to imply that social capital is associated with good social outcomes. For instance, Knack and Keefer (1997) and Tabellini (2010) find that social capital is associated with stronger economic performance and Guiso et al. (2004) show that social capital brings more trust in a community which improves its level of financial development. On the other hand, social capital can have negative effects on society (see discussion on results by Satyanath et al., 2017, above).

In an attempt to understand more deeply the distinct forms of social capital, Putnam (2000) makes the distinction between *bonding* and *bridging social capital*. The first type "bonds" similar individuals with each other and the second one "bridges" the gap between different individuals.[4] Different forms of social capital are implicitly present also in the seminal work of Granovetter (1973), who pointed out the importance of weak (versus strong) ties within a society. Strong and weak ties in the Granovetter sense can be seen as resulting from bonding and bridging social capital, respectively.

In general, different categorizations of social capital can be defined based on the characteristics that determine the perception of social distance among citizens. Although without referring to social capital, Iijima and Kamada (2017) look at the impact of different perceptions of social distance on network formation, with the two main cases of comparison coinciding with our definitions of bonding and bridging social capital. Nevertheless, the framework they study is totally different, as opinions are fixed and the agents form costly links whose benefits depend on the relative social distance between the connected agents.

With respect to non-consensus opinion formation Axelrod (1997) posed the question of why even though people tend to become more alike when they interact, the differences across them do not eventually disappear. Even though individuals are becoming more similar with every interaction, the fact that similarity itself leads to the interaction makes individuals who are quite different very unlikely to interact, thus making consensus unattainable. This observation pertains the key idea of homophily as a possible explanation for the persistence of disagreement, which is also present here. This is prominent in Melguizo (2018), where in a variation of the average-based updating model the network of interactions to coevolve with opinions, in a way that sharing of a common attribute strengthens the interaction between individuals. The driving force of these results is still the intensity, rather than the form, of homophily in the network. Other explanations for the persistence of disagreement contain anchoring to initial opinions (Friedkin and Johnsen, 1990), biased assimilation (Dandekar et al., 2013) and opinion fluctuations (Acemoglu et al., 2013).

Our analysis is based on the model of average-based updating introduced by DeGroot

[4]There are also two forms of "bridging", internal and external. Internal bridging brings together the members of a given association, whereas external bridging brings together members of different associations (see Geys and Murdoch, 2008). Moreover, Geys and Murdoch (2010) discuss how the bridging and bonding nature of networks can be measured. Following the discussion on social capital and redistribution, Borisova et al. (2015) show that it is in fact the bridging social capital that has positive effects on redistribution.

(1974), with the addition of bounded confidence. The seminal papers by Deffuant et al. (2000), Krause (2000) and Hegselmann and Krause (2002) introduce the concept of bounded confidence, according to which citizens whose opinions are too far away from each other do not take each other's opinions into account. This introduces implicitly a notion of homophily, not on interactions per se, but on interactions that influence individuals' opinions.[5] A recurrent feature of these processes is that multiple opinions can survive, with the fractionalization being negatively related to the size of confidence bound (Ben-Naim et al., 2003; Blondel et al., 2007; Lorenz, 2007).

The "island"-type distribution of opinions for the model of Deffuant et al. (2000) is proven analytically in Lorenz (2005). The main result of that paper provides a clear intuition of the general long-run properties of these processes, and its proof is essential for establishing Theorem 1 of this paper. Moreover, Fortunato et al. (2005) and Lorenz (2003, 2008) extend the original models of Deffuant et al. (2000) and Hegselmann and Krause (2002) to multidimensional opinions. More specifically, Lorenz (2003) and Fortunato et al. (2005) look at the multidimensional version of the model of Hegselmann and Krause (2002), both considering confidence bounds with similar shape as that of our bonding process. The former proves a similar theoretical result as Lorenz (2005), whereas the latter provides some additional theoretical results and an extensive set of simulations that supports the emergence of the "island"-type structure, focusing more on the impact of the size of the confidence bound on the number of islands. Lorenz (2008) looks at both types of models and allows for other shapes of confidence bounds, with the focus being again on the impact of the confidence bound on the number of islands. Finally, Kurahashi-Nakamura et al. (2016) allow individuals to interact with others who hold distant opinions, similarly to our bridging process. Yet, their main interest is on the distribution of opinions is affected by random changes that are not associated with the opinion formation process. Overall, this paper can be seen as *a revisit to the problems of opinion formation, fractionalization and consensus seeking from a social capital perspective.*

3 Empirical Motivation

Using the last wave of the European Values Study,[6] we can compare the attitudes of citizens of various European countries, given their participation or not in different kinds of associations.

[5]For an excellent survey on opinion dynamics and bounded confidence see Lorenz (2007) and for some empirical evidence see Lorenz (2017). Moreover, a stream of the literature on average-based updating looks at the shape of persisting disagreement without considering bounded confidence (see DeMarzo et al., 2003; Louis et al., 2017).

[6]EVS (2011): European Values Study 2008: Integrated Dataset (EVS 2008). GESIS Data Archive, Cologne. ZA4800 Data file Version 3.0.0, doi:10.4232/1.11004

The dataset contains a very rich set of variables spanning the whole spectrum of social life. The data has been collected around the whole of Europe and the place of residence of the participants is recorded at the regional level. This allows us to construct aggregated data points for each region, without restricting the size of our data too much, as it would be the case if location was measured at the national level.

For each country, we observe whether a participant has indicated to be an active member of a number of different associations, as well as whether a participant has declared spontaneously to be a member of no association at all.[7] We aggregate these observations at the regional level and obtain the share of the population that participates in each type of association at each region. The results for each association are presented separately.

In addition to participation in associations, we observe the self-declared position of the participants on the left–right political spectrum, which is measured on a scale from 1 (extreme left) to 10 (extreme right). Using this measure we are able to construct a fractionalization index for the given region. We use the standard version of the Ethnolinguistic Fractionalization Index $ELF = 1 - \sum_{i=1}^{10} s_i^2$, where $i \in \{1, \dots, 10\}$ corresponds to a stated position on the political spectrum and s_i is the share of participants from the region who declared to have position i.

The aim of this empirical exercise is to check whether the level of participation at a given type of association is correlated with the observed level of fractionalization in the given region. Note that the fractionalization index can be affected by the size of the population used for its calculation, as larger populations are expected to be more fractionalized than smaller ones just because of the fact that their calculation is based on a higher number of opinions. For this reason, the size of the sample for each region is taken into account in our regressions and indeed seems to have a positive and significant effect. In addition to this, we drop regions with fewer than twenty observations, as the results in such cases are affected very much by individual observations, thus being not representative of the entire population.[8]

We regress the calculated fractionalization indices on the participation level for each of the associations, including as regressors the sample population, the average age, education and wage levels and the share of each gender. All data are aggregated at the regional level. Table 1 presents the coefficients for each of the association in the relevant regression, together with their significance level.[9]

[7]We focus on the variable that indicates whether or not someone has provided voluntary work for the given organization, rather than just having participated, as this provides a stronger indication on the extent of involvement in the organization.

[8]The choice of twenty observations as a threshold is obviously ad hoc, as there is not standard way of making this choice. The idea is that on the one hand a small number of observations induces a lot of noise in the ELF, on the other hand setting a high threshold will lead us to drop too many observations, thus altering the nature of the total sample.

[9]The complete table of regression results is available upon request.

Religious	0.122**	Women	0.233*
	(0.0446)		(0.101)
Sports	0.132*	Environment	0.273***
	(0.0518)		(0.0808)
Arts	0.186**	Health	0.295***
	(0.0580)		(0.0722)
Professional	0.194	Political Party	0.300***
	(0.101)		(0.0826)
Trade Union	0.206*	Community	0.325***
	(0.0889)		(0.0870)
Welfare	0.213***	Human Rights	0.397***
	(0.0567)		(0.0895)
Youth	0.221**	Peace	0.404***
	(0.0756)		(0.116)
		None	-0.0749**
			(0.0235)
Observations	386		

Sample population ≥ 20. Standard errors in parentheses.
* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Table 1: OLS coefficients of participation in various associations on fractionalization.

Overall, we find that the share of participation in almost any type of association is positively related with the fractionalization of a society and analogously participating in no association is negatively related with it.[10] Regarding differences between bridging and bonding associations, despite the fact that there is no commonly accepted way of characterizing particular associations, some types of organizations are often associated with bridging social capital, such as sports, arts, religious and youth associations, whereas others are more often associated with bonding social capital, such as political parties and other politically involved associations, as well as activist organizations. For instance, sports or arts related associations are more likely to bring together enthusiasts of a specific activity (people who like the same sport, or who play the same instrument), without restricting participation based on other social characteristics. On the other hand, associations such as political parties and community groups have more strictly-defined requirements to allow a person's participation in them, either social or political. For a more detailed discussion of classification of associations see Geys and Murdoch (2008), Geys and Murdoch (2010), Coffé and Geys (2008) and Satyanath et al. (2017).

Having said that, we observe that associations of a more "bridging nature", such as, for example, arts and sports associations, are found to have the lowest impact among all others

[10] In a different set of regressions we found the same result to be true if one looks at partisanship levels instead of the ELF. By partisanship level we mean the average absolute distance from the average opinion of 5.5. This result could be connected with the findings of Satyanath et al. (2017) where the authors find participation in associations to be linked with increased entry level in the Nazi party before WWII.

and on the contrary associations of a more "bonding nature", such as peace and human rights associations, are found to have the largest impact among all others.

The result does not impose any causal effect, but indicates a different relation between different associations and the observed fractionalization in a society. Our subsequent theoretical analysis provides a framework that could potentially explain the observed differences based on the process of opinion formation that is induced by the participation in different types of associations. Note that we intend to explain the observed differences across associations and not why participation per se is associated with higher fractionalization.

4 Model

We examine a dynamic model of multidimensional opinion formation, á la Deffuant et al. (2000), in a population of n citizens, denoted by $N = \{1, \ldots, n\}$ with typical elements i, j. Citizens are considered to hold opinions over two issues x and y which can be represented as points on the Euclidean space $[0, 1] \times [0, 1]$. Citizens may meet via two different processes that are described later and upon communication they may adjust their opinions.

Initially, each citizen holds a pair of opinions (x_i^0, y_i^0). Initial opinions are summarized by the vectors $\mathbf{x}(0)$ and $\mathbf{y}(0)$ respectively. In each period, starting from period $t = 1$ onwards, two citizens i, j are randomly selected to communicate. If their opinion profiles are sufficiently "close" to each other then their communication can be successful, in which case the two citizens' opinions come closer on the issue they discussed. Otherwise, their opinions do not change. In our setup, successful communication means that the two citizens understand the opinions of each other and they are able and willing to find some common ground; thus, their opinions will become more similar. Unsuccessful communication means that the citizens are unwilling or unable to find common ground and as a result their opinions remain unchanged. Formally, if two citizens are selected to meet and discuss issue x on which they hold opinions x_i^t, x_j^t (with $x_i^t < x_j^t$) and communicate successfully, then their opinions become

$$x_i^{t+1} = x_i^t + \mu(x_j^t - x_i^t) \quad \text{and} \quad x_j^{t+1} = x_j^t - \mu(x_j^t - x_i^t)$$

respectively. The parameter $\mu \in (0, 1/2]$ denotes the extent to which opinions come closer to one another upon successful communication.

On the contrary, if the two citizens' opinion profiles are very "far" from each other then their communication can never be successful, in which case no adjustment is made in their opinion profiles.[11] This is a model of *bounded confidence*, where citizens are willing to exchange opinions only with those with whom there is some common ground for meaningful

[11] Alternatively, one could think that two citizens with very distant opinions can never be chosen to interact. The qualitative results of this alternative mechanism are identical and the only feature that is affected is the speed of convergence and some increased consistency of extreme opinions.

discussion.

We consider two distinct processes that differ with respect to the meaning of two citizens' opinions being "close" or "far". The two processes reflect the underlying effect of participation in certain associations on whether two citizens can meet and subsequently communicate successfully. The relationship between opinions and associations is the following: There are as many potential associations as there are opinions, and people can always find, and participate in, associations that are close to their opinions. Since there is a continuum of opinions there is a continuum of potential associations as well. As their opinions change, the associations that individuals can potentially participate in change as well. At the same time, each individual can potentially participate in any association that represents opinions that are "close" to hers.

Therefore, when we examine if two individuals can meet, we implicitly examine whether there are associations both individuals would potentially participate in. For instance, individual A might participate in a basketball association, but not in a volleyball association. While participating in the basketball association A might meet individual B who participates both in basketball and volleyball associations. On the other hand, individual B can also meet individual C who only participates in volleyball associations. Therefore A can meet and communicate with B, C can meet and communicate with B, but A cannot meet and communicate with C unless B convinces A to try volleyball, or C to try basketball.

We differentiate between two main types of associations, *bonding associations* versus *bridging associations* and analyze the respective processes in which citizens interact via one of these two types of associations.

Bonding process: Bonding associations are associations whose members tend to be quite similar to each other in all respects. Therefore, a citizen participating in a bonding association expects to communicate with citizens whose opinions are not too far away from her own in all issues. We formalize this by considering that citizens i, j with opinions (x_i^t, y_i^t) and (x_j^t, y_j^t) respectively at time t, may communicate successfully only as long as:

Confidence bounds in a bonding process: $|x_i^t - x_j^t| \leq d$ **and** $|y_i^t - y_j^t| \leq d$

where the parameter $d \in (0, 1]$ determines the confidence bounds. Geometrically, that means that a citizen may successfully communicate with others whose opinions on the two-dimensional Euclidean space lie on a square of size $2d$ centered at the citizen's opinion. Obviously, the higher the value of d the larger the set of citizens who may successfully communicate with each other. For a schematic representation see the left sub-figure of Figure 1. A citizen i located at the dot may successfully communicate with a citizen j if and only if j's opinion at the time of communication lies in the shaded area.

Bridging process: Bridging associations are associations whose members share a common interest, therefore they tend to share ex-ante similar opinions only on some issue. Therefore, a citizen participating in a bridging association expects to meet and discuss with citizens with whom she shares a very close opinion in the issue related to the association, but whose opinions might be very different on the other issue. For example, if x would represent sports associations in general, then if the citizen likes football, she will participate in a football association where she expects to meet other football enthusiasts. However, liking football is not necessarily a predictor for a citizen's stance on issue y. Football lovers come from all walks of life; therefore if issue y represented "politics" then we can reasonably think that football fans attending football associations span the entire political spectrum, from left to right.[12] As such, when meeting with other citizens in the football association, a citizen may end up discussing with someone with vastly different political opinions than her. This is exactly the reason why these types of associations are called "bridging".

To formalize this consider that citizens i, j with opinion profiles (x_i^t, y_i^t) and (x_j^t, y_j^t) may communicate successfully only as long as:

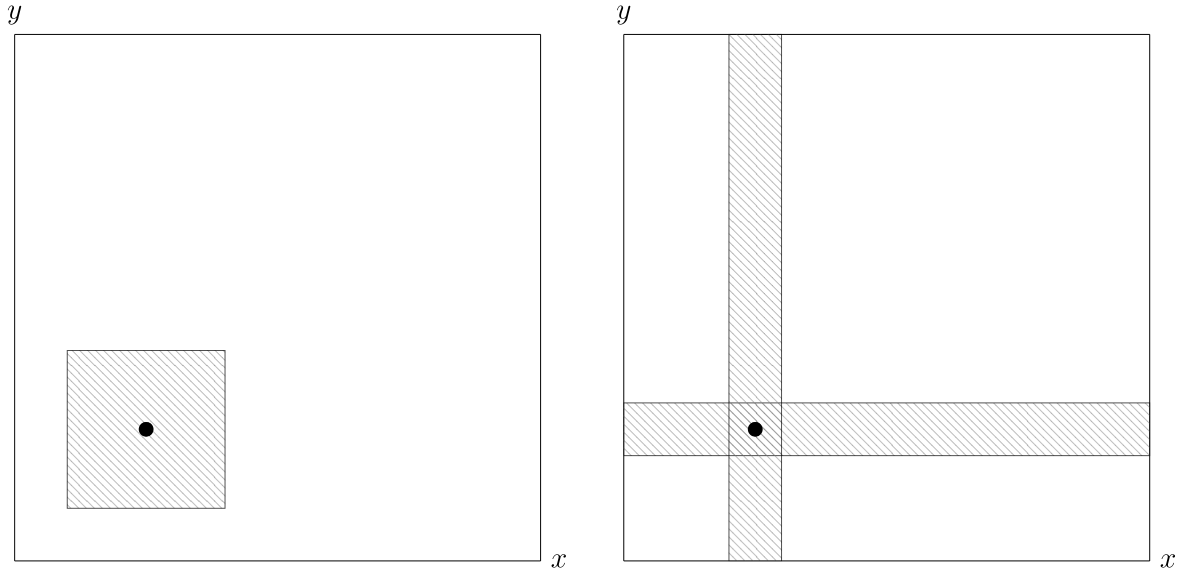
$$\textbf{\textit{Confidence bounds in a bridging process:}} \quad |x_i^t - x_j^t| \leq k \quad \textbf{or} \quad |y_i^t - y_j^t| \leq k$$

Schematically, this defines a "cross"-shaped area in the two-dimensional space centered at the citizen's opinion (see right subfigure of Figure 1). In general, it makes sense to assume that $k < d$, meaning that when opinions need to be similar only with respect to a single issue in order to make successful communication possible, this similarity should be stronger than when successful communication requires opinions to be similar in both issues. Intuitively, $k > d$ would mean that a bonding process would leave unambiguously smaller common ground for successful communication between two citizens compared to a bridging process, because it would require the citizens' ex-ante opinions in both issues to be more similar than they would need to be in even one of the issues in a bridging process.[13]

By comparing the two processes, the different shape of the areas within which successful communication is possible becomes apparent. On the one hand, the convex shape of the area in the bonding process induces some type of, loosely speaking, "monotonicity" in the likelihood of successful communication. On the other hand, the cross-shaped area of the bridging process can lead two agents who hold quite close opinions in both issues to be unable to successfully communicate, because of not being "too similar" in any of them,

[12]There might be cases where these two opinions might be ex-ante correlated, but this would not add anything to our model.

[13]Later in the paper, for some results we will impose the normalization $k = \frac{1}{2} - \sqrt{\frac{1}{4} - d^2}$ which ensures that the areas within which successful communication is possible cover the same area in each case. Observing at Figure 1 the area in which a citizen may find others to agree is $4d^2$ in the bonding case and $2 \cdot 2k - 4k^2$ in the bridging case. Equating the two quantities yields the normalized value of the parameter k.



(a) Bonding process **(b)** Bridging Process

Figure 1: Neighboring areas given the type of process.

while at the same time citizens with extremely distant opinions in one of the subjects may have the chance to communicate successfully.

In what follows, we define the set of citizens with whom citizen i can communicate successfully at period t as i's neighborhood. Formally:

Definition 1. The *neighborhood* of a citizen i at period t consists of all the citizens with whom i faces a strictly positive probability of communicating successfully. Formally:

- Bonding: $\mathcal{N}_i^t = \{j \in N \setminus \{i\} : |x_i^t - x_j^t| \leq d \text{ and } |y_i^t - y_j^t| \leq d\}$
- Bridging: $\mathcal{N}_i^t = \{j \in N \setminus \{i\} : |x_i^t - x_j^t| \leq k \text{ or } |y_i^t - y_j^t| \leq k\}$

The nature of both processes imposes an undirected relationship between citizens in the sense that if citizen i is in the neighborhood of citizen j, then j is in the neighborhood of i, i.e. $j \in \mathcal{N}_i^t \Leftrightarrow i \in \mathcal{N}_j^t$.

In both processes, the pair of citizens that is chosen to interact discusses one of the two issues, chosen randomly.[14] The probability of discussing each issue can depend on the citizens' current opinions or not, but this is inconsequential for our qualitative results, as long as each issue is chosen to be discussed with a strictly positive probability. Moreover, the probability that communication is successful is assumed to be non-increasing in the distance of opinions in the issue discussed and strictly positive as long as opinions are within

[14]We consider that the citizens discuss a single issue at a time, however all results directly extend to the case in which both issues are discussed simultaneously. Such an assumption would only speed up the convergence of the process

the respective confidence bounds of each process. Note that one might argue that the bridging process would be associated also with a lower frequency of interactions, as it is not uncommon that people with very different opinions might avoid interacting often. This assumption would also leave our process unaffected except for its impact on the speed of convergence.

Overall, the evolution of both processes depends on a randomly generated sequence of pairs of citizens who communicate in each period, the issue they discuss and whether communication is successful or not. Namely, the *sequences of opinions* $\{(x_i^t, y_i^t)\}_{t=1}^{\infty}$ for each $i \in N$ can be fully characterized from the *sequence of interactions* $\{z_t\}_{t=1}^{\infty}$, where $z_t = ((i_t, j_t), s_t, c_t)$. For each period t, (i_t, j_t) denotes the pair that communicates, $s_t \in \{x, y\}$ denotes the issue discussed and $c_t \in \{0, 1\}$ denotes whether communication was successful ($c_t = 1$) or not ($c_t = 0$). Hence, z_t is randomly drawn from a finite set $P \times \{x, y\} \times \{0, 1\}$, where $P := \{(i, j) \in N^2 : i \neq j\}$. However, the probability with which each element can be drawn is not fixed and depends on the state at which the process is. The only restriction we impose in all distributions is that at each period t each element that contains a pair of citizens who could potentially communicate successfully at this period is drawn with strictly positive probability that is bounded away from zero, i.e. if $i_t \in \mathcal{N}_j^t$, then $\mathbb{P}\left[z_t = ((i_t, j_t), \cdot, 1)\right] \geq \underline{\beta}$, for some $\underline{\beta} > 0$. Some further restrictions on these draws will be imposed in order to establish some of the results.

Having stated the basic elements of the model, we now present some additional definitions that will be used in subsequent results.

Definition 2. The *neighboring area of a point* (x, y) consists of all points $(\widehat{x}, \widehat{y})$ such that two citizens with opinion profiles (x, y) and $(\widehat{x}, \widehat{y})$ face a strictly positive probability of communicating successfully. Formally:

- Bonding: $\mathcal{N}_{(x,y)} = \{(\widehat{x}, \widehat{y}) \in [0, 1]^2 : |x - \widehat{x}| \leq d \text{ and } |y - \widehat{y}| \leq d\}$
- Bridging: $\mathcal{N}_{(x,y)} = \{(\widehat{x}, \widehat{y}) \in [0, 1]^2 : |x - \widehat{x}| \leq k \text{ or } |y - \widehat{y}| \leq k\}$

In a completely analogous manner, we can define the neighboring area of a set of multiple opinion profiles as follows:

Definition 3. The *neighboring area of set of points* S consists of all points $(\widehat{x}, \widehat{y})$ such that a citizen with some opinion profile $(x, y) \in S$ and another citizen with opinion profile $(\widehat{x}, \widehat{y})$ face a strictly positive probability of communicating successfully. Formally:

- Bonding: $\mathcal{N}_S = \{(\widehat{x}, \widehat{y}) \in [0, 1]^2 : |x - \widehat{x}| \leq d \text{ and } |y - \widehat{y}| \leq d \text{ for some } (x, y) \in S\}$
- Bridging: $\mathcal{N}_S = \{(\widehat{x}, \widehat{y}) \in [0, 1]^2 : |x - \widehat{x}| \leq k \text{ or } |y - \widehat{y}| \leq k \text{ for some } (x, y) \in S\}$

The last related area we are interested in is the smallest rectangle that surrounds the opinions of a group of citizens. Namely:

Definition 4. Consider a group of citizens $\mathcal{I} \subseteq N$ and let $S^t_{\mathcal{I}} = \{(x, y) \in \mathbb{R}^2 : (x, y) = (x^t_i, y^t_i)$ for some $i \in \mathcal{I}\}$, i.e. the set of opinions of $\mathcal{I}$'s members. Then, the *minimum bounding rectangle* of $S^t_{\mathcal{I}}$ is defined as follows:

$$MBR^t_{\mathcal{I}} = \{(x, y) \in \mathbb{R}^2 : \min_{i \in \mathcal{I}} x^t_i \leq x \leq \max_{i \in \mathcal{I}} x^t_i \text{ and } \min_{i \in \mathcal{I}} y^t_i \leq y \leq \max_{i \in \mathcal{I}} y^t_i\}$$

Observe that MBR is defined over a set of points, thus we can also consider the neighboring area of a minimum bounding rectangle, which is a quantity that will be proven handy in subsequent results.

Finally, it should be noted that Definitions 1 and 2 refer to different sets, but share some similarities. For instance, a citizen j belongs to the neighborhood of citizen i at a given period if and only if her opinion profile (x^t_j, y^t_j) belongs to the neighboring area $\mathcal{N}_{(x^t_i, y^t_i)}$ of i's current opinion profile, i.e. $j \in \mathcal{N}^t_i \Leftrightarrow \left(x^t_j, y^t_j\right) \in \mathcal{N}_{\left(x^t_i, y^t_i\right)}$.

5 Results

A useful initial observation is that both processes are mean preserving, meaning that the average opinion in the population remains the same in all periods. This is a well-known feature of processes that are variants of Deffuant et al. (2000). Moreover, the variance of opinions is weakly decreasing, as in each period either some opinions move closer to each other, or they remain unchanged.[15]

Note that mean preservation does not imply that all opinions will converge to this mean. In fact, our first result shows that for any finitely large population both processes converge to an "island"-type structure. That is, we observe the emergence of groups of citizens whose opinions converge to the same values in both issues. However, the opinions across groups may differ. When opinions within a group converge, we will say that this group reaches *consensus* and refer to these groups as *islands*. If a single island emerges, then we will say that a *society-wide consensus* is reached.

Notably, convergence is proven in a deterministic way, meaning that it is achieved for any sequence of interactions. Of course, both the number of islands and the opinions to which each group will converge will depend on the sequence of interactions. This result can be seen as an adaptation of the result of Lorenz (2005) in the current framework. The result provides a clear and intuitive sense of how the long run distributions of opinions in such processes will look like, as well as that the societies indeed move towards these long-run outcomes.

[15]The proofs of these arguments are omitted as they are rather straightforward and well–established in the literature and are available upon request.

Moreover, it shows that the geometry of the distribution of opinions does not depend on the actual process, as long as it satisfies certain mild conditions.

Definition 5. A group of citizens $\mathcal{I} \subseteq N$ reaches *consensus* if

$$\lim_{t\to\infty} x_i^t = \overline{x} \text{ and } \lim_{t\to\infty} y_i^t = \overline{y} \text{ for all } i \in \mathcal{I}$$

for some $\overline{x}, \overline{y} \in [0,1]$.

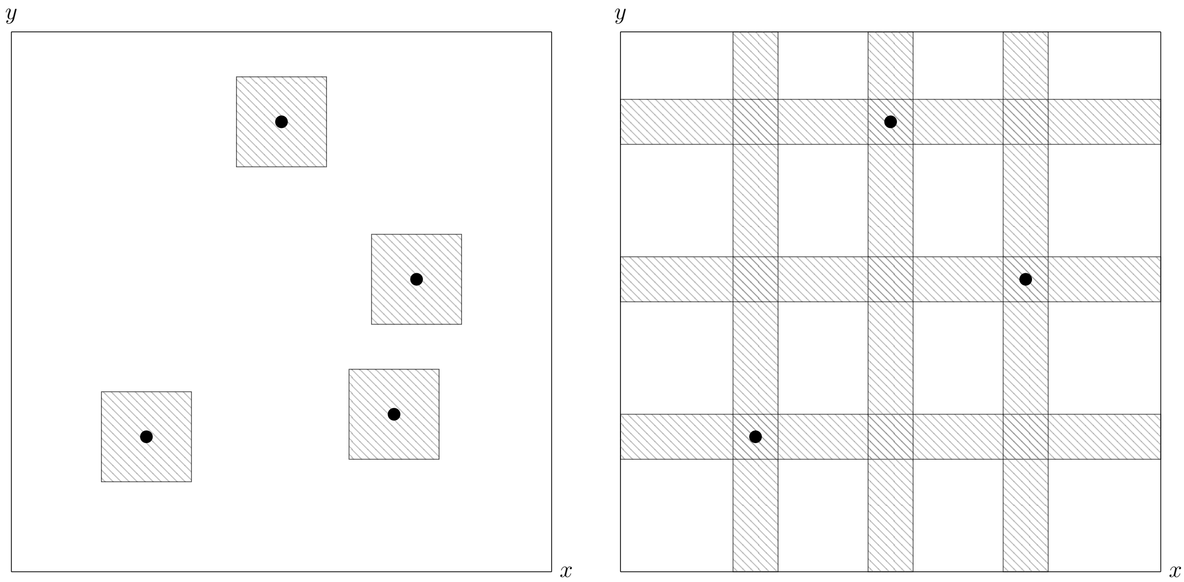
Theorem 1. *Both in a bonding and in a bridging process, for any vectors of initial opinions $\mathbf{x}(0)$ and $\mathbf{y}(0)$ and for any sequence of interactions $\{z_t\}_{t=1}^{\infty}$, opinions converge to form pairwise-disjoint groups of citizens $\mathcal{I}_1 \cup \cdots \cup \mathcal{I}_p = N$ such that each group of citizens, $g \in \{1, \ldots, p\}$, reaches consensus.*

The deterministic nature of the Theorem 1 allows it to capture even cases where little or no updating occurs. Yet, it does not say much about the relative position of the opinions to which different groups are expected to converge. Intuitively, we would expect islands to be located far from each other, in the sense that citizens located around different islands can no longer communicate successfully. Otherwise further updating between citizens located close to different islands could substantially affect the distribution of opinions. Nevertheless, although this is something we would expect to happen most of the times, it will always be possible to find sequences of interactions for which this would fail, thus the result should be stated probabilistically. Think, for instance, a trivial sequence in which all realized interactions are unsuccessful, thus the initial opinion profile remains unchanged forever. In the next result, we show that the probability of converging to an "island"-type structure with several islands located close to each other is zero.

Proposition 1. *Consider an arbitrary collection of non-empty pairwise-disjoint groups of citizens $\mathcal{I}_1, \ldots, \mathcal{I}_p$ such that $\mathcal{I}_1 \cup \cdots \cup \mathcal{I}_p = N$ and a collection of opinions $(\overline{x_1}, \overline{y_1}), \ldots, (\overline{x_p}, \overline{y_p})$ such that $(\overline{x_m}, \overline{y_m}) \in Int\left(\mathcal{N}_{(\overline{x_l}, \overline{y_l})}\right)$ for some $l, m \in \{1, \ldots, p\}$ and $l \neq m$. Both in a bonding and in a bridging process, the probability that opinions converge to form groups $\mathcal{I}_1, \ldots, \mathcal{I}_p$ such that citizens belonging to group $\mathcal{I}_g$ reach consensus at $(\overline{x_g}, \overline{y_g})$ for all $g \in \{1, \ldots, p\}$ is zero.*

$Int(\cdot)$ denotes the interior of a set.[16] The setup we describe in Proposition 1 is one in which the population is split into disjoint groups of citizens, where each group $\mathcal{I}_m$ consists of citizens whose opinions converge to a profile $(\overline{x_m}, \overline{y_m})$ –thus, creating an "island"-type structure– and some of these opinion profiles are sufficiently close to allow the possibility of further successful communication –i.e., for some $l, m \in \{1, \ldots, p\}$, it holds that $(\overline{x_m}, \overline{y_m})$

[16] A short discussion on why focusing on the interior of the neighboring area is provided in the Appendix after the proof of the proposition.



(a) Opinion distribution in a bonding process **(b)** Opinion distribution in a bridging process

Figure 2: The dots represent the consensus opinions of different sets of citizens and the hatched grey areas represent the respective neighboring areas for each set of citizens that has reached consensus.

is in the neighboring area of $(\overline{x_l}, \overline{y_l})$. Proposition 1 shows that the probability that such a configuration will emerge as a long-run outcome of the process is zero. Therefore, the islands that will eventually be created (as a consequence of Theorem 1) will almost certainly be located sufficiently far from each other. A graphical representation of the "island"-type structure presented in Theorem 1 and Proposition 1 is presented in Figure 2.

By this point, we have a clear idea about the opinion distributions societies are expected to converge to. Note that Proposition 1 suggests that after a sufficiently long number of periods we expect to have well-formed stable groups, whose members may communicate successfully only among themselves. This in turn means that, before opinions within each group converge to very similar values, the society passes through a phase of (what we will call) *neighborhood stabilization*. That is, there is some time period at which members of different groups come sufficiently close to each other and far from everyone else, which in turn means that for each citizen there is a group of fellow citizens with whom she can currently communicate successfully and this does not change from that period onwards. Formally,

Definition 6. A process reaches *neighborhood stabilization* at period t if for all $i \in N$ and $t' > t$ holds that $\mathcal{N}_i^t = \mathcal{N}_i^{t'}$.

A first (rather straightforward) remark regarding neighborhood stabilization is that it is associated with the emergence of common neighborhoods, meaning that after neighbors stabilize, all members of a groups are sufficiently close to each other to make successful communication within group possible.[17]

Remark 1. For both a bonding and a bridging process, neighborhood stabilization at period t implies that for all $t' > t$ and $i, j \in N$ such that $j \in \mathcal{N}_i^{t'}$ holds that $\mathcal{N}_i^{t'} \cup \{i\} = \mathcal{N}_j^{t'} \cup \{j\}$.

Therefore, a necessary condition for neighborhoods to stabilize is the emergence of disjoint groups of agents who have common neighbors. However, observing such groups at some time period is not sufficient to ensure neighborhood stabilization for two reasons, one of which applies only to the bridging process. First, in both processes and absent of any other conditions, successful communication between two citizens who belong to the same group might lead one or both of them inside the neighboring area of some other group, thus destabilizing the current neighborhood structure. See for instance an example of this for a bonding process in Figure 3. Second, only in the bridging process, successful communication between two citizens who belong to the same group might lead one of them outside of the neighboring area of some other member of the group. Formally,

Remark 2. Consider a group of citizens $\mathcal{I} \subseteq N$ such that, at some period t, $\mathcal{N}_i^t = \mathcal{I} \setminus \{i\}$ for all $i \in \mathcal{I}$. If citizens $j, l \in \mathcal{I}$ communicate successfully at period t, then:

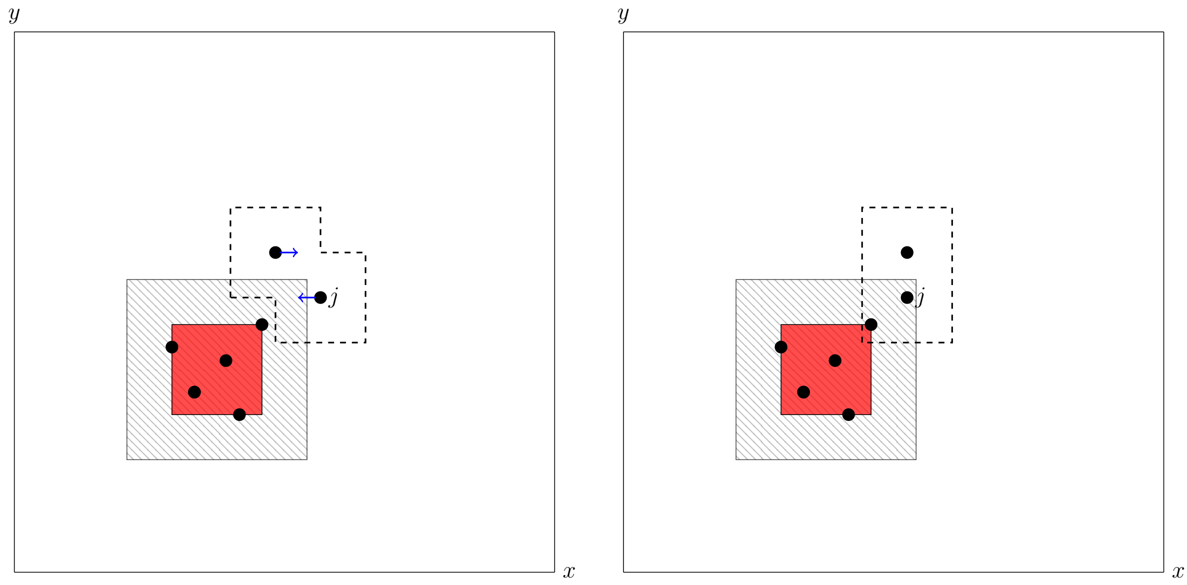
i In a bonding process, at period $t+1$ for sure $\mathcal{N}_i^{t+1} = \mathcal{I} \setminus \{i\}$ for all $i \in \mathcal{I}$.

ii In a bridging process, at period $t+1$ maybe $\mathcal{N}_i^{t+1} \neq \mathcal{I} \setminus \{i\}$ for some $i \in \mathcal{I}$.

Remark 2 is suggestive of the fact that neighborhood stabilization is tougher in bridging processes and therefore opinion updating might be more common in them. The reason why this remark holds has to do with the shape of the neighboring areas of groups of citizens and their minimum bounding rectangle (MBR). Note that, by construction, if two citizens of group $\mathcal{I}$ communicate successfully at period t then their opinions at $t+1$ will still trivially belong to $MBR_{\mathcal{I}}^t$, i.e. $MBR_{\mathcal{I}}^{t+1} \subseteq MBR_{\mathcal{I}}^t$. In a bonding process, the shape of the confidence bounds guarantees that $MBR_{\mathcal{I}}^t \subseteq \mathcal{N}_{(x_i^t, y_i^t)}$ for all $i \in \mathcal{I}$, therefore successful communication between members of the same group cannot unsettle the common neighborhood property. On the contrary, in a bridging process, this need not be the case, as it can be that $MBR_{\mathcal{I}}^t \nsubseteq \mathcal{N}_{(x_i^t, y_i^t)}$ for some $i \in \mathcal{I}$, which in turn means that successful communication between two members of the same group can lead one of them far from some other member of the group. A graphical representation of Remark 2 is presented in Figure 4.

Therefore, any condition that guarantees neighborhood stabilization must take into account these two factors. Intuitively, in order to overcome the additional restrictions for the bridging process, neighborhood stabilization would require a distribution of opinions that would induce the inclusion of the MBR of each group in its neighboring area. The following

[17]The proof is readily available upon request.



(a) Initial configurations

(b) Final configurations

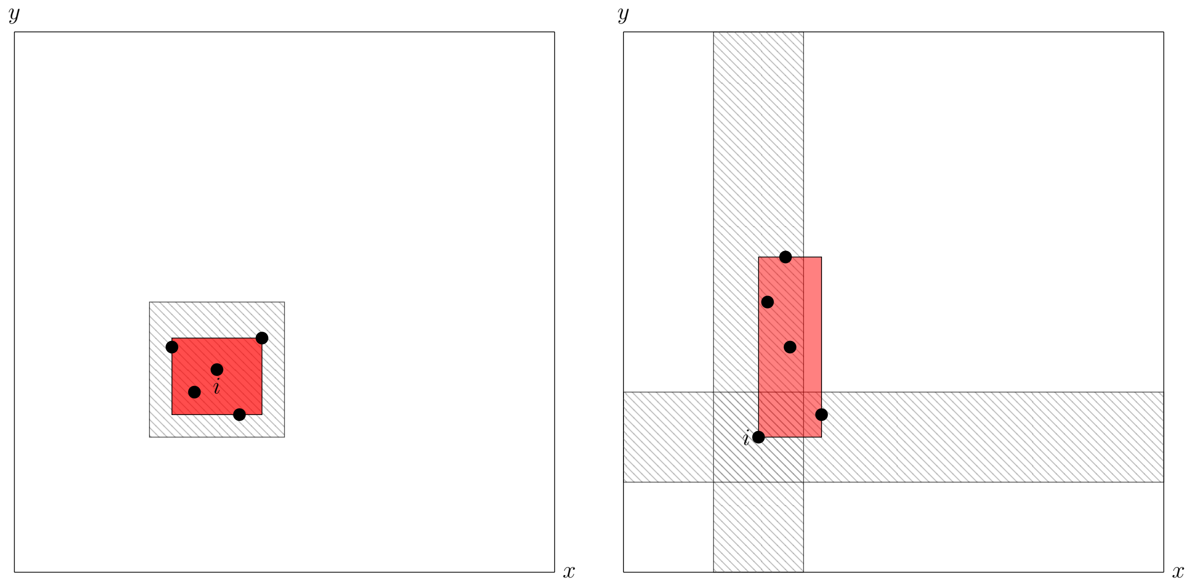
Figure 3: Bonding process: The red rectangle is the MBR of group $\mathcal{I}$ and the shaded area is the MBR's neighboring area. After successful communication, citizen j enters the neighboring area of group $\mathcal{I}$.

two propositions formalize these arguments. Note that, the conditions we present are not necessary, but sufficient. Nevertheless, they suffice for neighborhood stabilization to hold deterministically, i.e. irrespective of the realization of subsequent draws.

Proposition 2. *In a bonding process, a population reaches neighborhood stabilization at period t if the following two conditions hold at that period:*

- *there exist disjoint groups of citizens $\mathcal{I}_1 \cup \cdots \cup \mathcal{I}_p = N$ such that $\mathcal{N}_i^{t'} = \mathcal{I}_g \setminus \{i\}$ for all $g \in \{1, \ldots, p\}$ and for all $i \in \mathcal{I}_g$.*
- *$MBR^t_{\mathcal{I}_{g'}} \cap \mathcal{N}_{MBR^t_{\mathcal{I}_g}} = \emptyset$ for all $g, g' \in \{1, \ldots, p\}$ with $g \neq g'$.*

The first condition is necessary, as a consequence of Remarks 1 and 2. The second condition ensures that no citizen can potentially enter the neighboring area of a group of citizens other than the one she currently belongs to. A graphical example in which this might occur is presented in Figure 3. The provided condition is still not necessary, despite being quite strict, as there are opinion profiles within the MBR of a group which can never be reached by any citizen, for any sequence of interactions. Therefore, in some cases, although the MBR of a group intersects with the neighboring area of another group, no citizen may reach an opinion profile within this intersection. Notice that, the condition is stricter than considering the intersection between one group's MBR and the union of neighboring areas of another group's members. The reason is that successful interactions between members



(a) Common neighbors in bonding process

(b) Common neighbors in bridging process

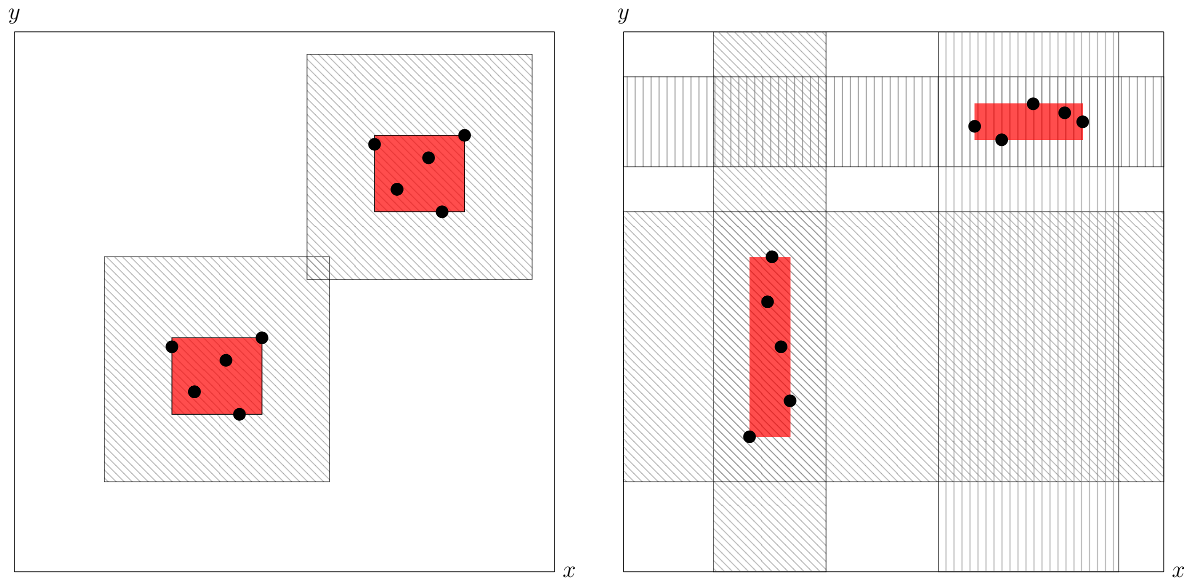
Figure 4: The red rectangles are the groups' MBRs, the grey hatched areas are the neighboring areas of citizen i's opinion. For the bonding process, any neighboring area would contain the MBR of the group.

of a group may change their neighboring areas, without necessarily shrinking them. This means that in subsequent periods the two sets may intersect. On the contrary, both $MBR^t_{\mathcal{I}_g}$ and $\mathcal{N}_{MBR^t_{\mathcal{I}_g}}$ shrink over time, i.e. $MBR^{t+1}_{\mathcal{I}_g} \subseteq MBR^t_{\mathcal{I}_g}$ and $\mathcal{N}_{MBR^{t+1}_{\mathcal{I}_g}} \subseteq \mathcal{N}_{MBR^t_{\mathcal{I}_g}}$, which is sufficient to guarantee neighborhood stabilization. A visual representation of a population that satisfies the conditions of Proposition 2 is presented in Figure 5(a).

As we have already said, neighborhood stabilization is tougher to be achieved in a bridging process, because a citizen may exit the neighboring area of some neighboring citizen, even if she interacts only with other members of the group. This is a consequence of Remark 2. For this reason, we need an additional condition to guarantee neighborhood stabilization in a bridging process.

Proposition 3. *In a bridging process, a population reaches neighborhood stabilization at period t if the following two conditions hold at that period:*

- *there exist disjoint groups of citizens $\mathcal{I}_1 \cup \cdots \cup \mathcal{I}_p = N$ such that $\mathcal{N}^t_i = \mathcal{I}_g \setminus \{i\}$ for all $g \in \{1, \ldots, p\}$ and for all $i \in \mathcal{I}_g$,*
- *for each group $\mathcal{I}_g$, there is a rectangular area R_g of size $k \times 1$ or $1 \times k$ such that $(x^t_i, y^t_i) \in R_g$ for all $i \in \mathcal{I}_g$ and*
- *$MBR^t_{\mathcal{I}_{g'}} \cap \mathcal{N}_{MBR^t_{\mathcal{I}_g}} = \emptyset$ for all $g, g' \in \{1, \ldots, p\}$ with $g \neq g'$.*



(a) Neighborhood stabilization (Bonding) **(b)** Neighborhood stabilization (Bridging)

Figure 5: The red rectangles are the groups' MBRs and the grey hatched areas are the neighboring areas of each MBR. The neighboring areas may intersect, but they cannot intersect with another group's MBR.

The other two conditions of Proposition 3 are the same as those of Proposition 2. A visual representation of the result for the bridging process is shown in Figure 5(b).

A final useful remark is that the conditions presented in Propositions 2 and 3 are sufficient to ensure that a consensus will be eventually reached among the members of each of these groups, as successful communication after period t would only be possible between members of each of the groups, which in turn can never break the existing neighborhoods. In fact, given this and the property of mean preservation in the process, we also know that the consensus opinions towards which each group will converge will be simply the average of its members opinions at t.

Remark 3. Consider a bonding (resp., bridging) process such that at some period t the conditions of Proposition 2 (resp., Proposition 3) hold. Then, the probability that for all $g \in \{1, \ldots, p\}$ opinions of citizens belonging to group $\mathcal{I}_g$ will converge towards consensus opinions $(\overline{x_g}, \overline{y_g}) = \left(\frac{1}{|\mathcal{I}_g|} \sum_{i \in \mathcal{I}_g} x_i^t, \frac{1}{|\mathcal{I}_g|} \sum_{i \in \mathcal{I}_g} y_i^t \right)$ is one.

Before moving on, it should be noted that none of the previous results depends on the probabilities with which different sequences of interactions appear. More specifically, nothing would change if instead of the probability of successful communication being a decreasing function of distance we would have assumed that communication is successful with probability one. Our idea of having the more general decreasing probability of successful communication is meant to capture the feature that people with too diverging opinions might communicate (because of the social capital that connects them) on an issue, but not necessarily agree.

Specific Results on a Restricted Setup

We have still not provided a conclusive comparison between the two processes, as far as the relative likelihood of consensus and the expected number of islands are concerned. In most of the related literature, these problems have been approached mainly via numerical simulations. The reason is that tractable analytical results are quite hard to obtain for two main reasons: First, the evolution of a process depends heavily not only on initial conditions, but also, especially in the case of bridging process, on the exact sequence of meetings. Second, bounded confidence naturally gives rise to discontinuities, which prevents the use of several standard results from Markov processes. Similar problems arise in our setup as well.

For this reason, in order to obtain some analytical results that still capture the main ideas of the two processes, we restrict our attention to a simpler case. More specifically, we analyze the simplest non-trivial case: a society with three citizens, $n = 3$, whose initial opinions are drawn uniformly at random from $[0, 1]$ for each issue, with confidence bounds $d \in (0, 1/2]$ and $k \in (0, 1/4]$, with $\mu = 1/2$, and in which interactions are chosen such that the pair that interacts can communicate successfully and communication is always successful.[18] Among the interactions that satisfy these assumptions, each one is chosen with the same probability, which also implies that each issue is discussed with probability 1/2. In what follows, we refer to this as *the Restricted Setup*.

The results we present for the Restricted Setup are obtained following constructive approach. More specifically, we intend to calculate at the best precision possible the probability with which the society ends up consisting of one, two or three islands respectively in each process. For the bonding process we provide the exact probabilities that each of these events occur, whereas for the bridging process we provide bounds, which are nevertheless sufficient to establish the desired results. We consider ex-ante probabilities and expectations, meaning that they are calculated before the realization of initial opinions.

The first result tackles the problem of inward versus outward-looking societies. Namely, one would expect that societies whose members have larger confidence bounds (i.e. are more outward looking) will end up with opinions concentrated in a smaller number of groups and would be more likely to reach consensus. It turns out that this is not necessarily true. More

[18] The choices regarding the confidence bounds are made in order to have a unique expression for the probabilities we calculate. Our analysis allows us to calculate the respective probabilities for larger values as well, but that would induce a larger computational cost without adding much to the intuition. The assumption regarding successful communication does not affect the result; it just eliminates all interactions that lead to no update which affect only the speed of convergence.

specifically, for the bonding process, when $n = 3$ one can show that for any vectors of initial opinions increasing the confidence bound weakly decreases the number of "islands" towards which opinions converge. However, for larger societies (for any $n \geq 4$) there exist vectors of initial opinions for which a larger confidence bound can lead to a larger number of "islands" (see Example 1). Interestingly, for the bridging process this non-monotonicity result appears even in societies consisting of as few as three citizens (see Example 2).

Example 1. Consider a bonding process with $\mu = 1/2$ in a society of four citizens A, B, C and D with initial opinions $(x_a^0, y_a^0) = (0, 0.1)$, $(x_b^0, y_b^0) = (0, 0.2)$, $(x_c^0, y_c^0) = (0, 0.32)$ and $(x_d^0, y_d^0) = (0, 0.42)$ respectively. Note that all citizens have the same opinion in issue x, so we can focus on communication on issue y.

If the confidence bound is $d = 0.1$, then only pairs (A,B) and (C,D) may communicate successfully. In fact, neighborhoods are already stabilized and opinions will converge towards two islands located at $(0, 0.15)$ and $(0, 0.37)$, with two citizens' opinions converging towards each of them.

If the confidence bound is increased to $d = 0.15$, observe that citizens B and C may also communicate successfully. If this occurs during the first period, then opinions at period $t = 1$ will be $(x_a^0, y_a^0) = (0, 0.1)$, $(x_b^0, y_b^0) = (0, 0.26)$, $(x_c^0, y_c^0) = (0, 0.26)$ and $(x_d^0, y_d^0) = (0, 0.42)$ after which no further updating is possible. In fact, no sequence of interactions can lead to fewer than two islands. Therefore, $d = 0.15$ leads always to at least as many islands as $d = 0.1$ and sometimes to strictly more.

Example 2. Consider a bridging process with $\mu = 1/2$ in a society of three citizens A, B and C with initial opinions $(x_a^0, y_a^0) = (0, 0.4)$, $(x_b^0, y_b^0) = (0.1, 0)$ and $(x_c^0, y_c^0) = (0.3, 0.14)$ respectively.

If the confidence bound is $k = 0.1$, then only citizens A and B may communicate successfully. If they first discuss issue y, then $(x_a^1, y_a^1) = (0, 0.2)$, $(x_b^1, y_b^1) = (0.1, 0.2)$ and $(x_c^1, y_c^1) = (0.3, 0.14)$, thus all three opinions in y are at a distance less than k, which means that they have reached neighborhood stabilization and will eventually converge to society-wide consensus. If they first discuss issue x then $(x_a^1, y_a^1) = (0.05, 0.4)$, $(x_b^1, y_b^1) = (0.05, 0)$ and $(x_c^1, y_c^1) = (0.3, 0.14)$, after which still only A and B can communicate successfully. The only non-trivial interaction that can lead to opinion updating is for A and B to discuss issue y, which then leads all agents to eventually reach society-wide consensus.

If the confidence bound is increased to $k = 0.15$, now observe that citizens B and C may also communicate successfully initially. Given this consider that the first two interactions are both between B and C, with issue x being discussed in one of them and issue y in the other. Opinions at period $t = 2$ will then be $(x_a^2, y_a^2) = (0, 0.4)$, $(x_b^2, y_b^2) = (0.2, 0.07)$ and $(x_c^2, y_c^2) = (0.2, 0.07)$, after which no further updating is possible. Therefore, the probability of society-wide consensus is strictly lower than one, the expected number of islands strictly higher than one and in fact the larger confidence bound yields a weakly larger number of islands in all sequences of interactions.

At this point it is important to clarify that in Examples 1 and 2 we look at specific realizations of the processes in which the opinion profiles are already drawn. However, this is not the way in which we proceed to prove our subsequent results. Instead, we focus on an ex-ante version of the process –i.e. before the realization of opinion profiles. Namely, we intend to characterize the probability of society-wide consensus and the expected value of the number of islands over the distribution of all initial opinion profiles, rather than comparing the outcomes of specific realized processes.

More formally, we approach the problem as follows: First, we identify collections of opinion profiles that lead to similar long-run configurations. More specifically, we group together opinion profiles for which the same sequences of meetings lead to the same number of islands in the long run. Then, we calculate the ex-ante probability with which one, two or three islands will be formed in the long run, by integrating over the distribution of initial opinion profiles and the sequences of meetings. With these probabilities in hand, we can directly calculate the expected number of islands. Importantly, for the second step it is very useful that the collections of opinion profiles we form in the first step have convenient geometric properties, as they can be represented as intersections of finitely many half-planes on a two-dimensional space (see Figures 9, 10 and 11 in the Appendix).

Proposition 4. *In the Restricted Setup, any increase of the confidence bound $d \in (0, 1/2]$ of a bonding process (i) increases the ex-ante probability of society-wide consensus and (ii) decreases the expected number of islands.*

Proposition 5. *In the Restricted Setup, any increase of the confidence bound $k \in (0, 1/4]$ of a bridging process (i) increases the lower bound of the ex-ante probability of society-wide consensus and (ii) decreases the upper bound of the expected number of islands.*

Despite not having a complete monotonicity result for the bridging process, Propositions 4 and 5 allow us to obtain an important comparison between the two processes.

Proposition 6. *In the Restricted Setup, consider an arbitrary bonding process with confidence bound $d \in (0, 1/4]$. There always exists a confidence bound $\widehat{k} \in (0, d)$ such that each bridging process with $k \in (\widehat{k}, 1/4]$ has (i) a higher ex-ante probability of reaching society-wide consensus and (ii) a lower expected number of islands compared to the bonding process with confidence bound d.*

The result follows immediately from Intermediate Value Theorem if one makes the following observations: First, the established lower bound of the probability of reaching society-wide consensus is an increasing and continuous function. Second, for $k = 0$ there are trivially always three islands meaning that, not only the lower bound, but also the actual probability of consensus is equal to zero. Third, for $k = d$ the lower bound for a bridging process is always higher than the actual probability of society-wide consensus for a bonding process with confidence bound d and the same holds for $k > d$. Therefore, there exists some $\bar{k}$ for which the lower bound for the bridging process with confidence bound $\bar{k}$ becomes equal to the actual probability for the bonding process with bound d, and it is higher for all $k > \bar{k}$. Hence, the same must also hold for the actual probability of society-wide consensus for any $k > \bar{k}$. The argument is identical for the expected number of islands, for a (possibly different) cutoff value $\widetilde{k}$. Thus, the argument can be completed by simply considering a common cutoff $\widetilde{k} = \max\{\bar{k}, \widetilde{k}\}$. A graphical illustration of the result is presented on Figure 6.

The result of Proposition 6 is quite intuitive if one observes that for $k = d$ the neighboring area of each point in the bonding process is included in its respective neighboring area in the bridging process. Therefore, one could argue that the comparison is not fair. For this reason, we also compare what would happen if one would consider a neighboring area of the bridging process that would be as large as the one of the bonding process (in terms of the area that it covers). Interestingly, we find that the same result as in Proposition 6 still holds. In fact, we can extend this result to larger values of d, much closer to $1/2$. Namely,

Proposition 7. *In the Restricted Setup, consider an arbitrary bonding process with confidence bound $d \in (0, \sqrt{3}/4]$ and the bridging process with $k = \frac{1}{2} - \sqrt{\frac{1}{4} - d^2}$. Then, the bridging process with k has: (i) a higher ex-ante probability of reaching society-wide consensus and (ii) a lower expected number of islands compared to the bonding process with d.*

Propositions 6 and 7 are probably the strongest results in terms of the implications they entail. Note that the normalization induces values of k that are much smaller than those of d and in practice close to the order of d^2.[19] This suggests that even minimally outward-looking bridging processes lead to more cohesive societies than modestly outward-looking bonding processes.

Simulations

The results obtained on the Restricted Setup yield the natural question on whether they would be robust under more general assumptions. We test the robustness of these results via numerical simulations in more general setups, with a higher number of citizens and we find the results to be robust and even be reinforced.

Our simulation exercise is performed as follows: we generate n two-dimensional opinion profiles (citizens) which are randomly drawn from the uniform distribution on $[0, 1]^2$. Each period two citizens are randomly chosen to communicate and new interactions are drawn

[19] A simple Taylor expansion gives $\frac{1}{2} - \sqrt{\frac{1}{4} - d^2} = d^2 + d^4 + 2d^6 + 5d^8 + O[d^{10}]$.

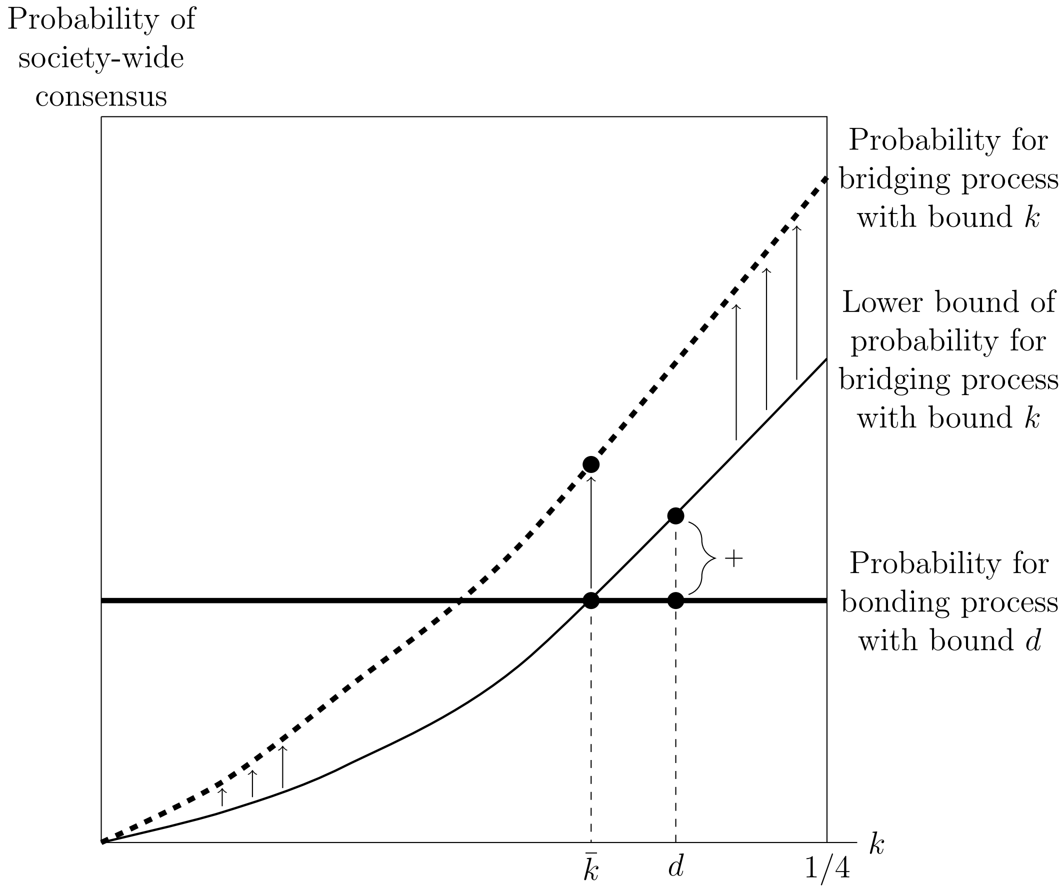


Figure 6: The figure provides a graphical representation of part (i) of Proposition 6. The axes of the diagram are the value of k (horizontal) and the probability of society-wide consensus (vertical). The thick horizontal line depicts the probability of society-wide consensus for a bonding process with confidence bound d. The thin solid curve depicts the lower bound of the probability of society-wide consensus for a bridging process with confidence bound k as a function of k, whereas the thick dashed curve depicts the actual probability of society-wide consensus as a function of k for the same processes. The arrows highlight the fact that the latter is always above the former. According to Proposition 6, for $k = 0$ the lower bound is equal to zero –thus below the horizontal line– and for $k = d$ it is above the horizontal line. The point $\bar{k}$ denotes the unique value of k at which the two intersect and above which the lower bound for the bridging processes is always above the horizontal line. Hence, the same must also hold for the actual probability, i.e. the thick dashed line is above the the thick horizontal line whenever $k > \bar{k}$.

until the process stabilizes. After each process stabilizes, we count the number of islands.[20] We repeat this process 1000 times for each type of process (bonding and bridging) and for different sizes of confidence bounds d and k that are chosen according to the normalization of Proposition 7 $\left(k = \frac{1}{2} - \sqrt{\frac{1}{4} - d^2}\right)$. The assumptions on the rest of the parameters used in the simulations are the same as those of the Restricted Setup.

In Table 2 we present descriptive statistics regarding the number of opinion islands that

[20]We use as conditions for stabilization those of Propositions 2 and 3, which are sufficient for us to count the number of islands, as explained in Remark 3.

		Bonding					Bridging		
d	Mean	Median	Sd	Consensus	k	Mean	Median	Sd	Consensus
				$n = 3$					
0.1	**2.90**	3	0.303	0.001	0.01	**2.90**	3	0.30	0.002
0.2	**2.63**	3	0.528	0.022	0.04	**2.53**	3	0.57	0.039
0.3	**2.30**	2	0.646	0.104	0.1	**2.04**	2	0.66	0.199
0.4	**1.98**	2	0.64	0.215	0.2	**1.56**	2	0.58	0.487
				$n = 10$					
0.1	**8.59**	9	1.012	0	0.01	**8.30**	8	1.19	0
0.2	**5.97**	6	1.209	0	0.04	**4.84**	5	1.36	0.006
0.3	**3.84**	4	1.009	0.002	0.1	**2.57**	3	0.94	0.119
0.4	**2.43**	2	0.899	0.161	0.2	**1.54**	1	0.58	0.507
				$n = 50$					
0.1	**24.12**	24	2.407	0	0.01	**15.36**	15	3.102	0
0.2	**8.05**	8	1.384	0	0.04	**5.79**	6	1.36	0
0.3	**3.87**	4	0.854	0.005	0.1	**3.05**	3	0.85	0.024
0.4	**1.99**	2	0.859	0.318	0.2	**1.80**	2	0.64	0.326
				$n = 100$					
0.1	**28.58**	29	2.604	0	0.01	**17.14**	17	2.82	0
0.2	**7.55**	8	1.329	0	0.04	**6.37**	6	1.36	0
0.3	**3.74**	4	0.990	0.016	0.1	**3.42**	3	0.83	0.009
0.4	**2.29**	2	0.917	0.190	0.2	**2.05**	2	0.67	0.2

Table 2: Mean, median and standard deviation of the number of islands and frequency of society-wide consensus for the two processes and for different population sizes.

arise in the long-run and the likelihood of society-wide consensus for each of the two processes. Table 3 refers to the same set of simulations and presents results on fractionalization. For each size of the society ($n = 3, 10, 50, 100$) we calculate the average number of islands and the frequency of society-wide consensus across all iterations for different values of d in the bonding processes, and the corresponding normalized k in the bridging processes.

The part that refers to $n = 3$ is a benchmark that refers to the case that corresponds to our theoretical results. Interestingly, the results become stronger as we increase the number of citizens: bridging processes give rise to a lower number of islands and are more likely to lead to society-wide consensus.

Table 3 shows the descriptive statistics for the levels of fractionalization in equilibrium in each case, using the Ethnolinguistic Fractionalization Index (ELF) and the Greenberg Index (GI).[21] Confirming the results of the previous table, the bridging process produces lower fractionalization compared to the bonding one.

Overall, the main theoretical predictions and intuitions are indeed verified by the numerical simulations. In fact, it is quite reassuring that the results, if something, are strengthened

[21] As a reminder, $ELF = 1 - \sum s_i^2$ and $GI = \sum \sum s_i s_j d_{ij}$, where s_i and s_j are shares of distinct opinion islands, and d_{ij} is the Euclidean distance between opinion islands i and j.

	Bonding						Bridging					
d	μ_{ELF}	$\widetilde{ELF}$	σ_{ELF}	μ_{GI}	$\widetilde{GI}$	σ_{GI}	k	μ_{ELF}	$\widetilde{ELF}$	σ_{ELF}	μ_{GI}	$\widetilde{GI}$	σ_{GI}
$n=3$												
0.1	0.644	0.667	0.069	0.000	0.000	0.000	0.01	0.645	0.667	0.07	0.000	0.000	0.000
0.2	0.579	0.667	0.135	0.000	0.000	0.000	0.04	0.554	0.667	0.155	0.000	0.000	0.000
0.3	0.487	0.444	0.196	0.000	0.000	0.000	0.1	0.409	0.444	0.223	0.000	0.000	0.000
0.4	0.392	0.444	0.222	0.000	0.000	0.000	0.2	0.238	0.444	0.236	0.000	0.000	0.000
$n=10$												
0.1	0.868	0.88	0.027	0.005	0.005	0.000	0.01	0.858	0.86	0.04	0.004	0.004	0.001
0.2	0.773	0.78	0.068	0.004	0.004	0.000	0.04	0.68	0.72	0.145	0.003	0.003	0.001
0.3	0.62	0.64	0.13	0.004	0.004	0.001	0.1	0.38	0.42	0.201	0.002	0.002	0.001
0.4	0.377	0.42	0.222	0.002	0.003	0.002	0.2	0.132	0	0.156	0.001	0	0.001
$n=50$												
0.1	0.94	0.941	0.009	0.126	0.126	0.006	0.01	0.817	0.832	0.077	0.067	0.067	0.014
0.2	0.799	0.805	0.042	0.112	0.113	0.009	0.04	0.45	0.458	0.139	0.038	0.038	0.013
0.3	0.533	0.586	0.178	0.074	0.08	0.027	0.1	0.18	0.151	0.111	0.019	0.017	0.012
0.4	0.077	0.039	0.105	0.011	0.006	0.0151	0.2	0.044	0.039	0.044	0.006	0.005	0.006
$n=100$												
0.1	0.945	0.945	0.007	0.506	0.506	0.017	0.010	0.680	0.701	0.107	0.185	0.184	0.034
0.2	0.781	0.786	0.041	0.430	0.432	0.030	0.040	0.323	0.305	0.127	0.102	0.099	0.038
0.3	0.435	0.506	0.022	0.225	0.254	0.124	0.100	0.124	0.114	0.070	0.051	0.047	0.029
0.4	0.041	0.039	0.040	0.022	0.019	0.022	0.200	0.033	0.020	0.027	0.017	0.013	0.014

Table 3: Fractionalization Indices – μ_x, $\widetilde{x}$ and σ_x correspond to the mean, median and standard deviation of the two fractionalization indices respectively, for the two processes for different population sizes. The values denoted by 0.000 refer to quantities that are positive but lower than 10^{-3}, whereas the values denoted by 0 refer to quantities with value exactly equal to zero.

when we look at more general setups, with larger populations.

In the Appendix, we include another robustness check, repeating the simulations with the same parameters except for the probability of successful communication, which is now considered to be quadratically decreasing in the distance of opinions in the issue discussed. For the bonding process, if citizens i, j, with opinion profiles (x_i^t, y_i^t) and (x_j^t, y_j^t) respectively, discuss issue x, their communication is successful with probability $1-(x_i^t-x_j^t)^2$ if $|x_i^t-x_j^t| \leq d$ and $|y_i^t - y_j^t| \leq d$ and 0 otherwise. For the bridging process, if citizens i, j, with opinion profiles (x_i^t, y_i^t) and (x_j^t, y_j^t) respectively, discuss issue x, their communication is successful with probability $1-(x_i^t-x_j^t)^2$ if $|x_i^t-x_j^t| \leq k$ or $|y_i^t-y_j^t| \leq k$ and 0 otherwise. It is easy to see that the results are virtually identical, confirming that whether the probability of successful communication is set to be equal to one or it is considered to be decreasing on the distance has no bearing in the results.

6 Conclusion

Social capital has been considered as a major factor that affects the establishment of predominant opinions within societies. The aim of this paper has been to identify the effect of different forms of social capital on the distribution of opinions within a society, as a result of the differences in the channels of communication they encourage.

It turns out that both of the considered processes converge towards distributions where groups of citizens are concentrated around common opinions, which are located sufficiently far from each other, with intermediate opinions gradually dying out. Other than this, there is very little we can say about the kind of opinions around which large groups are concentrated, except of the fact that both the distribution of initial opinions and the exact sequence of interactions matter. It could be the purpose of a different study to understand how different processes lead to more moderate or extreme opinions, as well as the importance of influential citizens to the prevalence of certain opinions.

Regarding the social dynamics, the common feature of the two processes is that more outward-looking societies tend be more cohesive, as opinions tend to be less fractionalized and there is a higher likelihood of reaching society-wide consensus. Nevertheless, this result is sensitive to the distribution of initial opinions. The separating feature of the two processes seems to be that the bridging process seems to lead most of the times in a lower expected number of opinions and a higher probability of society-wide consensus.

Finally, the fact that societies where citizens meet through bridging associations seem more prone to reach consensus indicates that societies in which communication takes place predominantly through bridging associations allow the interaction between citizens with very distant opinions, providing them the opportunity to come closer. This feature indirectly induces smaller distances between unconnected citizens (see Iijima and Kamada, 2017); an observation that can have further implications. If one represents meetings that yield positive probability of agreement as network links, then the result suggests that networks with smaller average distance between unconnected nodes are more likely to reach consensus.

A Proofs of General Results

Proof of Theorem 1. For either of the two processes, given two vectors of initial opinions $\mathbf{x}(0)$ and $\mathbf{y}(0)$ and a sequence of interactions $\{z_t\}_{t=1}^{\infty}$, the evolution of opinions can be described by the vectors of initial opinions and a sequence of transition matrices for each issue, $\{A_x(t)\}_{t=1}^{\infty}$ and $\{A_y(t)\}_{t=1}^{\infty}$ respectively, where $A_x(t) = A_x(\mathbf{x}(t-1), \mathbf{y}(t-1), z_t)$ and $A_y(t) = A_y(\mathbf{x}(t-1), \mathbf{y}(t-1), z_t)$, for which $\mathbf{x}(t) = A_x(t)\mathbf{x}(t-1)$ and $\mathbf{y}(t) = A_y(t)\mathbf{y}(t-1)$.

At each t, by $z_t = ((i_t, j_t), s_t, c_t)$ we know the pair (i_t, j_t) that communicates, the issue s_t discussed and whether communication was successful, $c_t \in \{0, 1\}$. If $c_t = 0$, then $A_x(t) = A_y(t) = I$, where I is the identity matrix, as no opinion is updated in this period. If $c_t = 1$ and $s_t = x$, then $A_y(t) = I$ and for $A_x(t)$ we have that for all $k \neq i, j$ it holds that $a^t_{x,kk} = a^t_{y,kk} = 1$ and $a^t_{x,kl} = a^t_{y,kl} = 1$ for all $l \neq k$. For citizens i, j though we get that $a^t_{x,ik} = a^t_{y,ik} = a^t_{x,jk} = a^t_{y,jk} = 0$ for all $k \neq i, j$, whereas $a^t_{y,ij} = a^t_{y,ji} = \mu$ and $a^t_{y,ii} = a^t_{y,jj} = 1 - \mu$. The same argument holds if $c_t = 1$ and $s_t = y$, with x and y reversed.

Therefore, for each sequence $\{z_t\}_{t=1}^{\infty}$, we can consider the evolution of opinions in each of the issues separately and for convergence to hold it is sufficient to show that the sufficient conditions provided by Lorenz (2005) hold for both issue–specific sequences of transition matrices. We show that indeed this is the case:

(i) Self–confidence: At each period t, each citizen $i \in N$ puts positive weight to her own opinion, i.e. it holds that $a^t_{\cdot,ii} > 0$, which is clearly true in our case.

(ii) Mutual confidence: Zero entries in the transition matrix are symmetric. For every pair $i, j \in N$ it holds that $a^t_{\cdot,ij} > 0 \Leftrightarrow a^t_{\cdot,ji} > 0$. This is again clearly true in both our processes, given that after successful communication both citizens revise their opinions.

(iii) Positive weights do not converge to zero: There is a $\delta > 0$ such that the lowest positive entry of the transition matrix is greater than δ. This is also true in our case, considering $\delta = \min\{\mu, 1-\mu\}$ which is strictly positive.

These three conditions ensure for each issue the emergence of pairwise disjoint groups of citizens $\mathcal{J}_{x,1} \cup \cdots \cup \mathcal{J}_{x,r}$ and $\mathcal{J}_{y,1} \cup \cdots \cup \mathcal{J}_{y,s}$ who reach consensus in this issue. To complete the argument, let $\mathcal{I}_{kl} = \mathcal{J}_{x,k} \cap \mathcal{J}_{y,l}$. There is a total of $p = r \times s$ such groups and within each of them citizens reach consensus in both issues.[22]

Note that the structure of the transition matrices does not depend on whether we consider a bonding or a bridging process, as these affect only the frequency of updating in each sequence z_t. $\square$

Proof of Proposition 1. For a bonding process, $(\overline{x}_m, \overline{y}_m) \in Int\left(\mathcal{N}_{(\overline{x}_l, \overline{y}_l)}\right) \Rightarrow |\overline{x}_m - \overline{x}_l| = \chi < d$ and $|\overline{y}_m - \overline{y}_l| = \psi < d$ and without loss of generality let $\chi > \psi$. Consider some $\widetilde{\epsilon} > 0$ that satisfies $\widetilde{\epsilon} < \frac{\delta-\chi}{2}$ and $\widetilde{\epsilon} < \frac{\mu\chi}{2(1+\mu)}$. If the probability that opinions converge to form groups $\mathcal{I}_1 \cup \cdots \cup \mathcal{I}_p$ such that citizens belonging to group $\mathcal{I}_g$ reach consensus at $(\overline{x_g}, \overline{y_g})$ for all $g \in \{1, \ldots, p\}$ is strictly positive, then for $\widetilde{\epsilon}$ (and in fact for all $\epsilon > 0$) there must exist some $\widehat{t_{\widetilde{\epsilon}}}$ such that the probability that for all $t > \widehat{t_{\widetilde{\epsilon}}}$ holds that $|x_i^t - \overline{x}_g| < \widetilde{\epsilon}$ and $|y_i^t - \overline{y}_g| < \widetilde{\epsilon}$, for all $i \in \mathcal{I}_g$ and for all $g \in \{1, \ldots, p\}$ is strictly positive.

Now focus on issue x and notice that $\widetilde{\epsilon} < \frac{\delta-\chi}{2}$ guarantees that as long as citizens of groups $\mathcal{I}_l$ and $\mathcal{I}_m$ remain at a distance less than $\widetilde{\epsilon}$ from $\overline{x}_l$ and $\overline{x_m}$ respectively (and the same on issue y), they face a strictly positive probability of communicating successfully. This is because the maximum distance between a citizen of $\mathcal{I}_m$ and another citizen of $\mathcal{I}_l$ is at most $\chi + 2\widetilde{\epsilon} < \delta$. Such a pair of citizens always exists because groups $\mathcal{I}_g$ are non–empty. On top of that, the condition $\widetilde{\epsilon} < \frac{\mu\chi}{2(1+\mu)}$ guarantees that a successful communication on issue x between some citizen in $\mathcal{I}_l$ and some citizen in $\mathcal{I}_m$ is sufficient to bring both of them at a distance larger than $\widetilde{\epsilon}$ from the respective consensus opinion. This is because any successful communication will incur an update of opinion between $\mu(\chi - 2\widetilde{\epsilon})$ and $\mu(\chi + 2\widetilde{\epsilon})$, which is always larger than the $2\widetilde{\epsilon}$. Therefore, as long as all citizens of groups $\mathcal{I}_l$ and $\mathcal{I}_g$ remain close to their consensus

[22] Lorenz (2005) states the result focusing on the limit of the backward product of transition matrices, which is equivalent to the consensus argument we present here.

opinions, there is at least one pair of citizens whose successful communication can lead the citizens more than $\widetilde{\epsilon}$ far from that opinion.

Hence, let H_t denote the event that all opinions in both issues are within $\widetilde{\epsilon}$ distance from the respective consensus opinion of the group at period t. Then, the event that for all $t > \widehat{t_{\widetilde{\epsilon}}}$ holds that $|x_i^t - \overline{x}_g| < \widetilde{\epsilon}$ and $|y_i^t - \overline{y}_g| < \widetilde{\epsilon}$, for all $i \in \mathcal{I}_g$ and for all $g \in \{1, \ldots, p\}$ is the infinite intersection of events H_t, i.e. $\bigcap\limits_{t=\widehat{t}+1}^{\infty} H_t$. And conditional on $H_{\widehat{t}}$ being true, $\mathbb{P}\left[\bigcap\limits_{t=\widehat{t}+1}^{\infty} H_t | H_{\widehat{t}}\right] = \lim\limits_{\tau\to\infty} \prod\limits_{t=\widehat{t}+1}^{\tau} \mathbb{P}\left[H_t|H_{t-1}\right]$. Yet, for each t, $\mathbb{P}\left[H_t|H_{t-1}\right] \leq 1 - \underline{b}$, where $\underline{b}$ is the strictly positive and independent of t (hence, bounded away from zero) lower bound of the probability of a pair that can communicate successfully to be actually drawn and indeed communicate successfully in one issue. We have already shown that such a pair exists. Therefore, $\mathbb{P}\left[\bigcap\limits_{t=\widehat{t}+1}^{\infty} H_t | H_{\widehat{t}}\right] = \lim\limits_{\tau\to\infty} \prod\limits_{t=\widehat{t}+1}^{\tau} \mathbb{P}\left[H_t|H_{t-1}\right] \leq \lim\limits_{\tau\to\infty} (1-\underline{b})^{\tau-\widehat{t}} = 0.$

The proof for the bridging process is identical, except of the difference in the definition of the neighboring area (which is inconsequential) and the value of confidence bound being k instead of d.

Explanatory Note: If $(\overline{x}_m, \overline{y}_m)$ is at the boundary of $\mathcal{N}_{(\overline{x}_l, \overline{y}_l)}$ then we can no longer find $\widetilde{\epsilon} < \frac{\delta - \chi}{2}$ to guarantee that all citizens of each group may communicate. If in particular it holds that $|\overline{x}_m - \overline{x}_l| = d$ and $|\overline{y}_m - \overline{y}_l| = d$ the result may not necessarily hold. Consider the following example with three citizens: Group $\mathcal{I}_1$ consists of a single citizen located at $(0, 0)$ and group $\mathcal{I}_2$ consists of two citizens located at $(d + \frac{\epsilon}{2}, d - \frac{\epsilon}{2})$ and $(d - \frac{\epsilon}{2}, d + \frac{\epsilon}{2})$ for some small $\epsilon > 0$, with consensus opinions (d, d). For any $\mu < 1/2$, the two citizens of group $\mathcal{I}_2$ can keep communicating successfully, converging towards (d, d), but they can never communicate successfully with the citizen of group $\mathcal{I}_1$. For the remaining parts of the boundaries in bonding process and for all boundaries in a bridging process the result can be shown to still hold. □

Proof of Proposition 2. The following two conditions are jointly equivalent to neighborhood stabilization: For each citizen $i \in N$ (i) i does not lose any of her current neighbors at a subsequent period (ii) no additional citizen enters i's neighborhood at a subsequent period.

Remark 1 ensures that the existence of disjoint groups of citizens with common neighborhoods are necessary for conditions (i) and (ii) to be satisfied. Absent of such groups, there would be a positive probability that some citizen's neighborhood will change. Given the existence of these disjoint groups of citizens, Remark 2 guarantees that a member of such a group will not face a change in her neighborhood, as long as the members of that group may communicate successfully only among themselves.

The second condition guarantees that the members of each of the disjoint groups may communicate successfully only among themselves. This is because at period t each citizen may only communicate successfully with another member of her own group. But successful communication between citizens $i, j \in \mathcal{I}_g$ weakly shrinks both the minimum bounding rectangle of the group and as a result also the neighboring area of the minimum bounding rectangle, i.e. $MBR^{t+1}_{\mathcal{I}_g} \subseteq MBR^t_{\mathcal{I}_g}$, which then implies that $\mathcal{N}_{MBR^{t+1}_{\mathcal{I}_g}} \subseteq \mathcal{N}_{MBR^t_{\mathcal{I}_g}}$. Therefore, if $MBR^t_{\mathcal{I}_{g'}} \cap \mathcal{N}_{MBR^t_{\mathcal{I}_g}} = \emptyset$, for all $g' \neq g$, it must also hold that $MBR^{t+1}_{\mathcal{I}_{g'}} \cap \mathcal{N}_{MBR^{t+1}_{\mathcal{I}_g}} = \emptyset$, for all $g' \neq g$. This completes the argument, as it guarantees that in all subsequent periods no additional citizen enters the neighborhood of any $i \in \mathcal{I}_g$. $\square$

Proof of Proposition 3. As explained in the proof of Proposition 2, the following two conditions are jointly equivalent with neighborhood stabilization: For each citizen $i \in N$ (i) i does not lose any of his current neighbors at a subsequent period (ii) no additional citizen enters i's neighborhood at a subsequent period. Moreover, the first condition of Proposition 3 is necessary for neighborhood stabilization because of Remark 1 and the third condition of Proposition 3 guarantees that the members of each group may communicate successfully only among themselves, as in Proposition 2.

Combining the first with the second condition guarantees that no citizen will face a change in her neighborhood, as long as the members of each group may communicate successfully only among themselves. This is because the second condition is sufficient to guarantee that $MBR^t_{\mathcal{I}} \subseteq \mathcal{N}_{(x^t_i, y^t_i)}$ for all $i \in \mathcal{I}$, which we show below.

Without loss of generality, we prove the result considering R_g to be of size $k \times 1$. $MBR^t_{\mathcal{I}_\}}$ is a rectangle with sizes $|\max_{i\in\mathcal{I}} x^t_i - \min_{i\in\mathcal{I}} x^t_i| \times |\max_{i\in\mathcal{I}} y^t_i - \min_{i\in\mathcal{I}} y^t_i|$. Given that all $i \in \mathcal{I}$ are such that $(x^t_i, y^t_i) \in R_g$, it must hold that $|\max_{i\in\mathcal{I}} x^t_i - \min_{i\in\mathcal{I}} x^t_i| = k - \epsilon$ for some $\epsilon \in [0, k]$ and also the left bound of R_g, denote it $\underline{x}$, must satisfy $\underline{x} \in [\min_{i\in\mathcal{I}} x^t_i - \epsilon, \min_{i\in\mathcal{I}} x^t_i]$. These last two conditions guarantee that the right bound of R_g, $\overline{x}$, will have to satisfy $\overline{x} \in [\max_{i\in\mathcal{I}} x^t_i, \min_{i\in\mathcal{I}} x^t_i + \epsilon]$. Therefore, recalling that R_g extends in the whole length of the vertical axis, it follows that $MBR^t_{\mathcal{I}} \subseteq R_g$. Furthermore, the neighboring area of the citizen $j = \operatorname{argmin}_{i\in\mathcal{I}} x^t_i$ extends for all y at least until $\min_{i\in\mathcal{I}} x^t_i - k \leq \underline{x}$. Analogously, the neighboring area of the citizen $l = \operatorname{argmin}_{i\in\mathcal{I}} x^t_i$ extends for all y at least until $\max_{i\in\mathcal{I}} x^t_i - k \leq \overline{x}$. Hence, it follows immediately that $R \subseteq \mathcal{N}_{(x^t_i, y^t_i)}$ for all $i \in \mathcal{I}$. Thus, also $MBR^t_{\mathcal{I}} \subseteq \mathcal{N}_{(x^t_i, y^t_i)}$ for all $i \in \mathcal{I}$. $\square$

B Proofs of Results on the Restricted Setup

Explanatory Note: For the results on the Restricted Setup we follow a constructive and rather primitive approach, which comes at a considerable computational cost. In particular, we derive analytically the exact probabilities (for the bonding process), or rather strict bounds of the probabilities (for the bridging process) with which each process converges to form one, two or three islands respectively. We provide the results for $d \in (0, 1/2]$ and $k \in (0, 1/4]$, nevertheless with similar calculations they could be extended to higher bounds, in which cases there could only exist either one or two islands. The reason we restrict k to be below 1/4 is that for higher values the closed form expressions of the probabilities we have calculated change. This would require to essentially repeat the same analysis, without adding much to our understanding of the problem. Finally, in order not to make the proofs excessively long, we present a concise version of the calculations. The complete and longer version of the calculations of all sequences of interactions is available upon request.

Lemma 1. *In the Restricted Setup, consider a bonding process with $d \in (0, 1/2]$. Opinions converge to form*

- *Three islands, with probability* $1 - 12d^2 + 12d^3 + 36d^4 - 68d^5 + \frac{88}{3}d^6$
- *Two islands, with probability* $\frac{1}{48}d^2(576 - 576d - 2700d^2 + 4884d^3 - 2083d^4)$
- *One island, with probability* $\frac{9}{16}d^4(6 - 5d)^2$

Proof of Lemma 1. It is useful to observe that the random draws of the initial opinions can be decomposed into three independent draws. Namely, first we draw independently three points, each from a uniform distribution in $[0, 1]$, which represent the opinions on x, call them $x_1 \leq x_2 \leq x_3$. Without loss of generality, let us identify citizens according to their opinions on x, i.e. $\widehat{x_a} = x_1$, $\widehat{x_b} = x_2$ and $\widehat{x_c} = x_3$. Then, we draw independently three points, each from a uniform distribution in $[0, 1]$, which represent the opinions on y, call them $y_1 \leq y_2 \leq y_3$. Finally, the three opinions on y are matched randomly with opinions on x, i.e. we draw one of the six permutations of (a, b, c) each with probability 1/6. If permutation (i, j, k) is selected, then $\widehat{y_i} = y_1$, $\widehat{y_j} = y_2$ and $\widehat{y_k} = y_3$. Importantly, the three sets of draws are independent, therefore we can perform the analysis separately for each issue.

There are five essentially different types of draws in each issue, depending on the relative position of the three points with respect to the confidence bound. They are presented graphically in Figure 7 and their probabilities of appearing are the following:

- All far: $6\int_0^{1-2d} \frac{(1-x_1-2d)^2}{2} dx_1 = (1-2d)^3$
- All close: $6\left(\int_0^{1-d} \frac{d^2}{2} dx_1 + \int_{1-d}^{1} \frac{(1-x_1)^2}{2} dx_1\right) = 3d^2 - 2d^3$
- 12, 23 close, 13 far: $6\left(\int_0^{1-2d} \frac{d^2}{2} dx_1 + \int_{1-2d}^{1-d} \frac{d^2}{2} - \frac{(x_1+2d-1)^2}{2} dx_1\right) = 3d^2 - 4d^3$
- 12 close, 23 far: $6\left(\int_0^{1-2d} \frac{d^2}{2} + d(1-x_1-2d) dx_1 + \int_{1-2d}^{1-d} \frac{(1-d-x_1)^2}{2} dx_1\right) = 3d - 9d^2 + 7d^3$
- 12 far, 23 close: $6\left(\int_0^{1-2d} \frac{d^2}{2} + d(1-x_1-2d) dx_1 + \int_{1-2d}^{1-d} \frac{(1-d-x_1)^2}{2} dx_1\right) = 3d - 9d^2 + 7d^3$

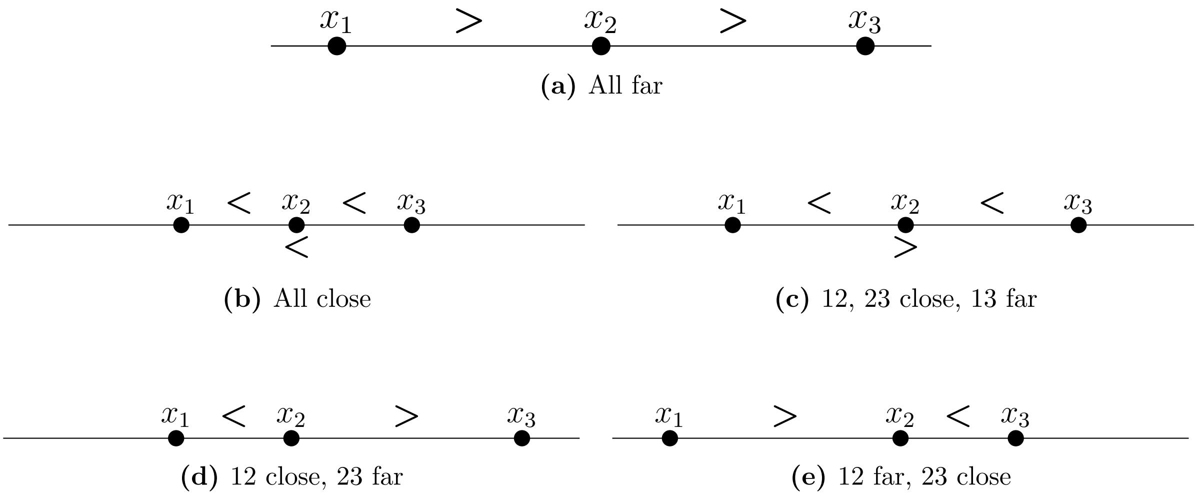
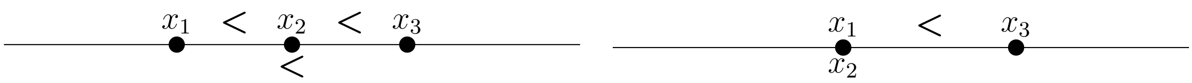


Figure 7: Cases of essentially different draws of three opinions on issue x. The signs ($>$ or $<$) above the segment that connects x_1 with x_2 (x_2 with x_3) refers to the distance between the opinions x_1 and x_2 (x_2 and x_3), whereas the sign below x_2 refers to the distance between x_1 and x_3. The sign $>$ ($<$) denotes that the respective distance is larger (smaller) than the confidence bound (d in bonding, k in bridging).

Let us focus first on the cases that lead to three islands. For this to happen, the initial opinions must be sufficiently far that no interaction can lead to an opinion updating.[23] We can calculate this probability by calculating and adding the probabilities of some mutually exclusive cases. To do so, we will use the fact that we have split the initial draws into three independent parts (draws for x, y and order), which allows to calculate the probabilities for each issue separately and then just multiply them. The (mutually exclusive) cases that lead to three islands in the bonding process are as follows:

- $(1-2d)^3$: All far in x.
- $(3d^2-2d^3)(1-2d)^3$: All close in x, all far in y.
- $(3d-9d^2+7d^3)(1-d)^2$: 12 close, 23 far in x, a and b far in y,
- $(3d-9d^2+7d^3)(1-d)^2$: 12 far, 23 close in x, b and c far in y,
- $(3d^2-4d^3)\left[(1-2d)^3+\frac{2}{3}(3d-9d^2+7d^3)\right]$
  - 12, 23 close, 13 far in x, all far in y,
  - 12, 23 close, 13 far in x, 12 close, 23 far in y, permutations (a,c,b) and (c,a,b),
  - 12, 23 close, 13 far in x, 12 far, 23 close in y, permutations (b,a,c) and (b,c,a).

[23]Otherwise, if at least two of the citizens are sufficiently close to interact successfully, this will be true in all subsequent periods and hence by Proposition 1 there will be at most two islands in the long run.



(a) Initially 12 and 23 close, but 13 far. **(b)** After 1 and 2 agree all close

Figure 8: Example of opinions after successful communication between citizens with x_1 and x_2.

For "12 close, 23 far in x" (resp., "12 far, 23 close in x") we simply need to consider that a and b (resp., b and c) are far in y, which occurs with probability $(1-d)^2$.

Overall, by adding those probabilities we get:

$$\begin{aligned}\mathbb{P}(3 \text{ islands}) =&(1-2d)^3+(3d^2-2d^3)(1-2d)^3+\dots\\ &\dots+2(3d-9d^2+7d^3)(1-d)^2+(3d^2-4d^3)\left[(1-2d)^3+\frac{2}{3}(3d-9d^2+7d^3)\right]=\\ =&1-12d^2+12d^3+36d^4-68d^5+\frac{88}{3}d^6\end{aligned}$$

Let us now turn our attention to the case of one island, i.e. the case in which the three citizens reach consensus in both issues. For this to happen, at some period all citizens must have opinions with distances at most d in both issues. This can happen under three circumstances: First, if they start with opinions that are at distance at most d in both issues. Second, if in one issue opinions are all close and in the other issue there is a distribution as in Figure 7(c) –x_1 close to x_2, x_2 close to x_3 but x_1 far from x_3, or analogously for y–. Third, if initial opinions are as in Figure 7(c) in both issues.

In the latter two cases, society-wide consensus is not certain, but it is possible. For instance, consider three citizens who start with all opinions close to each other in issue y and in issue x have opinions as in Figure 8. In this case, x_1 and x_3 are initially far (i.e., $x_3-x_1>d$), but get close to each other after x_1 and x_2 have been updated to $\frac{x_1+x_2}{2}$ (i.e., $x_3-\frac{x_1+x_2}{2}\leq d$). However, this does not occur necessarily, as the initial opinions could be such that not only x_1 remains far from x_3, but also brings x_2 far from x_3, in which case society-wide consensus becomes impossible. In general, these indirect interactions may happen in a number of complex ways, yet in the specific problem with three agents and $\mu=1/2$ these are only possible for initial conditions that at least in one issue are of the form presented in Figure 8(a). Hence, two additional conditions are relevant here: (i) $x_3-\frac{x_1+x_2}{2}\lesseqgtr d$ and (ii) $\frac{x_3+x_2}{2}-x_1\lesseqgtr d$, as well as the two respective conditions for issue y.

The two additional conditions divide the set of triplets (x_1,x_2,x_3) of Figure 7(c) to four subsets, which are depicted in Figure 9. We calculate the probability that a randomly drawn triplet will belong to each of these sets.

1. $x_2-x_1\leq d$, $x_3-x_2\leq d$, $x_3-x_1>d$ and $x_3-\frac{x_1+x_2}{2}\leq d$, $\frac{x_3+x_2}{2}-x_1\leq d$:
   - $6\left(\int_0^{1-4d/3}\frac{d^2}{6}dx_1+\int_{1-4d/3}^{1-d}\frac{d^2}{6}-\frac{3(x_1+\frac{4d}{3}-1)^2}{2}dx_1\right)=\frac{1}{9}(9d^2-10d^3)$

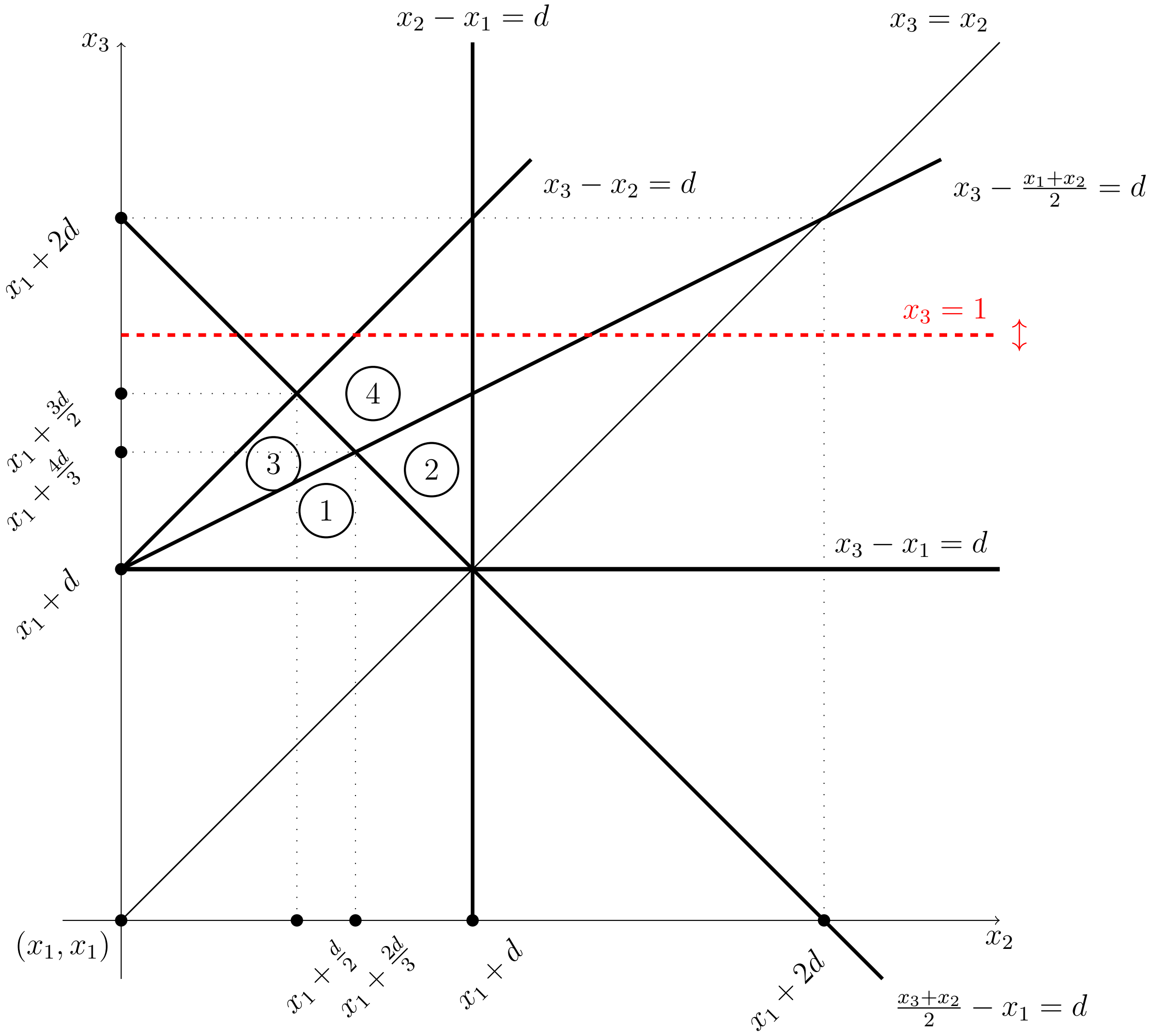


Figure 9: Two–dimensional projection of the area in which $x_2 - x_1 \leq d$, $x_3 - x_2 \leq d$ and $x_3 - x_1 > d$, for some $x_1 \in [0, 1-d]$. Areas 1–4 correspond to the four different areas with respect to the conditions $x_3 - \frac{x_1+x_2}{2} \lesseqgtr d$ and $\frac{x_3+x_2}{2} - x_1 \lesseqgtr d$. The origin of the axes is (x_1, x_1) and the location of the line where $x_3 = 1$ depends on x_1.

2. $x_2 - x_1 \leq d$, $x_3 - x_2 \leq d$, $x_3 - x_1 > d$ and $x_3 - \frac{x_1+x_2}{2} \leq d$, $\frac{x_3+x_2}{2} - x_1 > d$:
   - $6\left(\int_0^{1-3d/2} \frac{d^2}{12} dx_1 + \int_{1-3d/2}^{1-4d/3} \frac{d^2}{12} - (x_1 + \frac{3d}{2} - 1)^2 dx_1 + \int_{1-4d/3}^{1-d} \frac{(1-d-x_1)^2}{2} dx_1\right) = \frac{1}{36}(18d^2 - 23d^3)$
3. $x_2 - x_1 \leq d$, $x_3 - x_2 \leq d$, $x_3 - x_1 > d$ and $x_3 - \frac{x_1+x_2}{2} > d$, $\frac{x_3+x_2}{2} - x_1 \leq d$:
   - $6\left(\int_0^{1-3d/2} \frac{d^2}{12} dx_1 + \int_{1-3d/2}^{1-4d/3} \frac{d^2}{12} - (x_1 + \frac{3d}{2} - 1)^2 dx_1 + \int_{1-4d/3}^{1-d} \frac{(1-d-x_1)^2}{2} dx_1\right) = \frac{1}{36}(18d^2 - 23d^3)$
4. $x_2 - x_1 \leq d$, $x_3 - x_2 \leq d$, $x_3 - x_1 > d$ and $x_3 - \frac{x_1+x_2}{2} > d$, $\frac{x_3+x_2}{2} - x_1 > d$:
   - $6\left(\int_0^{1-2d} \frac{d^2}{6} dx_1 + \int_{1-2d}^{1-3d/2} \frac{d^2}{6} - \frac{(x_1+2d-1)^2}{2} dx_1 + \int_{1-3d/2}^{1-4d/3} \frac{3(1-\frac{4d}{3}-x_1)^2}{2} dx_1\right) = \frac{1}{18}(18d^2 - 29d^3)$

For these cases, the emergence or not of society-wide consensus does not depend only on the initial opinions, but also on the sequence of interactions. In fact, for the same profile of initial opinions, one sequence might lead to consensus, while another might not. Below, we present the probabilities with which each class of initial opinions leads to consensus. When we mention "case i" we refer to area $i = 1, \ldots, 4$ in Figure 9 for one issue and when we mention "case (i, j)" we refer to area i for issue x and to area j for issue y.

- $(3d^2 - 2d^3)^2$: All close in both x and y.
- $(3d^2 - 2d^3)\left[\frac{1}{9}(9d^2 - 10d^3) + \frac{1}{36}(18d^2 - 23d^3)\right]$: All close in x, 12, 23 close, 13 far in y.
  - Case 1, with probability 1.
  - Case 2, with probability 1/2.
  - Case 3, with probability 1/2.
  - same argument and probabilities hold for:
    * 12, 23 close, 13 far in x, all close in y.
- $\left[\frac{1}{9}(9d^2 - 10d^3) + \frac{1}{36}(18d^2 - 23d^3)\right]^2$: 12, 23 close, 13 far in both x and y.
  - Case $(1, 1)$, with probability 1.
  - Cases $(1, 2), (2, 1), (1, 3), (3, 1)$ with probability 1/2.
  - Cases $(2, 2), (2, 3), (3, 2), (3, 3)$ with probability 1/4.

Given that these cases are mutually exclusive, we can simply add the probabilities and get:

$$\mathbb{P}(1 \text{ island}) = \left[(3d^2 - 2d^3) + \frac{1}{9}(9d^2 - 10d^3) + \frac{1}{36}(18d^2 - 23d^3)\right]^2 = \frac{9}{16}d^4(6 - 5d)^2$$

Finally, two islands arise with the remaining probability. $\square$

Lemma 2. *In the Restricted Setup, consider a bridging process with $k \in (0, 1/4]$. Opinions converge to form*

- *Three islands, with probability* $(1 - 2k)^6$
- *Two islands, with probability at most* $(12 - 93k + 337k^2 - 725k^3 + 946k^4 - 535k^5)\, k$
- *One island, with probability at least* $(33 - 177k + 485k^2 - 754k^3 + 471k^4)\, k^2$

Proof of Lemma 2. In a bridging process the only way to end up with three islands is if all citizens' initial opinions are located further than the confidence bound in both issues. The probability that the three opinions are all far from each other is $(1-2k)^3$ in each issue, draws are independent across issues and the permutations that match opinions on x with opinions on y do not affect the result. Thus $\mathbb{P}(3 \text{ islands}) = (1-2k)^6$.

In all the other cases the society ends with either one or two islands and, contrary to the bonding process, it is possible to reach society-wide consensus in each of them. In particular, if all citizens are close (see Figure 7b) in at least one of the two issues, then society-wide consensus is reached with probability one, as this guarantees that any two of them can communicate successfully at any period. The probability of all citizens being close in at least one of the two issues is equal to $2(3k^2-2k^3)-(3-2k)^2k^4$.

For the remaining cases one island occurs with probability lower than one. For some of them, we provide their exact values, whereas for others we provide a lower bound.

We first present the results for the cases that care as in Figure 7(d), that is $x_2 - x_1 \leq k$ and $x_3 - x_2 > k$. Areas 5–8 on Figure 10 split these triplets into four areas within which the probabilities of reaching society-wide consensus are altered. Areas 7 and 8 are split into two parts each (7i, 7ii and 8i, 8ii respectively), where the additional splits are relevant only in some cases. Below we calculate explicitly the probability that three initial opinions in an issue belong to each of these areas. Areas 1–4 on Figure 10 correspond to draws as in Figure 7(c) and coincide with the areas presented in Figure 9, if one substitutes d with k, for which we have already calculated the probabilities of occurrence. Finally areas 9–13 on Figure 11 correspond to the case Figure 7(a) in which all opinions are far from each other and each of these areas is relevant in some cases. We also calculate explicitly their probabilities.

5. $x_2 - x_1 \leq k$, $x_3 - x_2 > k$, $x_3 - x_1 > k$ and $\frac{x_3+x_2}{2} - x_1 \leq k$:
   - $6\left(\int_0^{1-2k} \frac{k^2}{4} dx_1 + \int_{1-2k}^{1-\frac{3k}{2}} \frac{k^2}{4} - \frac{[x_1-(1-2k)]^2}{2} dx_1 + \int_{1-\frac{3k}{2}}^{1-k} \frac{[(1-k)-x_1]^2}{2} dx_1\right) = \frac{3}{4}(2-3k)k^2$
6. $x_2 - x_1 \leq k$, $x_3 - x_2 > k$, $x_3 - x_1 > k$, $\frac{x_3+x_2}{2} - x_1 > k$ and $\frac{1}{2}\left(x_3 - \frac{x_1+x_2}{2}\right) \leq k$:
   - $6\left(\int_0^{1-\frac{5k}{2}} \frac{k^2}{2} dx_1 + \int_{1-\frac{5k}{2}}^{1-2k} \frac{k^2}{2} - \left[x_1 - \left(1-\frac{5k}{2}\right)\right]^2 dx_1 + \int_{1-2k}^{1-\frac{3k}{2}} \left[\left(1-\frac{3k}{2}\right) - x_1\right]^2 dx_1\right) = 3(1-2k)k^2$
7. $x_2 - x_1 \leq k$, $x_3 - x_2 > k$, $x_3 - x_1 > k$, $\frac{1}{2}\left(x_3 - \frac{x_1+x_2}{2}\right) > k$ and $\frac{x_1+x_3}{2} - x_2 \leq k$:
   - (7) $6\left(\int_0^{1-4k} \frac{3k^2}{4} dx_1 + \int_{1-4k}^{1-\frac{5k}{2}} \frac{3k^2}{4} - \frac{[x_1-(1-4k)]^2}{4} dx_1 + \int_{1-\frac{5k}{2}}^{1-2k} \frac{3}{4}\left[(1-2k)-x_1\right]^2 dx_1\right) = \frac{3}{4}(6-17k)k^2$
   - (7ii) $6\left(\int_0^{1-4k} \frac{k^2}{6} dx_1 + \int_{1-4k}^{1-\frac{10k}{3}} \frac{k^2}{6} - \frac{[x_1-(1-4k)]^2}{4} dx_1 + \int_{1-\frac{10k}{3}}^{1-3k} \frac{1}{2}\left[(1-3k)-x_1\right]^2 dx_1\right) = \frac{1}{9}(9-31k)k^2$
   - (7i) $\frac{3}{4}(6-17k)k^2 - \frac{1}{9}(9-31k)k^2 = \frac{1}{36}(126-335k)k^2$
8. $x_2 - x_1 \leq k$, $x_3 - x_2 > k$, $x_3 - x_1 > k$ and $\frac{x_1+x_3}{2} - x_2 > k$:
   - (8) $6\left(\int_0^{1-4k} k^2 + k\left[(1-4k)-x_1\right] dx_1 + \int_{1-4k}^{1-2k} \frac{[(1-2k)-x_1]^2}{4} dx_1\right) = (3-18k+28k^2)k$

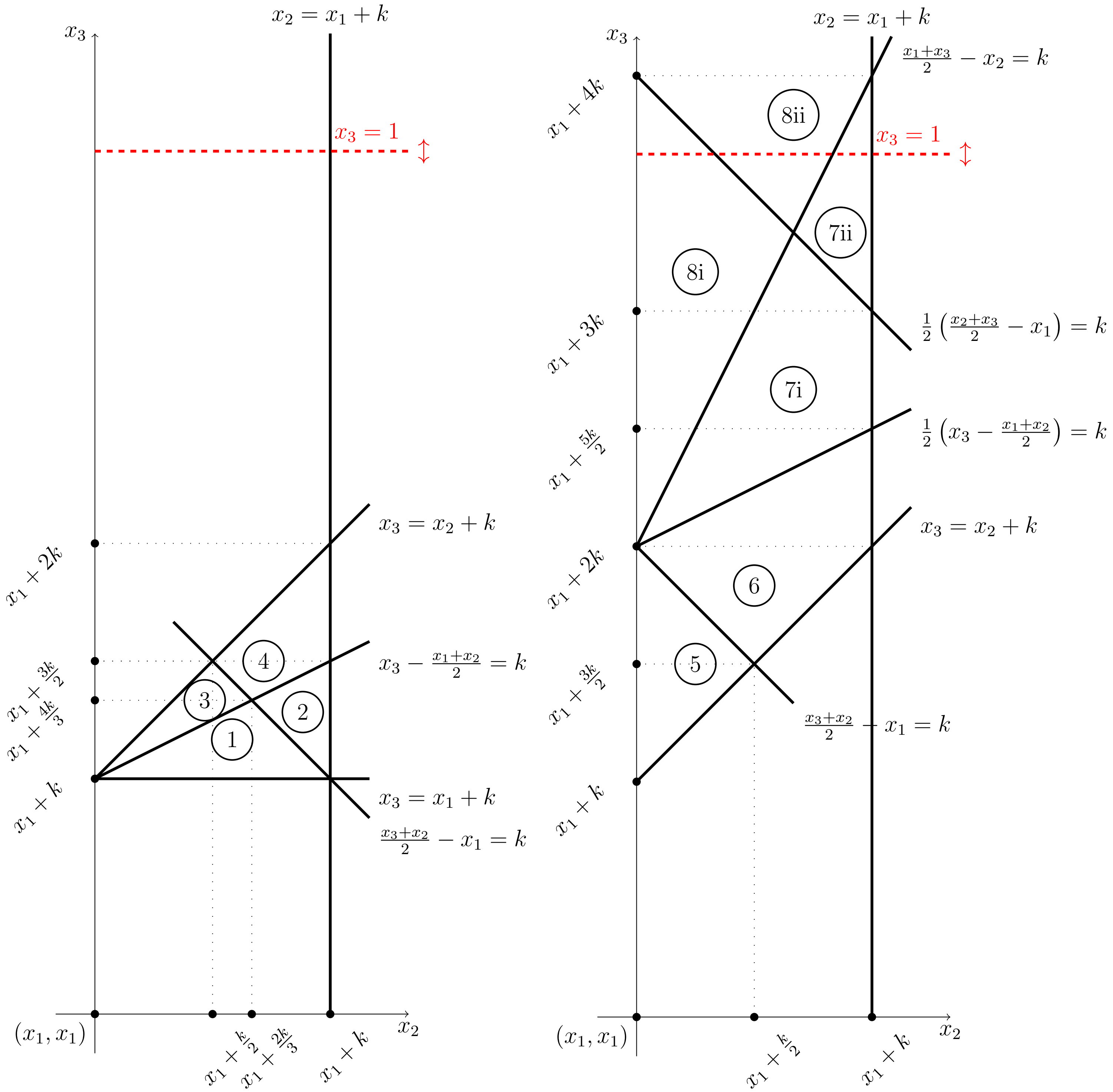


Figure 10: Two-dimensional projections of the following areas: (Left) $x_2 - x_1 \leq k$, $x_3 - x_2 \leq k$ and $x_3 - x_1 > k$, for some $x_1 \in [0, 1-k]$. Areas 1–4 correspond to the four different areas with respect to the conditions $x_3 - \frac{x_1+x_2}{2} \lessgtr k$ and $\frac{x_3+x_2}{2} - x_1 \lessgtr k$. (Right) $x_2 - x_1 \leq k$ and $x_3 - x_2 > k$, for some $x_1 \in [0, 1-k]$. Areas 5–8 correspond to the four different areas with respect to the conditions $\frac{x_3+x_2}{2} - x_1 \lessgtr k$, $\frac{1}{2}\left(x_3 - \frac{x_1+x_2}{2}\right) \lessgtr k$ and $\left|\frac{x_1+x_3}{2} - x_2\right| \lessgtr k$. For the last condition only $\frac{x_1+x_3}{2} - x_2 \lessgtr k$ is relevant. The origins of the axes are (x_1, x_1) in both subfigures and the location of the line where $x_3 = 1$ depends on x_1.

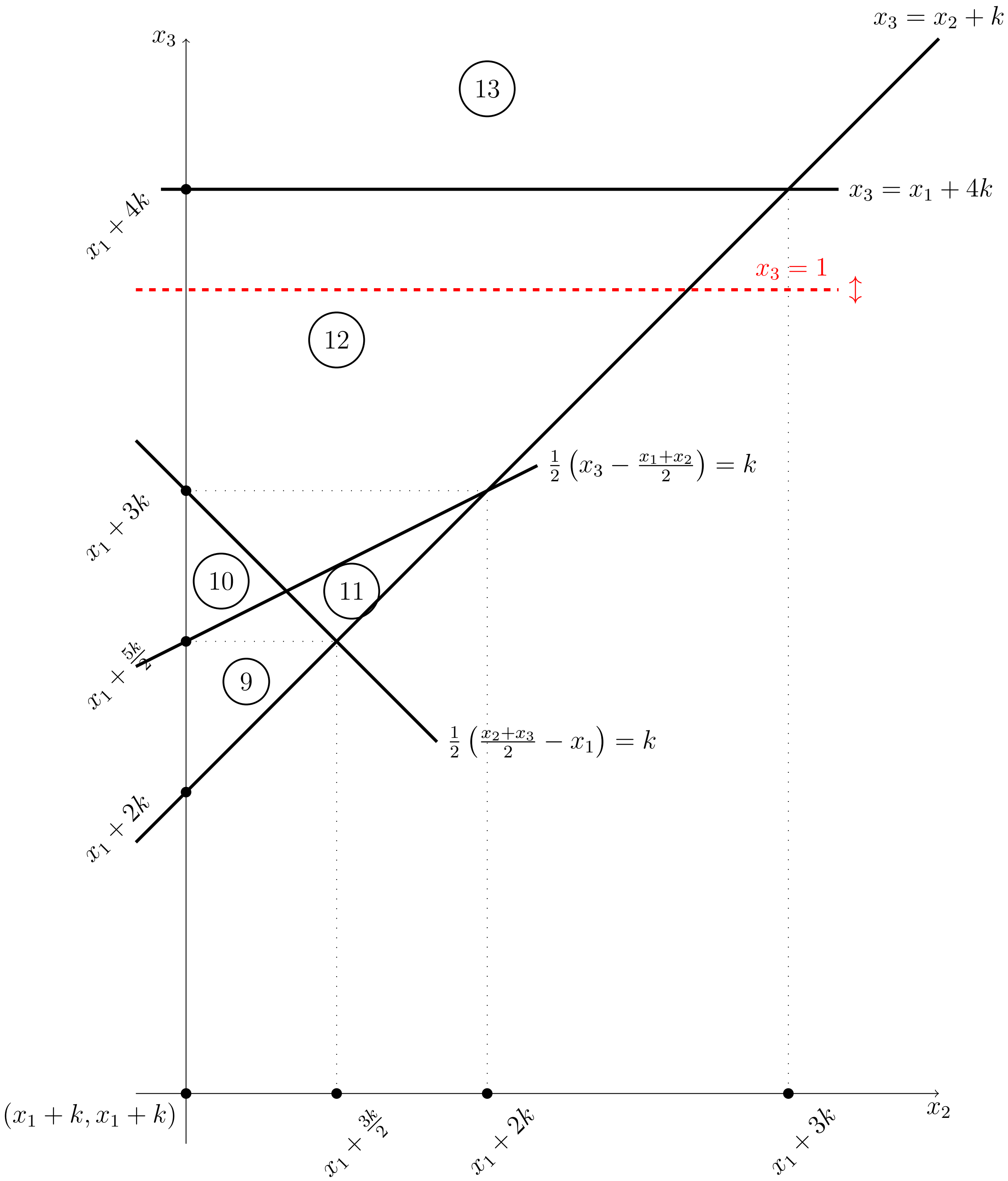


Figure 11: Two-dimensional projections of areas that satisfy $x_2 - x_1 > k$ and $x_3 - x_2 > k$. Areas 9–13 are obtained by taking into account the additional conditions $\frac{1}{2}\left(x_3 - \frac{x_1+x_2}{2}\right) \leqslant k$, $\frac{1}{2}\left(\frac{x_2+x_3}{2} - x_1\right) \leqslant k$ and $x_3 \leqslant x_1 + 4k$.

- (8ii) $6\left(\int_0^{1-4k}\frac{k^2}{3}+k\left[(1-4k)-x_1\right]dx_1+\int_{1-4k}^{1-\frac{10k}{3}}\frac{3\left[\left(1-\frac{10k}{3}\right)-x_1\right]^2}{4}dx_1\right)=(3-22k+\frac{364}{9}k^2)k$
- (8i) $(3-18k+28k^2)k-(3-22k+\frac{364}{9}k^2)k=\frac{4}{9}(9-28k)k^2$

9. $x_2-x_1>k$, $x_3-x_2>k$, $\frac{1}{2}\left(\frac{x_3+x_2}{2}-x_1\right)\leq k$ and $\frac{1}{2}\left(x_3-\frac{x_1+x_2}{2}\right)\leq k$:
   - $6\left(\int_0^{1-\frac{8k}{3}}\frac{k^2}{6}dx_1+\int_{1-\frac{8k}{3}}^{1-\frac{5k}{2}}\frac{k^2}{6}-\frac{3\left(x_1-(1-\frac{8k}{3})\right]^2}{2}dx_1+\int_{1-\frac{5k}{2}}^{1-2k}\frac{[(1-2k)-x_1]^2}{2}dx_1\right)=\frac{1}{18}(18-43k)k^2$
10. $x_2-x_1>k$, $x_3-x_2>k$, $\frac{1}{2}\left(\frac{x_3+x_2}{2}-x_1\right)\leq k$ and $\frac{1}{2}\left(x_3-\frac{x_1+x_2}{2}\right)>k$:
    - $6\left(\int_0^{1-3k}\frac{k^2}{12}dx_1+\int_{1-3k}^{1-\frac{8k}{3}}\frac{k^2}{12}-\frac{[(1-3k)-x_1]^2}{2}dx_1+\int_{1-\frac{8k}{3}}^{1-\frac{5k}{2}}\left[\left(1-\frac{5k}{2}\right)-x_1\right]^2dx_1\right)=\frac{1}{36}(18-49k)k^2$
11. $x_2-x_1>k$, $x_3-x_2>k$, $\frac{1}{2}\left(\frac{x_3+x_2}{2}-x_1\right)>k$ and $\frac{1}{2}\left(x_3-\frac{x_1+x_2}{2}\right)<k$:
    - $6\left(\int_0^{1-3k}\frac{k^2}{12}dx_1+\int_{1-3k}^{1-\frac{8k}{3}}\frac{k^2}{12}-\frac{[(1-3k)-x_1]^2}{2}dx_1+\int_{1-\frac{8k}{3}}^{1-\frac{5k}{2}}\left[\left(1-\frac{5k}{2}\right)-x_1\right]^2dx_1\right)=\frac{1}{36}(18-49k)k^2$
12. $x_2-x_1>k$, $x_3-x_2>k$, $\frac{1}{2}\left(\frac{x_3+x_2}{2}-x_1\right)>k$, $\frac{1}{2}\left(x_3-\frac{x_1+x_2}{2}\right)>k$ and $x_3\leq x_1+4k$:
    - $6\left(\int_0^{1-4k}\frac{5k^2}{3}dx_1+\int_{1-4k}^{1-3k}\frac{k^2}{6}+\frac{(1-k-x_1)(1-3k-x_1)}{2}dx_1+\int_{1-3k}^{1-\frac{8k}{3}}\frac{3\left(1-\frac{8k}{3}-x_1\right)^2}{2}\right)=\frac{2}{9}(45-157k)k^2$
13. $x_2-x_1>k$, $x_3-x_2>k$ and $x_3>x_1-4k$:
    - $6\int_0^{1-4k}\frac{(1-x_1)(1-x_1-4k)}{2}dx_1=1-6k+32k^3$

We can now calculate the probabilities of society-wide consensus for each of the cases.

- $\left(6-54k+178k^2-252k^3+\frac{392}{3}k^4\right)k^2$: 12 close, 23 far in x and all far in y
  - if $\left|\frac{y_1+y_3}{2}-y_2\right|\leq k$, with probability 1/3 –permutations (a,c,b) or (b,c,a)–.
  - same argument and probabilities hold for:
    * 12 far, 23 close in x and all far in y,
    * all far in x and 12 close, 23 far in y and
    * all far in x and 12 far, 23 close in y.
- $(10-51k+\frac{259}{3}k^2-49k^3)k^3$: 12 close, 23 far in x and 12 close, 23 far in y
- Case $(5,5)$, with probability 2/3.
- Cases $(5,6),(6,5)$ with probability 5/9.
- Cases $(5,7),(7,5),(6,6)$ with probability 4/9.
- Cases $(5,8),(8,5),(6,7),(7,6)$ with probability 1/3.
- Cases $(6,8),(8,6),(7,7)$ with probability 2/9.

- Cases $(7,8),(8,7)$ with probability 1/9.
- same argument and probabilities hold for:
  - 12 close, 23 far in x and 12 far, 23 close in y
  - 12 far, 23 close in x and 12 close, 23 far in y
  - 12 far, 23 close in x and 12 far, 23 close in y

- $> \left(\frac{58067}{8424} - \frac{3769369}{151362}k + \frac{22810741}{909792}k^2 + \frac{6168103}{1259712}k^3\right)k^3$: 12 close, 23 far in x and 12, 23 close, 13 far in y
  - For permutation (a,b,c) - Cases: $(\cdot,1)$ w.p. 1, $(5,2)$ w.p. 1, $(6,2)$ w.p. 5/6, $(7i,2)$ w.p. 5/6, $(8i,2)$ w.p. 5/6, $(7ii,2)$ w.p. 39/54, $(8ii,2)$ w.p. $> 55/81$, $(5,3)$ w.p. 2/3, $(6,3)$ w.p. 2/3, $(7i,3)$ w.p. 2/3, $(8i,3)$ w.p. 2/3, $(7ii,3)$ w.p. 43/72, $(8ii,3)$ w.p. $> 359/648$, $(5,4)$ w.p. 2/3, $(6,4)$ w.p. 1/2, $(7i,4)$ w.p. 7/18, $(8i,4)$ w.p. 7/18, $(7ii,4)$ w.p. 23/72, $(8ii,4)$ w.p. $> 161/468$.
  - For permutation (b,a,c) - Cases: $(\cdot,1)$ w.p. 1, $(5,2)$ w.p. 1, $(6,2)$ w.p. 11/12, $(7i,2)$ w.p. 29/36, $(8i,2)$ w.p. 13/18, $(7ii,2)$ w.p. 29/36, $(8ii,2)$ w.p. $> 55/81$, $(5,3)$ w.p. 2/3, $(6,3)$ w.p. 2/3, $(7i,3)$ w.p. 2/3, $(8i,3)$ w.p. 2/3, $(7ii,3)$ w.p. 2/3, $(8ii,3)$ w.p. $> 359/648$, $(5,4)$ w.p. 2/3, $(6,4)$ w.p. 7/12, $(7i,4)$ w.p. 17/36, $(8i,4)$ w.p. 7/18, $(7ii,4)$ w.p. 17/36, $(8ii,4)$ w.p. $> 161/468$.
  - For permutation (a,c,b) - Cases: $(\cdot,1)$ w.p. 1, $(5,2)$ w.p. 1, $(6,2)$ w.p. 8/9, $(7i,2)$ w.p. 91/108, $(8i,2)$ w.p. 91/108, $(7ii,2)$ w.p. 91/108, $(8ii,2)$ w.p. $> 733/972$, $(5,3)$ w.p. 1, $(6,3)$ w.p. 1, $(7i,3)$ w.p. 35/36, $(8i,3)$ w.p. 31/36, $(7ii,3)$ w.p. 25/27, $(8ii,3)$ w.p. $> 755/972$, $(5,4)$ w.p. 1, $(6,4)$ w.p. 8/9, $(7i,4)$ w.p. 85/108, $(8i,4)$ w.p. 73/108, $(7ii,4)$ w.p. 80/108, $(8ii,4)$ w.p. $> 14/27$.
  - By symmetry, these give us also the total probabilities for the other three permutations. (c,b,a) is analogous to (a,b,c), (b,c,a) is analogous to (b,a,c) and (c,a,b) to (a,c,b).
  - same argument and probabilities hold for:
    - 12 far, 23 close in x and 12, 23 close, 13 far in y
    - 12, 23 close, 13 far in x and 12 close, 23 far in y
    - 12, 23 close, 13 far in x and 12 far, 23 close in y

- $> \left(\frac{545}{324} - \frac{70663}{5832}k + \frac{75203}{1944}k^2 - \frac{811}{9}k^3 + \frac{22145}{288}k^4\right)k^2$: 12 close, 23 far in x and all far in y
  - For permutation (a,b,c) - Cases: $(\cdot,1)$ w.p. 1, $(9,2)$ w.p. 2/3, $(10,2)$ w.p. 43/72, $(11,2)$ w.p. 2/3, $(12,2)$ w.p. 43/72, $(13,2)$ w.p. $> 181/324$, $(9,3)$ w.p. 2/3, $(10,3)$ w.p. 2/3, $(11,3)$ w.p. 43/72, $(12,3)$ w.p. 43/72, $(13,3)$ w.p. $> 181/324$, $(9,4)$ w.p. 1/3, $(10,4)$ w.p. 19/72, $(11,4)$ w.p. 19/72, $(12,4)$ w.p. 7/36, $(13,4)$ w.p. $> 10/81$.

- For permutation (b,a,c) - Cases: $(\cdot,1)$ w.p. 1, $(9,2)$ w.p. 1, $(10,2)$ w.p. 67/72, $(11,2)$ w.p. 1, $(12,2)$ w.p. 67/72, $(13,2)$ w.p. $>181/324$, $(9,3)$ w.p. 2/3, $(10,3)$ w.p. 2/3, $(11,3)$ w.p. 2/3, $(12,3)$ w.p. 2/3, $(13,3)$ w.p. $>181/324$, $(9,4)$ w.p. 2/3, $(10,4)$ w.p. 43/72, $(11,4)$ w.p. 2/3, $(12,4)$ w.p. 43/72, $(13,4)$ w.p. $>10/81$.
- By symmetry, these give us also the total probabilities for the remaining permutations. (c,b,a) is analogous to (a,b,c), and all others are analogous (b,a,c) and (c,a,b).
- same argument and probabilities hold for:
  * all far in x and 12, 23 close, 13 far in y

- $\frac{1}{48}\left(412-1092k+723k^2\right)k^4$: 12, 23 close, 13 far in both x and y
  - Cases $(1,\cdot)$, $(\cdot,1)$ with probability 1.
  - Cases $(2,2)$, $(2,3)$, $(3,2)$, $(3,3)$ with probability 103/108.
  - Cases $(2,4)$, $(4,2)$, $(3,4)$, $(4,3)$ with probability 98/108.
  - Case $(4,4)$ with probability 88/108.

Given that these cases are mutually exclusive, we can simply add the probabilities (recalling that some of the cases appear multiple times and we get the lower bound:

$$\begin{aligned}\mathbb{P}(1\text{ island}) &> \left(\frac{5405}{162}-\frac{6696853}{37908}k+\frac{18405017}{37908}k^2-\frac{171347717}{227448}k^3+\frac{148623383}{314928}k^4\right)k^2 > \\ &> (33-177k+485k^2-754k^3+471k^4)k^2\end{aligned}$$

□

Proof of Proposition 4. The result is implied immediately by Lemma 1. $\mathbb{P}(1\text{ island})$ increases in d, whereas $\mathbb{P}(3\text{ islands})$ decreases in d, both for all $d\in(0,1/2]$. Thus, part (i) is obtained immediately and part (ii) is obtained by observing that $\mathbb{E}(\text{islands})=\mathbb{P}(1\text{ island})+2\mathbb{P}(2\text{ islands})+3\mathbb{P}(3\text{ islands})=2-\mathbb{P}(1\text{ island})+\mathbb{P}(3\text{ islands})$. □

Proof of Proposition 5. The result is implied immediately by Lemma 2. The lower bound of $\mathbb{P}(1\text{ island})$ increases in k, whereas $\mathbb{P}(3\text{ islands})$ decreases in k, both for all $k\in(0,1/4]$. Thus, part (i) is obtained immediately and part (ii) is obtained by recalling that $\mathbb{E}(\text{islands})=2-\mathbb{P}(1\text{ island})+\mathbb{P}(3\text{ islands})$, thus substituting the lower bound of $\mathbb{P}(1\text{ island})$ and the actual value of $\mathbb{P}(3\text{ islands})$ gives an upper bound of $\mathbb{E}(\text{islands})$, which then decreases in k. □

Proof of Proposition 6. The proofs of both parts are almost identical and both make use of the Intermediate Value Theorem.

We start by proving the result on the probability of society-wide consensus. For $k = 0$ the number of islands is equal to three with probability one.[24] Therefore, the probability of society-wide consensus in such a bridging process is zero, which is strictly lower than the respective probability for the bonding process with confidence bound d, for any $d \in (0, 1/4]$, which is always strictly positive. For $k = d$, we know from Lemmas 1 and 2 that the probability of ending up with one island in a bonding process with bound d is lower than the lower bound of the probability of ending up with one island in a bridging process with bound $k = d$, hence it is obviously also lower than the actual probability. Moreover, given that the lower bound is continuous in k (as a polynomial) and it is also strictly increasing for all $k \in (0, 1/4]$, we know from Intermediate Value Theorem that there will be some $\overline{k} \in (0, 1/4)$ such that the lower bound of the probability in the bridging process becomes equal to the probability in the bonding process with confidence bound d and for all $k > \overline{k}$ the lower bound, thus also the actual probability, in the bridging process will be strictly larger than the respective probability in the bonding process.

The arguments for the expected number of islands is analogous. For $k = 0$, the expected number of islands in the bridging process is equal to three, which is strictly higher than for that of the bonding process with confidence bound d, for any $d \in (0, 1/2]$. For $k = d$ we already mentioned that the probability of having 1 island in the bonding process with d is lower even than the lower bound of the probability of having 1 island in the bridging process with $k = d$. We also know (again from Lemmas 1 and 2) that the probability of having 3 islands is larger in the bonding than in the bridging process. Thus, recalling that $\mathbb{E}(\text{islands}) = 2 - \mathbb{P}(1 \text{ island}) + \mathbb{P}(3 \text{ islands})$, the expected number of islands in the bonding process is larger than in the bridging process. Therefore, given that the upper bound of the expected number of islands in the bridging process is continuous, again from Intermediate Value Theorem, there will be some $\tilde{k} \in (0, 1/4)$ such that the upper bound of the expected number of islands in the bridging process will be exactly equal to the expected number of islands in the bonding process with confidence bound d. Moreover, given that it is also strictly decreasing in k, for all $k > \tilde{k}$ the upper bound for the bridging process, thus also the actual expected number of islands, will be strictly smaller than the expected number of islands in the bonding process with confidence bound d. Both $\overline{k}$ and $\tilde{k}$ are in the interior of $(0, 1/4]$, hence both results hold simultaneously for any $k > \widehat{k} = \max\{\overline{k}, \tilde{k}\} \in (0, 1/4)$. □

Proof of Proposition 7. The result can be obtained immediately by simple calculations, using the values obtained in Lemmas 1 and 2. □

[24] In order to have two or fewer islands, at least two citizens must have exactly the same initial opinions, which has probability zero of occurring given the assumed distribution initial opinions are drawn from.

	Bonding					Bridging			
d	Mean	Median	Sd	Consensus	k	Mean	Median	Sd	Consensus
$n=3$									
0.1	2.981	3	0.318	0.002	0.01	2.886	3	0.33	0.004
0.2	2.608	3	0.543	0.028	0.04	2.564	3	0.576	0.043
0.3	2.276	2	0.625	0.095	0.1	2.035	2	0.672	0.209
0.4	1.954	2	0.656	0.239	0.2	1.572	2	0.599	0.485
$n=10$									
0.1	8.515	9	1.05	0	0.01	8.363	8	1.166	0
0.2	5.862	6	1.153	0	0.04	4.827	5	1.365	0.002
0.3	3.877	4	0.97	0.006	0.1	2.547	2	0.882	0.100
0.4	2.595	3	0.929	0.131	0.2	1.532	1	0.583	0.513
$n=50$									
0.1	24.249	24	2.448	0	0.01	15.342	15	3.236	0
0.2	8.002	8	1.345	0	0.04	5.65	6	1.302	0
0.3	3.908	4	0.87	0.009	0.1	3.123	3	0.852	0.017
0.4	2.089	2	0.86	0.267	0.2	1.817	2	0.616	0.297
$n=100$									
0.1	28.691	29	2.616	0	0.01	16.937	17	2.852	0
0.2	7.609	8	1.278	0	0.04	6.262	6	1.35	0
0.3	3.782	4	0.96	0.015	0.1	3.357	3	0.848	0.012
0.4	2.248	2	0.886	0.212	0.2	1.998	2	0.658	0.217

Table 4: Quadratic probability of successful communication. Mean, median and standard deviation of number of islands and frequency of society-wide consensus for the two processes for different population sizes.

	Bonding							Bridging					
d	μ_{ELF}	$\widetilde{ELF}$	σ_{ELF}	μ_{GI}	$\widetilde{GI}$	σ_{GI}	k	μ_{ELF}	$\widetilde{ELF}$	σ_{ELF}	μ_{GI}	$\widetilde{GI}$	σ_{GI}
$n=3$													
0.1	0.642	0.667	0.074	0.000	0.000	0.000	0.01	0.64	0.667	0.08	0.000	0.000	0.000
0.2	0.573	0.667	0.143	0.000	0.000	0.000	0.04	0.56	0.667	0.158	0.000	0.000	0.000
0.3	0.485	0.444	0.188	0.000	0.000	0.000	0.1	0.406	0.444	0.228	0.000	0.000	0.000
0.4	0.381	0.444	0.23	0.000	0.000	0.000	0.2	0.241	0.444	0.24	0.000	0.000	0.000
$n=10$													
0.1	0.866	0.88	0.032	0.005	0.005	0.001	0.01	0.86	0.86	0.037	0.004	0.004	0.001
0.2	0.771	0.78	0.068	0.004	0.004	0.001	0.04	0.68	0.72	0.139	0.003	0.003	0.001
0.3	0.621	0.66	0.131	0.004	0.004	0.001	0.1	0.383	0.42	0.201	0.002	0.002	0.001
0.4	0.414	0.46	0.222	0.003	0.003	0.002	0.2	0.136	0	0.161	0.001	0	0.001
$n=50$													
0.1	0.94	0.942	0.01	0.126	0.126	0.006	0.01	0.823	0.837	0.072	0.069	0.069	0.015
0.2	0.797	0.805	0.043	0.112	0.113	0.009	0.04	0.455	0.459	0.141	0.04	0.039	0.014
0.3	0.539	0.59	0.175	0.075	0.0806	0.027	0.1	0.189	0.183	0.111	0.02	0.018	0.012
0.4	0.092	0.039	0.117	0.013	0.006	0.017	0.2	0.045	0.039	0.043	0.006	0.005	0.006
$n=100$													
0.1	0.945	0.946	0.007	0.507	0.507	0.018	0.01	0.685	0.69	0.104	0.191	0.191	0.038
0.2	0.784	0.787	0.041	0.432	0.433	0.029	0.04	0.338	0.328	0.134	0.109	0.105	0.041
0.3	0.462	0.523	0.213	0.241	0.266	0.12	0.1	0.125	0.114	0.072	0.051	0.048	0.029
0.4	0.04	0.039	0.041	0.021	0.018	0.022	0.2	0.032	0.02	0.027	0.017	0.012	0.014

Table 5: Quadratic probability of successful communication. Fractionalization Indices – μ_x, $\widetilde{x}$ and σ_x correspond to the mean, median and standard deviation of the two fractionalization indices respectively, for the two processes and for different populations sizes. The values denoted by 0.000 refer to quantities that are positive but lower than 10^{-3}, whereas the values denoted by 0 refer to quantities with value exactly equal to zero.

References

Acemoglu, D., G. Como, F. Fagnani, and A. Ozdaglar (2013). Opinion fluctuations and disagreement in social networks. *Mathematics of Operations Research 38*(1), 1–27.

Acemoglu, D. and A. Ozdaglar (2011, Mar). Opinion dynamics and learning in social networks. *Dynamic Games and Applications 1*(1), 3–49.

Axelrod, R. (1997). The Dissemination of Culture. A Model with Local Convergence and Global Polarization. *Journal of Conflict Resolution 41*(2), 203–226.

Bala, V. and S. Goyal (1998). Learning from neighbours. *The Review of Economic Studies 65*(3), 595–621.

Banerjee, A. and D. Fudenberg (2004). Word-of-mouth learning. *Games and Economic Behavior 46*(1), 1 – 22.

Ben-Naim, E., P. L. Krapivsky, and S. Redner (2003). Bifurcations and patterns in compromise processes. *Physica D: Nonlinear Phenomena 183*(3-4), 190–204.

Bjørnskov, C. (2006). The multiple facets of social capital. *European Journal of Political Economy 22*, 22–40.

Blondel, V. D., J. M. Hendrickx, and J. N. Tsitsiklis (2007, July). On the 2r conjecture for multi-agent systems. In *2007 European Control Conference (ECC)*, pp. 874–881.

Borisova, E., A. Govorun, and D. Ivanov (2015). Bridging or Bonding? Preferences for Redistribution and Social Capital in Russia. *Mimeo*.

Coffé, H. and B. Geys (2008). Measuring the bridging nature of voluntary organizations: The importance of association size. *Sociology 42*(2), 357–369.

Dandekar, P., A. Goel, and D. T. Lee (2013). Biased assimilation, homophily, and the dynamics of polarization. *Proceedings of the National Academy of Sciences 110*(15), 5791–5796.

Deffuant, G., D. Neau, F. Amblard, and G. Weisbuch (2000). Mixing beliefs among interacting agents. *Advances in Complex Systems 03*(01n04), 87–98.

DeGroot, M. H. (1974). Reaching a Consensus. *Journal of the American Statistical Association 69*(345), 118–121.

DeMarzo, P. M., D. Vayanos, and J. Zwiebel (2003). Persuasion bias, social influence, and uni-dimensional opinions. *The Quarterly Journal of Economics 118*(3), 909–968.

Durlauf, S. N. and M. Fafchamps (2005). Social Capital. In P. Aghion and S. N. Durlauf (Eds.), *Handbook of Economic Growth. Vol 1, Part B.*, pp. 1639–99. Amsterdam: Elsevier.

Fortunato, S., V. Latora, A. Pluchino, and A. Rapisarda (2005). Vector Opinion Dynamics in a Bounded Confidence Consensus Model. *International Journal of Modern Physics C 16*(10), 1535–1551.

Friedkin, N. E. and E. C. Johnsen (1990). Social influence and opinions. *The Journal of Mathematical Sociology 15*(3-4), 193–206.

Gale, D. and S. Kariv (2003). Bayesian learning in social networks. *Games and Economic Behavior 45*(2), 329–346.

Geys, B. and Z. Murdoch (2008). How to make head or tail of 'bridging' and 'bonding'?: addressing the methodological ambiguity. *The British Journal of Sociology 59*(3), 435–454.

Geys, B. and Z. Murdoch (2010). Measuring the 'Bridging' versus 'Bonding' Nature of Social Networks: A Proposal for Integrating Existing Measures. *Sociology 44*(3), 523–540.

Golub, B. and M. O. Jackson (2010). Naïve Learning in Social Networks and the Wisdom of Crowds. *American Economic Journal: Microeconomics 2*(1), 112–149.

Golub, B. and M. O. Jackson (2012). How homophily affects the speed of learning and best-response dynamics. *The Quarterly Journal of Economics 127*(3), 1287–1338.

Golub, B. and E. Sadler (2017). Learning in social networks. In B. Yann, A. Galeotti, and B. Rogers (Eds.), *Oxford Handbook of the Economics of Networks*. Oxford University Press.

Granovetter, M. S. (1973). The strength of weak ties. *American Journal of Sociology 78*(6), 1360–1380.

Guiso, L., P. Sapienza, and L. Zingales (2004). The Role Of Social Capital In Financial Development. *American Economic Review 94*(3), 526–556.

Hegselmann, R. and U. Krause (2002). Opinion Dynamics and Bounded Confidence Models, Analysis and Simulation. *Journal of Artificial Societies and Social Simulation 5*(3).

Holme, P. and M. E. Newman (2006). Nonequilibrium phase transition in the coevolution of networks and opinions. *Physical Review E 74*(5), 056108.

Iijima, R. and Y. Kamada (2017). Social distance and network structures. *Theoretical Economics 12*(2), 655–689.

Knack, S. and P. Keefer (1997). Does social capital have an economic pay-off? A Cross-Country Investigation. *The Quartery Journal of Economics 112*(4), 1251–1288.

Krause, U. (2000). A discrete nonlinear and non-autonomous model of consensus formation. *Communications in difference equations* (1), 227–236.

Kurahashi-Nakamura, T., M. Mäs, and J. Lorenz (2016). Robust clustering in generalized bounded confidence models. *Journal of Artificial Societies and Social Simulation 19*(4), 7.

Lorenz, J. (2003). Multidimensional Opinion Dynamics when Confidence Changes. pp. 107–112.

Lorenz, J. (2005). A stabilization theorem for dynamics of continuous opinions. *Physica A: Statistical Mechanics and its Applications 355*(1), 217–223.

Lorenz, J. (2007). Continuous Opinion Dynamics Under Bounded Confidence: A Survey. *International Journal of Modern Physics C 18*(12).

Lorenz, J. (2008). *Fostering Consensus in Multidimensional Continuous Opinion Dynamics* *under Bounded Confidence*, pp. 321–334. Berlin, Heidelberg: Springer Berlin Heidelberg.

Lorenz, J. (2017). *Modeling the Evolution of Ideological Landscapes Through Opinion Dynamics*, pp. 255–266. Cham: Springer International Publishing.

Louis, P., O. Troumpounis, and N. Tsakas (2017). Communication and the emergence of a unidimensional world. *SSRN Working Paper*.

Melguizo, I. (2018). Homophily and the persistence of disagreement. *The Economic Journal 129*(619), 1400–1424.

Mueller-Frank, M. (2013). A general framework for rational learning in social networks. *Theoretical Economics 8*(1), 1–40.

Mueller-Frank, M. (2015). Reaching consensus in social networks. *SSRN Working Paper*.

Putnam, R. D. (1995). Bowling Alone: America's Declining Social Capital. *Journal of Democracy 6*(1), 65–78.

Putnam, R. D. (2000). *Bowling Alone: The Collapse and Revival of American Community.* New York: Simon & Schuster.

Satyanath, S., N. Voigtländer, and H.-J. Voth (2017). Bowling for Fascism: Social Capital and the Rise of the Nazi Party. *Journal of Political Economy 125*(2), 478–526.

Tabellini, G. (2010). Culture and Institutions: Economic Development in the Regions of Europe. *Journal of the European Economic Association 8*(4), 677–716.