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FINITENESS FOR HECKE ALGEBRAS OF p-ADIC GROUPS

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ABSTRACT. Let G be a reductive group over a non-archimedean local field F of residue characteristic p. We prove that the Hecke algebras of $G(F)$ with coefficients in any noetherian $\mathbb{Z}_\ell$-algebra R with $\ell \neq p$, are finitely generated modules over their centers, and that these centers are finitely generated R-algebras. Following Bernstein's original strategy, we then deduce that "second adjointness" holds for smooth representations of $G(F)$ with coefficients in any $\mathbb{Z}[\frac{1}{p}]$-algebra. These results had been conjectured for a long time. The crucial new tool that unlocks the problem is the Fargues-Scholze morphism between a certain "excursion algebra" defined on the Langlands parameters side and the Bernstein center of $G(F)$. Using this bridge, our main results are representation theoretic counterparts of the finiteness of certain morphisms between coarse moduli spaces of local Langlands parameters that we also prove here, which may be of independent interest.

CONTENTS

1. MAIN RESULTS

Let $\mathbb{G}$ be a reductive group over a non-archimedean local field F of residue characteristic p. The group $G := \mathbb{G}(F)$ is then locally profinite, hence for any open compact subgroup H of G, the free abelian group $\mathbb{Z}[H\backslash G/H]$ carries a structure of an associative ring, called a Hecke ring. One of the main results of this paper is the following statement.

Theorem 1.1. *For any prime $\ell \neq p$ and any noetherian $\mathbb{Z}_\ell$-algebra R, the base change $R[H\backslash G/H]$ is a finitely generated module over its center, which is a finitely generated (commutative) R-algebra.*

Here is an equivalent formulation in terms of smooth representations. Denote by $\mathrm{Rep}_R(G)$ the category of all smooth RG-modules and by $\mathfrak{Z}_R(G)$ the center of this category. We define an RG-module to be $\mathfrak{Z}$-*finite* if

- the image of $\mathfrak{Z}_R(G) \longrightarrow \mathrm{End}_{RG}(V)$ is a finitely generated R-algebra, and
- V is admissible over $\mathfrak{Z}_R(G)$, i.e. V^H is a finitely generated $\mathfrak{Z}_R(G)$-module for any compact open subgroup H of G.

Then the above theorem is equivalent to the following one (see Lemma 3.2).

Theorem 1.2. *(Same hypothesis on R.) Any finitely generated smooth RG-module V is $\mathfrak{Z}$-finite.*

When $R = \overline{\mathbb{Q}}_\ell$, these statements are famous theorems of Bernstein. For $R = \mathbb{Z}_\ell$ or $R = \mathbb{F}_\ell$, the only previously known case was for $G = \mathrm{GL}_n$ in [Hel16], and the proof there relies on very specific features of GL_n such as Vigneras' "uniqueness of supercuspidal support". There have been also partial results

for more general groups. For example, in [Dat09], the fact that $R[H\backslash G/H]$ is a noetherian ring (albeit with no control on the center) was proved to be implied by the so-called "second adjointness" between parabolic functors, and the latter property was established for groups having a suitable form of type theory, such as classical groups or "very tame" groups. However, this second adjointness property was first discovered by Bernstein for complex representations as a consequence of his theorem on finiteness of Hecke algebras. Following his argument, we also prove :

Corollary 1.3 (Second adjointness). *For any $\mathbb{Z}[\frac{1}{p}]$-algebra R, and for all pairs of opposite parabolic subgroups $(P, \bar{P})$ in G with common Levi component $M = P \cap \bar{P}$, the twisted opposite Jacquet functor $\delta_P.\mathrm{R}_{\bar{P}} : \mathrm{Rep}_R(G) \longrightarrow \mathrm{Rep}_R(M)$ is right adjoint to the parabolic induction functor $\mathrm{I}_P : \mathrm{Rep}_R(M) \longrightarrow \mathrm{Rep}_R(G)$, where $\delta_P : M \longrightarrow \mathbb{Z}[\frac{1}{p}]^\times$ denotes the modulus character of P.*

Note that when one can fix a square root of q in R, the parabolic functors can be normalized by putting $i_P := \mathrm{I}_P \delta_P^{\frac{1}{2}}$ and $r_P := \delta_P^{-\frac{1}{2}}.\mathrm{R}_P$, and the twist by the modulus character disappears from the statement of Corollary 1.3, as in Bernstein's original result.

It is perhaps surprising that we can deduce second adjointness over $\mathbb{Z}[\frac{1}{p}]$ from finiteness results that we can only establish over $\mathbb{Z}_\ell$ for $\ell \neq p$. The key point is to deduce a certain "stability" property for objects of $\mathrm{Rep}_{\mathbb{Z}[\frac{1}{p}]}(G)$; we reduce this problem to establishing "stability" for certain injective objects I_ℓ of $\mathrm{Rep}_{\mathbb{Z}[\frac{1}{p}]}(G)$ that naturally have the additional structure of a $\mathbb{Z}_\ell$-module, allowing us to apply the finiteness results we have proven over $\mathbb{Z}_\ell$.

Let us quote three further consequences, for which only partial results have been known so far.

Corollary 1.4. *For any noetherian $\mathbb{Z}[\frac{1}{p}]$-algebra R, and any compact open subgroup H of G, the Hecke algebra $R[H\backslash G/H]$ is noetherian.*

Our arguments in this paper fall short of establishing that the rings $R[H\backslash G/H]$ are finitely generated over their centers when R is not a $\mathbb{Z}_\ell$-algebra. We will address this question in forthcoming work. Indeed, we expect to be able to prove this, for R an arbitrary noetherian $\mathbb{Z}[\frac{1}{p}]$-algebra, by first proving it over $\mathbb{Z}[\frac{1}{pN}]$ for some integer N, and then applying flat descent (see Lemma 3.3 (2)) to the map $\mathbb{Z}[\frac{1}{p}] \longrightarrow \mathbb{Z}[\frac{1}{pN}] \times \prod_{\ell | N} \mathbb{Z}_\ell$. Here N will be the l.c.m. of the orders of torsion elements in G. In this "banal" setting, with the help of Corollary 1.3, one can construct quite explicit projective objects of $\mathrm{Rep}_R(G)$ and thus one has good control over the center $\mathfrak{Z}_R(G)$.

Corollary 1.5. *Let P be a parabolic subgroup with Levi component M, and let R be a noetherian $\mathbb{Z}[\frac{1}{p}]$-algebra.*

    (1) The parabolic induction functor I_P takes projective, resp. finitely generated, smooth RM-modules to projective, resp. finitely generated, smooth RG-modules.

    (2) The Jacquet functor R_P takes admissible RG-modules to admissible RM-modules.

    (3) Suppose R is an algebraically closed field and let σ be an irreducible RM-module.

        (a) The representation $i_P(\sigma\psi)$ is irreducible for ψ in a Zariski-dense open subset of the R-torus of unramified characters of M.

        (b) If Q is an other parabolic subgroup of G with Levi component M, we have $[i_P(\sigma)] = [i_Q(\sigma)]$ in the Grothendieck group of finite length RG-modules.

In turn, item (2) above is the main ingredient to prove the next corollary, which was not known for general groups even when ℓ is a banal prime (that is, when $\bar{\ell}$ is a prime not dividing the pro-order of any compact open subgroup of G).

Corollary 1.6. *An irreducible $\overline{\mathbb{Q}}_\ell G$-representation is integral (i.e. admits an admissible G-stable $\overline{\mathbb{Z}}_\ell$-lattice) if and only if its supercuspidal support is integral.*

Let us outline our strategy to prove Theorems 1.1 and 1.2. The details are given in Section 3.

We first observe that it is enough to prove them in the case where R is $\mathbb{Z}_\ell$ or any finite flat extension of $\mathbb{Z}_\ell$. For several reasons, it will be convenient to have a fixed square root of q in R, so we will work over $\mathbb{Z}'_\ell := \mathbb{Z}_\ell[\sqrt{q}]$ where $\sqrt{q} \in \overline{\mathbb{Q}}_\ell$ is fixed, and we will use normalized parabolic functors i_P and r_P.

Since $\mathfrak{Z}$-finite objects are stable under taking finite direct products and quotients, in order to prove Theorem 1.2, it is enough to prove that each $\mathbb{Z}'_\ell G$-module of the form $\mathbb{Z}'_\ell[G/H]$ for some open pro-p-subgroup H of G is $\mathfrak{Z}$-finite. For such an H, using Bernstein's decomposition over $\overline{\mathbb{Q}}_\ell$, we show that $\mathbb{Z}'_\ell[G/H]$ can be embedded in a $\mathbb{Z}'_\ell G$-module of the form $V_H = \bigoplus_P i_P(W_P)$, where $P = U_P M_P$ runs through a finite set of parabolic subgroups and W_P is a *cuspidal* finitely generated $\mathbb{Z}'_\ell M_P$-module. Since $\mathfrak{Z}$-finite objects are stable under taking subobjects, it is enough to prove that V_H is $\mathfrak{Z}$-finite.

So we are left with proving that any V of the form $i_P(W)$ for some finitely generated *cuspidal* $\mathbb{Z}'_\ell M$-module W, is $\mathfrak{Z}$-finite (even though, at this point, it is not even clear whether such a V is finitely generated). Now, by [Dat09, Lemme 4.2], we know that such a W is admissible over $\mathfrak{Z}_{\mathbb{Z}'_\ell}(Z_M)$, where Z_M denotes the maximal central torus of M.

This is where Fargues and Scholze's local version of V. Lafforgue's theory of excursion operators comes in. Denote by $\hat{G}$ the dual pinned reductive group scheme of $\mathbb{G}$ over $\mathbb{Z}[\frac{1}{p}]$, endowed with its pinning-preserving action of the Weil group W_F of F. Recall the subgroup W_F^0 of W_F obtained in [DHKM20] by "discretizing tame inertia", and choose a separated decreasing filtration $(P_F^e)_{e\in\mathbb{N}}$ of the wild inertia subgroup $P_F \subset W_F$ by normal subgroups. The "excursion" $\mathbb{Z}[\frac{1}{p}]$-algebra is defined as

$$\mathrm{Exc}(W_F, \hat{G}) := \lim_e \mathrm{Exc}(W_F^0/P_F^e, \hat{G}) \text{ where } \mathrm{Exc}(W_F^0/P_F^e, \hat{G}) := \mathrm{colim}_{n, F_n \to W_F^0/P_F^e} \mathcal{O}(Z^1(F_n, \hat{G}))^{\hat{G}}$$

and F_n denotes a free group on n generators. It can be thought of as the ring of functions on the space of "$\hat{G}$-valued continuous pseudo-characters" of W_F^0. Moreover, each $\mathrm{Exc}(W_F^0/P_F^e, \hat{G})$ is known to be a finitely generated (commutative) $\mathbb{Z}[\frac{1}{p}]$-algebra. Using Hecke operators on spaces of G-bundles on the Fargues-Fontaine curve, Fargues and Scholze have constructed in [FS21, Ch. IX] a map of $\mathbb{Z}'_\ell$-algebras

$$\mathrm{FS}_G : \mathrm{Exc}(W_F, \hat{\mathbb{G}})_{\mathbb{Z}'_\ell} \longrightarrow \mathfrak{Z}_{\mathbb{Z}'_\ell}(G)$$

that enjoys the following important properties (details and references are given in section 3.1) :

(1) Compatibility with parabolic induction : for any parabolic subgroup P with Levi component M and any $\mathbb{Z}'_\ell M$-module W, the following diagram is commutative

$$\begin{array}{ccc} \mathrm{Exc}(W_F, \hat{G})_{\mathbb{Z}'_\ell} & \xrightarrow{\mathrm{FS}_G} & \mathrm{End}_{\mathbb{Z}'_\ell G}(i_P W) \\ \downarrow & & \uparrow{\scriptstyle i_P} \\ \mathrm{Exc}(W_F, \hat{M})_{\mathbb{Z}'_\ell} & \xrightarrow{\mathrm{FS}_M} & \mathrm{End}_{\mathbb{Z}'_\ell M}(W) \end{array}$$

where the left vertical map is given by pushforward of 1-cocycles along $\hat{M} \hookrightarrow \hat{G}$.

(2) Compatibility with central characters : upon identifying $\widehat{Z_M}$ with $\hat{M}_{\mathrm{ab}}$, the following diagram commutes :

$$\begin{array}{ccc} \mathrm{Exc}(W_F, \hat{M})_{\mathbb{Z}'_\ell} & \xrightarrow{\mathrm{FS}_M} & \mathfrak{Z}_{\mathbb{Z}'_\ell}(M) \\ \uparrow & & \uparrow \\ \mathrm{Exc}(W_F, \hat{M}_{\mathrm{ab}})_{\mathbb{Z}'_\ell} & \xrightarrow{\mathrm{FS}_{Z_M}} & \mathfrak{Z}_{\mathbb{Z}'_\ell}(Z_M) \end{array} \quad .$$

(3) Isomorphism for tori : the map $\mathrm{FS}_{Z_M} : \mathrm{Exc}(W_F, \hat{M}_{\mathrm{ab}})_{\mathbb{Z}'_\ell} \longrightarrow \mathfrak{Z}_{\mathbb{Z}'_\ell}(Z_M)$ is an isomorphism.

(4) Continuity : for any finitely generated $\mathbb{Z}'_\ell M$-module W, the map $\mathrm{Exc}(W_F, \hat{M})_{\mathbb{Z}'_\ell} \longrightarrow \mathrm{End}_{\mathbb{Z}'_\ell M}(W)$ factors over $\mathrm{Exc}(W_F^0/P_F^e, \hat{M})_{\mathbb{Z}'_\ell}$ for some e.

Let us return to our representation $V = i_P(W)$ with W a finitely generated cuspidal $\mathbb{Z}'_\ell M$-module. Since W is $\mathfrak{Z}_{\mathbb{Z}'_\ell}(Z_M)$-admissible, the last three properties above show that W is admissible over $\mathrm{Exc}(W_F^0/P_F^e, \hat{M})_{\mathbb{Z}'_\ell}$ for some $e \in \mathbb{N}$. Since parabolic induction preserves admissibility, property (1) above shows that, in order to prove that V is $\mathfrak{Z}$-finite (and thus conclude the proof of Theorem 1.2), it suffices to prove that the map $\mathrm{Exc}(W_F^0/P_F^e, \hat{G}) \longrightarrow \mathrm{Exc}(W_F^0/P_F^e, \hat{M})$ is finite.

This leads us to our results on the dual side. Instead of excursion algebras, we will first focus on the moduli space of Langlands parameters $Z^1(W_F^0/P_F^e, \hat{G})$ over $\mathbb{Z}[\frac{1}{p}]$, as defined in [DHKM20] and its GIT quotient. Let us pick a lift of Frobenius Fr in W_F^0 and let $\bar{\text{Fr}}$ denote its image in $\text{Aut}(\hat{G})$. We write $\hat{G} \rtimes \bar{\text{Fr}}$ to denote $\hat{G}$ with the $\bar{\text{Fr}}$-twisted conjugacy action of $\hat{G}$. Our main result in this context is :

Theorem 1.7. *For any $e \geq 0$, the map $Z^1(W_F^0/P_F^e, \hat{G}) \longrightarrow \hat{G}$, $\varphi \mapsto \varphi(\text{Fr})$ induces a finite morphism $Z^1(W_F^0/P_F^e, \hat{G}) /\!\!/ \hat{G} \longrightarrow \hat{G} \rtimes \bar{\text{Fr}} /\!\!/ \hat{G}$.*

From this result, we will easily deduce the following corollary.

Corollary 1.8. *Suppose $\hat{H} \subset \hat{G}$ is a closed reductive subgroup scheme of $\hat{G}$ stable under W_F. Then*
  (1) the natural morphism $Z^1(W_F^0/P_F^e, \hat{H}) /\!\!/ \hat{H} \longrightarrow Z^1(W_F^0/P_F^e, \hat{G}) /\!\!/ \hat{G}$ is finite.
  (2) the natural morphism of algebras $\text{Exc}(W_F^0/P_F^e, \hat{G})_{\text{red}} \longrightarrow \text{Exc}(W_F^0/P_F^e, \hat{H})_{\text{red}}$ is finite.

In (2) above, the subscript red denotes the reduced quotient. Since $\mathfrak{Z}_{\mathbb{Z}_\ell'}(G)$ is a subring of $\mathfrak{Z}_{\overline{\mathbb{Q}}_\ell}(G)$, it is reduced, by Bernstein's description. Hence the Fargues-Scholze morphism FS_G factors through the reduced excursion algebra anyway, and this version of finiteness is sufficient to finish the proof of Theorems 1.2 and 1.1.

We also mention the following corollary of Theorem 1.7, which may have independent interest.

Corollary 1.9. *Let F' be a finite extension of F and put $P_{F'}^e := P_F^e \cap W_{F'}$. Then the restriction morphism $Z^1(W_F^0/P_F^e, \hat{G}) /\!\!/ \hat{G} \longrightarrow Z^1(W_{F'}^0/P_{F'}^e, \hat{G}) /\!\!/ \hat{G}$ is finite.*

These results are proved in Section 2. The proofs of the finiteness results on the group side are given in Section 3.

2. Finiteness on the parameters side

In this section, R denotes a ring of the form $\bar{\mathbb{Z}}[\frac{1}{N}]$, where N is any positive integer. Recall that any reductive group scheme over R is split (Prop A.13 of [DHKM20]). For a diagonalizable group scheme D over R, we denote by D^0 its maximal subtorus (which may not be the maximal connected subgroup scheme). The following technical lemma makes working over R a bit more convenient than working over finite extensions of $\mathbb{Z}[\frac{1}{N}]$.

Lemma 2.1. *Let D be a diagonalizable group scheme over R acting on an R-scheme X, and let $X \xrightarrow{\varphi} Y$ be a relative D-torsor for the fppf topology. Then the map $X(R) \longrightarrow Y(R)$ is surjective.*

Proof. The fiber over a point $y \in Y(R)$ is a D-torsor over $S := \text{Spec}(R)$. So we need to show that any D-torsor over S is trivial. Writing the character group $X^*(D)$ as a product of cyclic abelian groups, we may assume that D is $\mathbb{G}_m$ or μ_m for some $m \in \mathbb{N}^*$. In the case $D = \mathbb{G}_m$, we have $H^1_{fppf}(S, \mathbb{G}_m) = \text{Pic}(S) = \text{colim}_{K \subset \overline{\mathbb{Q}}} \text{Pic}(\mathcal{O}_K[\frac{1}{N}]) = \{1\}$ since, for any number fields $K \subset K'$, the map $\text{Pic}(\mathcal{O}_K[\frac{1}{N}]) \longrightarrow \text{Pic}(\mathcal{O}_{K'}[\frac{1}{N}])$ is trivial whenever K' contains the Hilbert class field of K. In the case $D = \mu_m$, the exact sequence $R^\times \xrightarrow{(-)^m} R^\times \longrightarrow H^1_{fppf}(S, \mu_m) \longrightarrow \text{Pic}(R) \xrightarrow{(-)^m} \text{Pic}(R)$ and the surjectivity of $R^\times \xrightarrow{(-)^m} R^\times$ show that $H^1_{fppf}(S, \mu_m) = \{1\}$ as desired. $\square$

Lemma 2.2. *Let $\hat{G}$ be a reductive group scheme over R and θ an automorphism of $\hat{G}$ with finite order. Let $\hat{H}$ be a reductive subgroup scheme of $\hat{G}$ over R, and suppose it is $\text{Int}_g \circ \theta$-stable for some $g \in \hat{G}(R)$, where Int_g denotes the automorphism of $\hat{G}$ given by conjugation by g. Then the canonical morphism $\hat{H}g \rtimes \theta /\!\!/ \hat{H} \longrightarrow \hat{G} \rtimes \theta /\!\!/ \hat{G}$ is finite.*

Proof. After maybe multiplying g by some element $h \in \hat{H}(R)$ on the left, we may assume that $\text{Int}_g \circ \theta$ stabilizes a pinning $(T_{\hat{H}}, B_{\hat{H}}, X_{\hat{H}})$ in $\hat{H}$; indeed, the set of such h is the set of R-points of a $Z(\hat{H})$-torsor, hence it is not empty by the above lemma. Then, the "twisted version" of the Chevalley-Steinberg theorem (see e.g. Prop 6.6 in [DHKM20]) implies that the inclusion $T_{\hat{H}} \subset \hat{H}$ induces a finite

and surjective morphism $(T_{\hat{H}})_{g\theta} \longrightarrow \hat{H}g \rtimes \theta \sslash \hat{H}$, where $(T_{\hat{H}})_{g\theta}$ denotes the co-invariants of $T_{\hat{H}}$ under $\mathrm{Int}_g \circ \theta$. Note that the restriction of $\mathrm{Int}_g \circ \theta$ to $T_{\hat{H}}$ has finite order since the centralizer of $T_{\hat{H}}$ in $\hat{G}$ has finite index in its normalizer. Denoting by $S := (T_{\hat{H}})^{g\theta,0}$ the maximal subtorus of $T_{\hat{H}}$ that is fixed under $\mathrm{Int}_g \circ \theta$, it follows that the inclusion $S \subset \hat{H}$ also induces a finite surjective morphism $Sg \rtimes \theta \longrightarrow \hat{H}g \rtimes \theta \sslash \hat{H}$. In particular, the ring $\mathcal{O}(\hat{H}g \rtimes \theta)^{\hat{H}}$, which is reduced, embeds in $\mathcal{O}(Sg \rtimes \theta)$.

Therefore, if we can prove that $\mathcal{O}(Sg \rtimes \theta)$ is finite over $\mathcal{O}(\hat{G} \rtimes \theta)^{\hat{G}}$, we get that $\mathcal{O}(\hat{H}g \rtimes \theta)^{\hat{H}}$ is integral over $\mathcal{O}(\hat{G} \rtimes \theta)^{\hat{G}}$, hence also finite since $\mathcal{O}(\hat{H}g \rtimes \theta)^{\hat{H}}$ is a finitely generated R-algebra. So we are left with proving that the map $Sg \rtimes \theta \longrightarrow \hat{G} \rtimes \theta \sslash \hat{G}$ is finite. Let n be the order of θ, and consider the diagram

$$
\begin{array}{ccc}
Sg \rtimes \theta & \longrightarrow & \hat{G} \rtimes \theta \sslash \hat{G} \\
{\scriptstyle (-)^n}\downarrow & & \downarrow{\scriptstyle (-)^n} \\
Sg_n & \longrightarrow & \hat{G} \sslash \hat{G}
\end{array}
$$

Here the horizontal maps are the natural ones, and the vertical maps are induced by raising to the power n and we have put $g_n := g\theta(g) \cdots \theta^{n-1}(g)$. Note that, since S is centralized by $\mathrm{Int}_g \circ \theta$, the left vertical map is given by $sg \rtimes \theta \mapsto s^n g_n$. In particular it is a finite morphism, hence the finiteness of the top horizontal map will follow if we can prove finiteness of the bottom horizontal map.

Denote by $M := C_{\hat{G}}(S)$ the centralizer of S in $\hat{G}$, which is a Levi subgroup. Then we have $g_n \in M(R)$, whence a factorization $Sg_n \longrightarrow M \sslash M \longrightarrow \hat{G} \sslash \hat{G}$. The second map is finite by the Chevalley-Steinberg theorem. In order to prove that the first map is finite, consider the isogeny $Z(M)^0 \times M' \longrightarrow M$ where $Z(M)^0$ denotes the maximal central torus in M and M' the derived subgroup. Another application of the Chevalley-Steinberg theorem shows that the morphism $(Z(M)^0 \times M') \sslash M = Z(M)^0 \times M' \sslash M \longrightarrow M \sslash M$ is finite. Thanks to the above lemma, we may write $g_n = zm'$ with $z \in Z(M)^0(R)$ and $m' \in M'(R)$. Then we get a factorization

$$
Sg_n \longrightarrow Z(M)^0 \times \{m'\} \longrightarrow Z(M)^0 \times M' \sslash M \longrightarrow M \sslash M
$$

where the first two maps are closed immersions and the last one is finite. This implies that $Sg_n \longrightarrow \hat{G} \sslash \hat{G}$ is finite and completes the proof. $\qquad\square$

We now fix a prime p and assume further that $p|N$ and that the reductive group scheme $\hat{G}$ over $R = \bar{\mathbb{Z}}[\frac{1}{N}]$ is endowed with a finite action of the Weil group W_F of a local field F of residue characteristic p. As usual, we denote by Fr a lift of Frobenius, and by $\bar{\mathrm{Fr}}$ its image in $\mathrm{Aut}(\hat{G})$.

Theorem 2.3. *For any $e \geq 0$, the map $Z^1(W_F^0/P_F^e, \hat{G}) \longrightarrow \hat{G}$, $\varphi \mapsto \varphi(\mathrm{Fr})$ induces a finite morphism $Z^1(W_F^0/P_F^e, \hat{G}) \sslash \hat{G} \longrightarrow \hat{G} \rtimes \bar{\mathrm{Fr}} \sslash \hat{G}$.*

Before giving the proof, let us draw some consequences.

Corollary 2.4. *Suppose $\hat{H} \subset \hat{G}$ is a closed reductive subgroup scheme of $\hat{G}$ stable under W_F. Then the natural morphism $Z^1(W_F^0/P_F^e, \hat{H}) \sslash \hat{H} \longrightarrow Z^1(W_F^0/P_F^e, \hat{G}) \sslash \hat{G}$ is finite.*

Proof. Consider the commutative diagram

$$
\begin{array}{ccc}
Z^1(W_F^0/P_F^e, \hat{G}) \sslash \hat{G} & \longrightarrow & \hat{G} \rtimes \bar{\mathrm{Fr}} \sslash \hat{G} \\
\uparrow & & \uparrow \\
Z^1(W_F^0/P_F^e, \hat{H}) \sslash \hat{H} & \longrightarrow & \hat{H} \rtimes \bar{\mathrm{Fr}} \sslash \hat{H}
\end{array}
$$

The theorem says that the bottom horizontal map is finite, and Lemma 2.2 says that the right vertical map is finite. It follows that the left vertical map is finite. $\qquad\square$

Similarly, a W_F-equivariant isogeny $\hat{G} \longrightarrow \hat{G}'$ induces a finite morphism $Z^1(W_F^0/P_F^e, \hat{G}) \sslash \hat{G} \longrightarrow Z^1(W_F^0/P_F^e, \hat{G}') \sslash \hat{G}'$. Now, let us change F instead of $\hat{G}$.

Corollary 2.5. *Let F' be a finite extension of F and put $P_{F'}^e := P_F^e \cap W_{F'}$. Then the restriction morphism $Z^1(W_F^0/P_F^e, \hat{G}) \sslash \hat{G} \longrightarrow Z^1(W_{F'}^0/P_{F'}^e, \hat{G}) \sslash \hat{G}$ is finite.*

Proof. Let us choose the Frobenius lifts $\mathrm{Fr} \in W_F$ and $\mathrm{Fr}' \in W_{F'}$ such that $\mathrm{Fr}' = \mathrm{Fr}^f$, and consider the commutative diagram

$$
\begin{array}{ccc}
Z^1(W_{F'}^0/P_{F'}^e, \hat{G}) \sslash \hat{G} & \longrightarrow & \hat{G} \rtimes \bar{\mathrm{Fr}}' \sslash \hat{G} \ . \\
{\scriptstyle \text{restr.}} \uparrow & & \uparrow {\scriptstyle (-)^f} \\
Z^1(W_F^0/P_F^e, \hat{G}) \sslash \hat{G} & \longrightarrow & \hat{G} \rtimes \bar{\mathrm{Fr}} \sslash \hat{G}
\end{array}
$$

Again the desired finiteness of the left vertical map follows from the finiteness of the bottom horizontal map, provided by the theorem, and the finiteness of the right vertical map, which follows from the twisted version of the Chevalley-Steinberg map, since raising to power f on a torus is a finite isogeny. $\qquad\square$

Finally, we can also prove a statement similar to that of Theorem 2.3 for the excursion algebra

$$
\mathrm{Exc}(W_F^0/P_{F^e}, \hat{G}) = \mathrm{colim}_{n, F_n \to W_F^0/P_F^e} \mathcal{O}(Z^1(F_n, \hat{G}))^{\hat{G}}
$$

from Definition VIII.3.4 of [FS21]. Indeed, we have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Exc}(W_F^0/P_{F^e}, \hat{G}) & \longrightarrow & \mathcal{O}(Z^1(W_F^0/P_F^e, \hat{G}))^{\hat{G}} \\
\uparrow & & \uparrow \\
\mathrm{Exc}(\langle \mathrm{Fr} \rangle, \hat{G}) & \xrightarrow{\ \simeq\ } & \mathcal{O}(Z^1(\langle \mathrm{Fr} \rangle, \hat{G}))^{\hat{G}} = \mathcal{O}(\hat{G} \rtimes \mathrm{Fr})^{\hat{G}}
\end{array}
$$

where the horizontal maps follow from the construction of the excursion algebra, and the vertical maps are induced by restriction to the free abelian subgroup of W_F^0/P_F^e generated by Fr. Note that, as with any free group, the bottom horizontal map is an isomorphism. Theorem 2.3 tells us that the right vertical map is finite. On the other hand, we know that the top horizontal map induces bijections on L-valued points for any algebraically closed field R-algebra. In particular, this map induces an embedding of the *reduced* excursion algebra $\mathrm{Exc}(W_F^0/P_{F^e}, \hat{G})_{\mathrm{red}}$ in $\mathcal{O}(Z^1(W_F^0/P_F^e, \hat{G}))^{\hat{G}}$ (which was proved to be reduced in [DHKM20]). Observe also that the whole commutative square above is base-changed from the same square over some finitely generated subring $R' \subset R$. Over such a subring, the finitely generated R'-algebra $\mathrm{Exc}(\langle \mathrm{Fr} \rangle, \hat{G})$ is noetherian, so we infer that the map $\mathrm{Exc}(\langle \mathrm{Fr} \rangle, \hat{G}) \longrightarrow \mathrm{Exc}(W_F^0/P_{F^e}, \hat{G})_{\mathrm{red}}$ is finite.

The following interesting corollary is then proved like Corollary 2.4.

Corollary 2.6. *Suppose $\hat{H} \subset \hat{G}$ is a closed reductive subgroup scheme of $\hat{G}$ stable under W_F. Then the natural morphism of algebras $\mathrm{Exc}(W_F^0/P_F^e, \hat{G})_{\mathrm{red}} \longrightarrow \mathrm{Exc}(W_F^0/P_F^e, \hat{H})_{\mathrm{red}}$ is finite.*

Proof of Theorem 2.3. Step 1 : reduction to the tame case. We use the notation of Proposition 1.2 of [DHKM20]. This proposition provides us with a decomposition

$$
\coprod_{(\phi, \tilde{\varphi})} Z^1_{\mathrm{Ad}_{\tilde{\varphi}}}(W_F^0/P_F, C_{\hat{G}}(\phi)^\circ) \sslash C_{\hat{G}}(\phi)_{\tilde{\varphi}} \xrightarrow{\ \sim\ } Z^1(W_F^0/P_F^e, \hat{G}) \sslash \hat{G}.
$$

For any pair $(\phi, \tilde{\varphi})$ as above, consider the diagram

$$
\begin{array}{ccc}
Z^1(W_F^0/P_F^e, \hat{G}) \sslash \hat{G} & \xrightarrow{\ \varphi \mapsto \varphi(\mathrm{Fr}) \rtimes \bar{\mathrm{Fr}}\ } & \hat{G} \rtimes \bar{\mathrm{Fr}} \sslash \hat{G} \\
{\scriptstyle \eta \mapsto \eta \cdot \tilde{\varphi}} \uparrow & & \uparrow \\
Z^1_{\mathrm{Ad}_{\tilde{\varphi}}}(W_F^0/P_F, C_{\hat{G}}(\phi)^\circ) \sslash C_{\hat{G}}(\phi)^\circ & \longrightarrow & C_{\hat{G}}(\phi)^\circ \tilde{\varphi}(\mathrm{Fr}) \rtimes \bar{\mathrm{Fr}} \sslash C_{\hat{G}}(\phi)^\circ
\end{array}
$$

in which the bottom horizontal map is given by $\eta \mapsto \eta(\mathrm{Fr})\tilde{\varphi}(\mathrm{Fr}) \rtimes \bar{\mathrm{Fr}}$. By the above decomposition, the left vertical map is a finite covering of a summand of $Z^1(W_F^0/P_F^e, \hat{G}) \mathbin{/\!\!/} \hat{G}$. By Lemma 2.2, the right vertical map is finite. So, if we can prove finiteness of the horizontal bottom map, then we get finiteness of the horizontal top map restricted to the summand associated to $(\phi, \tilde{\varphi})$. Varying $(\phi, \tilde{\varphi})$ we then get finiteness of the top map. Since the action of $\mathrm{Ad}_{\tilde{\varphi}}(W_F)$ on $C_{\hat{G}}(\phi)^\circ$ is finite, tamely ramified, and stabilizes a Borel pair, we are reduced to the "tame case" below.

Step 2 : the tame case. Suppose here that the action of W_F on $\hat{G}$ is trivial on P_F and stabilizes a Borel pair $(\hat{B}, \hat{T})$ of $\hat{G}$. We are going to prove that the map

$$Z_{\mathrm{tame}}^1 := Z^1(W_F^0/P_F, \hat{G}) \mathbin{/\!\!/} \hat{G} \longrightarrow \hat{G} \rtimes \bar{\mathrm{Fr}} \mathbin{/\!\!/} \hat{G}, \ \varphi \mapsto \varphi(\mathrm{Fr}) \rtimes \bar{\mathrm{Fr}}$$

is finite. Let s be a generator of I_F^0/P_F (as a $\mathbb{Z}[\frac{1}{q}]$-module). We have

$$Z^1(W_F^0/P_F, \hat{G}) = \left\{ (\sigma, F) \in (\hat{G} \rtimes \bar{s}) \times (\hat{G} \rtimes \bar{\mathrm{Fr}}), \ F\sigma F^{-1} = \sigma^q \right\}.$$

Let $\Omega := N_{\hat{G}}(\hat{T})/\hat{T}$ denote the Weyl group of $\hat{G}$ and $\pi : N_{\hat{G}}(\hat{T}) \longrightarrow \Omega$ the projection. Denote by $\Omega^s = \Omega^{s^q}$ the subgroup of Ω fixed by s (equivalently, by s^q), and by $N_{\hat{G}}(\hat{T})_s$ its inverse image in $N_{\hat{G}}(\hat{T})$. We further put

$$N_s := \{ n \in N_{\hat{G}}(\hat{T})_s, \ ns^q(n)^{-1} \in \hat{T}^{s,0} \}.$$

This is a closed subgroup scheme of $N_{\hat{G}}(\hat{T})_s$ whose intersection with $\hat{T}$ is the diagonalizable group scheme

$$\hat{T} \cap N_s = \{ t \in \hat{T}, \ ts^q(t)^{-1} \in \hat{T}^{s,0} \}.$$

Lemma 2.7. *With the foregoing notation,*

(1) the sequence $(\hat{T} \cap N_s)(R) \hookrightarrow N_s(R) \twoheadrightarrow \Omega_s$ is exact.

(2) $(\hat{T} \cap N_s)^0 = \hat{T}^{s,0}$ (recall that ^{0} denotes a maximal subtorus here).

Proof. (1) Let us first prove that the morphism $N_s(R) \longrightarrow \Omega_s$ is surjective. Let $w \in \Omega_s$ and start with any lift n of w in $N_{\hat{G}}(\hat{T})(R)$. Since $\hat{T}^{s^q,0} = \hat{T}^{s,0} \longrightarrow \hat{T}_{s^q}$ is a finite covering, there exists $t_s \in \hat{T}^{s,0}(R)$ whose image in $\hat{T}_{s^q}(R)$ coincides with that of $ns^q(n)^{-1}$. Applying Lemma 2.1 to the morphism $\hat{T} \longrightarrow \hat{T}_{s^q}$, we thus can find $t \in \hat{T}(R)$ such that $t_s = ts^q(t)^{-1}ns^q(n)^{-1}$. Then the element $\dot{w} := tn$ satisfies $\dot{w}s^q(\dot{w})^{-1} = t_s \in \hat{T}^{s,0}(R)$, hence $\dot{w} \in N_s(R)$ and maps to w. On the other hand, we see from the definitions that the fiber of N_s over $w \in \Omega_s$ is $(\hat{T} \cap N_s)(R)\dot{w}$.

(2) Consider the sequence of morphisms :

$$\hat{T}^s = \hat{T}^{s^q} \overset{}{\longhookrightarrow} \hat{T} \cap N_s \xrightarrow{t \mapsto ts^q(t)^{-1}} \hat{T}^{s,0} \longtwoheadrightarrow \hat{T}_s = \hat{T}_{s^q}$$

It may not be exact but it is at least a complex. Since the last map is finite, this implies that $(\hat{T} \cap N_s)/\hat{T}^s$ is finite, hence $\hat{T}^{s,0} = (\hat{T} \cap N_s)^0$. $\qquad \square$

We are now interested in the closed subscheme A of $\hat{T}^{s,0} \times N_s$ defined on R-algebras by
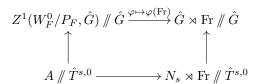
$$A(R') := \left\{ (t, n) \in \hat{T}^{s,0}(R') \times N_s(R'), \ n\mathrm{Fr}(t)n^{-1}t^{-q} = s^q(n)n^{-1} \right\}.$$

The reason is that this condition is equivalent to asking that $(t \rtimes s, n \rtimes \mathrm{Fr})$ be a point of $Z^1(W_F^0/P_F, \hat{G})$, so that A identifies to a closed subscheme of $Z^1(W_F^0/P_F, \hat{G})$.

Lemma 2.8. *A is finite over N_s.*

Proof. The morphism $\alpha : \hat{T}^{s,0} \times N_s \longrightarrow \hat{T}^{s,0} \times N_s$, $(t, n) \mapsto (n\mathrm{Fr}(t)n^{-1}t^{-q}, n)$ is an endo-isogeny of the relative torus $\hat{T}^{s,0} \times N_s$ over N_s. In particular, the kernel $\ker(\alpha)$ of this isogeny is a finite group scheme over N_s. Now observe that A is the preimage under the map α of the section $N_s \longrightarrow \hat{T}^{s,0} \times N_s$ given by $n \mapsto (s^q(n)n^{-1}, n)$, so it is a torsor under $\ker(\alpha)$, hence it is finite over N_s. $\qquad \square$

Let us identify A to a closed subscheme of $Z^1(W_F^0/P_F, \hat{G})$ through $(t, n) \mapsto (t \rtimes s, n \rtimes \mathrm{Fr})$. We note that A is then stable under conjugation by $\hat{T}^{s,0}$. We have therefore a commutative diagram

$$
\begin{array}{ccc}
Z^1(W_F^0/P_F, \hat{G}) /\!\!/ \hat{G} & \xrightarrow{\varphi \mapsto \varphi(\mathrm{Fr})} & \hat{G} \rtimes \mathrm{Fr} /\!\!/ \hat{G} \\
\uparrow & & \uparrow \\
A /\!\!/ \hat{T}^{s,0} & \longrightarrow & N_s \rtimes \mathrm{Fr} /\!\!/ \hat{T}^{s,0}
\end{array}
$$

The last lemma shows that the horizontal bottom map is finite. Up to writing $N_s(R)$ as a union of finitely many left orbits under $\hat{T}^{s,0}(R)$ (thanks to lemmas 2.7 and 2.1), Lemma 2.2 implies that the right vertical map is finite. By the next lemma, the left vertical map is dominant. Hence $\mathcal{O}(Z^1(W_F^0/P_F, \hat{G}))^{\hat{G}}$, being reduced, is a subalgebra of $\mathcal{O}(A)^{\hat{T}^{s,0}}$, and thus it is integral over $\mathcal{O}(\hat{G} \rtimes \mathrm{Fr})^{\hat{G}}$ by what we have proved above. But since it is also of finite type as an $\mathcal{O}(\hat{G} \rtimes \mathrm{Fr})^{\hat{G}}$-algebra (even as a R-algebra), it is finite. Therefore, the next lemma finishes the proof of the theorem.

Lemma 2.9. *The morphism $A \longrightarrow Z^1(W_F^0/P_F, \hat{G}) /\!\!/ \hat{G}$ is surjective on L points for any algebraically closed field R-algebra.*

Proof. Recall that L-points of $Z^1(W_F^0/P_F, \hat{G}) /\!\!/ \hat{G}$ correspond to closed $\hat{G}$-orbits in $Z^1(W_F^0/P_F, \hat{G})(L)$. So let $\varphi = (\sigma, F) \in Z^1(W_F^0/P_F, \hat{G}(L))$ have closed orbit under $\hat{G}$. We must prove it is $\hat{G}(L)$-conjugate to a point in $A(L)$. By a theorem of Richardson (see Theorem 4.13 of [DHKM20] in this context), the subgroup $\varphi(W_F^0) = \langle \sigma, F \rangle$ of ${}^L\mathrm{G}(L)$ is completely reducible, and this implies that the normal subgroup $\langle \sigma \rangle$ is also completely reducible. In turn, this implies that σ stabilizes a Borel pair of $\hat{G}$, that we may assume to be $(\hat{B}, \hat{T})$ after conjugating φ (compare the proof of Prop. 4.19 (2) of [DHKM20]). Since s also stabilizes $(\hat{B}, \hat{T})$, it follows that $\sigma \in \hat{T}(L) \rtimes s$. Using surjectivity of $\hat{T}^{s,0} \longrightarrow \hat{T}_s$ as above, we can further conjugate φ by an element of $\hat{T}(L)$ to achieve $\sigma \in \hat{T}^{s,0}(L) \rtimes s$. Now, the (reduced) centralizer $\hat{G}_\sigma$ is a possibly non-connected reductive algebraic subgroup of $\hat{G}_L$ with maximal torus $\hat{T}^{s,0}$ and Borel subgroup $\hat{B}^s$, and it is stable under conjugation by F. Moreover we have an isomorphism from $\hat{G}_\sigma F$ to the (reduced) closed subscheme $Z^1(W_F^0/P_F, \hat{G})_{L,\sigma}$ of $Z^1(W_F^0/P_F, \hat{G})_L$ defined by the condition $\varphi(s) = \sigma$. The $\hat{G}_\sigma$-orbit of F in $\hat{G}_\sigma F$ then identifies to the intersection of $Z^1(W_F^0/P_F, \hat{G})_\sigma$ with the $\hat{G}$-orbit of (σ, F) in $Z^1(W_F^0/P_F, \hat{G})$. So it is closed, hence $\langle F \rangle$ is a completely reducible subgroup of $\hat{G}_\sigma.\langle F \rangle$ and, as above, F normalizes a Borel pair of $\hat{G}_\sigma^\circ$. After conjugation by an element of $\hat{G}_\sigma^\circ$, we may assume that F normalizes the Borel pair $(\hat{B}^s, \hat{T}^{s,0})$. Then F also normalizes the centralizer of $\hat{T}^{s,0}$ in $\hat{G}$, which is $\hat{T}$. So F is of the form $n \rtimes \mathrm{Fr}$ for some $n \in N_{\hat{G}}(\hat{T})$. Writing $\sigma = t \rtimes s$, we have the equality $n\mathrm{Fr}(t)n^{-1}(ns^q(n)^{-1}) = t^q$. Since Fr and F both normalize $\hat{T}^{s,0}$, so does n, and this equality thus implies that $n \in N_s(L)$, and finally that $(\sigma, F) \in A(L)$.   □

□

We close this section with the following corollary of the above arguments. This result is stated for future reference but is not used in this paper.

Corollary 2.10. *Let R be either of the form $\bar{\mathbb{Z}}[\frac{1}{N}]$ as above or of the form $\overline{\mathbb{Z}_\ell}$ for some prime $\ell \neq p$. Then the canonical morphism $Z^1(W_F^0/P_F^e, \hat{G}) \longrightarrow Z^1(W_F^0/P_F^e, \hat{G}) /\!\!/ \hat{G}$ is surjective on R-points.*

Proof. We may reduce to the tame case as in Step 1 of the proof of Theorem 2.3. So we focus on the setting of Step 2 of that proof and use the same notation, including the closed subscheme A of $Z^1(W_F^0/P_F, \hat{G})$, which is stable under the action of the torus $\hat{T}^{s,0}$. We have just proved that the morphism $A /\!\!/ \hat{T}^{s,0} \longrightarrow Z^1(W_F^0/P_F, \hat{G}) /\!\!/ \hat{G}$ is finite and surjective. Since R is integrally closed and $\mathrm{Frac}(R)$ is algebraically closed, this morphism is thus surjective on R-points, and we are left to prove that the morphism $A \longrightarrow A /\!\!/ \hat{T}^{s,0}$ is surjective on R-points. It is certainly surjective on $\mathrm{Frac}(R)$-points. So let $\bar{a}$ be an R-point of $A /\!\!/ \hat{T}^{s,0}$ and let $\tilde{a} = (\tilde{t}, \tilde{n}) \in A(\mathrm{Frac}(R))$ be a $\mathrm{Frac}(R)$-point above

$\bar{a}$. By Lemma 2.7 (1), the morphism $N_s \longrightarrow (N_s \rtimes \mathrm{Fr}) /\!\!/ \hat{T}^{s,0}$ is a disjoint union, indexed by $\omega \in \Omega^s$, of morphisms of the form $(\hat{T} \cap N_s)\dot\omega \longrightarrow ((\hat{T} \cap N_s)\dot\omega \rtimes \mathrm{Fr}) /\!\!/ \hat{T}^{s,0}$. But each such morphism is a torsor over some quotient of $\hat{T}^{s,0}$, namely $\hat{T}^{s,0}/(\hat{T}^{s,0})^{\dot\omega \mathrm{Fr}}$ respectively. In particular, the morphism $N_s \longrightarrow (N_s \rtimes \mathrm{Fr}) /\!\!/ \hat{T}^{s,0}$ is surjective on R-points, and its geometric fibers are $\hat{T}^{s,0}$-orbits. This implies that we can conjugate $(\tilde{t}, \tilde{n})$ by an element of $\hat{T}^{s,0}(\mathrm{Frac}(R))$ so that $\tilde{n}$ becomes R-valued. But now, Lemma 2.9 tells us that the morphism $A \longrightarrow N_s$, $a = (t, n) \mapsto n$ is finite, so this implies that $\tilde{a}$ is R-valued. $\square$

3. Finiteness on the group side

We take up the setting of the introduction.

Lemma 3.1. *Let R be any noetherian ring with $p \in R^\times$. The full subcategory of $\mathrm{Rep}_R(G)$ formed by $\mathfrak{Z}$-finite RG-modules is closed under taking subobjects and quotients.*

Proof. Let us denote by $\mathfrak{Z}_V$ the image of the map $\mathfrak{Z}_R(G) \longrightarrow \mathrm{End}_{RG}(V)$. If V' is either a subobject or a quotient object of V, the kernel of the map $\mathfrak{Z}_R(G) \longrightarrow \mathrm{End}_{RG}(V')$ contains the kernel of $\mathfrak{Z}_R(G) \longrightarrow \mathrm{End}_{RG}(V)$, and therefore $\mathfrak{Z}_{V'}$ is a quotient of $\mathfrak{Z}_V$. Suppose that V is $\mathfrak{Z}$-finite. In particular $\mathfrak{Z}_V$ is a finitely generated R-algebra, so $\mathfrak{Z}_{V'}$ has the same property. Moreover, $\mathfrak{Z}_V$ is noetherian, hence V' is admissible over $\mathfrak{Z}_V$ since V is. Therefore V' is $\mathfrak{Z}$-finite. $\square$

The next lemma justifies our claim in the introduction that Theorems 1.1 and 1.2 are equivalent.

Lemma 3.2. *Let R be any noetherian ring with $p \in R^\times$. The following are equivalent :*

(1) For all open compact subgroups of G, the Hecke algebra $R[H\backslash G/H]$ is finitely generated over its center, and its center is a finitely generated R-algebra.

(2) Any finitely generated RG-module is $\mathfrak{Z}$-finite.

Proof. Assume (2), let H be an open compact subgroup and put $V := R[G/H]$. Let us identify $\mathrm{End}_{RG}(V) = R[H\backslash G/H]$, so that the action of $\mathfrak{Z}_R(G)$ on V is given by a morphism $\mathfrak{Z}_R(G) \longrightarrow Z(R[H\backslash G/H])$, the image of which we denote by $\mathfrak{Z}_V$, as above. Since V is finitely generated, it is $\mathfrak{Z}$-finite. In particular, $V^H = R[H\backslash G/H]$ is finitely generated as a $\mathfrak{Z}_V$-module, hence also as a $Z(R[H\backslash G/H])$-module. Moreover, $\mathfrak{Z}_V$ being noetherian, $Z(R[H\backslash G/H])$ is also a finitely generated $\mathfrak{Z}_V$-module, hence it is a finitely generated R-algebra.

Now assume (1) and let us prove (2). Recall from [Dat09, Appendix] the decomposition $\mathrm{Rep}_R(G) = \prod_r \mathrm{Rep}_R(G)_r$ according to depth, in the sense of Moy and Prasad. If H is any open pro-p-subgroup of G, we thus have a canonical decomposition of $R[G/H]$ as a sum of mutually orthogonal subrepresentations $R[G/H] = \bigoplus_r R[G/H]_r$. Then the ring $R[H\backslash G/H] = \mathrm{End}_{RG}(R[G/H])$ and its center $Z(R[H\backslash G/H])$ decompose accordingly, and we have

$$Z(R[H\backslash G/H])_r = Z(\mathrm{End}_{RG}(R[G/H]_r)).$$

Let us fix a depth r. Since the factor subcategory $\mathrm{Rep}_R(G)_r$ is generated by a finitely generated projective generator, we can find H such that $R[G/H]_r$ is such a (finitely generated) projective generator of $\mathrm{Rep}_R(G)_r$. It follows that the map $\mathfrak{Z}_R(G) \longrightarrow \mathrm{End}_{RG}(R[G/H]_r)$ induces an isomorphism

$$\mathfrak{Z}_R(G)_r \overset{\sim}{\longrightarrow} Z(\mathrm{End}_{RG}(R[G/H]_r)).$$

The two displayed isomorphisms above and our assumption (1) imply that $R[G/H]_r$ is $\mathfrak{Z}$-finite. Since it is true for all r, the last lemma implies that any finitely generated V is $\mathfrak{Z}$-finite. $\square$

Lemma 3.3. *Let R be a noetherian ring with $p \in R^\times$ and let R' be a noetherian R-algebra.*

(1) If Theorem 1.1 holds for R, then it holds for R'.

(2) If Theorem 1.1 holds for R' and R' is faithfully flat over R, then it holds for R.

Proof. (1) Suppose Theorem 1.1 holds over R. The isomorphism $R' \otimes_R R[H\backslash G/H] \overset{\sim}{\longrightarrow} R'[H\backslash G/H]$ shows that $R'[H\backslash G/H]$ is a finitely generated module over $R' \otimes_R Z(R[H\backslash G/H])$. It also induces a morphism $R' \otimes_R Z(R[H\backslash G/H]) \longrightarrow Z(R'[H\backslash G/H])$ through which the action of $R' \otimes_R Z(R[H\backslash G/H])$ on

$R'[H\backslash G/H]$ factors. Hence $R'[H\backslash G/H]$ is a fortiori a finitely generated module over $Z(R'[H\backslash G/H])$. Moreover, since $R' \otimes_R Z(R[H\backslash G/H])$ is noetherian, $Z(R'[H\backslash G/H])$ is also a finitely generated $R' \otimes_R Z(R[H\backslash G/H])$-module, so it is a finitely generated R'-algebra because $Z(R[H\backslash G/H])$ is a finitely generated R-algebra by assumption.

(2) Since R' is flat over R, the map $R' \otimes_R Z(R[H\backslash G/H]) \longrightarrow Z(R'[H\backslash G/H])$ is an isomorphism. Since it is even faithfully flat, the finite type property of $Z(R'[H\backslash G/H])$ as a R'-algebra implies that of $Z(R[H\backslash G/H])$ as an R-algebra, see [Sta22, Lemma 00QP]. For the same reason, the finite type property of $R'[H\backslash G/H]$ as a $Z(R'[H\backslash G/H])$-module implies that of $R[H\backslash G/H]$ as a $Z(R[H\backslash G/H])$-module, see [Sta22, Lemma 03C4]. $\square$

In the next lemma, we use our notation $\mathbb{Z}'_\ell := \mathbb{Z}_\ell[\sqrt{q}]$ from the introduction. Recall that i_P denotes normalized parabolic induction. Although at this point normalization is not important, it will be more convenient in the main argument of Section 3.1.

Lemma 3.4. *Let Q be a finitely generated projective $\mathbb{Z}'_\ell G$-module. Then there is an embedding $Q \hookrightarrow V$ with V of the form $V = \bigoplus_P i_P(W_P)$ where P runs among a finite set of parabolic subgroups of G, and each W_P is a cuspidal and finitely generated ℓ-torsion free $\mathbb{Z}'_\ell M_P$-module.*

Proof. The $\overline{\mathbb{Q}}_\ell G$-module $Q \otimes \overline{\mathbb{Q}}_\ell$ is projective and finitely generated, so it is a direct factor of a finite direct sum of projective generators of Bernstein blocks of $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G)$. By Bernstein's theory, such projective generators can be taken of the form $i_P(\pi \otimes \overline{\mathbb{Q}}_\ell[M/M^0])$, where

- P is a parabolic subgroup of G with Levi component M,
- π is a supercuspidal irreducible $\overline{\mathbb{Q}}_\ell M$-module, and
- M^0 is the subgroup of M generated by compact elements.

Note that π can be defined over a finite extension E_π of $\mathbb{Q}_\ell$. Moreover, we may and will choose π so that the central character of π takes values in $\overline{\mathbb{Z}_\ell}$, in which case, π admits an M-invariant lattice which can be defined over $\mathcal{O}_{E_\pi}$. More precisely, there is an admissible and cyclic $\mathcal{O}_{E_\pi} M$-module L_π such that $\pi \simeq L_\pi \otimes_{\mathcal{O}_{E_\pi}} \overline{\mathbb{Q}}_\ell$.

So $Q \otimes \overline{\mathbb{Q}}_\ell$ is contained in a direct sum of representations of the form $i_P(L_\pi \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell[M/M^0])$. Since Q is finitely generated, it follows that there is a finite extension E of $\mathbb{Q}_\ell$, containing $\sqrt{q}$ and such that $Q \otimes \mathbb{Q}_\ell$ is contained in a direct sum of representations of the form $i_P(L_\pi \otimes_{\mathbb{Z}_\ell} E[M/M^0])$. For the same reason, after maybe scaling the embedding by a power of ℓ, we see that Q is contained in a direct sum of representations of the form $i_P(L_\pi \otimes_{\mathbb{Z}_\ell} \mathcal{O}_E[M/M^0])$. But $L_\pi \otimes_{\mathbb{Z}_\ell} \mathcal{O}_E[M/M^0] = \mathrm{ind}_{M^0}^M((L_\pi)_{|M^0})$ is a finitely generated cuspidal $\mathbb{Z}_\ell M$-module, as desired. $\square$

3.1. **Proof of Theorem 1.2.** We now start over the proof of Theorem 1.2 as outlined in the introduction. Thanks to Lemmas 3.2 and 3.3, we may assume $R = \mathbb{Z}'_\ell$, and thanks to Lemmas 3.1 and 3.4, we are reduced to proving that if V is a $\mathbb{Z}'_\ell G$-module of the form $V = i_P(W)$ where $P = MU$ is a parabolic subgroup and W is a finitely generated cuspidal ℓ-torsion free $\mathbb{Z}'_\ell M$-module, then V is $\mathfrak{Z}$-finite. As in the introduction, denote by Z_M the maximal central torus of M. We know by [Dat09, Lemme 4.2] that W is admissible as a $\mathfrak{Z}_{\mathbb{Z}'_\ell}(Z_M)M$-module.

Now we use Fargues and Scholze excursion theorem as explained in the introduction. We refer to Definition VIII.3.4 of [FS21] for the excursion algebra and to [FS21, Thm VIII.4.1] and [FS21, Thm IX.0.1] for the construction of the map FS_G. The compatibility with parabolic induction (property (1) in the introduction) is proved in [FS21, Cor. IX.7.3]. However, there, ordinary parabolic induction is used while the map on the excursion side is twisted by a cyclotomic central cocycle. Using normalized parabolic induction cancels out this cyclotomic twist. The compatibility "with central characters" (property (2)) follows from [FS21, Thm IX.6.2] and [FS21, Thm IX.6.1] applied to the isogeny $M_{\mathrm{der}} \times Z_M \longrightarrow M$. The fact that FS_{Z_M} is an isomorphism (property (3)) follows from [FS21, Prop IX.6.5]. Finally, the continuity property (property (4)) follows from [FS21, Prop IX.5.1] (and in fact is an important ingredient of the construction of the map FS_G).

Now, we return to our $V = i_P(W)$. The smooth $\mathbb{Z}'_\ell M$-module W is admissible over $\mathfrak{Z}_{\mathbb{Z}'_\ell}(Z_M)$, hence, by property (3) and property (2), it is admissible over $\mathrm{Exc}(W_F, \hat{M})_{\mathbb{Z}'_\ell}$. Since induction preserves admissibility, it follows that V is admissible as a $\mathrm{Exc}(W_F, \hat{M})_{\mathbb{Z}'_\ell} G$-module. Moreover, property (4) tells us that the action of $\mathrm{Exc}(W_F, \hat{M})_{\mathbb{Z}'_\ell}$ on W factors through some $\mathrm{Exc}(W_F^0/P_F^e, \hat{M})_{\mathbb{Z}'_\ell}$. Since W is ℓ-torsion free, and since the nilradical of $\mathrm{Exc}(W_F^0/P_F^e, \hat{M})_{\mathbb{Z}'_\ell}$ is ℓ-torsion (because $\mathrm{Exc}(W_F^0/P_F^e, \hat{M})_{\mathbb{Z}_\ell}[\frac{1}{\ell}] \simeq \mathcal{O}(Z^1(W_F^0/P_F^e, \hat{M})_{\mathbb{Z}_\ell})^{\hat{M}}[\frac{1}{\ell}]$ which is reduced), this action actually factors through $\mathrm{Exc}(W_F^0/P_F^e, \hat{M})_{\mathbb{Z}'_\ell, \mathrm{red}}$. It then follows from property (1) that there is $e \in \mathbb{N}$ such that the action of $\mathrm{Exc}(W_F, \hat{G})_{\mathbb{Z}'_\ell}$ on V factors through $\mathrm{Exc}(W_F^0/P_F^e, \hat{G})_{\mathbb{Z}'_\ell, \mathrm{red}}$. We can now apply Corollary 1.8, which says that $\mathrm{Exc}(W_F^0/P_F^e, \hat{M})_{\mathbb{Z}'_\ell, \mathrm{red}}$ is finite over $\mathrm{Exc}(W_F^0/P_F^e, \hat{G})_{\mathbb{Z}'_\ell, \mathrm{red}}$. This implies that V is admissible over $\mathrm{Exc}(W_F^0/P_F^e, \hat{G})_{\mathbb{Z}'_\ell}$ and, a fortiori, that it is admissible over $\mathfrak{Z}_{\mathbb{Z}'_\ell}(G)$.

It remains to prove that the image $\mathfrak{Z}_V$ of the map $\mathfrak{Z}_{\mathbb{Z}'_\ell}(G) \longrightarrow \mathrm{End}_{\mathbb{Z}'_\ell G}(V)$ is a finitely generated $\mathbb{Z}'_\ell$-algebra. Denote by $\mathfrak{E}_V \subset \mathfrak{Z}_V$ the image of $\mathrm{Exc}(W_F, \hat{G})_{\mathbb{Z}'_\ell}$ in $\mathrm{End}_{\mathbb{Z}'_\ell G}(V)$. We know that $\mathfrak{E}_V$ is a finitely generated $\mathbb{Z}'_\ell$-algebra, hence it is noetherian and it suffices to prove that $\mathrm{End}_{\mathbb{Z}'_\ell G}(V)$ is a finitely generated module over $\mathfrak{E}_V$. Note that $\mathrm{End}_{\mathbb{Z}'_\ell G}(V) = \mathrm{End}_{\mathfrak{E}_V G}(V)$, since $\mathfrak{E}_V$ acts on V through the Bernstein center of $\mathrm{Rep}_{\mathbb{Z}'_\ell}(G)$, so that any endomorphism of V commutes with $\mathfrak{E}_V$. Suppose we can find an open compact subgroup H such that the restriction map $\mathrm{End}_{\mathfrak{E}_V G}(V) \longrightarrow \mathrm{End}_{\mathfrak{E}_V}(V^H)$ is injective, then we are done since V^H is a finitely generated $\mathfrak{E}_V$-module. In order to find such an H, observe that V belongs to some bounded depth category $\mathrm{Rep}_{\mathbb{Z}'_\ell}(G)_{\leq r}$, since $V \otimes \mathbb{Q}_\ell$ is finitely generated and contains V, which is ℓ-torsion free. So it suffices to pick H such that some finitely generated projective generator of $\mathrm{Rep}_{\mathbb{Z}'_\ell}(G)_{\leq r}$ is generated by its H-invariants. Indeed, the $\mathbb{Z}'_\ell G$-module V is then generated by V^H, so the restriction map $\mathrm{End}_{\mathfrak{E}_V G}(V) \longrightarrow \mathrm{End}_{\mathfrak{E}_V}(V^H)$ is injective.

We end this subsection by noting that the above proof actually shows the following :

Corollary 3.5. (of the proof)

(1) *Any finitely generated smooth $\mathbb{Z}'_\ell G$-module is admissible over $\mathrm{Exc}(W_F, \hat{G})_{\mathbb{Z}'_\ell}$ (through FS_G).*

(2) *For any $r > 0$, there is $e \in \mathbb{N}$ such that the composition $\mathrm{Exc}(W_F, \hat{G}) \longrightarrow \mathfrak{Z}_{\mathbb{Z}'_\ell}(G)_{\leq r}$ factors over $\mathrm{Exc}(W_F^0/P_F^e, \hat{G})_{\mathrm{red}}$ and makes $\mathfrak{Z}_{\mathbb{Z}'_\ell}(G)_{\leq r}$ a finite module over $\mathrm{Exc}(W_F^0/P_F^e, \hat{G})_{\mathbb{Z}'_\ell, \mathrm{red}}$.*

We also have the following easy converse to Theorem 1.2.

Remark 3.6. *Let V be a $\mathfrak{Z}$-finite RG-module of bounded depth. Then V is finitely generated.*

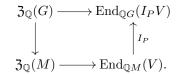*Proof.* It suffices to prove this when $V \in \mathrm{Rep}_R(G)_r$ for some depth r. Let P_r be a finitely generated projective generator of $\mathrm{Rep}_R(G)_r$. It is generated by its H-invariants for a sufficiently small open pro-p-subgroup. Therefore, any object in $\mathrm{Rep}_R(G)_r$ is also generated by its H-invariants. In particular V is generated by V^H, and it follows that any generating set of V^H as a $R[H\backslash G/H]$-module is a generating set as an RG-module. But our assumption says that V^H is a finite $\mathfrak{Z}_R(G)$-module, hence it is a fortiori a finite $R[H\backslash G/H]$-module. $\square$

4. Second Adjointness

We now study consequences of the above results for parabolic induction and restriction. Let R be a noetherian $\mathbb{Z}[\frac{1}{p}]$-algebra, and let $P = MU$ be a parabolic subgroup of G. It is an easy consequence of the Bernstein-Deligne description of $\mathfrak{Z}_{\overline{\mathbb{Q}}}(G)$ in [BD84] that one has a unique map:

$$\mathfrak{Z}_{\mathbb{Q}}(G) \to \mathfrak{Z}_{\mathbb{Q}}(M)$$

such that for any smooth $\mathbb{Q}M$-module V, one has a commutative diagram:

$$\begin{array}{ccc}
\mathfrak{Z}_\mathbb{Q}(G) & \longrightarrow & \mathrm{End}_{\mathbb{Q}G}(I_P V) \\
\downarrow & & \uparrow {\scriptstyle I_P} \\
\mathfrak{Z}_\mathbb{Q}(M) & \longrightarrow & \mathrm{End}_{\mathbb{Q}M}(V).
\end{array}$$

In fact, it is not hard to deduce a similar result over $\mathbb{Z}[\frac{1}{p}]$, or indeed over any flat $\mathbb{Z}[\frac{1}{p}]$-algebra R:

Theorem 4.1. *Let R be a noetherian flat $\mathbb{Z}[\frac{1}{p}]$-algebra. Then there is a unique map $\mathfrak{Z}_R(G) \to \mathfrak{Z}_R(M)$ such that for any smooth RM-module V, one has a commutative diagram:*

$$\begin{array}{ccc}
\mathfrak{Z}_R(G) & \longrightarrow & \mathrm{End}_{RG}(I_P V) \\
\downarrow & & \uparrow {\scriptstyle I_P} \\
\mathfrak{Z}_R(M) & \longrightarrow & \mathrm{End}_{RM}(V).
\end{array}$$

Proof. We first prove this for $R = \mathbb{Z}[\frac{1}{p}]$. Recall from [MP96, Thm 5.2] and [Vig96, 5.12] that the parabolic induction and restriction functors preserve depth, so that for each depth r, the map $\mathfrak{Z}_\mathbb{Q}(G) \to \mathfrak{Z}_\mathbb{Q}(M)$ takes $\mathfrak{Z}_\mathbb{Q}(G)_r$ into $\mathfrak{Z}_\mathbb{Q}(M)_r$. We begin by showing that this map takes $\mathfrak{Z}_{\mathbb{Z}[\frac{1}{p}]}(G)_r$ to $\mathfrak{Z}_{\mathbb{Z}[\frac{1}{p}]}(M)_r$. To do so we identify $\mathfrak{Z}_{\mathbb{Z}[\frac{1}{p}]}(M)_r$ with the center of the endomorphism ring of $\mathbb{Z}[\frac{1}{p}][M/H]_r$ for some sufficiently small subgroup H of M. Note that the action of $z \in \mathfrak{Z}_{\mathbb{Z}[\frac{1}{p}]}(G)_r$ on $I_P\mathbb{Q}[M/H]_r$ is given by $I_P z_M$ for some $z_M \in \mathfrak{Z}_\mathbb{Q}(M)_r$, and it suffices to show that z_M lies in $\mathfrak{Z}_{\mathbb{Z}[\frac{1}{p}]}(M)_r$; that is, that it preserves the $\mathbb{Z}[\frac{1}{p}][M]$-submodule $\mathbb{Z}[\frac{1}{p}][M/H]_r$ of $\mathbb{Q}[M/H]_r$. Certainly for a sufficiently large integer a not divisible by p, the element $a z_M$ of $\mathfrak{Z}_\mathbb{Q}(M)_r$ lies in $\mathfrak{Z}_{\mathbb{Z}[\frac{1}{p}]}(M)_r$, and the action of $a z_G$ on $I_P \mathbb{Z}_\ell[M/H]_r$ is given by $I_P a z_M$.

On the other hand, for any prime $\ell \neq p$, the map

$$I_P : \mathrm{End}_{\mathbb{F}_\ell M}(\mathbb{F}_\ell[M/H]_r) \to \mathrm{End}_{\mathbb{F}_\ell G}(I_P\mathbb{F}_\ell[M/H]_r)$$

is injective, as I_P is faithful. This implies that if x is an endomorphism of $\mathbb{Z}[\frac{1}{p}][M/H]_r$ such that $I_P x$ is divisible by ℓ in the endomorphism ring of $I_P\mathbb{Z}[\frac{1}{p}][M/H]_r$, then x is divisible by ℓ in the endomorphism ring of $\mathbb{Z}[\frac{1}{p}][M/H]_r$. In particular it follows that z_M gives rise to a (necessarily central) endomorphism of $\mathbb{Z}[\frac{1}{p}][M/H]_r$, and thus that z_M lies in $\mathfrak{Z}_{\mathbb{Z}[\frac{1}{p}]}(M)$.

Now let V be a smooth $\mathbb{Z}[\frac{1}{p}]M$-module, and let V_r denote its depth r summand, for each r. It suffices to verify the commutativity of the diagram for each r, and thus, since V_r admits a projective resolution by direct sums of copies of $\mathbb{Z}[\frac{1}{p}][M/H]_r$, it suffices to check the commutativity of the diagram in the case $V = \mathbb{Z}[\frac{1}{p}][M/H]_r$. In this case the commutativity of the diagram is a direct consequence of the construction in the previous paragraphs.

We finally turn to the case of R a flat $\mathbb{Z}[\frac{1}{p}]$-algebra. In such a case we have natural isomorphisms:

$$\mathfrak{Z}_R(M)_r \cong Z(\mathrm{End}_{RM}(R[M/H]_r)) \cong Z(\mathrm{End}_{\mathbb{Z}[\frac{1}{p}]M}(\mathbb{Z}[\frac{1}{p}][M/H]_r) \otimes_{\mathbb{Z}[\frac{1}{p}]} R)$$

and since R is flat the right hand side is isomorphic to $\mathfrak{Z}_{\mathbb{Z}[\frac{1}{p}]}(M) \otimes_{\mathbb{Z}[\frac{1}{p}]} R$. We have a similar morphism with G in place of M, and we take the map $\mathfrak{Z}_R(G) \to \mathfrak{Z}_R(M)$ to be the one obtained, via these identifications, by base change of the map $\mathfrak{Z}_{\mathbb{Z}[\frac{1}{p}]}(G) \to \mathfrak{Z}_{\mathbb{Z}[\frac{1}{p}]}(M)$. The commutativity of the diagram can be verified as above, by reducing to the case of $V = R[M/H]_r$, where it is clear. $\square$

Note that if the ring R contains a square root of q (whose choice must be fixed), then one has an exactly analogous result with normalized parabolic induction in place of unnormalized.

If R is not flat over $\mathbb{Z}[\frac{1}{p}]$, then it is not clear that the natural map $\mathfrak{Z}_{\mathbb{Z}[\frac{1}{p}]}(G) \otimes_{\mathbb{Z}[\frac{1}{p}]} R \to \mathfrak{Z}_R(G)$ is surjective (although we expect that it is). For such R one certainly has a map $\mathfrak{Z}_{\mathbb{Z}[\frac{1}{p}]}(G) \otimes_{\mathbb{Z}[\frac{1}{p}]} R \to$

$\mathfrak{Z}_R(M)$ making the diagram of Theorem 4.1 commute, but it is not clear that this map descends to a map from $\mathfrak{Z}_R(G)$ to $\mathfrak{Z}_R(M)$.

When R is a $\mathbb{Z}_\ell$-algebra for some ℓ, we can combine Theorem 4.1 with the results of the previous section to prove:

Lemma 4.2. *Let R be a noetherian $\mathbb{Z}_\ell$-algebra. For each r, the ring $\mathfrak{Z}_R(G)_r$ is finitely generated as a $\mathfrak{Z}_{\mathbb{Z}_\ell}(G)_r \otimes_{\mathbb{Z}_\ell} R$-module.*

Proof. We identify $\mathfrak{Z}_R(G)_r$ with the center of the endomorphism ring of $R[G/H]_r$ for some sufficiently small subgroup H of G. Since $\mathbb{Z}_\ell[G/H]_r$ is admissible over $\mathfrak{Z}_{\mathbb{Z}_\ell}(G)_r$ by Theorem 1.2, $R[G/H]_r$ is admissible over $\mathfrak{Z}_{\mathbb{Z}_\ell}(G)_r \otimes_{\mathbb{Z}_\ell} R$, and the result follows. $\square$

Combining this with Theorem 4.1 we obtain:

Theorem 4.3. *Let R be a noetherian $\mathbb{Z}_\ell$-algebra. The map $\mathfrak{Z}_{\mathbb{Z}_\ell}(G)_r \otimes_{\mathbb{Z}_\ell} R \to \mathfrak{Z}_R(M)_r$ makes $\mathfrak{Z}_R(M)_r$ into a finitely generated $\mathfrak{Z}_{\mathbb{Z}_\ell}(G)_r \otimes_{\mathbb{Z}_\ell} R$-module.*

Proof. The previous lemma allows us to reduce to the case $R = \mathbb{Z}_\ell$. In this case, the claim follows from Corollary 3.5 and Corollary 1.8 (2).

$\square$

With these results in hand we now turn to the question of establishing second adjointness. The surprising thing is that in spite of relying on the finiteness theorems of the previous sections, which require one to work over $\mathbb{Z}_\ell$ for some ℓ, our results suffice to establish second adjointness over $\mathbb{Z}[\frac{1}{p}]$. Our basic approach is closely related to that used to establish second adjointness in section 11 of [Hel16]. We begin by showing:

Lemma 4.4. *Let $P = MU$ be a parabolic subgroup of G, and H a compact open subgroup of G. Then the parabolic restriction $R_P \mathbb{Z}_\ell[G/H]_r$ is admissible over $\mathfrak{Z}_{\mathbb{Z}_\ell}(G)$.*

Proof. It is an easy consequence of the Iwasawa decomposition that the parabolic restriction of any finitely generated $\mathbb{Z}_\ell[G]$-module is finitely generated as a $\mathbb{Z}_\ell[M]$-module; in particular $R_P \mathbb{Z}_\ell[G/H]_r$ is finitely generated as a $\mathbb{Z}_\ell[M]$-module, and hence admissible over $\mathfrak{Z}_{\mathbb{Z}_\ell}(M)$. Since $\mathfrak{Z}_{\mathbb{Z}_\ell}(M)$ is finitely generated as a module over $\mathfrak{Z}_{\mathbb{Z}_\ell}(G)$, the result follows. $\square$

We will use this lemma to establish "stability" for the modules $\mathbb{Z}_\ell[G/H]_r$, and later deduce stability results for an arbitrary object of $\mathrm{Rep}_{\mathbb{Z}[\frac{1}{p}]}(G)_r$. The notion of stability originated in unpublished lecture notes of Bernstein (see also [Hel16, Def 11.9]); let us recall what it means. Fix $P = MU$ and $\overline{P} = M\overline{U}$ parabolic subgroups of G, such that $P \cap \overline{P} = M$, let λ be a totally positive central element of M, and let K a compact open subgroup of G that is decomposed with respect to $P, \overline{P}$; that is, such that $K = K^- K_M K^+$, where $K_M = K \cap M$, $K^+ = K \cap U$, and $K^- = K \cap \overline{U}$. Recall that a $\mathbb{Z}_\ell G$-module V is called K, P-stable if there exists an integer $c_{K,P,\lambda} \geq 1$, and a direct sum decomposition $V^K = (V^K)_0 \oplus (V^K)_*$ such that:

- $T_\lambda = 1_{K\lambda K}$ acts invertibly on $(V^K)_*$, and
- $T_\lambda^{c_{K,P,\lambda}}$ annihilates $(V^K)_0$.

We will call such an integer $c_{K,P,\lambda}$ a *constant of K, P, λ-stability* for V, or simply a *constant of stability* if we wish to supress the dependence on K, P and λ. Note that these constants depend on λ, although the notion of K, P-stablility is independent of the choice of λ.

The key point is the following,

Lemma 4.5. *Let R be a noetherian $\mathbb{Z}_\ell$-algebra, and let V be an admissible RG-module such that $R_P V$ is an admissible RM-module. Then V is K, P-stable.*

Proof. This is proven in the case $G = \mathrm{GL}_n(F)$ in [Hel16], Lemmas 11.12 and 11.13. The proof carries over, with only minor changes such as the use of non-normalized parabolic restriction in place of normalized, to the current setting. $\square$

In particular we immediately deduce from Lemma 4.4 and the above that:

Corollary 4.6. *For all H and r, $\mathbb{Z}_\ell[G/H]_r$ is K,P-stable for any pair K,P such that K is decomposed with respect to P.*

We will need tools to deduce K,P-stability of other $\mathbb{Z}[\frac{1}{p}]G$-modules from that of $\mathbb{Z}_\ell[G/H]_r$. We first observe that if V and W are both K,P-stable, and $f : V \to W$ is any morphism, then f maps V^K into W^K, and then $(V^K)_0$ into $(W^K)_0$ and $(V^K)_*$ into $(W^K)_*$. In particular any endomorphism of a K,P-stable representation V preserves the direct sum decomposition $V^K = (V^K)_0 \oplus (V^K)_*$.

From this one immediately deduces that if $f : V \to W$ is a map of K,P-stable modules, then the kernel and cokernel of f are also K,P-stable, and the constants of (K,P,λ)-stability of the kernel and cokernel of f are bounded by those of V and W, respectively. Moreover, if $\{V_i\}_{i\in I}$ is a collection of $\mathbb{Z}[\frac{1}{p}]G$-modules that are K,P-stable, and the constants of stability of the V_i are uniformly bounded by a constant c, then the direct sum and product of the V_i is also K,P-stable, with c a constant of stability.

Since every object of $\mathrm{Rep}_{\mathbb{Z}_\ell}(G)_r$ has a projective resolution by direct sums of copies of $\mathbb{Z}_\ell[G/H]_r$, it follows that every object of $\mathrm{Rep}_{\mathbb{Z}_\ell}(G)_r$ is K,P-stable, and that if c_ℓ is a constant of (K,P,λ)-stability for $\mathbb{Z}_\ell[G/H]_r$, then it is also a constant of (K,P,λ)-stability for every object of $\mathrm{Rep}_{\mathbb{Z}_\ell}(G)_r$.

From this it is not hard to deduce:

Lemma 4.7. *Fix K, P, and λ as above. There exists an absolute constant c such that for any prime $\ell \neq p$, and any object V of $\mathrm{Rep}_{\mathbb{Z}_\ell}(G)_r$, the constant c is a constant of (K,P,λ)-stability for V.*

Proof. Let c_ℓ be the smallest positive integer that is a constant of (K,P,λ)-stability for $\mathbb{Z}_\ell[G/H]_r$; the discussion of the previous paragraph shows in particular that c_ℓ is then also a constant of stability for $\overline{\mathbb{Q}_\ell}[G/H]_r$. Conversely, since $\mathbb{Z}_\ell[G/H]_r \subseteq \mathbb{Q}_\ell[G/H]_r$, we have that c_ℓ is the smallest constant of stability for $\overline{\mathbb{Q}_\ell}[G/H]_r$. On the other hand, fixing an isomorphism of $\overline{\mathbb{Q}_\ell}$ with $\mathbb{C}$, and noting that K,P-stability is a purely algebraic notion, we find that c_ℓ is the smallest constant of stability of $\mathbb{C}[G/H]_r$. In particular c_ℓ does not depend on ℓ, and the claim follows. $\qquad\square$

We now introduce a duality on $\mathrm{Rep}_{\mathbb{Z}_\ell}(G)$. For V a smooth $\mathbb{Z}_\ell G$-module, let $V^{\vee,\ell}$ denote the G-smooth elements of $\mathrm{Hom}_{\mathbb{Z}_\ell}(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$. Note that we have natural isomorphisms:

$$\mathrm{Hom}_{\mathbb{Z}_\ell G}(V, W^{\vee,\ell}) \cong \mathrm{Hom}_{\mathbb{Z}_\ell}(V \otimes_{\mathbb{Z}_\ell G} W, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \cong \mathrm{Hom}_{\mathbb{Z}_\ell G}(W, V^{\vee,\ell});$$

in particular the functor $V \mapsto V^{\vee,\ell}$ takes projectives to injectives. Moreover, the "double dual" map $V \mapsto (V^{\vee,\ell})^{\vee,\ell}$ is an isomorphism for V a simple $\mathbb{Z}_\ell G$-module. From this one deduces that (for H compact open and sufficiently small) $\mathbb{Z}_\ell[G/H]_r^{\vee,\ell}$ is an injective $\mathbb{Z}_\ell G$-module that admits every simple object of $\mathrm{Rep}_{\mathbb{Z}_\ell}(G)_r$ as a submodule. Since the "forgetful functor" from $\mathbb{Z}_\ell G$-modules to $\mathbb{Z}[\frac{1}{p}]$-modules is right adjoint to the exact functor $V \mapsto V \otimes_{\mathbb{Z}[\frac{1}{p}]} \mathbb{Z}_\ell$, we find that $\mathbb{Z}_\ell[G/H]_r^{\vee,\ell}$ remains injective as a $\mathbb{Z}[\frac{1}{p}]G$-module. We will denote this module by I_ℓ.

Lemma 4.8. *Let V be a simple object of $\mathrm{Rep}_{\mathbb{Z}[\frac{1}{p}]}(G)_r$. Then V embeds in I_ℓ for some ℓ. In particular, every object of $\mathrm{Rep}_{\mathbb{Z}[\frac{1}{p}]}(G)_r$ admits an injective resolution by direct products of copies of I_ℓ for varying ℓ.*

Proof. If V is killed by multiplication by ℓ for some ℓ then we can regard V as a $\mathbb{Z}_\ell G$-module and the claim is clear. Otherwise multiplication by ℓ is invertible on V for all ℓ and we can regard V as a $\mathbb{Q}G$-module. Then V embeds in $V \otimes_\mathbb{Q} \mathbb{Q}_\ell$, and the latter embeds in a direct product of copies of I_ℓ. We thus obtain an embedding of V in a product of copies of I_ℓ; the projection of V to at least one of these copies is nonzero and thus injective.

To prove the final claim it suffices to show that every object of $\mathrm{Rep}_{\mathbb{Z}[\frac{1}{p}]}(G)_r$ embeds in a direct product of copies of I_ℓ for varying ℓ. Fix an object V of this category, and for each pair of subobjects (W, W') of V such that W is contained in W' and the quotient W'/W is simple, fix an embedding $\iota_{W,W'}$ of W'/W into an injective $I_{W,W'}$ that is isomorphic to I_ℓ for some $\ell \neq p$. Then we may regard

$\iota_{W,W'}$ as a map from W' to $I_{W,W'}$, and injectivity of $I_{W,W'}$ allows us to extend this map to a map: $\hat{\iota}_{W,W'} : V \to I_{W,W'}$. It suffices to show that the product of the maps $\hat{\iota}_{W,W'}$ is an embedding of V in $\prod_{W,W'} I_{W,W'}$; suppose otherwise. Then the kernel of this product contains a nonzero, finitely generated subobject W'; this subobject admits a simple quotient, so we may fix a further subobject W of W' with W'/W simple. But the map $\hat{\iota}_{W,W'}$ is nonzero on W', contradicting the fact that W' is contained in the kernel of the product map. $\qquad\square$

Corollary 4.9. *Every object of* $\mathrm{Rep}_{\mathbb{Z}[\frac{1}{p}]}(G)_r$ *is* K, P*-stable.*

For V a smooth $\mathbb{Z}[\frac{1}{p}]G$-module, define $V^\vee$ to be the set of G-smooth elements of $\mathrm{Hom}_{\mathbb{Z}[\frac{1}{p}]}(V, \mathbb{Q}/\mathbb{Z}[\frac{1}{p}])$. This gives a faithful duality functor on $\mathrm{Rep}_{\mathbb{Z}[\frac{1}{p}]}(G)_r$. As with the latter functors $V \mapsto V^{\vee,\ell}$ introduced earlier, the functor $V \mapsto V^\vee$ takes projectives to injectives and the "double dual" map $V \mapsto (V^\vee)^\vee$ is an isomorphism for V a simple $\mathbb{Z}[\frac{1}{p}]$-module. Using this duality and our stability results, we can prove the following version of Jacquet's lemma for arbitrary objects of $\mathrm{Rep}_{\mathbb{Z}[\frac{1}{p}]}(G)_r$:

Lemma 4.10. *Let V be an object of* $\mathrm{Rep}_{\mathbb{Z}[\frac{1}{p}]}(G)_r$*, and $P = MU$, $\overline{P} = M\overline{U}$ parabolic subgroups of G with $P \cap \overline{P} = M$. There is a natural isomorphism:*

$$R_P(V^\vee) \cong (R_{\overline{P}}V)^\vee$$

where $\overline{P}$ denotes the opposite parabolic to P.

Proof. The proof follows the basic strategy of Bernstein, using K, P-stability of V for a cofinal family of compact open subgroups K of G; for $\mathrm{GL}_n(F)$ the details can be found in the proof of Lemma 11.16 of [Hel16], which carries over to the current setting with only minor changes, such as replacing the duality over $\mathbb{Z}_\ell$ with that over $\mathbb{Z}[\frac{1}{p}]$, and normalized parabolic restriction with non-normalized parabolic restriction. $\qquad\square$

The proof of Corollary 1.3 is now more-or-less immediate; we only give a sketch as the proof closely follows the argument of [Hel16], Theorem 11.18 (which itself is a slight modification of an argument of Bernstein). We first note that it suffices to prove the theorem for $R = \mathbb{Z}[\frac{1}{p}]$. In this case, one wishes to construct a natural isomorphism:

$$\mathrm{Hom}_{\mathbb{Z}[\frac{1}{p}]G}(I_P V, W) \cong \mathrm{Hom}_{\mathbb{Z}[\frac{1}{p}]M}(V, \delta_P R_{\overline{P}} W),$$

for all objects V and W of $\mathrm{Rep}_{\mathbb{Z}[\frac{1}{p}]}(M)$ and $\mathrm{Rep}_{\mathbb{Z}[\frac{1}{p}]}(G)$, respectively. When W is of the form $(W')^\vee$ for some object W' of $\mathrm{Rep}_{\mathbb{Z}[\frac{1}{p}]}(G)$, such an isomorphism can be constructed formally from Jacquet's lemma (Lemma 4.10) using the fact that R_P is left adjoint to I_P, together with the natural isomorphism $(I_P V)^\vee \cong I_P(\delta_P V^\vee)$.

In particular the result holds when $W = \mathbb{Z}[\frac{1}{p}][G/H]_r^\vee$. More generally, it holds for any direct product $\prod_{i \in I} \mathbb{Z}[\frac{1}{p}][G/H_i]_{r_i}^\vee$, as such a product is dual to the direct sum $\bigoplus_{i \in I} \mathbb{Z}[\frac{1}{p}][G/H_i]_{r_i}$. Since any object of $\mathrm{Rep}_{\mathbb{Z}[\frac{1}{p}]}(G)$ has a resolution by direct products of copies of $\mathbb{Z}[\frac{1}{p}][G/H]_r^\vee$ for various H and r we can deduce the result in general.

Once we have Corollary 1.3, Corollaries 1.4 and 1.5 (1) and (2) follow immediately from Corollaire 4.4 and Lemme 4.6 of [Dat09], respectively. A proof of assertion (3) (a) of Corollary 1.5 is given in Theorem 5.1 of [Dat05], under the assumption that there exist cocompact lattices in G. However, this hypothesis is only used in order to get property i) of Proposition 3.14 of loc.cit, and apply the implication i) $\Rightarrow$ iii) of that proposition. But property i) of that proposition also follows from second adjointness, by [Dat09, Lemme 4.12]. Finally, (3) (b) of Corollary 1.5 is [Dat09, Lemme 4.13].

Let us now prove Corollary 1.6. Fix $\pi \in \mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(G)$ and let (M, σ) belong to its supercuspidal support. Suppose first that σ is integral, and let L_σ be an M-stable $\overline{\mathbb{Z}}_\ell$-lattice in σ. Then for any parabolic subgroup P with Levi M, the $\overline{\mathbb{Z}}_\ell G$-module $i_P(L_\sigma)$ is known to be admissible. Therefore, if we choose P such that $\pi \hookrightarrow i_P(\sigma)$, the $\overline{\mathbb{Z}}_\ell G$-module $i_P(L_\sigma) \cap \pi$ is a lattice in π. Conversely, suppose

that π is integral and L_π be a G-stable $\overline{\mathbb{Z}_\ell}$-lattice in π, and pick any parabolic subgroup P of G with Levi M. As recalled above, we know from Iwasawa decomposition that $r_P(L_\pi)$ is a finitely generated $\overline{\mathbb{Z}_\ell}M$-module. Moreover, Corollary 1.5 (2) tells us that it is an admissible $\overline{\mathbb{Z}_\ell}M$-module. Since it is contained in, and generates, $r_P(\pi)$, this is a $\overline{\mathbb{Z}_\ell}M$-lattice in $r_P(\pi)$. In particular, the socle of $r_P(\pi)$ is integral. But any irreducible component of this socle is conjugate to σ by uniqueness of cuspidal support.

Here is a consequence of Corollary 1.6 that may be worth mentioning.

Corollary 4.11. *Let π be a simple integral $\overline{\mathbb{Q}}_\ell G$-module such that $r_\ell(\pi)$ is cuspidal. Then π is cuspidal.*

Proof. Let $P \subset G$ be a parabolic subgroup and $M \subset P$ a Levi factor. As explained above, if L_π is a lattice in π, then $r_P(L_\pi)$, being a finitely generated $\overline{\mathbb{Z}_\ell}M$-module, has to be a lattice in $r_P(\pi)$. In particular, $r_\ell(r_P(\pi)) = r_P(r_\ell\pi) = 0$, and therefore $r_P(\pi) = 0$. $\square$

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