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# EXISTENCE AND STABILITY THEORIES FOR A COUPLED SYSTEM INVOLVING p-LAPLACIAN OPERATOR OF A NONLINEAR ATANGANA–BALEANU FRACTIONAL DIFFERENTIAL EQUATIONS

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## Abstract

We investigate the appropriate and sufficient conditions for the existence and uniqueness of a solution for a coupled system of Atangana–Baleanu fractional equations with a p-Laplacian operator. We also study the HU-stability of the solution by using the Atangana–Baleanu–Caputo (ABC) derivative. To achieve these goals, we convert the coupled system of Atangana–Baleanu fractional equations into an integral equation form with the help of Green functions. The existence of the solution is proven by using topological degree theory and Banach's fixed point theorem, with which we analyze the solution's continuity, equicontinuity and boundedness. Then, we use Arzela–Ascolli theory to ensure that the solution is completely continuous. Uniqueness is established using the Banach contraction principle. We also investigate several adequate conditions for HU-stability and generalized HU-stability of the solution. An illustrative example is presented to verify our results.

*Keywords*: Fractional Differential Equation; Topological Degree Theorem; Hyers–Ulam Stability; Atangana–Baleanu–Caputo (ABC) Derivative; Existence and Uniqueness.



## 1. INTRODUCTION

Fractional calculus has become a popular method of solving various physical problems in recent years. So, many scientists and researchers have shown great interest in fractional differential equations (FDEs). FDEs are widely applied in different fields of engineering, science and technology for the mathematical modeling of structures and processes, polymer rheology, physics, economics, biophysics, control theory, blood flow phenomena and chemistry due to their theoretical development and high precision. References 1–7 have provided details on these applications. Researchers have studied different aspects of FDEs, including the existence, stability and uniqueness of solutions with the help of fixed-point theories, Banach contraction theorem and different techniques. Such studies include Refs. 8–20. One of the most important reasons for this expansion is the findings obtained by several researchers when they used fractional operators to model real-world problems. Another interesting fact is that there are varieties of fractional operators with different kernels. This fact allows scientists to select the most suitable operator for the model they are investigating. In recent years, scientists and researchers have studied several AB-fractional equations from theoretical aspects, such as the existence and uniqueness of solutions for FDEs. Related studies include Refs. 21–24. We are interested in studying the coupled system of ABC-fractional because it can help researchers solve a wide range of problems in applied fields by studying fractional modeling and its related existence results as well as the qualitative behaviors of solutions for distinct fractional problems. To the best of our knowledge, no study has investigated the existence and uniqueness of a positive solution for a coupled system of ABC-fractional differential equations. We analyze stability in our study because stability analysis is crucial for verifying the stability of FDE solutions, and stable solutions always provide essential information in the prescribed domain. Researchers have studied several stability theories to fractional differential equations such as Lyapunov, exponential and Mittag-Leffler. Other important kinds of stability analysis are HU-stability and UH-Rassias, which we use for our system. Numerous new ideas and concepts have been introduced to the AB-fractional derivative. For example, Atangana and Baleanu[25] proposed a new definition of fractional differential derivative with a non-singular kernel. Mehmood *et al.*[26] studied the existence theorem for fractional equations solution involving AB-derivative with boundary conditions. Khan *et al.*[27] discussed the existence and numerical simulation of the solution for fractional-fuzzy-Volterra IDE. Alomari *et al.*[28] studied the approximate solutions using HAM depending on the fractional operators involving ABR derivative equations and ABC derivative equations with kernels of Mittag Leffler in three parameters. Abdeljawad[29] studied the existence and uniqueness theories for initial value problems in Atangana–Baleanu–Caputo (ABC) and Atangana–Baleanu–Riemann (ABR) type. Abdeljawad *et al.*[30] proved existence theories for ABC-differential equations with Mittag-Leffler kernels by perturbing the equilibrium points. Further information can be found in Refs. 31–38. Recently, Jarad *et al.*[39] investigated the existence and uniqueness of a solution for the ABC-fractional equation as

$$\begin{cases} {}_0^{\mathrm{ABC}}\mathcal{D}^{\lambda_0}\mathcal{Q}(\zeta) \\ \qquad = \mathcal{Z}_1^*(\zeta,\mathcal{Q}(\zeta)), \quad \lambda_0 \in (0,1), \ \ \mathcal{Q}(0) = \mathcal{Q}_0, \end{cases}$$

where ${}_0^{\mathrm{ABC}}\mathcal{D}^{\lambda_0}$ is Caputo AB-fractional derivative and $\lambda_0 \in (0,1)$, ${}_0^{\mathrm{ABC}}\mathcal{D}^{\lambda_0}\mathcal{Q}(\zeta), \mathcal{Z}_1^*(\zeta,\mathcal{Q}(\zeta)) \in C[a,b]$. Ali *et al.*[40] suggested some conditions for existence theorem to boundary value problems of FDEs involving ABC-derivative.

$$\begin{cases} {}^{\mathrm{ABC}}\mathcal{D}_\zeta^\lambda \kappa(\zeta) \\ \qquad = g(\zeta,\kappa(\zeta),\kappa(\eta\zeta)), \quad 0<\eta<1, \ \ 1<\lambda\le 2, \\ \qquad\qquad \kappa(0)=\kappa_0, \ \ \kappa(\beta)=\kappa_1, \end{cases}$$

where ${}^{\mathrm{ABC}}\mathcal{D}_\zeta^\lambda$ is ABC-derivative, $g : J\times R\times R \longrightarrow R$ is a continuous function and $J=[0,\beta]$. Sutar and Kucche[41] studied the uniqueness and existence of a maximal and minimal solution for FDEs involving the Atangana–Baleanu–Caputo (ABC) derivative:

$$\begin{cases} {}_0^{\mathrm{ABC}}\mathcal{D}_\zeta^\lambda \left(\dfrac{\delta(\zeta)}{g(\zeta,\delta(\zeta))}\right) \\ \qquad = \mathcal{Q}(\zeta,\delta(\zeta)), \quad \text{a.e. } \zeta\in J, \ \ \delta(0)=\delta_0\in R, \end{cases}$$

where (i) $J=[0,\beta], \beta>0$ and $0<\lambda<1$, (ii) ${}_0^{\mathrm{ABC}}\mathcal{D}_\zeta^\lambda$ is ABC-derivative of order $\lambda$ with lower terminal 0, (iii) $g\in C(J\times R\times R\backslash\{0\})$, $\delta\in C(J)$ and ${}_0^{\mathrm{ABC}}\mathcal{D}_\zeta^\lambda h\in C(J)$, where $h(\zeta)=\frac{\delta(\zeta)}{g(\zeta,\delta(\zeta))}, \zeta\in J$, (iv) $\mathcal{Q}\in C$ is such that $\mathcal{Q}(0,\delta(0))=0$ where $C=\{h\,|\,h : J\times R\longrightarrow R$ is continuous, $h(\zeta,\cdot)$ is measurable and $h(\cdot,\delta)$ is continuous$\}$. For the coupled system of AB-fractional equations, it has not been investigated until yet. Inspired by the aforementioned literatures, the main goal of our work is to investigate existence and uniqueness of the solution for a coupled system of AB-fractional equations with a p-Laplacian operator, as well as to study the HU-stability and generalized HU-stability of the solution by using the ABC-fractional derivative given by

$$\begin{cases} {}_0^{\mathrm{ABC}}\mathcal{D}^{\lambda_1}(\phi_p({}_0^{\mathrm{ABC}}\mathcal{D}^{\sigma_1}(\mu(\zeta))) \\ \qquad = -\psi_1(\zeta)\mathcal{Y}_1^*(\zeta,\nu(\zeta-\tau)), \\ {}_0^{\mathrm{ABC}}\mathcal{D}^{\lambda_2}(\phi_p({}_0^{\mathrm{ABC}}\mathcal{D}^{\sigma_2}(\nu(\zeta))) \\ \qquad = -\psi_2(\zeta)\mathcal{Y}_2^*(\zeta,\mu(\zeta-\tau)), \\ \phi_p({}_0^{\mathrm{ABC}}\mathcal{D}^{\sigma_1}(\mu(\zeta)))|_{\zeta=\rho_1}=0, \\ \phi_p({}_0^{\mathrm{ABC}}\mathcal{D}^{\sigma_2}(\nu(\zeta)))|_{\zeta=\rho_2}=0, \\ \mu(\gamma_1) = {}_0^{\mathrm{AB}}\,\mathcal{I}_a^{\sigma_1+1}\mathcal{Y}_1^*(\vartheta,\nu(\vartheta)), \\ \nu(\gamma_2) = {}_0^{\mathrm{AB}}\,\mathcal{I}_a^{\sigma_2+1}\mathcal{Y}_2^*(\vartheta,\mu(\vartheta)), \end{cases} \tag{1.1}$$

where ${}_0^{\mathrm{ABC}}\mathcal{D}^\lambda$ and ${}_0^{\mathrm{ABC}}\mathcal{D}^\sigma$ are ABC-fractional order differential operations, $0<\lambda_i,\sigma_i\le 1$ for $i=1,2, a\in[1,2]$ and $\mathcal{Y}_1^*,\mathcal{Y}_2^*,\psi\in C[0,1]$ are continuous functions. The $\phi_p(\vartheta)=\vartheta|\vartheta|^{(p-2)}$ is the p-Laplacian operator and $\phi_p^{-1}=\phi_q$, such that $1/p+1/q=1$. By solution $\mu(\zeta),\nu(\zeta)$ of (1.1), we mean $\mu(\zeta),\nu(\zeta)>0$ for $\zeta\in(0,1]$ satisfying (1.1). Our proposed problem AB-fractional DEs involving the operator $\phi_p$, is more complicated and more general than the abovementioned. To investigate the EUS of the solution to our problem and its stability in the sense of HU-stability, we will convert our problem into an equivalent AB-fractional integral equations with the help of the classical results, ${}_0^{\mathrm{AB}}\mathcal{I}^{\sigma_i}$, ${}_0^{\mathrm{AB}}\mathcal{I}^{\lambda_i}$ for $i=1,2$ and Green functions. For application, we will give an interpretative example.

## 2. PRELIMINARIES

We provide basic definitions and desired lemmas in this section, which are required in this paper.

**Definition 1 (Refs. 21 and 25).** The ABC-fractional derivative of the function $\mathcal{Q}\in\mathcal{H}^*(h,r)$ and $r>h, \epsilon^*\in[0,1]$ is defined by

$$\begin{aligned} {}_h^{\mathrm{ABC}}\mathcal{D}_\zeta^{\epsilon^*}\mathcal{Q}(\zeta) = \frac{\mathbb{B}(\epsilon^*)}{1-\epsilon^*}\int_h^\zeta \mathcal{Q}'(\theta_0)\mathbb{E}_{\epsilon^*} \\ \times\left[\frac{-\epsilon^*(\zeta-\theta_0)^{\epsilon^*}}{1-\epsilon^*}\right]d\theta_0 \end{aligned}$$

and

$$\begin{aligned} {}^{\mathrm{ABC}}\mathcal{D}_r^{\epsilon^*}\mathcal{Q}(\zeta) = \frac{\mathbb{B}(\epsilon^*)}{1-\epsilon^*}\int_\zeta^r \mathcal{Q}'(\theta_0)\mathbb{E}_{\epsilon^*} \\ \times\left[\frac{-\epsilon^*(\theta_0-\zeta)^{\epsilon^*}}{1-\epsilon^*}\right]d\theta_0, \end{aligned}$$

where $\mathbb{B}(\epsilon^*) = 1-\epsilon^* + \frac{\epsilon^*}{\Gamma(\epsilon^*)}$, $\mathbb{B}(0) = \mathbb{B}(1) = 1$. $\mathbb{B}(\epsilon^*)$ is the normalization function.

**Definition 2 (Ref. 21).** The ABR-fractional derivative of the function $\mathcal{Q} \in \mathcal{H}^*(h,r)$ and $r > h, \epsilon^* \in [0,1]$ is defined as

$$ {}_{h}^{\mathrm{ABR}}\mathcal{D}_{\zeta}^{\epsilon^*}\mathcal{Q}(\zeta) = \frac{\mathbb{B}(\epsilon^*)}{1-\epsilon^*}\frac{d}{d\zeta}\int_h^{\zeta}\mathcal{Q}(\theta_0)\mathbb{E}_{\epsilon^*} \times \left[\frac{-\epsilon^*(\zeta-s)^{\epsilon^*}}{1-\epsilon^*}\right]d\theta_0 $$

and

$$ {}^{\mathrm{ABR}}\mathcal{D}_{r}^{\epsilon^*}\mathcal{Q}(\zeta) = \frac{\mathbb{B}(\epsilon^*)}{1-\epsilon^*}\frac{-d}{d\zeta}\int_{\zeta}^{r}\mathcal{Q}(\theta_0)\mathbb{E}_{\epsilon^*} \times \left[\frac{-\epsilon^*(\zeta-s)^{\epsilon^*}}{1-\epsilon^*}\right]d\theta_0. $$

**Definition 3 (Ref. 42).** The AB-fractional integral of the function $\mathcal{Q} \in \mathcal{H}^*(h,r)$, $r > h, 0 < \epsilon^* < 1$ is given by

$$ {}_{0}^{\mathrm{AB}}\mathcal{I}_{\zeta}^{\epsilon^*}\mathcal{Q}(\zeta) = \frac{1-\epsilon^*}{\mathbb{B}(\epsilon^*)}\mathcal{Q}(\zeta) + \frac{\epsilon^*}{\mathbb{B}(\epsilon^*)\Gamma(\epsilon^*)} \times \int_h^{\zeta}\mathcal{Q}(\theta_0)(\zeta-\theta_0)^{\epsilon^*-1}d\theta_0 $$

and

$$ {}^{\mathrm{AB}}\mathcal{I}_{\zeta}^{\epsilon^*}\mathcal{Q}(\zeta) = \frac{1-\epsilon^*}{\mathbb{B}(\epsilon^*)}\mathcal{Q}(\zeta) + \frac{\epsilon^*}{\mathbb{B}(\epsilon^*)\Gamma(\epsilon^*)} \times \int_{\zeta}^{r}\mathcal{Q}(\theta_0)(\theta_0-\zeta)^{\epsilon^*-1}d\theta_0. $$

**Lemma 1 (Refs. 42 and 43).** *Let* $\mathcal{Q} \in \mathcal{H}^*(h,r)$, $r > h, \epsilon^* \in [0,1]$, *we obtain*

$$ {}_{h}^{\mathrm{AB}}\mathcal{I}^{\epsilon^*}({}_{h}^{\mathrm{ABc}}\mathcal{D}_{\zeta}^{\epsilon^*})\mathcal{Q}(\zeta) = \mathcal{Q}(\zeta) - \mathcal{Q}(h) $$

*and*

$$ {}^{\mathrm{AB}}\mathcal{I}_{r}^{\epsilon^*}({}^{\mathrm{ABc}}\mathcal{D}_{r}^{\epsilon^*})\mathcal{Q}(\zeta) = \mathcal{Q}(\zeta) - \mathcal{Q}(r). $$

The discrete version of Lemma 1 was announced in Refs. 34 and 44.

**Definition 4 (Ref. 21).** The Riemann–Liouville fractional integral of a function $\chi$ of order $\alpha^* > 0, \chi : (0,+\infty) \longrightarrow R$ is defined by

$$ {}_{0}\mathcal{I}^{\epsilon^*}\chi(\zeta) = \frac{1}{\Gamma(\epsilon^*)}\int_0^{\zeta}(\zeta-s)^{\epsilon^*-1}\chi(s)ds, $$

such that $\mathrm{Re}(\epsilon^*) > 0$ we have

$$ \Gamma(\epsilon^*) = \int_0^{+\infty} e^{-s}s^{\epsilon^*-1}ds, $$

consider the space $\mathfrak{F} = C([0,1],R)$ is real and continuous functions and topological with a norm $\|\alpha(\zeta)\| = \sup\{|\alpha(\zeta)| : \zeta \in (0,1)\}$. The product space $\varpi^* = \mathfrak{F} \times \mathfrak{F}$ with norms $\|(\alpha,\omega)(\zeta)\| = \|\alpha(\zeta)\| + \|\omega(\zeta)\|$ is also a Banach space.

**Definition 5 (Ref. 45).** The mapping $\mathcal{L} : \mathbb{K} \to (0,+\infty)$ for Kuratowski measure of non-compactness is defined as

$$ \mathcal{L}(u) = \inf\{a > 0 : u \text{ is finite cover for sets of diameter} \le a\}, $$

where $u \in \mathbb{K}$ and all bounded sets of $\mathbb{Z}(\mathfrak{F})$ be denoted by $\mathbb{K}$.

**Definition 6 (Ref. 45).** Let $\psi : \theta \to \mathfrak{L}$ be a continuous and bounded mapping such that $\varrho \subset \mathfrak{L}$. Then $\psi$ is an $\mathcal{L}$-Lipschitz, where $\eta \ge 0$ such that

$$ \mathcal{L}(\psi(\nu) \le \eta\xi(\nu) \quad \forall \text{ bounded } \nu \subset \varrho. $$

Then $\psi$ is a strict $\mathcal{L}$-contraction under the condition $\eta < 1$.

**Definition 7 (Ref. 45).** The continuous function $\psi$ is $\mathcal{L}$-condensing if $\mathcal{L}(\psi(\nu) \le \mathcal{L}(\nu)$, for all bounded $u \subset \varrho$ such that $\mathcal{L}(\nu) > 0$. Therefore, $\mathcal{L}(\psi(\nu)) \ge \xi(\nu)$ yields $\xi(\nu) = 0$. Further we have $\psi : \theta \to \mathfrak{L}$ is Lipschitz for $\eta > 0$, such that

$$ \|\psi(\vartheta) - \psi(\bar{\vartheta})\| \le \eta\|\vartheta - \bar{\vartheta}\|, \quad \text{forall } \vartheta, \bar{\vartheta} \in \varrho. $$

If $\eta < 1$, then $\psi$ is strict contraction.

**Proposition 1 (Refs. 45 and 48).** *The mapping* $\psi$ *is* $\mathcal{L}$*-Lipschitz with constant* $\eta = 0$ *if and only if* $\psi : \theta \to \mathfrak{L}$ *is compact.*

**Proposition 2 (Ref. 45).** *The operator* $\psi$ *is* $\mathcal{L}$*-Lipschitz for some constant* $\eta$ *if and only if* $\psi : \theta \to \mathfrak{L}$ *is Lipschitz with constant* $\eta$.

**Theorem 1 (Refs. 45 and 47).** *Let* $\psi : \mathfrak{L} \to \mathfrak{L}$ *is a* $\mathcal{L}$*-contraction and*

$$ E = \{v \in \mathfrak{L} : \textit{there is an existence } 0 \le \mu \le 1 \textit{ such that } v = \mu\psi(v)\}. $$

*if* $E \subset v_h(0)$, *is bounded in* $\mathfrak{L}$ *there is an existence* $h > 0$ *and* $E \subset u_h(0)$ *with degree* $\deg(I - \mu\psi, v_h(0), 0) = 1$, *for* $0 \le \mu \le 1$.

*Then,* $\psi$ *has at least one fixed point.*

**Lemma 2 (Refs. 41 and 46).** *Let $\phi_q$ be p-Laplacian. Then we have*

($\xi_1$) *If $1 < q \leq 2, \varrho_1, \varrho_2 > 0$ and $|\varrho_1|, |\varrho_2| \geq \rho > 0$, then*

$$|\phi_q(\varrho_1) - \phi_q(\varrho_2)| \leq (q-1)\rho^{(q-2)}|\varrho_1 - \varrho_2|.$$

($\xi_2$) *If $q > 2$ and $|\varrho_1|, |\varrho_2| \leq \rho^*$, then*

$$|\phi_q(\varrho_1) - \phi_q(\varrho_2)| \leq (q-1)\rho^{*q-2}|\varrho_1 - \varrho_2|.$$

## 3. EXISTENCE OF THE SOLUTION

**Theorem 2.** *For $\lambda_1, \sigma_1 \in (0,1]$, a function $\mathcal{Y}_1^* \in C[0,1]$ satisfies (1.1). Then the solution of*

$$\begin{cases} {}_0^{\mathrm{ABC}}\mathcal{D}^{\lambda_1}(\phi_p({}_0^{\mathrm{ABC}}\mathcal{D}^{\sigma_1}(\mu(\zeta))) \\ \qquad = -\psi_1(\zeta)\mathcal{Y}_1^*(\zeta,\nu(\zeta-\tau)), \\ \phi_p({}_0^{\mathrm{ABC}}\mathcal{D}^{\sigma_1}(\mu(\zeta)))|_{\zeta=\rho_1} = 0, \\ \mu(\gamma_1) = {}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_1+1}\mathcal{Y}_1^*(\vartheta,\nu(\vartheta)), \end{cases} \tag{3.1}$$

*is given by the integral equation*

$$\mu_1(\zeta) = \int_0^1 \mathcal{G}^{\sigma_1}(\zeta,\vartheta)\phi_q\biggl(\int_0^1 \mathcal{G}^{\lambda_1}(\vartheta,\eta)(\psi_1(\eta) \times \mathcal{Y}_1^*(\eta,\nu(\eta-\tau)))\biggr)d\vartheta + {}^{\mathrm{AB}}\mathcal{I}_a^{\sigma_1+1}\mathcal{Y}_1^*(\vartheta,\nu(\vartheta), \tag{3.2}$$

*where $\mathcal{G}^{\sigma_1}(\zeta,\vartheta), \mathcal{G}^{\lambda_1}(\vartheta,\eta)$ are Green functions defined by*

$$\mathcal{G}^{\sigma_1}(\zeta,\vartheta) = \begin{cases} \dfrac{\sigma_1}{\mathbb{B}(\sigma_1)}\dfrac{(\zeta-\vartheta)^{\sigma_1-1}}{\Gamma(\sigma_1)} - \dfrac{\sigma_1}{\mathbb{B}(\sigma_1)}\dfrac{(\gamma_1-\vartheta)^{\sigma_1-1}}{\Gamma(\sigma_1)}, & \vartheta \leq \zeta \leq \gamma_1, \\ -\dfrac{\sigma_1}{\mathbb{B}(\sigma_1)}\dfrac{(\gamma_1-\vartheta)^{\sigma_1-1}}{\Gamma(\sigma_1)}, & \zeta \leq \vartheta \leq \gamma_1, \end{cases} \tag{3.3}$$

$$\mathcal{G}^{\lambda_1}(\zeta,\vartheta) = \begin{cases} \dfrac{\lambda_1}{\mathbb{B}(\lambda_1)}\dfrac{(\rho_1-\vartheta)^{\lambda_1-1}}{\Gamma(\lambda_1)} - \dfrac{\lambda_1}{\mathbb{B}(\lambda_1)}\dfrac{(\zeta-\vartheta)^{\lambda_1-1}}{\Gamma(\lambda_1)}, & \vartheta \leq \zeta \leq \rho_1, \\ -\dfrac{\lambda_1}{\mathbb{B}(\lambda_1)}\dfrac{(\zeta-\vartheta)^{\lambda_1-1}}{\Gamma(\lambda_1)}, & \zeta \leq \vartheta \leq \rho_1. \end{cases} \tag{3.4}$$

**Proof.** By applying AB-fractional integral ${}^{\mathrm{AB}}\mathcal{I}_0^{\lambda_1}$ and using Lemma 1 on (3.1), we get

$$\phi_p({}_0^{\mathrm{ABC}}\mathcal{D}^{\sigma_1}(\mu(\zeta))) = -{}^{\mathrm{AB}}\mathcal{I}_0^{\lambda_1}(\psi_1(\zeta)\mathcal{Y}_1^*(\zeta,\nu(\zeta-\tau))) + c_0, \tag{3.5}$$

by using the condition Eq. (3.5) is zero, we obtain

$$c_0 = {}^{\mathrm{AB}}\mathcal{I}_0^{\lambda_1}(\psi_1(\eta)\mathcal{Y}_1^*(\zeta,\nu(\zeta-\tau)))|_{\zeta=\rho_1}. \tag{3.6}$$

Putting the value $c_0$ in (3.5), we get

$$\phi_p({}_0^{\mathrm{ABC}}\mathcal{D}^{\sigma_1}(\mu(\zeta))) = -{}_0^{\mathrm{AB}}\mathcal{I}_\zeta^{\lambda_1}(\psi_1(\zeta)\mathcal{Y}_1^*(\zeta,\nu(\zeta-\tau))) + {}_0^{\mathrm{AB}}\mathcal{I}_{\zeta=\rho_1}^{\lambda_1}(\psi_1(\zeta)\mathcal{Y}_1^*(\zeta,\nu(\zeta-\tau))). \tag{3.7}$$

Applying $\phi_p^{-1} = \phi_q$ on sides of (3.7), we get

$${}_0^{\mathrm{ABC}}\mathcal{D}^{\sigma_1}(\mu(\zeta)) = \phi_q({}_0^{\mathrm{AB}}\mathcal{I}_{\zeta=\rho_1}^{\lambda_1}(\psi(\zeta)\mathcal{Y}_1^*(\zeta,\mu(\zeta-\tau))) - {}_0^{\mathrm{AB}}\mathcal{I}_\zeta^{\lambda_1}(\psi(\zeta)\mathcal{Y}_1^*(\zeta,\nu(\zeta-\tau)))). \tag{3.8}$$

Applying integral of fractional ${}^{\mathrm{AB}}\mathcal{I}_0^{\sigma_1}$ in Eq. (3.8), we get

$$\mu(\zeta) = {}_0^{\mathrm{AB}}\mathcal{I}_\zeta^{\sigma_1}\phi_q({}_0^{\mathrm{AB}}\mathcal{I}_{\zeta=\rho_1}^{\lambda_1}(\psi_1(\zeta)\mathcal{Y}_1^*(\zeta,\nu(\zeta-\tau))) - {}_0^{\mathrm{AB}}\mathcal{I}_\zeta^{\lambda_1}(\psi_1(\zeta)\mathcal{Y}_1^*(\zeta,\nu(\zeta-\tau)))) + c_1. \tag{3.9}$$

The condition $\mu(\gamma_1) = {}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_1+1}\mathcal{Y}_2^*(\vartheta,\mu(\vartheta))$ implies that

$$c_1 = {}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_1+1}\mathcal{Y}_1^*(\vartheta,\nu(\vartheta)) - {}_0^{\mathrm{AB}}\mathcal{I}_{\zeta=\gamma_1}^{\sigma_1}\phi_q({}_0^{\mathrm{AB}}\mathcal{I}_{\zeta=\rho_1}^{\lambda}(\psi_1(\zeta)\mathcal{Y}_1^*(\zeta,\nu(\zeta-\tau))) - {}_0^{\mathrm{AB}}\mathcal{I}_\zeta^{\lambda_1}(\psi_1(\zeta)\mathcal{Y}_1^*(\zeta,\nu(\zeta-\tau)))). \tag{3.10}$$

Putting the value $c_1$ in (3.9), we get

$$\begin{aligned}\mu(\zeta) &= ({}_0^{\mathrm{AB}}\mathcal{I}_\zeta^{\sigma_1} - {}_0^{\mathrm{AB}}\mathcal{I}_{\zeta=\gamma_1}^{\sigma_1}) \times \phi_q({}_0^{\mathrm{AB}}\mathcal{I}_{\zeta=\rho_1}^{\lambda_1}(\psi_1(\zeta)\mathcal{Y}_1^*(\zeta,\nu(\zeta-\tau))) - {}_0^{\mathrm{AB}}\mathcal{I}_\zeta^{\lambda_1}(\psi(\eta)\mathcal{Y}_1^*(\zeta,\nu(\zeta-\tau)))) + {}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_1+1}\mathcal{Y}_1^*(\vartheta,\nu(\vartheta)) \\ &= ({}_0^{\mathrm{AB}}\mathcal{I}_\zeta^{\sigma_1} - {}_0^{\mathrm{AB}}\mathcal{I}_{\zeta=\gamma_1}^{\sigma_1})(\phi_q({}_0^{\mathrm{AB}}\mathcal{I}_{\zeta=\rho_1}^{\lambda_1} - {}_0^{\mathrm{AB}}\mathcal{I}_\zeta^{\lambda_1})\end{aligned}$$

$$\times(\psi_1(\zeta)\mathcal{Y}_1^*(\zeta,\nu(\zeta-\tau))))$$
$$+{}_0^{AB}\mathcal{I}_a^{\sigma_1+1}\mathcal{Y}_1^*(\vartheta,\nu(\vartheta))$$
$$=\left[\frac{1-\sigma_1}{\mathbb{B}(\sigma_1)}+\frac{\sigma_1}{\mathbb{B}(\sigma_1)}\mathcal{I}_\zeta^{\sigma_1}\right]-\left[\frac{1-\sigma_1}{\mathbb{B}(\sigma_1)}\right.$$
$$\left.+\frac{\sigma_1}{\mathbb{B}(\sigma_1)}\mathcal{I}_{\zeta=\gamma_1}^{\sigma_1}\right]\phi_q\left(\left[\frac{1-\lambda_1}{\mathbb{B}(\lambda_1)}+\frac{\lambda_1}{\mathbb{B}(\lambda_1)}\mathcal{I}_{\zeta=\rho_1}^{\lambda_1}\right]\right.$$
$$-\left[\frac{1-\lambda_1}{\mathbb{B}(\lambda_1)}+\frac{\lambda_1}{\mathbb{B}(\lambda_1)}\mathcal{I}_\zeta^{\lambda_1}\right]$$
$$\left.\times(\psi_1(\zeta)\mathcal{Y}_1^*(\zeta,\nu(\zeta-\tau)))\right)$$
$$+{}_0^{AB}\mathcal{I}_a^{\sigma_1+1}\mathcal{Y}_1^*(\vartheta,\nu(\vartheta)), \tag{3.11}$$
$$\mu(\zeta)=\int_0^1\mathcal{G}^{\sigma_1}(\zeta,\vartheta)\phi_q\left(\int_0^1\mathcal{G}^{\lambda_1}(\vartheta,\eta)(\psi_1(\eta)\right.$$
$$\left.\times\mathcal{Y}_1^*(\eta,\nu(\eta-\tau)))d\eta\right)d\vartheta$$
$$+{}_0^{AB}\mathcal{I}_a^{\sigma_1+1}\mathcal{Y}_1^*(\vartheta,\nu(\vartheta)). \tag{3.12}$$

□

Theorem 2 implies that our system of AB-fractional differential with p-Laplacian (1.1) is equivalent to a coupled system of integral equations

$$\mu(\zeta)=\int_0^1\mathcal{G}^{\sigma_1}(\zeta,\vartheta)\phi_q\left(\int_0^1\mathcal{G}^{\lambda_1}(\vartheta,\eta)(\psi_1(\eta)\right.$$
$$\left.\times\mathcal{Y}_1^*(\eta,\nu(\eta-\tau)))d\eta\right)d\vartheta$$
$$+{}_0^{AB}\mathcal{I}_a^{\sigma_1+1}\mathcal{Y}_1^*(\vartheta,\nu(\vartheta)), \tag{3.13}$$
$$\nu(\zeta)=\int_0^2\mathcal{G}^{\sigma_2}(\zeta,\vartheta)\phi_q\left(\int_0^1\mathcal{G}^{\lambda_2}(\vartheta,\eta)(\psi_2(\eta)\right.$$
$$\left.\times\mathcal{Y}_2^*(\eta,\mu(\eta-\tau)))d\eta\right)d\vartheta$$
$$+{}_0^{AB}\mathcal{I}_a^{\sigma_2+1}\mathcal{Y}_2^*(\vartheta,\mu(\vartheta)), \tag{3.14}$$

where $\mathcal{G}^{\sigma_2}(\zeta,\vartheta),\mathcal{G}^{\lambda_2}(\zeta,\vartheta)$ are the following Green functions:

$$\mathcal{G}^{\sigma_2}(\zeta,\vartheta)=\begin{cases}\dfrac{\sigma_2}{\mathbb{B}(\sigma_2)}\dfrac{(\zeta-\vartheta)^{\sigma_2-1}}{\Gamma(\sigma_2)}-\dfrac{\sigma_2}{\mathbb{B}(\sigma_2)}\dfrac{(\gamma_2-\vartheta)^{\sigma_2-1}}{\Gamma(\sigma_2)}, & \vartheta\le\zeta\le\gamma_2,\\ -\dfrac{\sigma_2}{\mathbb{B}(\sigma_2)}\dfrac{(\gamma_2-\vartheta)^{\sigma_2-1}}{\Gamma(\sigma_2)}, & \zeta\le\vartheta\le\gamma_2,\end{cases} \tag{3.15}$$

$$\mathcal{G}^{\lambda_2}(\zeta,\vartheta)=\begin{cases}\dfrac{\lambda_2}{\mathbb{B}(\lambda_2)}\dfrac{(\rho_2-\vartheta)^{\lambda_2-1}}{\Gamma(\lambda_2)}-\dfrac{\lambda_2}{\mathbb{B}(\lambda_2)}\dfrac{(\zeta-\vartheta)^{\lambda_2-1}}{\Gamma(\lambda_2)}, & \vartheta\le\zeta\le\rho_2,\\ -\dfrac{\lambda_2}{\mathbb{B}(\lambda_2)}\dfrac{(\zeta-\vartheta)^{\lambda_2-1}}{\Gamma(\lambda_2)}, & \zeta\le\vartheta\le\rho_2.\end{cases} \tag{3.16}$$

Define $\mathcal{W}_i^*:\mathfrak{F}\to\mathfrak{F}$ for $(i=1;2)$ by

$$\mathcal{W}_1^*\mu(\zeta)=\int_0^1\mathcal{G}^{\sigma_1}(\zeta,\vartheta)\phi_q\left(\int_0^1\mathcal{G}^{\lambda_1}(\vartheta,\eta)\right.$$
$$\left.\times(\psi_1(\eta)\mathcal{Y}_1^*(\eta,\nu(\eta-\tau)))d\eta\right)d\vartheta$$
$$+{}_0^{AB}\mathcal{I}_a^{\sigma_1+1}\mathcal{Y}_1^*(\vartheta,\nu(\vartheta)), \tag{3.17}$$
$$\mathcal{W}_2^*\nu(\zeta)=\int_0^1\mathcal{G}^{\sigma_2}(\zeta,\vartheta)\phi_q\left(\int_0^1\mathcal{G}^{\lambda_2}(\vartheta,\eta)\right.$$
$$\left.\times(\psi_2(\eta)\mathcal{Y}_2^*(\eta,\mu(\eta-\tau)))d\eta\right)d\vartheta$$
$$+{}_0^{AB}\mathcal{I}_a^{\sigma_2+1}\mathcal{Y}_2^*(\vartheta,\mu(\vartheta)), \tag{3.18}$$

with the help of theory 2, the positive solution for the coupled system of AB-fractional differential of integral equations (3.13), (3.14) is equivalent to the fixed point, say $(\mu,\nu)$, of the operator equation

$$(\mu,\nu)=\mathcal{W}^*(\mu,\nu)=(\mathcal{W}_1^*(\mu),\mathcal{W}_2^*(\nu))(\zeta)$$

for $\mathcal{W}=(\mathcal{W}_1^*,\mathcal{W}_2^*)$. To proceed further, we need some propositions:

$(A_1)$ $\psi:(0,1)\longrightarrow[0,+\infty]$ is discontinuous on $(0,1)$ and nonvanishing such that

$$\|\psi_1\|=\max_{\zeta\in[0,1]}\psi_1<+\infty\quad\text{and}$$
$$\|\psi_2\|=\max_{\zeta\in[0,1]}\psi_2<+\infty.$$

$(A_2)$ With positive constant value $a,b,\mathcal{T}_{\mathcal{Y}_1^*}^*,\mathcal{T}_{\mathcal{Y}_2^*}^*$ and $\varsigma_1,\varsigma_2\in[0,1]$, the functions $\mathcal{Y}_1^*,\mathcal{Y}_2^*$ satisfy the growth conditions

$$|\mathcal{Y}_1^*(\zeta,\nu(\zeta-\tau))|\le\phi_p(a\|\nu\|^{\varsigma_1}+\mathcal{T}_{\mathcal{Y}_1^*}^*),$$
$$|\mathcal{Y}_2^*(\zeta,\mu(\zeta-\tau))|\le\phi_p(b\|\mu\|^{\varsigma_2}+\mathcal{T}_{\mathcal{Y}_2^*}^*),$$
$$|\mathcal{Y}_1^*(\zeta,\nu(\zeta)|\le\phi_p(a\|\nu\|^{\varsigma_1}+\mathcal{T}_{\mathcal{Y}_1^*}^*),$$
$$|\mathcal{Y}_2^*(\zeta,\mu(\zeta))|\le\phi_p(b\|\mu\|^{\varsigma_2}+\mathcal{T}_{\mathcal{Y}_2^*}^*).$$

$(A_3)$ There exist real valued constants $\Omega_{\mathcal{Y}_1^*}, \Omega_{\mathcal{Y}_2^*}$, for all $\mu, \nu, \xi, \omega \in \mathfrak{F}$,

$$|\mathcal{Y}_1^*(\zeta, \nu(\zeta-\tau)) - \mathcal{Y}_1^*(\zeta, \xi)| \leq \Omega_{\mathcal{Y}_1^*}|\nu - \xi|,$$
$$|\mathcal{Y}_2^*(\zeta, \mu(\zeta-\tau)) - \mathcal{Y}_2^*(\zeta, \xi)| \leq \Omega_{\mathcal{Y}_2^*}|\mu - \xi|,$$
$$|\mathcal{Y}_1^*(\zeta, \nu) - \mathcal{Y}_1^*(\zeta, \omega)| \leq \Omega_{\mathcal{Y}_1^*}|\nu - \omega|,$$
$$|\mathcal{Y}_2^*(\zeta, \mu) - \mathcal{Y}_2^*(\zeta, \omega)| \leq \Omega_{\mathcal{Y}_2^*}|\mu - \omega|.$$

**Theorem 3.** *Under assumption $(A_1)$, $(A_2)$, the operator $\mathcal{W}^* : \varpi^* \to \varpi^*$ is continuous and satisfies the growth condition*

$$\|\mathcal{W}^*(\mu, \nu)\| \leq \Upsilon\|(\mu, \nu)\|^{\varepsilon} + \mathbb{F}^* \tag{3.19}$$

*for each $(\mu, \nu) \in \varrho_h \subset \varpi*$, where $\Upsilon = \nabla(a+b), \mathbb{F}^* = \nabla(\mathcal{T}_{\mathcal{Y}_1^*}) + \mathcal{T}_{\mathcal{Y}_2^*})$ and*

$$\nabla = \max\Bigg[\left(\frac{\sigma_1}{\mathbb{B}(\sigma_1)\Gamma(\sigma_1+1)} + \frac{\sigma_1\gamma^{\sigma_1}}{\mathbb{B}(\sigma_1)\Gamma(\sigma_1+1)}\right) \times \left(\frac{\lambda_1\rho_1^{\lambda_1}}{\mathbb{B}(\lambda_1)\Gamma(\lambda_1+1)} + \frac{\lambda_1}{\mathbb{B}(\lambda_1)\Gamma(\lambda_1+1)}\right)^{q-1} \times \|\psi\|^{q-1} + \phi_p\left[\frac{-\sigma_1}{\mathbb{B}(\sigma_1+1)} + \frac{(\sigma_1+1)a^{\sigma_1+2}}{\mathbb{B}(\sigma_1+1)(\sigma_1+2)\Gamma(\sigma_1+1)}\right]$$
$$\times \left(\frac{\sigma_2}{\mathbb{B}(\sigma_2)\Gamma(\sigma_2+1)} + \frac{\sigma_2\gamma^{\sigma_2}}{\mathbb{B}(\sigma_2)\Gamma(\sigma_2+1)}\right) \times \left(\frac{\lambda_2\rho_1^{\lambda_2}}{\mathbb{B}(\lambda_2)\Gamma(\lambda_2+1)} + \frac{\lambda_2}{\mathbb{B}(\lambda_2)\Gamma(\lambda_2+1)}\right)^{q-1} \times \|\psi\|^{q-1} + \phi_p\left[\frac{-\sigma_2}{\mathbb{B}(\sigma_2+1)} + \frac{(\sigma_2+1)a^{\sigma_2+2}}{\mathbb{B}(\sigma_2+1)\Gamma(\sigma_2+2)(\sigma_2+1)}\right]\Bigg].$$

**Proof.** Assume that a bounded set $\varrho_h = \{(\mu, \nu) \in \varpi : \|(\mu, \nu)\| \leq h\}$ with sequence $(\mu_n, \nu_n)$ converging to $(\mu, \nu)$ in $\varrho_h$. To show that $\|\mathcal{W}^*(\mu_n, \nu_n) - \mathcal{W}^*(\mu, \nu)\| \to 0$ as $n \to +\infty$, let us consider

$$\begin{aligned}
&|\mathcal{W}_1^*\mu_n(\zeta) - \mathcal{W}_1^*\mu(\zeta)|\\
&= \bigg|\int_0^1 \mathcal{G}^{\sigma_1}(\zeta, \vartheta)\phi_q\bigg(\int_0^1 \mathcal{G}^{\lambda_1}(\vartheta, \eta) \times \psi_1(\eta)\mathcal{Y}_1^*(\eta, \nu_n(\eta-\tau))d\eta\bigg)d\vartheta\\
&\quad + {}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_1+1}\mathcal{Y}_1^*(\vartheta, \nu_n(\vartheta)) - \int_0^1 \mathcal{G}^{\sigma_1}(\zeta, \vartheta)\phi_q \times \bigg(\int_0^1 \mathcal{G}^{\lambda_1}(\vartheta, \eta)\psi_1(\eta)\mathcal{Y}_1^*(\eta, \nu(\eta-\tau))d\eta\bigg)d\vartheta\\
&\quad + {}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_1+1}\mathcal{Y}_1^*(\vartheta, \nu(\vartheta))\bigg|\\
&\leq \int_0^1|\mathcal{G}^{\sigma_1}(\zeta, \vartheta)|\bigg|\phi_q\bigg(\int_0^1 \mathcal{G}^{\lambda_1}(\vartheta, \eta)\psi_1(\eta) \times \mathcal{Y}_1^*(\eta, \nu_n(\eta-\tau))d\eta\bigg)d\vartheta - \phi_q\bigg(\int_0^1 \mathcal{G}^{\lambda_1}(\vartheta, \eta) \times \psi_1(\eta)\mathcal{Y}_1^*(\eta, \nu(\eta-\tau))d\eta\bigg)\bigg|d\vartheta\\
&\quad + {}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_1+1}|\mathcal{Y}_1^*(\vartheta, \nu_n(\vartheta)) - \mathcal{Y}_1^*(\vartheta, \nu(\vartheta))|\\
&\leq (q-1)\Theta_1^{q-2}\int_0^1|\mathcal{G}^{\sigma_1}(\zeta, \vartheta)|\bigg[\int_0^1|\mathcal{G}^{\lambda_1}(\vartheta, \eta)| \times |\mathcal{Y}_1^*(\eta, \nu_n(\eta-\tau)) - \mathcal{Y}_1^*(\eta, \nu(\eta-\tau))|d\eta\bigg]d\vartheta\\
&\quad + {}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_1+1}|\mathcal{Y}_1^*(\vartheta, \nu_n(\vartheta)) - \mathcal{Y}_1^*(\vartheta, \nu(\vartheta))|
\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
&|\mathcal{W}_2^*\nu_n(\zeta) - \mathcal{W}_2^*\nu(\zeta)|\\
&= \bigg|\int_0^1 \mathcal{G}^{\sigma_2}(\zeta, \vartheta)\phi_q\bigg(\int_0^1 \mathcal{G}^{\lambda_2}(\vartheta, \eta)\psi_2(\eta) \times \mathcal{Y}_2^*(\eta, \mu_n(\eta-\tau))d\eta\bigg)d\vartheta + {}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_2+1} \times \mathcal{Y}_2^*(\vartheta, \mu_n(\vartheta)) - \int_0^1 \mathcal{G}^{\sigma_2}(\zeta, \vartheta)\phi_q\\
&\quad \times \bigg(\int_0^1 \mathcal{G}^{\lambda_2}(\vartheta, \eta)\psi_2(\eta)\mathcal{Y}_2^*(\eta, \mu(\eta-\tau))d\eta\bigg)d\vartheta + {}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_2+1}\mathcal{Y}_2^*(\vartheta, \mu(\vartheta))\bigg|\\
&\leq \int_0^1|\mathcal{G}^{\sigma_2}(\zeta, \vartheta)|\bigg|\phi_q\bigg(\int_0^1 \mathcal{G}^{\lambda_2}(\vartheta, \eta)\psi_2(\eta) \times \mathcal{Y}_2^*(\eta, \mu_n(\eta-\tau))d\eta\bigg)d\vartheta\\
&\quad - \phi_q\bigg(\int_0^1 \mathcal{G}^{\lambda_2}(\vartheta, \eta)\psi_2(\eta)
\end{aligned}$$

$$\times \mathcal{Y}_2^*(\eta,\mu(\eta-\tau))d\eta\Big)\Big|d\vartheta$$
$$+{}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_2+1}|\mathcal{Y}_2^*(\vartheta,\mu_n(\vartheta))-\mathcal{Y}_2^*(\vartheta,\mu(\vartheta))|$$
$$\leq (q-1)\Theta_2^{q-2}\int_0^1|\mathcal{G}^{\sigma_2}(\zeta,\vartheta)|\Big[\int_0^1|\mathcal{G}^{\lambda_2}(\vartheta,\eta)|$$
$$\times|\mathcal{Y}_2^*(\eta,\mu_n(\eta-\tau))-\mathcal{Y}_2^*(\eta,\mu(\eta-\tau))|d\eta\Big]d\vartheta$$
$$+{}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_2+1}|\mathcal{Y}_2^*(\vartheta,\mu_n(\vartheta))-\mathcal{Y}_2^*(\vartheta,\mu(\vartheta))|. \tag{3.21}$$

From (3.20) and (3.21), we have

$$|\mathcal{W}^*(\mu_n,\nu_n)(\zeta)-\mathcal{W}^*(\mu,\nu)(\zeta)|$$
$$\leq (q-1)\Theta_1^{q-2}\int_0^1|\mathcal{G}^{\sigma_1}(\zeta,\vartheta)|$$
$$\times\Big[\int_0^1|\mathcal{G}^{\lambda_1}(\vartheta,\eta)||\mathcal{Y}_1^*(\eta,\nu_n(\eta-\tau))$$
$$-\mathcal{Y}_1^*(\eta,\nu(\eta-\tau))d\eta\Big]|d\vartheta$$
$$+{}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_1+1}|\mathcal{Y}_1^*(\vartheta,\nu_n(\vartheta))-\mathcal{Y}_1^*(\vartheta,\nu(\vartheta))|$$
$$+{}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_2+1}|\mathcal{Y}_2^*(\vartheta,\mu_n(\vartheta))-\mathcal{Y}_2^*(\vartheta,\mu(\vartheta))|$$
$$+(q-1)\Theta_2^{q-2}\int_0^1|\mathcal{G}^{\sigma_2}(\zeta,\vartheta)|\Big[\int_0^1|\mathcal{G}^{\lambda_2}(\vartheta,\eta)|$$
$$\times|\mathcal{Y}_2^*(\eta,\mu_n(\eta-\tau))-\mathcal{Y}_2^*(\eta,\mu(\eta-\tau))|d\eta\Big]d\vartheta, \tag{3.22}$$

by the estimate (3.22) and continuity of the function $\mathcal{Y}_1^*,\mathcal{Y}_2^*$, we have $|\mathcal{W}^*(\mu_n,\nu_n)(\zeta)-\mathcal{W}^*(\mu,\nu)(\zeta)|\longrightarrow 0$ as $n\longrightarrow+\infty$. Consequently, the operator $\mathcal{W}^*$ is a continuous. Now, by using (3.13) and presumptions $(A_1-A_2)$, we have

$$|\mathcal{W}_1^*\mu(\zeta)|$$
$$=\Big|\int_0^1\mathcal{G}^{\sigma_1}(\zeta,\vartheta)\phi_q\Big(\int_0^1\mathcal{G}^{\lambda_1}(\vartheta,\eta)\psi_1(\eta)$$
$$\times\mathcal{Y}_1^*(\eta,\nu(\eta-\tau))d\eta\Big)d\vartheta$$
$$+{}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_1+1}\mathcal{Y}_1^*(\vartheta,\nu(\vartheta))\Big|$$
$$\leq\int_0^1|\mathcal{G}^{\sigma_1}(\zeta,\vartheta)|\phi_q\Big(\int_0^1\mathcal{G}^{\lambda_1}(\vartheta,\eta)\|\psi_1\|$$
$$\times|\mathcal{Y}_1^*(\eta,\nu(\eta-\tau))|d\eta\Big)d\vartheta$$
$$+{}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_1+1}|\mathcal{Y}_1^*(\vartheta,\nu(\vartheta))|$$
$$\leq\int_0^1|\mathcal{G}^{\sigma_1}(1,\vartheta)|\phi_q\Big(\int_0^1\mathcal{G}^{\lambda_1}(\vartheta,\eta)\|\psi_1\|$$
$$\times\phi_p(a_1\|\nu\|^{\varsigma_1}+\mathcal{T}_{\mathcal{Y}_1^*}^*)d\eta\Big)d\vartheta$$
$$+\left[\frac{-\sigma_1}{\mathbb{B}(\sigma_1+1)}+\frac{\sigma_1+1}{\mathbb{B}(\sigma_1+1)}{}_0\mathcal{I}_a^{\sigma_1+1}\right]$$
$$\times\phi_p(a_1\|\nu\|^{\varsigma_1}+\mathcal{T}_{\mathcal{Y}_1^*}^*)$$
$$\leq\left(\frac{\sigma_1}{\mathbb{B}(\sigma_1)\Gamma(\sigma_1+1)}+\frac{\sigma_1\gamma^{\sigma_1}}{\mathbb{B}(\sigma_1)\Gamma(\sigma_1+1)}\right)$$
$$\times\left(\frac{\lambda_1\rho_1^{\lambda_1}}{\mathbb{B}(\lambda_1)\Gamma(\lambda_1+1)}+\frac{\lambda_1}{\mathbb{B}(\lambda_1)\Gamma(\lambda_1+1)}\right)^{q-1}$$
$$\times\|\psi\|^{q-1}(a_1\|\nu\|^{\varsigma_1}+\mathcal{T}_{\mathcal{Y}_1^*}^*)+\left[\frac{-\sigma_2}{\mathbb{B}(\sigma_1+1)}\right.$$
$$\left.+\frac{(\sigma_1+1)a^{\sigma_1+2}}{\mathbb{B}(\sigma_1+1)(\sigma_1+2)\Gamma(\sigma_1+1)}\right]$$
$$\times\phi_p(a_1\|\nu\|^{\varsigma_1}+\mathcal{T}_{\mathcal{Y}_1^*}^*)$$
$$\leq\nabla(a_1\|\nu\|^{\varsigma_1}+\mathcal{T}_{\mathcal{Y}_1^*}^*), \tag{3.23}$$

from (3.14) and presumptions $(A_1-A_2)$, we get

$$|\mathcal{W}_2^*\nu(\zeta)|$$
$$=\Big|\int_0^1\mathcal{G}^{\sigma_2}(\zeta,\vartheta)\phi_q\Big(\int_0^1\mathcal{G}^{\lambda_2}(\vartheta,\eta)\psi_2(\eta)$$
$$\times\mathcal{Y}_2^*(\eta,\mu(\eta-\tau))d\eta\Big)d\vartheta$$
$$+{}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_2+1}\mathcal{Y}_2^*(\vartheta,\mu(\vartheta))\Big|$$
$$\leq\int_0^1|\mathcal{G}^{\sigma_2}(\zeta,\vartheta)|\phi_q\Big(\int_0^1\mathcal{G}^{\lambda_2}(\vartheta,\eta)$$
$$\times|\|\psi_2\|\mathcal{Y}_2^*(\eta,\mu(\eta-\tau))|d\eta\Big)d\vartheta$$
$$+{}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_2+1}|\mathcal{Y}_2^*(\vartheta,\mu(\vartheta))|$$
$$\leq\int_0^1|\mathcal{G}^{\sigma_2}(1,\vartheta)|\phi_q\Big(\int_0^1\mathcal{G}^{\lambda_2}(\vartheta,\eta)\|\psi_2\|$$
$$\times\phi_p(a_2\|\mu\|^{\varsigma_2}+\mathcal{T}_{\mathcal{Y}_2^*}^*)d\eta\Big)d\vartheta$$

$$
\begin{aligned}
&+\left[\frac{-\sigma_2}{\mathbb{B}(\sigma_2+1)}+\frac{\sigma_2+1}{\mathbb{B}(\sigma_2+1)}{}_0\mathcal{I}_a^{\sigma_2+1}\right]\\
&\times\phi_p(a_2\|\mu\|^{\varsigma_2}+\mathcal{T}_{\mathcal{Y}_2^*}^*)\\
&\leq\left(\frac{\sigma_2}{\mathbb{B}(\sigma_2)\Gamma(\sigma_2+1)}+\frac{\sigma_2\gamma^{\sigma_2}}{\mathbb{B}(\sigma_2)\Gamma(\sigma_2+1)}\right)\\
&\times\left(\frac{\lambda_2\rho_1^{\lambda_2}}{\mathbb{B}(\lambda_2)\Gamma(\lambda_2+1)}+\frac{\lambda_2}{\mathbb{B}(\lambda_2)\Gamma(\lambda_2+1)}\right)^{q-1}\\
&\times\|\psi\|^{q-1}(a_2\|\mu\|^{\varsigma_2}+\mathcal{T}_{\mathcal{Y}_2^*}^*)\\
&+\left[\frac{-\sigma_2}{\mathbb{B}(\sigma_2+1)}+\frac{(\sigma_2+1)a^{\sigma_2+2}}{\mathbb{B}(\sigma_2+1)(\sigma_2+2)\Gamma(\sigma_2+1)}\right]\\
&\times\phi_p(a_2\|\mu\|^{\varsigma_2}+\mathcal{T}_{\mathcal{Y}_2^*}^*)\\
&\leq\nabla(a_2\|\mu\|^{\varsigma_2}+\mathcal{T}_{\mathcal{Y}_2^*}^*). \qquad (3.24)
\end{aligned}
$$

From (3.23) and (3.24), we obtain

$$
\begin{aligned}
\mathcal{W}^*(\mu,\nu)(\zeta)&=\mathcal{W}_1^*\mu(\zeta)+\mathcal{W}_2^*\nu(\zeta)\\
&\leq\nabla(a_1\|\nu\|^{\varsigma_1}+\mathcal{T}_{\mathcal{Y}_1^*}^*)\\
&\quad+\nabla(a_2\|\mu\|^{\varsigma_2}+\mathcal{T}_{\mathcal{Y}_2^*}^*)\\
&\leq\nabla(a_1+a_2)(|\nu|+|\mu|)+(\mathcal{T}_{\mathcal{Y}_1^*}^*+\mathcal{T}_{\mathcal{Y}_2^*}^*)\\
&=\Upsilon\|(\mu,\nu)\|^{\varepsilon}+\mathbb{F}^*. \qquad (3.25)
\end{aligned}
$$

□

**Theorem 4.** *Let suppositions $(A_1)$ and $(A_2)$ hold, then the operator $\mathcal{W}^*:\varpi^*\to\varpi^*$ is compact and $\xi$-Lipschitz with constant zero.*

**Proof.** By using Theorem 3, we conclude that $\mathcal{W}^*:\omega\to\omega$ is bounded. Next, by suppositions $(A_1)$ and $(A_2)$, Lemma 2 and Eq. (3.13), for any $\zeta_1,\zeta_2\in[0,1]$, we get

$$
\begin{aligned}
&|\mathcal{W}_1^*\mu(\zeta_1)-\mathcal{W}_1^*\mu(\zeta_2)|\\
&=\bigg|\int_0^1\mathcal{G}^{\sigma_1}(\zeta_1,\vartheta)\phi_q\bigg(\int_0^1\mathcal{G}^{\lambda_1}(\vartheta,\eta)\psi_1(\eta)\\
&\times\mathcal{Y}_1^*(\eta,\nu(\eta-\tau))d\eta\bigg)d\vartheta\\
&+{}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_1+1}\mathcal{Y}_1^*(\vartheta,\nu(\vartheta))\\
&-\int_0^1\mathcal{G}^{\sigma_1}(\zeta_2,\vartheta)\phi_q\bigg(\int_0^1\mathcal{G}^{\lambda_1}(\vartheta,\eta)\psi_1(\eta)\\
&\times\mathcal{Y}_1^*(\eta,\nu(\eta-\tau))d\eta\bigg)d\vartheta\\
&+{}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_1+1}\mathcal{Y}_1^*(\vartheta,\nu(\vartheta))\bigg|\\
&\leq\int_0^1|\mathcal{G}^{\sigma_1}(\zeta_1,\vartheta)-\mathcal{G}^{\sigma_1}(\zeta_2,\vartheta)|\phi_q\\
&\times\bigg(\int_0^1\mathcal{G}^{\lambda_1}(\vartheta,\eta)\|\psi_1\|\\
&\times|\mathcal{Y}_1^*(\eta,\nu(\eta-\tau))|d\eta\bigg)d\vartheta+\bigg[\frac{-\sigma_1}{\mathbb{B}(\sigma_1+1)}\\
&+\frac{\sigma_1+1}{\mathbb{B}(\sigma_1+1)}{}_0\mathcal{I}_a^{\sigma_1+1}\bigg]|\mathcal{Y}_1^*(\vartheta,\nu(\vartheta))|\\
&\leq(q-1)\Theta_1^{q-2}\bigg(\frac{|\zeta_1^{\sigma_1}-\zeta_2^{\sigma_1}|}{\mathbb{B}(\sigma_1)\Gamma(\sigma_1)}\bigg)\\
&\times\bigg(\frac{2}{\mathbb{B}(\lambda_1)\Gamma(\lambda_1)}\bigg)^{q-1}\|\psi_1\|^{q-1}(a|\nu|^{\varsigma_1}+\mathcal{T}_{\mathcal{Y}_1^*}^*)\\
&+\bigg[\frac{-\sigma_1}{\mathbb{B}(\sigma_1+1)}+\frac{(\sigma_1+1)a^{\sigma_1+2}}{\mathbb{B}(\sigma_1+1)(\sigma_1+2)\Gamma(\sigma_1+1)}\bigg]\\
&\times\phi_p(a|\nu|^{\varsigma_1}+\mathcal{T}_{\mathcal{Y}_1^*}^*)\\
&\leq\bigg[(q-1)\Theta_1^{q-2}\bigg(\frac{|\zeta_1^{\sigma_1}-\zeta_2^{\sigma_1}|}{\mathbb{B}(\sigma_1)\Gamma(\sigma_1)}\bigg)\\
&\times\bigg(\frac{2}{\mathbb{B}(\lambda_1)\Gamma(\lambda_1)}\bigg)^{q-1}\|\psi_1\|^{q-1}+\bigg[\frac{-\sigma_1}{\mathbb{B}(\sigma_1+1)}\\
&+\frac{(\sigma_1+1)a^{\sigma_1+2}}{\mathbb{B}(\sigma_1+1)(\sigma_1+2)\Gamma(\sigma_1+1)}\bigg]\phi_p\bigg]\\
&\times(a|\nu|^{\varsigma_1}+\mathcal{T}_{\mathcal{Y}_1^*}^*) \qquad (3.26)
\end{aligned}
$$

and

$$
\begin{aligned}
&|\mathcal{W}_2^*\nu(\zeta_1)-\mathcal{W}_2^*\nu(\zeta_2)|\\
&=\bigg|\int_0^1\mathcal{G}^{\sigma_2}(\zeta_1,\vartheta)\phi_q\bigg(\int_0^1\mathcal{G}^{\lambda_2}(\vartheta,\eta)\psi_2(\eta)\\
&\times\mathcal{Y}_2^*(\eta,\mu(\eta-\tau))d\eta\bigg)d\vartheta\\
&+{}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_2+1}\mathcal{Y}_2^*(\vartheta,\mu(\vartheta))\\
&-\int_0^1\mathcal{G}^{\sigma_2}(\zeta_2,\vartheta)\phi_q\bigg(\int_0^1\mathcal{G}^{\lambda_2}(\vartheta,\eta)\psi_2(\eta)\\
&\times\mathcal{Y}_2^*(\eta,\mu(\eta-\tau))d\eta\bigg)d\vartheta\\
&+{}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_2+1}\mathcal{Y}_2^*(\vartheta,\mu(\vartheta))\bigg|\\
&\times\int_0^1|\mathcal{G}^{\sigma_2}(\zeta_1,\vartheta)-\mathcal{G}^{\sigma_2}(\zeta_2,\vartheta)|\phi_q
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(\int_0^1 \mathcal{G}^{\lambda_2}(\vartheta,\eta)\|\psi_2\||\mathcal{Y}_2^*(\eta,\mu(\eta-\tau))|d\eta\right)d\vartheta\\
&+\left[\frac{-\sigma_2}{\mathbb{B}(\sigma_2+1)}+\frac{\sigma_2+1}{\mathbb{B}(\sigma_2+1)}\,{}_0\mathcal{I}_a^{\sigma_2+1}\right]\\
&\times|\mathcal{Y}_2^*(\vartheta,\mu(\vartheta))|\\
\leq{}&(q-1)\Theta_2^{q-2}\left(\frac{|\zeta_1^{\sigma_2}-\zeta_2^{\sigma_2}|}{\mathbb{B}(\sigma_2)\Gamma(\sigma_2)}\right)\\
&\times\left(\frac{2}{\mathbb{B}(\lambda_2)\Gamma(\lambda_2)}\right)^{q-1}\|\psi_2\|^{q-1}(a|\mu|^{\varsigma_2}+\mathcal{T}^*_{\mathcal{Y}_2^*})\\
&+\left[\frac{-\sigma_2}{\mathbb{B}(\sigma_2+1)}+\frac{(\sigma_2+1)a^{\sigma_2+2}}{\mathbb{B}(\sigma_2+1)(\sigma_2+2)\Gamma(\sigma_2+1)}\right]\\
&\times\phi_p(a|\nu|^{\varsigma_1}+\mathcal{T}^*_{\mathcal{Y}_2^*})\\
\leq{}&\left[(q-1)\Theta_2^{q-2}\left(\frac{|\zeta_1^{\sigma_2}-\zeta_2^{\sigma_2}|}{\mathbb{B}(\sigma_2)\Gamma(\sigma_2)}\right)\right.\\
&\times\left(\frac{2\|\psi_2\|}{\mathbb{B}(\lambda_2)\Gamma(\lambda_2)}\right)^{q-1}+\left[\frac{-\sigma_2}{\mathbb{B}(\sigma_2+1)}\right.\\
&\left.\left.+\frac{(\sigma_2+1)a^{\sigma_2+2}}{\mathbb{B}(\sigma_2+1)(\sigma_2+2)\Gamma(\sigma_2+1)}\right]\phi_p\right]\\
&\times(a|\mu|^{\varsigma_2}+\mathcal{T}^*_{\mathcal{Y}_2^*}).
\end{aligned}
\tag{3.27}
$$

From (3.26) and (3.27), we get

$$
\begin{aligned}
&|\mathcal{W}^*(\mu,\nu)(\zeta_1)-\mathcal{W}^*(\mu,\nu)(\zeta_2)|\\
&=\left[(q-1)\Theta_1^{q-2}\left(\frac{|\zeta_1^{\sigma_1}-\zeta_2^{\sigma_1}|}{\mathbb{B}(\sigma_1)\Gamma(\sigma_1)}\right)\right.\\
&\quad\times\left(\frac{2\|\psi_1\|}{\mathbb{B}(\lambda_1)\Gamma(\lambda_1)}\right)^{q-1}+\left[\frac{-\sigma_1}{\mathbb{B}(\sigma_1+1)}\right.\\
&\quad\left.\left.+\frac{(\sigma_1+1)a^{\sigma_1+2}}{\mathbb{B}(\sigma_1+1)(\sigma_1+2)\Gamma(\sigma_1+1)}\right]\phi_p\right]\\
&\quad\times(a|\nu|^{\varsigma_1}+\mathcal{T}^*_{\mathcal{Y}_1^*})\\
&\leq\left[(q-1)\Theta_2^{q-2}\left(\frac{|\zeta_1^{\sigma_2}-\zeta_2^{\sigma_2}|}{\mathbb{B}(\sigma_2)\Gamma(\sigma_2)}\right)\right.\\
&\quad\times\left(\frac{2\|\psi_2\|}{\mathbb{B}(\lambda_2)\Gamma(\lambda_2)}\right)^{q-1}+\left[\frac{-\sigma_2}{\mathbb{B}(\sigma_2+1)}\right.\\
&\quad\left.\left.+\frac{(\sigma_2+1)a^{\sigma_2+2}}{\mathbb{B}(\sigma_2+1)(\sigma_2+2)\Gamma(\sigma_2+1)}\right]\phi_p\right]\\
&\quad\times(a|\mu|^{\varsigma_2}+\mathcal{T}^*_{\mathcal{Y}_2^*}).
\end{aligned}
\tag{3.28}
$$

As $\zeta_1\to\zeta_2$, the right-hand side of (3.28) approaches zero. Thus, the operator $\mathcal{W}^*=(\mathcal{W}_1^*,\mathcal{W}_2^*)$ is an equicontinuous on $\mathcal{S}$. Arela–Ascoli theorem implies that $\mathcal{W}^*(\mathcal{S})$ is compact. Subsequently, $\mathcal{S}$ is $\xi$-Lipschitz with constant zero. □

**Theorem 5.** *Under the suppositions $(A_2)$ and $(A_3)$ and $\Upsilon<1$. Then the coupled system of AB-fractional equation* (1.1) *has solution and the set containing solutions of AB-fractional equation is bounded in $\varpi^*$.*

**Proof.** For EUS of the AB-fractional equation for problem (1.1), with the help of Theorem 1, let us consider that $\mathcal{U}^*=\{(\mu,\nu)\in\varpi^*$ : there exist $\mathbb{O}\in[0,1]$, where $(\mu,\nu)=\mathbb{O}\mathbb{X}(\mu,\nu)\}$, to prove to that $\mathcal{U}^*$ is bounded, for this, we presume a contrary way. Presume that $(\mu,\nu)\in\mathcal{U}^*$, such that $\|(\mu,\nu)\|=\mathcal{N}\to\infty$ forms Theorem 3, we get

$$
\begin{aligned}
\|(\mu,\nu)\|&=\|\mathbb{O}\mathbb{X}(\mu,\nu)\|\\
&\leq\|\mathbb{X}(\mu,\nu)\|\\
&\leq\|\mathcal{W}_1^*(\mu)+\mathcal{W}_2^*(\nu)\|\\
&\leq\mathbb{F}^*+\Upsilon\|(\mu,\nu)\|^{\varepsilon}
\end{aligned}
\tag{3.29}
$$

as $\|(\mu,\nu)\|=\mathcal{N}$, then (3.29) implies that

$$
\begin{aligned}
&\|(\mu,\nu)\|\leq\Upsilon\|(\mu,\nu)\|^{\varepsilon}+\mathbb{F}^*,\\
&1\leq\Upsilon\frac{\|(\mu,\nu)\|^{\varepsilon}}{\|(\mu,\nu)\|}+\frac{\mathbb{F}^*}{\|(\mu,\nu)\|},\\
&1\leq\Upsilon\frac{1}{\mathcal{N}^{1-\varepsilon}}+\frac{\mathbb{F}^*}{\mathcal{N}}\to 0,\quad\text{as }\mathcal{N}\to\infty.
\end{aligned}
$$

This is a contradiction. In the end, $\|(\mu,\nu)\|<\infty$ which means that $\mathcal{U}^*$ is bounded and by theory 1, the operator $\mathcal{U}^*$ has at least one solution to the suggested problem (1.1). Thus, the solution $\mathcal{U}^*$ for the coupled system of AB-fractional equation (1.1) is bounded. □

**Theorem 6.** *Let suppositions $(A_1)$ and $(A_3)$ hold. Then the coupled system of AB-fractional equation* (1.1) *has a unique positive solution provided that $\kappa^*<1$ such that*

$$
\begin{aligned}
\kappa^*=\Bigg[&(q-1)\Theta_1^{q-2}\\
&\times\left(\frac{\sigma_1}{\mathbb{B}(\sigma_1)\Gamma(\sigma_1+1)}+\frac{\sigma_1\gamma^{\sigma_1}}{\mathbb{B}(\sigma_1)\Gamma(\sigma_1+1)}\right)\\
&\times\left(\frac{\lambda_1\rho_1^{\lambda_1}}{\mathbb{B}(\lambda_1)\Gamma(\lambda_1+1)}+\frac{\lambda_1}{\mathbb{B}(\lambda_1)\Gamma(\lambda_1+1)}\right)
\end{aligned}
$$

$$\times \|\psi_1\|\Omega_{\mathcal{Y}_1^*} + \left[\frac{-\sigma_1}{\mathbb{B}(\sigma_1+1)} + \frac{(\sigma_1+1)a^{\sigma_1+2}}{\mathbb{B}(\sigma_1+1)(\sigma_1+2)\Gamma(\sigma_1+1)}\right]\Omega_{\mathcal{Y}_1^*}\Bigg] + \Bigg[(q-1)\Theta_2^{q-2}\left(\frac{\sigma_2}{\mathbb{B}(\sigma_2)\Gamma(\sigma_2+1)} + \frac{\sigma_2\gamma^{\sigma_2}}{\mathbb{B}(\sigma_1)\Gamma(\sigma_2+1)}\right)\left(\frac{\lambda_2\rho_1^{\lambda_2}}{\mathbb{B}(\lambda_2)\Gamma(\lambda_2+1)} + \frac{\lambda_2}{\mathbb{B}(\lambda_2)\Gamma(\lambda_2+1)}\right) \times \|\psi_2\|\Omega_{\mathcal{Y}_2^*} + \left[\frac{-\sigma_2}{\mathbb{B}(\sigma_2+1)} + \frac{(\sigma_2+1)a^{\sigma_2+2}}{\mathbb{B}(\sigma_2+1)(\sigma_2+2)\Gamma(\sigma_2+1)}\right]\Omega_{\mathcal{Y}_2^*}\Bigg].$$

**Proof.** From (3.13), suppositions $(A_1)$ and $(A_3)$ and Lemma 2, we have

$$\begin{aligned}
&|\mathcal{W}_1^*\mu(\zeta) - \mathcal{W}_1^*\bar{\mu}(\zeta)| \\
&= \Bigg| \int_0^1 \mathcal{G}^{\sigma_1}(\zeta,\vartheta)\phi_q\left(\int_0^1 \mathcal{G}^{\lambda_1}(\vartheta,\eta)\psi_1(\eta) \times \mathcal{Y}_1^*(\eta,\nu(\eta-\tau))d\eta\right)d\vartheta + {}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_1+1}\mathcal{Y}_1^*(\vartheta,\nu(\vartheta)) \\
&\quad - \int_0^1 \mathcal{G}^{\sigma_1}(\zeta,\vartheta)\phi_q\left(\int_0^1 \mathcal{G}^{\lambda_1}(\vartheta,\eta)\psi_1(\eta) \times \mathcal{Y}_1^*(\eta,\bar{\nu}(\eta-\tau))d\eta\right)d\vartheta + {}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_1+1}\mathcal{Y}_1^*(\vartheta,\bar{\nu}(\vartheta))\Bigg| \\
&\le (q-1)\Theta_1^{q-2}\int_0^1 |\mathcal{G}^{\sigma_1}(1,\vartheta)| \times \int_0^1 \mathcal{G}^{\lambda_1}(\vartheta,\eta)|\|\psi_1\|\mathcal{Y}_1^*(\eta,\nu(\eta-\tau) - \mathcal{Y}_1^*(\eta,\bar{\nu}(\eta-\tau))|d\eta d\vartheta \\
&\quad + \left[\frac{-\sigma_1}{\mathbb{B}(\sigma_1+1)} + \frac{\sigma_1+1}{\mathbb{B}(\sigma_1+1)}\mathcal{I}_a^{\sigma_1+1}\right] \times |\mathcal{Y}_1^*(\vartheta,\nu(\vartheta)) - \mathcal{Y}_1^*(\vartheta,\bar{\nu}(\vartheta))| \\
&\le \Bigg[(q-1)\Theta_1^{q-2}\left(\frac{\sigma_1}{\mathbb{B}(\sigma_1)\Gamma(\sigma_1+1)} + \frac{\sigma_1\gamma^{\sigma_1}}{\mathbb{B}(\sigma_1)\Gamma(\sigma_1+1)}\right)\left(\frac{\lambda_1\rho_1^{\lambda_1}}{\mathbb{B}(\lambda_1)\Gamma(\lambda_1+1)} + \frac{\lambda_1}{\mathbb{B}(\lambda_1)\Gamma(\lambda_1+1)}\right)\|\psi_1\|\Omega_{\mathcal{Y}_1^*} \\
&\quad + \left[\frac{-\sigma_1}{\mathbb{B}(\sigma_1+1)} + \frac{(\sigma_1+1)a^{\sigma_1+2}}{\mathbb{B}(\sigma_1+1)(\sigma_1+2)\Gamma(\sigma_1+1)}\right]\Omega_{\mathcal{Y}_1^*}\Bigg] \times |\nu(\zeta) - \bar{\nu}(\zeta)|,
\end{aligned} \tag{3.30}$$

from (3.14), suppositions $(A_1)$ and $(A_3)$ and Lemma 2, we get

$$\begin{aligned}
&|\mathcal{W}_2^*\nu(\zeta) - \mathcal{W}_2^*\bar{\nu}(\zeta)| \\
&= \Bigg| \int_0^1 \mathcal{G}^{\sigma_2}(\zeta,\vartheta)\phi_q\left(\int_0^1 \mathcal{G}^{\lambda_2}(\vartheta,\eta)\psi_2(\eta) \times \mathcal{Y}_1^*(\eta,\mu(\eta-\tau))d\eta\right)d\vartheta + {}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_2+1}\mathcal{Y}_2^*(\vartheta,\mu(\vartheta)) \\
&\quad - \int_0^1 \mathcal{G}^{\sigma_2}(\zeta,\vartheta)\phi_q\left(\int_0^1 \mathcal{G}^{\lambda_2}(\vartheta,\eta)\psi_2(\eta) \times \mathcal{Y}_2^*(\eta,\bar{\mu}(\eta-\tau))d\eta\right)d\vartheta + {}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_2+1}\mathcal{Y}_2^*(\vartheta,\bar{\mu}(\vartheta))\Bigg| \\
&\le (q-1)\Theta_2^{q-2}\int_0^1 |\mathcal{G}^{\sigma_2}(1,\vartheta)| \times \int_0^1 \mathcal{G}^{\lambda_2}(\vartheta,\eta)|\|\psi_2\|\mathcal{Y}_2^*(\eta,\mu(\eta-\tau) - \mathcal{Y}_2^*(\eta,\bar{\mu}(\eta-\tau))|d\eta d\vartheta \\
&\quad + \left[\frac{-\sigma_2}{\mathbb{B}(\sigma_2+1)} + \frac{\sigma_2+1}{\mathbb{B}(\sigma_2+1)}{}_0\mathcal{I}_a^{\sigma_2+1}\right] \times |\mathcal{Y}_2^*(\vartheta,\mu(\vartheta)) - \mathcal{Y}_2^*(\vartheta,\bar{\mu}(\vartheta))| \\
&\le \Bigg[(q-1)\Theta_2^{q-2}\left(\frac{\sigma_2}{\mathbb{B}(\sigma_2)\Gamma(\sigma_2+1)} + \frac{\sigma_2\gamma^{\sigma_2}}{\mathbb{B}(\sigma_1)\Gamma(\sigma_2+1)}\right)\left(\frac{\lambda_2\rho_1^{\lambda_2}}{\mathbb{B}(\lambda_2)\Gamma(\lambda_2+1)} + \frac{\lambda_2}{\mathbb{B}(\lambda_2)\Gamma(\lambda_2+1)}\right)\|\psi_2\|\Omega_{\mathcal{Y}_2^*} + \Bigg[\frac{-\sigma_2}{\mathbb{B}(\sigma_2+1)}
\end{aligned}$$

$$+\frac{(\sigma_2+1)a^{\sigma_2+2}}{\mathbb{B}(\sigma_2+1)(\sigma_2+2)\Gamma(\sigma_2+1)}\Bigg]\Omega_{\mathcal{Y}_2^*}\Bigg] \times|\mu(\zeta)-\bar{\mu}(\zeta)|, \tag{3.31}$$

with the help of (3.30) and (3.31), we obtain

$$\begin{aligned}
&|\mathcal{W}^*(\mu,\nu)(\zeta)-\mathcal{W}^*(\bar{\mu},\bar{\nu})(\zeta)| \\
&\le \Bigg[(q-1)\Theta_1^{q-2}\Bigg(\frac{\sigma_1}{\mathbb{B}(\sigma_1)\Gamma(\sigma_1+1)} + \frac{\sigma_1\gamma^{\sigma_1}}{\mathbb{B}(\sigma_1)\Gamma(\sigma_1+1)}\Bigg)\Bigg(\frac{\lambda_1\rho_1^{\lambda_1}}{\mathbb{B}(\lambda_1)\Gamma(\lambda_1+1)} + \frac{\lambda_1}{\mathbb{B}(\lambda_1)\Gamma(\lambda_1+1)}\Bigg)\|\psi_1\|\Omega_{\mathcal{Y}_1^*} + \Bigg[\frac{-\sigma_1}{\mathbb{B}(\sigma_1+1)} \\
&\quad + \frac{(\sigma_1+1)a^{\sigma_1+2}}{\mathbb{B}(\sigma_1+1)(\sigma_1+2)\Gamma(\sigma_1+1)}\Bigg]\Omega_{\mathcal{Y}_1^*}\Bigg] \times|\nu(\zeta)-\bar{\nu}(\zeta)| \\
&\quad + \Bigg[(q-1)\Theta_2^{q-2}\Bigg(\frac{\sigma_2}{\mathbb{B}(\sigma_2)\Gamma(\sigma_2+1)} + \frac{\sigma_2\gamma^{\sigma_2}}{\mathbb{B}(\sigma_1)\Gamma(\sigma_2+1)}\Bigg)\Bigg(\frac{\lambda_2\rho_1^{\lambda_2}}{\mathbb{B}(\lambda_2)\Gamma(\lambda_2+1)} + \frac{\lambda_2}{\mathbb{B}(\lambda_2)\Gamma(\lambda_2+1)}\Bigg)\times\|\psi\|\Omega_{\mathcal{Y}_2^*}| + \Bigg[\frac{-\sigma_2}{\mathbb{B}(\sigma_2+1)} \\
&\quad + \frac{(\sigma_2+1)a^{\sigma_2+2}}{\mathbb{B}(\sigma_2+1)(\sigma_2+2)\Gamma(\sigma_2+1)}\Bigg]\Omega_{\mathcal{Y}_2^*}\Bigg] \times|\mu(\zeta)-\bar{\mu}(\zeta)| \\
&\le \Bigg[(q-1)\Theta_1^{q-2}\Bigg(\frac{\sigma_1}{\mathbb{B}(\sigma_1)\Gamma(\sigma_1+1)} + \frac{\sigma_1\gamma^{\sigma_1}}{\mathbb{B}(\sigma_1)\Gamma(\sigma_1+1)}\Bigg)\Bigg(\frac{\lambda_1\rho_1^{\lambda_1}}{\mathbb{B}(\lambda_1)\Gamma(\lambda_1+1)} + \frac{\lambda_1}{\mathbb{B}(\lambda_1)\Gamma(\lambda_1+1)}\Bigg)\times\|\psi\|\Omega_{\mathcal{Y}_1^*} + \Bigg[\frac{-\sigma_1}{\mathbb{B}(\sigma_1+1)} \\
&\quad + \frac{(\sigma_1+1)a^{\sigma_1+2}}{\mathbb{B}(\sigma_1+1)(\sigma_1+2)\Gamma(\sigma_1+1)}\Bigg]\Omega_{\mathcal{Y}_1^*}\Bigg] \\
&\quad + \Bigg[(q-1)\Theta_2^{q-2}\Bigg(\frac{\sigma_2}{\mathbb{B}(\sigma_2)\Gamma(\sigma_2+1)} + \frac{\sigma_2\gamma^{\sigma_2}}{\mathbb{B}(\sigma_1)\Gamma(\sigma_2+1)}\Bigg)\Bigg(\frac{\lambda_2\rho_1^{\lambda_2}}{\mathbb{B}(\lambda_2)\Gamma(\lambda_2+1)} + \frac{\lambda_2}{\mathbb{B}(\lambda_2)\Gamma(\lambda_2+1)}\Bigg)\times\|\psi\|\Omega_{\mathcal{Y}_2^*} + \Bigg[\frac{-\sigma_2}{\mathbb{B}(\sigma_2+1)} \\
&\quad + \frac{(\sigma_2+1)a^{\sigma_2+2}}{\mathbb{B}(\sigma_2+1)(\sigma_2+2)\Gamma(\sigma_2+1)}\Bigg]\Omega_{\mathcal{Y}_2^*}\Bigg] \times(\|(\mu,\nu)(\zeta)-(\bar{\mu},\bar{\nu})(\zeta)\|) \\
&= \kappa^*(\|(\mu,\nu)(\zeta)-(\bar{\mu},\bar{\nu})(\zeta)\|),
\end{aligned} \tag{3.32}$$

with $\kappa^* < 1$. Banach's contraction principle implies that $\mathcal{W}^*$ has a unique fixed point. Consequently, the coupled system (1.1) of AB-fractional equation has a unique positive solution. □

## 4. HYERS–ULAM STABILITY OF SYSTEM

Here we prove appropriate and needed conditions for HU- stability and GUH-stability for the solution to the coupled system of AB-fractional equation in this section of the manuscript. We provide the important definitions and the desired remark; for more details, see Refs. 49 and 50. Let $\mathfrak{J} = (C[0,1],\mathbb{R})$ be a Banach space.

**Definition 8.** The coupled system of AB-fractional equation (1.1) is Hyers–Ulam stable if we can find a positive number $C_{\mathcal{Y}^*}$, such that, for $\epsilon > 0$ and for any solution $\mu,\nu \in \mathfrak{J}$ of the inequality

$$\begin{aligned}
&|{}_0^{\mathrm{ABC}}\mathcal{D}^{\lambda_1}(\phi_p({}_0^{\mathrm{ABC}}\mathcal{D}^{\sigma_1}(\mu(\zeta)))) \\
&\quad = -\psi_1(\zeta)\mathcal{Y}_1^*(\zeta,\nu(\zeta-\tau))| \le \epsilon_1, \quad \forall\,\zeta\in[0,1], \\
&|{}_0^{\mathrm{ABC}}\mathcal{D}^{\lambda_2}(\phi_p({}_0^{\mathrm{ABC}}\mathcal{D}^{\sigma_2}(\mu(\zeta)))) \\
&\quad = -\psi_2(\zeta)\mathcal{Y}_2^*(\zeta,\nu(\zeta-\tau))| \le \epsilon_2, \quad \forall\,\zeta\in[0,1],
\end{aligned} \tag{4.1}$$

there is the unique positive solution $\aleph,\wp \in \mathfrak{J}$ of suggested problem (1.1) such that

$$|\mu(\zeta)-\aleph(\zeta)| \le C_{\mathcal{Y}^*}\epsilon, \quad \forall\,\zeta\in[0,1],$$

$$|\nu(\zeta)-\wp(\zeta)| \le C_{\mathcal{Y}^*}\epsilon, \quad \forall\,\zeta\in[0,1],$$

it is going to be generalized Hyers–Ulam (GHU) stable, if there exists

$$\mathcal{Z}:(0,\infty)\longrightarrow(0,\infty), \quad \mathcal{Z}(0)=0,$$

such that

$$\|\mu-\aleph\| \le C_{\mathcal{Y}^*}\mathcal{Z}(\epsilon),$$

$$\|\nu-\varrho\| \le C_{\mathcal{Y}^*}\mathcal{Z}(\epsilon).$$

**Remark 1.** A function $\mu(\zeta) \in \mathfrak{J}$ is a solution of inequality (4.1) if and only if there exists a function $\delta \in \mathfrak{J}$ such that

($\iota$) $|\delta_1(\zeta)| \leq \epsilon_1, |\delta_2(\zeta)| \leq \epsilon_2$;

($\iota\iota$)
$$\begin{aligned}&{}_0^{\mathrm{ABC}}\mathcal{D}^{\lambda_1}(\phi_p({}_0^{\mathrm{ABC}}\mathcal{D}^{\sigma_1}(\mu(\zeta))))\\&\quad = -\psi_1(\zeta)\mathcal{Y}_1^*(\zeta,\nu(\zeta-\tau))| + \delta_1(\zeta),\\&{}_0^{\mathrm{ABC}}\mathcal{D}^{\lambda_2}(\phi_p({}_0^{\mathrm{ABC}}\mathcal{D}^{\sigma_2}(\mu(\zeta))))\\&\quad = -\psi_2(\zeta)\mathcal{Y}_2^*(\zeta,\nu(\zeta-\tau))| + \delta_2(\zeta).\end{aligned}$$

**Definition 9.** The coupled system of AB-fractional equation of integral equations (3.13) and (3.14) is HU-stable if there exist constants $\mathbb{V}^*_{1,\mathcal{Y}^*}, \mathbb{V}^*_{2,\mathcal{Y}^*}$ that achieve these conditions.

For each $\mathfrak{y}_1, \mathfrak{y}_2 > 0$, if

$$\begin{aligned}&\left|\mu(\zeta) - \int_0^1 \mathcal{G}^{\sigma_1}(\zeta,\vartheta)\phi_q\left(\int_0^1 \mathcal{G}^{\lambda_1}(\vartheta,\eta)(\psi_1(\eta)\right.\right.\\&\quad\left.\times\mathcal{Y}_1^*(\eta,\nu(\eta-\tau)))d\eta\right)d\vartheta\\&\quad\left.+{}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_1+1}\mathcal{Y}_1^*(\vartheta,\nu(\vartheta))\right| \leq \mathfrak{y}_1, \qquad (4.2)\end{aligned}$$

$$\begin{aligned}&\left|\nu(\zeta) - \int_0^2 \mathcal{G}^{\sigma_2}(\zeta,\vartheta)\phi_q\left(\int_0^1 \mathcal{G}^{\lambda_2}(\vartheta,\eta)(\psi_2(\eta)\right.\right.\\&\quad\left.\times\mathcal{Y}_2^*(\eta,\mu(\eta-\tau)))d\eta\right)d\vartheta\\&\quad\left.+{}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_2+1}\mathcal{Y}_2^*(\vartheta,\mu(\vartheta))\right| \leq \mathfrak{y}_2. \qquad (4.3)\end{aligned}$$

**Lemma 3.** *Under the suppositions* ($A_1$) *and* ($A_3$) *hold, Remark* 1 *and the function* $\mu,\nu \in \mathfrak{J}$ *corresponding to the solution of the coupled system of our problem*

$$\begin{cases}{}_0^{\mathrm{ABC}}\mathcal{D}^{\lambda_1}(\phi_p({}_0^{\mathrm{ABC}}\mathcal{D}^{\sigma_1}(\mu(\zeta))))\\\quad = -\psi_1(\zeta)\mathcal{Y}_1^*(\zeta,\nu(\zeta-\tau)) + \delta_1(\zeta),\\{}_0^{\mathrm{ABC}}\mathcal{D}^{\lambda_2}(\phi_p({}_0^{\mathrm{ABC}}\mathcal{D}^{\sigma_2}(\nu(\zeta))))\\\quad = -\psi_2(\zeta)\mathcal{Y}_2^*(\zeta,\mu(\zeta-\tau)) + \delta_2(\zeta),\\\phi_p({}_0^{\mathrm{ABC}}\mathcal{D}^{\sigma_1}(\mu(\zeta)))|_{\zeta=\rho_1} = 0,\\\quad\phi_p({}_0^{\mathrm{ABC}}\mathcal{D}^{\sigma_2}(\nu(\zeta)))|_{\zeta=\rho_2} = 0,\\\mu(\gamma_1) = {}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_1+1}\mathcal{Y}_1^*(\vartheta,\nu(\vartheta)),\\\quad\nu(\gamma_2) = {}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_2+1}\mathcal{Y}_2^*(\vartheta,\mu(\vartheta)),\end{cases} \qquad (4.4)$$

*satisfies the relation given by*

$$|\mu(\zeta) - \mathbb{F}\mu^*(\zeta)| \leq \mathbb{V}^*_{1,\mathcal{Y}^*}\mathfrak{y}_1, \qquad (4.5)$$

$$|\nu(\zeta) - \mathbb{F}\nu^*(\zeta)| \leq \mathbb{V}^*_{2,\mathcal{Y}^*}\mathfrak{y}_2, \qquad (4.6)$$

*where*

$$\begin{aligned}\mathbb{F}\mu^*(\zeta) &= \int_0^1 \mathcal{G}^{\sigma_1}(\zeta,\vartheta)\phi_q\left(\int_0^1 \mathcal{G}^{\lambda_1}(\vartheta,\eta)\right.\\&\quad\left.\times(\psi_1(\eta)\mathcal{Y}_1^*(\eta,\nu^*(\eta-\tau)))d\eta\right)d\vartheta\\&\quad+{}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_1+1}\mathcal{Y}_1^*(\vartheta,\nu^*(\vartheta)), \qquad (4.7)\\\mathbb{F}\nu^*(\zeta) &= \int_0^2 \mathcal{G}^{\sigma_2}(\zeta,\vartheta)\phi_q\left(\int_0^1 \mathcal{G}^{\lambda_2}(\vartheta,\eta)\right.\\&\quad\left.\times(\psi_2(\eta)\mathcal{Y}_2^*(\eta,\mu^*(\eta-\tau)))d\eta\right)d\vartheta\\&\quad+{}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_2+1}\mathcal{Y}_2^*(\vartheta,\mu^*(\vartheta)). \qquad (4.8)\end{aligned}$$

**Theorem 7.** *The suppositions* ($A_1$) *and* ($A_3$) *hold and Remark* 1 *with the condition that* $1 > \mathbb{V}^*_{1,\mathcal{Y}^*}$ *and* $1 > \mathbb{V}^*_{2,\mathcal{Y}^*}$ *then the coupled system AB-fractional equation with p-Laplacian operator* (1.1) *is HU-stable.*

**Proof.** From Theory 6 and Definition 8, let $\mu(\zeta), \nu(\zeta)$ be a positive solution for coupled system of integral equations (3.13), (3.14) and $\mathbb{F}\mu^*(\zeta), \mathbb{F}\nu^*(\zeta)$ be any other approximation achieving (4.7) and (4.8), we get

$$\begin{aligned}&|\mu(\zeta) - \mathbb{F}\mu^*(\zeta)|\\&\quad= \left|\int_0^1 \mathcal{G}^{\sigma_1}(\zeta,\vartheta)\phi_q\left(\int_0^1 \mathcal{G}^{\lambda_1}(\vartheta,\eta)\right.\right.\\&\qquad\left.\times(\psi_1(\eta)\mathcal{Y}_1^*(\eta,\nu(\eta-\tau)))d\eta\right)d\vartheta\\&\qquad+{}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_1+1}\mathcal{Y}_1^*(\vartheta,\nu(\vartheta))\\&\qquad-\int_0^1 \mathcal{G}^{\sigma_1}(\zeta,\vartheta)\phi_q\left(\int_0^1 \mathcal{G}^{\lambda_1}(\vartheta,\eta)\right.\\&\qquad\left.\times(\psi_1(\eta)\mathcal{Y}_1^*(\eta,\nu^*(\eta-\tau)))d\eta\eta\right)d\vartheta\\&\qquad\left.+{}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_1+1}\mathcal{Y}_1^*(\vartheta,\nu^*(\vartheta))\right|\\&\quad\leq \int_0^1 \mathcal{G}^{\sigma_1}(\zeta,\vartheta)\phi_q\left(\int_0^1 \mathcal{G}^{\lambda_1}(\vartheta,\eta)(\|\psi_1\|\right.\\&\qquad\left.\times|\mathcal{Y}_1^*(\vartheta,\nu(\vartheta)) - \mathcal{Y}_1^*(\eta,\nu^*(\eta-\tau))|)d\eta\right)d\vartheta\end{aligned}$$

$$+\left[\frac{-\sigma_1}{\mathbb{B}(\sigma_1+1)}+\frac{\sigma_1+1}{\mathbb{B}(\sigma_1+1)}\,{}_0\mathcal{I}_a^{\sigma_1+1}\right]$$
$$\times|\mathcal{Y}_1^*(\vartheta,\nu(\vartheta))-\mathcal{Y}_1^*(\vartheta,\nu^*(\vartheta))|$$
$$\leq\Bigg[(q-1)\Theta_1^{q-2}\bigg(\frac{\sigma_1}{\mathbb{B}(\sigma_1)\Gamma(\sigma_1+1)}$$
$$+\frac{\sigma_1\gamma^{\sigma_1}}{\mathbb{B}(\sigma_1)\Gamma(\sigma_1+1)}\bigg)\bigg(\frac{\lambda_1\rho_1^{\lambda_1}}{\mathbb{B}(\lambda_1)\Gamma(\lambda_1+1)}$$
$$+\frac{\lambda_1}{\mathbb{B}(\lambda_1)\Gamma(\lambda_1+1)}\bigg)\times\|\psi\|\Omega_{\mathcal{Y}_1^*}+\bigg[\frac{-\sigma_1}{\mathbb{B}(\sigma_1+1)}$$
$$+\frac{(\sigma_1+1)a^{\sigma_1+2}}{\mathbb{B}(\sigma_1+1)(\sigma_1+2)\Gamma(\sigma_1+1)}\bigg]\Omega_{\mathcal{Y}_1^*}\Bigg]$$
$$\times|\nu(\zeta)-\nu^*(\zeta)|$$
$$=\mathbb{V}_{1,\mathcal{Y}^*}^*\mathfrak{y}_1 \tag{4.9}$$

and

$$|\nu(\zeta)-\mathbb{F}\nu^*(\zeta)|$$
$$=\bigg|\int_0^1\mathcal{G}^{\sigma_2}(\zeta,\vartheta)\phi_q\bigg(\int_0^1\mathcal{G}^{\lambda_2}(\vartheta,\eta)(\psi_2(\eta)$$
$$\times\mathcal{Y}_2^*(\eta,\mu(\eta-\tau)))d\eta\bigg)d\vartheta$$
$$+{}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_2+1}\mathcal{Y}_2^*(\vartheta,\mu(\vartheta))$$
$$-\int_0^1\mathcal{G}^{\sigma_2}(\zeta,\vartheta)\phi_q\bigg(\int_0^1\mathcal{G}^{\lambda_2}(\vartheta,\eta)(\psi_2(\eta)$$
$$\times\mathcal{Y}_2^*(\eta,\mu^*(\eta-\tau)))d\eta\eta\bigg)d\vartheta$$
$$+{}_0^{\mathrm{AB}}\mathcal{I}_a^{\sigma_2+1}\mathcal{Y}_2^*(\vartheta,\mu^*(\vartheta))\bigg|$$
$$\leq\int_0^1\mathcal{G}^{\sigma_2}(\zeta,\vartheta)\phi_q\bigg(\int_0^1\mathcal{G}^{\lambda_2}(\vartheta,\eta)(\|\psi_2\|$$
$$\times|\mathcal{Y}_2^*(\vartheta,\mu(\vartheta))-\mathcal{Y}_2^*(\eta,\mu^*(\eta-\tau))|)d\eta\bigg)d\vartheta$$
$$+\left[\frac{-\sigma_2}{\mathbb{B}(\sigma_2+1)}+\frac{\sigma_2+1}{\mathbb{B}(\sigma_2+1)}\,{}_0\mathcal{I}_a^{\sigma_2+1}\right]$$
$$\times|\mathcal{Y}_2^*(\vartheta,\mu(\vartheta))-\mathcal{Y}_2^*(\vartheta,\mu^*(\vartheta))|$$
$$\leq\Bigg[(q-1)\Theta_2^{q-2}\bigg(\frac{\sigma_2}{\mathbb{B}(\sigma_2)\Gamma(\sigma_2+1)}$$
$$+\frac{\sigma_2\gamma^{\sigma_2}}{\mathbb{B}(\sigma_2)\Gamma(\sigma_2+1)}\bigg)\bigg(\frac{\lambda_1\rho_2^{\lambda_2}}{\mathbb{B}(\lambda_2)\Gamma(\lambda_2+1)}$$
$$+\frac{\lambda_2}{\mathbb{B}(\lambda_2)\Gamma(\lambda_2+1)}\bigg)\|\psi\|\Omega_{\mathcal{Y}_2^*}+\bigg[\frac{-\sigma_2}{\mathbb{B}(\sigma_2+1)}$$
$$+\frac{(\sigma_2+1)a^{\sigma_2+2}}{\mathbb{B}(\sigma_2+1)(\sigma_2+2)\Gamma(\sigma_2+1)}\bigg]\Omega_{\mathcal{Y}_2^*}\Bigg]$$
$$\times|\mu(\zeta)-\mu^*(\zeta)|$$
$$=\mathbb{V}_{2,\mathcal{Y}^*}^*\mathfrak{y}_2, \tag{4.10}$$

where

$$V_{1,\mathcal{Y}^*}^*=\Bigg[(q-1)\Theta_1^{q-2}\bigg(\frac{\sigma_1+\sigma_1\gamma^{\sigma_1}}{\mathbb{B}(\sigma_1)\Gamma(\sigma_1+1)}\bigg)$$
$$\times\bigg(\frac{\lambda_1\rho_1^{\lambda_1}+\lambda_1}{\mathbb{B}(\lambda_1)\Gamma(\lambda_1+1)}\bigg)\|\psi\|\Omega_{\mathcal{Y}_1^*}+\bigg[\frac{-\sigma_1}{\mathbb{B}(\sigma_1+1)}$$
$$+\frac{(\sigma_1+1)a^{\sigma_1+2}}{\mathbb{B}(\sigma_1+1)(\sigma_1+2)\Gamma(\sigma_1+1)}\bigg]\Omega_{\mathcal{Y}_1^*}\Bigg],$$
$$V_{2,\mathcal{Y}^*}^*=\Bigg[(q-1)\Theta_2^{q-2}\bigg(\frac{\sigma_2+\sigma_2\gamma^{\sigma_2}}{\mathbb{B}(\sigma_2)\Gamma(\sigma_2+1)}\bigg)$$
$$\times\bigg(\frac{\lambda_1\rho_2^{\lambda_2}+\lambda_2}{\mathbb{B}(\lambda_2)\Gamma(\lambda_2+1)}\bigg)\|\psi\|\Omega_{\mathcal{Y}_2^*}+\bigg[\frac{-\sigma_2}{\mathbb{B}(\sigma_2+1)}$$
$$+\frac{(\sigma_2+1)a^{\sigma_2+1}}{\mathbb{B}(\sigma_2+1)(\sigma_2+2)\Gamma(\sigma_2+1)}\bigg]\Omega_{\mathcal{Y}_2^*}\Bigg].$$

As a result, with the assistance of Eqs. (4.9) and (4.10), the coupled system of AB-fractional equation (3.13) and (3.14) is HU-stable. Therefore, the problem (1.1) is HU-stable. □

**Theorem 8.** *On the basis of the assumption* ($A_3$) *along with Lemma* 3, *the positive solution for a coupled system of AB-fractional equation* (1.1) *is HU-stable and GHU-stable if* $1\neq\mathbb{V}_{1,\mathcal{Y}^*}^*$ *and* $1\neq\mathbb{V}_{2,\mathcal{Y}^*}^*$ *holds.*

**Proof.** If $\mu,\nu$ is any solution and $\mu^{\mathfrak{e}},\nu^{\mathfrak{e}}$ is a unique positive solution of suggested problem (1.1), we have

$$\begin{aligned}|\mu(\zeta)-\mu^{\mathfrak{e}}(\zeta)|&=|\mu(\zeta)-\mathbb{F}\mu^{\mathfrak{e}}(\zeta)|\\&=|\mu(\zeta)-\mathbb{F}\mu(\zeta)+\mathbb{F}\mu(\zeta)+\mathbb{F}\mu^{\mathfrak{e}}(\zeta)|\\&\leq|\mu(\zeta)+\mathbb{F}\mu(\zeta)|+|\mathbb{F}\mu(\zeta)+\mathbb{F}\mu^{\mathfrak{e}}(\zeta)|\\&\leq\mathcal{U}_{1,\iota,Q}\mathfrak{y}_1+2K_\xi\mathcal{U}_{1,\iota,Q}\|\mu-\mu^{\mathfrak{e}}\|.\end{aligned}$$

Similarly, we have $\|\mu - \mu^{\mathfrak{e}}\| \le \frac{\mathcal{U}_{1,\iota,Q}}{1-2K_\xi \mathcal{U}_{1,\iota,Q}}\mathfrak{y}_1$ and

$$\begin{aligned}|\nu(\zeta) - \nu^{\mathfrak{e}}(\zeta)| &= |\nu(\zeta) - \mathbb{F}\nu^{\mathfrak{e}}(\zeta)| \\ &= |\nu(\zeta) - \mathbb{F}\nu(\zeta) + \mathbb{F}\nu(\zeta) + \mathbb{F}\nu^{\mathfrak{e}}(\zeta)| \\ &\le |\nu(\zeta) + \mathbb{F}\nu(\zeta)| + |\mathbb{F}\nu(\zeta) + \mathbb{F}\nu^{\mathfrak{e}}(\zeta)| \\ &\le \mathcal{V}_{1,\iota,Q}\mathfrak{y}_2 + 2K_\xi \mathcal{V}_{1,\iota,Q}\|\nu - \nu^{\mathfrak{e}}\|.\end{aligned}$$

This further implies that $\|\nu - \nu^{\mathfrak{e}}\| \le \frac{\mathcal{V}_{1,\iota,Q}}{1-2K_\xi \mathcal{V}_{1,\iota,Q}}\mathfrak{y}_2$. Let $\mathcal{V}^{\xi} = \frac{\mathcal{V}_{1,\iota,Q}}{1-2K_\xi \mathcal{V}_{1,\iota,Q}}$ and $\mathcal{U}^{\xi} = \frac{\mathcal{U}_{1,\iota,Q}}{1-2K_\xi \mathcal{U}_{1,\iota,Q}}$, then the positive solution of the suggested problem (1.1) is HU-stable. Moreover, if $\mathcal{Z}(\varepsilon) = \varepsilon$, then the positive solution is GHU-stable. □

## 5. ILLUSTRATIVE EXAMPLE

To further explain our findings from the previous two sections, we will introduce an example to prove our results of the proposed problem in Secs. 3 and 4.

**Example 1.** Let the coupled system with p-Laplacian of AB-fractional equation be as follows:

$$\begin{cases} {}_0^{\mathrm{ABC}}\mathcal{D}^{\frac{1}{4}}(\phi_p({}_0^{\mathrm{ABC}}\mathcal{D}^{\frac{1}{3}}(\mu(\zeta)))) \\ \qquad = -\psi_1(\zeta)\mathcal{Y}_1^*(\zeta, \nu(\zeta-\tau)), \\ {}_0^{\mathrm{ABC}}\mathcal{D}^{\frac{1}{4}}(\phi_p({}_0^{\mathrm{ABC}}\mathcal{D}^{\frac{1}{3}}(\nu(\zeta)))) \\ \qquad = -\psi_2(\zeta)\mathcal{Y}_2^*(\zeta, \mu(\zeta-\tau)), \\ \phi_p({}_0^{\mathrm{ABC}}\mathcal{D}^{\frac{1}{3}}(\mu(\zeta)))|_{\zeta=\frac{1}{4}} = 0, \\ \phi_p({}_0^{\mathrm{ABC}}\mathcal{D}^{\frac{1}{3}}(\nu(\zeta)))|_{\zeta=\frac{1}{4}} = 0, \\ \mu\left(\frac{1}{6}\right) = {}_0^{\mathrm{AB}}\,\mathcal{I}_a^{\frac{1}{3}+1}\mathcal{Y}_1^*(\vartheta, \nu(\vartheta)), \\ \nu\left(\frac{1}{6}\right) = {}_0^{\mathrm{AB}}\,\mathcal{I}_a^{\frac{1}{3}+1}\mathcal{Y}_2^*(\vartheta, \mu(\vartheta)), \end{cases} \tag{5.1}$$

where $\zeta \in [0,1], p = 5, \Theta_i = \frac{1}{2}, \sigma_i = \frac{1}{3}, \lambda_i = \frac{1}{4}$, for $i = 1,2$, $\mathcal{Y}_1^*(\zeta, \nu(\zeta-\tau)) = -25/17 + 1/15\sin(\nu), \mathcal{Y}_2^*(\zeta, \mu(\zeta-\tau)) = 26/16 + 1/13\cos(\mu)$, which implies $\Omega_{\mathcal{Y}_1^*} = \Omega_{\mathcal{Y}_2^*} = \frac{1}{2}, \rho_1 = \rho_2 = \frac{1}{4}, \gamma_1 = \gamma_2 = \frac{1}{6}, a = 1.6, \psi_1 = \psi_2 = 0.1$. By simple mathematical computations, we get

$$\begin{aligned}\kappa^* = \Bigg[&(q-1)\Theta_1^{q-2}\left(\frac{\sigma_1}{\mathbb{B}(\sigma_1)\Gamma(\sigma_1+1)}\right. \\ &\left.+ \frac{\sigma_1\gamma^{\sigma_1}}{\mathbb{B}(\sigma_1)\Gamma(\sigma_1+1)}\right)\left(\frac{\lambda_1\rho_1^{\lambda_1}}{\mathbb{B}(\lambda_1)\Gamma(\lambda_1+1)}\right. \\ &\left.+ \frac{\lambda_1}{\mathbb{B}(\lambda_1)\Gamma(\lambda_1+1)}\right) \times \|\psi\|\Omega_{\mathcal{Y}_1^*} + \left[\frac{-\sigma_2}{\mathbb{B}(\sigma_1+1)}\right. \\ &\left.+ \frac{(\sigma_1+1)a^{\sigma_1+2}}{\mathbb{B}(\sigma_1+1)(\sigma_1+2)\Gamma(\sigma_1+1)}\right]\Omega_{\mathcal{Y}_1^*}\Bigg] \\ &+ \Bigg[(q-1)\Theta_2^{q-2}\left(\frac{\sigma_2}{\mathbb{B}(\sigma_2)\Gamma(\sigma_2+1)}\right. \\ &\left.+ \frac{\sigma_2\gamma^{\sigma_2}}{\mathbb{B}(\sigma_1)\Gamma(\sigma_2+1)}\right)\left(\frac{\lambda_2\rho_1^{\lambda_2}}{\mathbb{B}(\lambda_2)\Gamma(\lambda_2+1)}\right. \\ &\left.+ \frac{\lambda_2}{\mathbb{B}(\lambda_2)\Gamma(\lambda_2+1)}\right) \times \|\psi\|\Omega_{\mathcal{Y}_2^*}| + \left[\frac{-\sigma_2}{\mathbb{B}(\sigma_2+1)}\right. \\ &\left.+ \frac{(\sigma_2+1)a^{\sigma_2+2}}{\mathbb{B}(\sigma_2+1)(\sigma_2+2)\Gamma(\sigma_2+1)}\right]\Omega_{\mathcal{Y}_2^*}\Bigg] \\ =\ & 0.424821 < 1. \end{aligned} \tag{5.2}$$

By Theory 6 and Eq. (5.2), we deduce that (5.1) has a unique positive solution. Thus, the conditions of Theory 7, Theory 8 can be verified simply. Similarly, the coupled system AB-fractional equation (5.1) is HU-stable and consequently GHU-stable. Similarly, by using a nondecreasing function, the conditions of HUR and GHUR stability can be easily derived.

## 6. CONCLUSION OF THE PAPER

By using topological degree theory, we provided sufficient conditions for the existence of a solution for a coupled system of Atangana–Baleanu fractional differential equations with the p-Laplacian operator and ABC derivative of order $\sigma, \lambda \in [0,1]$. For these aims, we used Green functions to convert the proposed problem (1.1) into an integral equation. Then, the uniqueness of the solution was proven with the aid of the Banach contraction principle. We also discussed the HU-stability and generalized HU-stability of the solution by using the HU-stability technique. An example was provided to illustrate the results. In future studies, the concept can be applied to highly complex problems, including hybrid differential equations for multiplicity solutions. We emphasize that the findings of this study are novel.

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