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*Research article*

# On Maia type fixed point results via implicit relation

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**Abstract:** In order to study Maia type fixed point results for several well-known contractions, we suggest two novel contractions called $\mathcal{A}$-contraction and $\mathcal{A}'$-contraction. The majority of the Maia type fixed point results for various contractions can now be unified through these, which eliminate the need to manage various contractions individually. The advantage of including such contractions in the study of Maia type fixed point results has been demonstrated in suitable examples. We present an application of one of our established results towards the conclusion of the paper.

**Keywords:** metric space; fixed point; $\mathcal{A}$-contractions; $\mathcal{A}'$-contraction; enriched contractions
**Mathematics Subject Classification:** 47H10, 54H25

## 1. Introduction

The Banach contraction principle plays the fundamental role in the development of metric fixed point theory since this theory is integrated with the Banach contraction principle due to S. Banach [8] in the year 1922. This principle confirms that a mapping $T$ defined on a metric space $(S, d)$ into itself admits a unique fixed point, i.e., there is a exactly one $s \in S$ such that $Ts = s$ provided that

$$d(Ts, Tt) \leq c\, d(s, t) \tag{1.1}$$

for all $s, t \in S$ with $0 \leq c < 1$, and the metric space $(S, d)$ is complete.

A self-mapping $T$ satisfying (1.1) is known as a Banach contraction mapping. After this fundamental result, metric fixed point theory has become a research field of extensive interest in the literature. Consequently, this theory has encouraged a number of renowned mathematicians to contribute to it. As a result, we have witnessed a large number of fixed point results. Some remarkable results of these are due to Kannan [22], Chatterjea [14], Reich [30], Ćirić [15, 16], Bianchini [13], Khan [23] etc. (see [31] for more contractions of this type). For some recent items in this field of study, one can consult [17].

On the other hand, fixed point theory is also attractive for its vast applications in different fields of mathematics and engineering. This theory is an indispensable tool for obtaining existence and/or uniqueness criteria of solutions of several types of differential equations, integral equations, matrix equations etc. Also, this theory has wide applications in nonlinear optimization problems, variational inequality problems, split feasibility problems, equilibrium problems etc. (for example see [3,5,20,21] and references therein).

If we go through all the aforementioned results and also some other related results, we can notice one similarity among the results. The similarity is that all the results deal with the affirmation of a unique fixed point of a self-mapping if the space under consideration is complete. This similarity compels us to think about the affirmation of fixed points of self-mappings if the space under consideration is not necessarily complete. It can be shown easily that if the underlying space is incomplete, then a self-map may not have a fixed point. In these circumstances, a question arises about what modifications can be made to guarantee that a self-map on an incomplete metric space admits a fixed point. It was Maia [24] who came up with a positive answer to this question in 1968. The author obtained the following interesting modification to the Banach contraction principle:

**Theorem 1.1.** *Let $S$ be a non-empty set endowed with two metrics $d_1$ and $d_2$. Let*

$$d_1(s,t) \leq d_2(s,t)$$

*for all $s,t \in S$ and let $T$ be a self-map on $S$. Suppose that*

*(i) the metric space $(S,d_1)$ is complete;*
*(ii) $T$ is continuous with respect to $d_1$;*
*(iii) $T$ is a contraction mapping with respect to $d_2$, i.e.,*

$$d_2(Ts,Tt) \leq k\, d_2(s,t)$$

*for all $s,t \in S$ with $0 \leq k < 1$. Then $T$ has a unique fixed point.*

After such an interesting result by Maia, many authors have generalized this result in a variety of ways. Most of them were constituted by changing the contraction condition, see for example Albu [2], Ansari et al. [4], Balazs [6,7], Sadiq Basha [9], Berinde [10], Dhage [18], Dhage and Dhobale [19], V. Mureşan [26], A.S. Mureşan [27,28], Pathak and Dubey [29], Rus [32,33], Rzepecki [34,35], Shukla and Radenović [36], Trif [37]. As a result, we received several well-known contractions with Maia type modifications. However, in order to earn all such modifications, the authors have to compose separate results for every contraction which is very laborious. Thus, it becomes legitimate to think about some contractions whose Maia type modifications will award us most of the aforementioned modifications without formulating separate results for every contraction. These facts motivate us to introduce $\mathcal{A}$-contraction and $\mathcal{A}'$-contraction using two different implicit classes of functions and obtain the Maia type modifications of these two contractions. With the help of these contractions and their Maia type results, we obtain the Maia type results of many well-known contractions as special cases. After that, we come up with some non-trivial examples to endorse our established results. Furthermore, we study some extensions of Maia type theorems for enriched contractions using implicit relations. Finally, we conclude the paper with an application of one of our established results.

## 2. Maia type results of $\mathcal{A}$-contractions and $\mathcal{A}'$-contractions

Throughout the paper, $\mathbb{R}_+$ refers to the set of all non-negative real numbers. For any two non-empty sets $U$ and $V$, let us denote by $V^U$ the collection of all mappings $f : U \to V$. We use the notation $Fix(T)$ to denote the set of all fixed points of a mapping $T$ and $\mathbb{N}_0$ to denote the set $\mathbb{N} \cup \{0\}$.

Before going to establish our main results, we now introduce $\mathcal{A}$-contraction and $\mathcal{A}'$-contraction, first one is very similar to Akram et al. [1].

Let us first define two subsets $\mathcal{A}$ and $\mathcal{A}'$ of $\mathbb{R}^{\mathbb{R}^3_+}$. Let $\mathcal{A}$ be the collection of all mappings $f \in \mathbb{R}^{\mathbb{R}^3_+}$ satisfying the following conditions:

($\mathcal{A}_1$) there exists a real number $\mu$ with $0 \le \mu < 1$ such that if $u \le f(v, u, v)$ or $u \le f(u, v, v)$, then $u \le \mu v$ for all $u, v \in \mathbb{R}_+$;
($\mathcal{A}_2$) for $k > 0$ and for all $u, v \in \mathbb{R}_+$, $kf(u, v, w) \le f(ku, kv, kw)$.

Let $\mathcal{A}'$ be the collection of all mappings $f \in \mathbb{R}^{\mathbb{R}^3_+}$ satisfying the following conditions:

($\mathcal{A}'_1$) there exists a real number $\mu$ with $0 \le \mu < 1$ such that if $u \le f(v, 0, u + v)$, then $u \le \mu v$ for all $u, v \in \mathbb{R}_+$;
($\mathcal{A}'_2$) if $w \le w_1$, then $f(u, v, w) \le f(u, v, w_1)$;
($\mathcal{A}'_3$) if $u \le f(u, u, u)$, then $u = 0$.

As for example, the class $\mathcal{A}$ includes the following mappings:

(i) $f(u, v, w) = \alpha(v + w)$, where $0 \le \alpha < \frac{1}{2}$;
(ii) $f(u, v, w) = \alpha \max\{v, w\}$, where $0 \le \alpha < 1$;
(iii) $f(u, v, w) = \alpha \max\{u, v, w\}$, where $0 \le \alpha < 1$;
(iv) $f(u, v, w) = \alpha_1 u + \alpha_2 v + \alpha_3 w$, where $0 \le \alpha_1, \alpha_2, \alpha_3 < 1$ and $\alpha_1 + \alpha_2 + \alpha_3 < 1$.

The following list includes some examples of mappings from the class $\mathcal{A}'$:

(i) $f(u, v, w) = \alpha(v + w)$, where $0 \le \alpha < \frac{1}{2}$;
(ii) $f(u, v, w) = \alpha(u + v + w)$, where $0 \le \alpha < \frac{1}{3}$;
(iii) $f(u, v, w) = \alpha \max\{v, w\}$, where $0 \le \alpha < 1$.

Both of the aforementioned lists are not all-inclusive.

We are now in a position to introduce $\mathcal{A}$-contraction and $\mathcal{A}'$-contraction.

**Definition 2.1.** *Let $(S, d)$ be a metric space and let $T \in S^S$. Then $T$ is said to be an $\mathcal{A}$-contraction (with respect to $d$) if there exists $f \in \mathcal{A}$ such that*

$$d(Ts, Tt) \le f(d(s, t), d(s, Ts), d(t, Tt)) \tag{2.1}$$

*for all $s, t \in S$.*

**Definition 2.2.** *Let $(S, d)$ be a metric space and let $T \in S^S$. Then $T$ is said to be an $\mathcal{A}'$-contraction (with respect to $d$) if there exists $f \in \mathcal{A}'$ such that*

$$d(Ts, Tt) \le f(d(s, t), d(s, Tt), d(t, Ts)) \tag{2.2}$$

*for all $s, t \in S$.*

Now we put in place our first result.

**Theorem 2.1.** *Let $S$ be a non-empty set endowed with two metrics $d_1$ and $d_2$ satisfying*

$$d_1(s,t) \leq d_2(s,t) \tag{2.3}$$

*for all $s,t \in S$ and let $T \in S^S$. Suppose that*

*(i)* $(S, d_1)$ *is complete;*
*(ii)* $T$ *is continuous with respect to $d_1$;*
*(iii)* $T$ *is an $\mathcal{A}$-contraction with respect to $d_2$.*

*Then*

*(a)* $T$ *is a Picard operator.*
*(b) There exists a real number $\mu$ with $0 \leq \mu < 1$ such that*

$$d_1(s_n, s) \leq \frac{\mu^n}{1-\mu} d_2(s_1, s_0)$$

*and*

$$d_1(s_n, s) \leq \frac{\mu}{1-\mu} d_2(s_n, s_{n-1})$$

*hold for all $n \in \mathbb{N}$, where $\{s_n\}$ is a Picard sequence based at the point $s_o \in S$.*

*Proof.* Let $s_0 \in S$ be arbitrary. For all $n \in \mathbb{N}_0$, we define $s_{n+1} = T s_n$.

If $s_{n+1} = s_n$ for some $n$, then $T s_n = s_n$ and therefore $s_n \in Fix(T)$. So, we suppose that $s_{n+1} \neq s_n$ for all $n \in \mathbb{N}_0$. Now

$$\begin{aligned} d_2(s_{n+2}, s_{n+1}) &= d_2(T s_{n+1}, T s_n) \\ &\leq f(d_2(s_{n+1}, s_n), d_2(s_{n+1}, s_{n+2}), d_2(s_n, s_{n+1})). \end{aligned}$$

Then there exists a real number $\mu$ with $0 \leq \mu < 1$ such that

$$d_2(s_{n+2}, s_{n+1}) \leq \mu d_2(s_{n+1}, s_n)$$

for all $n \in \mathbb{N}_0$. Therefore,

$$d_2(s_{n+1}, s_n) \leq \mu d_2(s_n, s_{n-1}). \tag{2.4}$$

Continuing this process, we get

$$d_2(s_{n+2}, s_{n+1}) \leq \mu^{n+1} d_2(s_1, s_0)$$

for all $n \in \mathbb{N}_0$.

Now for all $m, n \in \mathbb{N}_0$, we have

$$\begin{aligned} d_2(s_{m+n}, s_n) &\leq d_2(s_{m+n}, s_{m+n-1}) + d_2(s_{m+n-1}, s_{m+n-2}) + \cdots + d_2(s_{n+1}, s_n) \\ &\leq (\mu^{m+n-1} + \mu^{m+n-2} + \cdots + \mu^n) d_2(s_1, s_0) \\ &= \mu^n \frac{1-\mu^m}{1-\mu} d_2(s_1, s_0). \end{aligned} \tag{2.5}$$

This implies that

$$d_2(s_{m+n}, s_n) \to 0$$

as $m, n \to +\infty$ which indicates the Cauchyness of the sequence $\{s_n\}$ in $(S, d_2)$. Cauchyness of the sequence $\{s_n\}$ in $(S, d_1)$ follows from the fact that

$$d_1(s_{m+n}, s_n) \leq d_2(s_{m+n}, s_n)$$

for all $m, n \in \mathbb{N}_0$. Now, completeness of $(S, d_1)$ produces an element $s \in S$ such that

$$s_n \to s \quad \text{as} \quad n \to +\infty. \tag{2.6}$$

Since $s_{n+1} = Ts_n$, applying continuity of $T$ in the metric space $(S, d_1)$ we get $s = Ts$ which, in turn, implies that $s \in Fix(T)$. To prove the uniqueness of $s$, let us take $s' \in Fix(T)$ i.e., $Ts' = s'$. Then

$$\begin{aligned} d_2(s, s') &= d_2(Ts, Ts') \\ &\leq f(d_2(s, s'), d_2(s, Ts), d_2(s', Ts')) \\ &= f(d_2(s, s'), 0, 0) \end{aligned}$$

which implies that $d_2(s, s') = 0$. Hence $s = s'$ and the uniqueness is confirmed.

Now, from (2.3) and (2.5), it follows that

$$d_1(s_{m+n}, s_n) \leq \mu^n \frac{1 - \mu^m}{1 - \mu} d_2(s_1, s_0).$$

Taking limit as $m \to +\infty$, we see that

$$d_1(s_n, s) \leq \frac{\mu^n}{1 - \mu} d_2(s_1, s_0).$$

Again, from (2.3) and (2.4), we find that

$$d_1(s_{n+1}, s_n) \leq d_2(s_{n+1}, s_n) \leq \mu d_2(s_n, s_{n-1})$$

which implies that

$$\begin{aligned} d_1(s_{n+m}, s_n) &\leq d_2(s_{n+m}, s_n) \\ &\leq d_2(s_{n+m}, s_{n+m-1}) + d_2(s_{n+m-1}, s_{n+m-2}) + \cdots + d_2(s_{n+1}, s_n) \\ &\leq (\mu^m + \mu^{m-1} + \cdots + \mu) d_2(s_n, s_{n-1}) \\ &= \mu \frac{1 - \mu^m}{1 - \mu} d_2(s_n, s_{n-1}). \end{aligned}$$

Taking limit as $m \to +\infty$, we get

$$d_1(s_n, s) \leq \frac{\mu}{1 - \mu} d_2(s_n, s_{n-1}).$$

Next, we establish the analogous version of the above result in case of $\mathcal{A}'$-contraction.

**Theorem 2.2.** *Let $S$ be a non-empty set endowed with two metrics $d_1$ and $d_2$ satisfying*

$$d_1(s,t) \leq d_2(s,t) \tag{2.7}$$

*for all $s,t \in S$ and let $T \in S^S$. Suppose that*

*(i)* $(S, d_1)$ *is complete;*
*(ii)* $T$ *is continuous with respect to* $d_1$*;*
*(iii)* $T$ *is an* $\mathcal{A}'$*-contraction with respect to* $d_2$.

*Then*

*(a)* $T$ *is a Picard operator.*
*(b) There exists a real number* $\mu$ *with* $0 \leq \mu < 1$ *such that*

$$d_1(s_n, s) \leq \frac{\mu^n}{1-\mu} d_2(s_1, s_0)$$

*and*

$$d_1(s_n, s) \leq \frac{\mu}{1-\mu} d_2(s_n, s_{n-1})$$

*hold for all* $n \in \mathbb{N}$*, where* $\{s_n\}$ *is a Picard sequence based at the point* $s_o \in S$.

*Proof.* Let $s_0 \in S$ be arbitrary. Now, for all $n \in \mathbb{N}_0$, we define $s_{n+1} = Ts_n$.
If $s_{n+1} = s_n$ for some $n$, then $Ts_n = s_n$ and therefore $s_n \in Fix(T)$.
Let us now suppose that $s_{n+1} \neq s_n$ for all $n \in \mathbb{N}_0$. Now

$$\begin{aligned} d_2(s_{n+2}, s_{n+1}) &= d_2(Ts_{n+1}, Ts_n) \\ &\leq f(d_2(s_{n+1}, s_n), d_2(s_{n+1}, s_{n+1}), d_2(s_n, s_{n+2})) \\ &\leq f(d_2(s_n, s_{n+1}), 0, d_2(s_n, s_{n+1}) + d_2(s_{n+1}, s_{n+2})). \end{aligned}$$

Then there exists a real number $\mu$ with $0 \leq \mu < 1$ such that

$$d_2(s_{n+2}, s_{n+1}) \leq \mu d_2(s_{n+1}, s_n)$$

for all $n \in \mathbb{N}_0$.

Now, proceeding similarly as in the above theorem, we can draw a conclusion to the Cauchyness of the sequence $\{s_n\}$ in $(S, d_2)$. Cauchyness of the sequence $\{s_n\}$ in $(S, d_1)$ can be obtained from

$$d_1(s_{m+n}, s_n) \leq d_2(s_{m+n}, s_n).$$

Using completeness of $(S, d_1)$, we find an element $s \in S$ such that

$$s_n \to s \quad \text{as} \quad n \to +\infty. \tag{2.8}$$

Using $s_{n+1} = Ts_n$ and continuity of $T$ in the metric space $(S, d_1)$, we see that $s = Ts$ which shows that $s \in Fix(T)$.

To prove the uniqueness of $s$, let us take $s' \in Fix(T)$ i.e., $Ts' = s'$. Then

$$\begin{aligned} d_2(s, s') &= d_2(Ts, Ts') \\ &\leq f(d_2(s, s'), d_2(s, Ts'), d_2(s', Ts)) \\ &= f(d_2(s, s'), d_2(s, s'), d_2(s, s')), \end{aligned}$$

which implies that $d_2(s, s') = 0$. Hence $s = s'$ and the uniqueness is established.

The remaining parts follow from the same arguments as in the above theorem.

Below, we present a couple of examples in support of our proven results. The following one is in support of Theorem 2.1.

**Example 2.1.** *Let $S = \mathbb{N}_0$. Define $d_1, d_2 : S \times S \to [0, +\infty)$ as follows:*
*For $m, n \in S$,*

$$d_1(m, n) = \begin{cases} 0 & \text{if } m = n, \\ \left|\frac{1}{m^2} - \frac{1}{n^2}\right| & \text{if } m \neq n \neq 0, \\ \frac{1}{n^2} & \text{if } n \neq 0, m = 0, \\ \frac{1}{m^2} & \text{if } m \neq 0, n = 0, \end{cases}$$

*and*

$$d_2(m, n) = \begin{cases} 0 & \text{if } m = n, \\ \left|\frac{1}{m} - \frac{1}{n}\right| & \text{if } m \neq n \neq 0, \\ 1 & \text{if one of } m, n \text{ is zero and the other is non-zero.} \end{cases}$$

*Then it is easy to verify that the metric space $(S, d_1)$ is complete whereas the metric space $(S, d_2)$ is incomplete. It is also to be noted that*

$$d_1(m, n) \leq d_2(m, n)$$

*for all $m, n \in S$.*

*Let us now define $T \in S^S$ by $Ts = 0$ for all $s \in S$. Let us choose $f \in \mathcal{A}$ where $f(u, v, w) = \frac{1}{4}(u+v+w)$ for all $u, v, w \in \mathbb{R}_+$. Then it is an easy task to conclude that $T$ is an $\mathcal{A}$-contraction with respect to $d_2$. It is also to be noted that $T$ is continuous with respect to the metric $d_1$. Thus all the assumptions of the Theorem 2.1 are met and as a result $T$ admits a unique fixed point say, $s = 0$ i.e., $Fix(T) = \{0\}$.*

We now present the following example to support Theorem 2.2:

**Example 2.2.** *Let $S = \{(s, t) \in \mathbb{R}^2 : 0 \leq s \leq 1, 0 \leq t \leq 1\}$. Define following mappings on $S$:*
*For $s = (s_1, t_1), t = (s_2, t_2) \in S$,*

$$d_1(s, t) = |s_1 - s_2| + |t_1 - t_2|,$$

*and*

$$d_2(s, t) = \begin{cases} 2 & \text{if exactly one of } s, t \text{ is } (0, 0), \\ |s_1 - s_2| + |t_1 - t_2| & \text{otherwise.} \end{cases}$$

*Then $d_1$ and $d_2$ define two metrics on $S$ such that $(S, d_1)$ is complete whereas $(S, d_2)$ is incomplete and also*

$$d_1(s, t) \leq d_2(s, t)$$

*for all $s, t \in S$.*

*We now define a map $T \in S^S$ as follows:*

$$T(s_1, t_1) = \left(\frac{s_1 + 1}{2}, \frac{1}{2}\right)$$

*for all $(s_1, t_1) \in S$. Let us choose $f \in \mathcal{A}'$ given by $f(u, v, w) = \frac{4}{5}\max\{u, v\}$ for all $u, v, w \in \mathbb{R}_+$.*

*Let $s = (s_1, t_1), t = (s_2, t_2) \in S$ be arbitrary. Now, let's look at the following cases:*
*Case-1. Let us now assume that $s, t$ are non-zero. i.e., $s, t \neq (0, 0)$. Then*

$$\begin{aligned}
d_2(Ts, Tt) &= d_2\left(\left(\frac{s_1 + 1}{2}, \frac{1}{2}\right), \left(\frac{s_2 + 1}{2}, \frac{1}{2}\right)\right) \\
&= \frac{1}{2}|s_1 - s_2| \\
&\leq \frac{4}{5}(|s_1 - s_2| + |t_1 - t_2|) \\
&= \frac{4}{5}d_2(s, t) \\
&\leq f(d_2(s, t), d_2(s, Tt), d_2(t, Ts)).
\end{aligned}$$

*Case-2. In this case, let $s = (0, 0)$ and $t \neq (0, 0)$. Then*

$$\begin{aligned}
d_2(Ts, Tt) &= d_2\left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{s_2 + 1}{2}, \frac{1}{2}\right)\right) \\
&= \frac{1}{2}s_2 \\
&\leq \frac{8}{5} \\
&= \frac{4}{5}d_2(s, t) \\
&\leq f(d_2(s, t), d_2(s, Tt), d_2(t, Ts)).
\end{aligned}$$

*Thus, in both the cases we find that*

$$d_2(Ts, Tt) \leq f(d_2(s, t), d_2(s, Tt), d_2(t, Ts))$$

*for every $s, t$ in $S$, i.e., $T$ is an $\mathcal{A}'$-contraction with respect to $d_2$. Also $T$ is continuous with respect to the metric $d_1$. Hence all the assumptions of Theorem 2.2 are met and so we conclude that there is exactly one element $s \in S$ such that $Ts = s$ say, $s = \left(1, \frac{1}{2}\right)$.*

It is now natural to ask, if we flip the roles of $d_1$ and $d_2$ in Theorem 2.1 and in Theorem 2.2, then does there exist a fixed point of the mapping under consideration? The following example shows that the answer is negative.

**Example 2.3.** *Let $S = (0, 1)$. Let $d_1$ be the usual metric of $\mathbb{R}$ and $d_2$ be the trivial metric on $S$. Then it is clear that*

$$d_1(s, t) \leq d_2(s, t)$$

*for every $s, t$ in $S$. It is well known that $(S, d_1)$ is an incomplete metric space whereas $(S, d_2)$ is a complete metric space.*

*Let us define $T \in S^S$ by $Ts = \frac{s+1}{2}$ for all $s \in S$. Let us choose $f \in \mathcal{A}$ given by $f(u, v, w) = \frac{4}{5}\max\{u, v, w\}$ for all $u, v, w \in \mathbb{R}_+$. Then for all $s, t \in S$, we have*

$$\begin{aligned} d_1(Ts, Tt) &= d_1\left(\frac{s+1}{2}, \frac{t+1}{2}\right) \\ &= \frac{1}{2}|s-t| \\ &\leq \frac{4}{5}|s-t| = \frac{4}{5}d_1(s,t) \\ &\leq f(d_1(s,t), d_1(s,Ts), d_1(t,Tt)). \end{aligned}$$

*This implies that $T$ is an $\mathcal{A}$-contraction with respect to the metric $d_1$. Hence we have*

*(i) $(S, d_2)$ is complete.*
*(ii) $T$ is continuous with respect to $d_2$.*
*(iii) $T$ is an $\mathcal{A}$-contraction with respect to the metric $d_1$.*

*However, it is to be noted that $T$ admits no fixed point in $S$.*

**Remark 2.1.** *However, if we replace the completeness of $(S, d_2)$ in (i) with compactness or boundedly compactness, it is a simple exercise to establish that $T \in S^S$ is a Picard operator and the other conclusions remain true. Therefore, it is natural to ask, what weaker condition(s) might be imposed on the space under discussion in addition to the other conditions to ensure that the results are valid?*

## 3. Maia type results of enriched contractions

Recently in 2020, Berinde and Păcurar [11] introduced an interesting type of contractions, known as enriched contractions. Berinde and Păcurar again extend such enriched contractions in various ways in the next two years, see [12]. Following such introduction and extensions, enriched contractions are further studied extensively by Mondal et al., see [25]. After this, Berinde [10] demonstrates the Maia type results of such enriched contractions. However, it should be noted that in order to demonstrate such results he has to compose separate results for every such enriched type contractions. So it again becomes legitimate to establish a Maia type result for an enriched contraction which will give us all the Maia type results of enriched contractions due to Berinde. We now discuss such creations in brief without giving detailed proofs.

Let $(S, \|\cdot\|)$ be a Banach space. Let $d$ be the metric defined on $S$ induced via $\|\cdot\|$, i.e.,

$$d(s,t) = \|s-t\|$$

for all $s, t \in S$.

Let $T \in S^S$ be a mapping such that there is $b \in [0, +\infty)$ and an $f \in \mathcal{A}$ with

$$\|b(s-t) + Ts - Tt\| \leq f((b+1)\|s-t\|, \|s-Ts\|, \|t-Tt\|) \tag{3.1}$$

for all $s, t \in S$, i.e., $T$ is an enriched $\mathcal{A}$-contraction due to Mondal et al. [25].

Let us put $\lambda = \frac{1}{b+1}$ so that $\lambda \in (0,1)$. Then with the help of the average mapping $T_\lambda$ defined by $T_\lambda s = (1-\lambda)s + \lambda Ts$ for all $s \in S$, (3.1) becomes

$$\|T_\lambda s - T_\lambda t\| \leq f(\|s-t\|, \|s - T_\lambda s\|, \|t - T_\lambda t\|),$$

i.e.,

$$d(T_\lambda s, T_\lambda t) \leq f(d(s,t), d(s, T_\lambda s), d(t, T_\lambda t))$$

for every $s, t$ in $S$. This implies that the mapping $T_\lambda$ is an $\mathscr{A}$-contraction. Hence we get the implicit version of Maia type theorems for enriched contractions. Henceforth Maia type fixed point theorems for enriched contractions [10, p. 37, Theorem 2.4], Maia type fixed point theorems for enriched Kannan contractions [10, p. 39, Theorem 3.6] and Maia type fixed point theorems for enriched Ćirić-Reich-Rus contractions [10, p. 42, Theorem 4.8] can be derived as particular cases of our proven results. More specifically, if we take $f(u,v,w) = \frac{\theta}{b+1}u$ with $\theta \in [0, b+1)$, then [10, p. 37, Theorem 2.4] follows; if we take $f(u,v,w) = \alpha(v+w)$ with $\alpha \in \left[0, \frac{1}{2}\right)$, then we get [10, p. 39, Theorem 3.6] and if we take $f(u,v,w) = \alpha u + \beta(v+w)$ with $\alpha, \beta \geq 0$ such that $\alpha + 2\beta \in [0,1)$, then we get [10, p. 42, Theorem 4.8].

## 4. An Application

In this section, we deal with an application of one of our obtained results. To be more specific, we deal with the existence of unique solution of the following boundary value problem:

$$\frac{d^2x}{dt^2} + g(t, x(t)) = 0, \quad t \in [0,1], \quad x(0) = x(1) = 0, \tag{4.1}$$

where $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

The above boundary value problem is equivalent to the integral equation

$$x(t) = \int_0^1 G(t,s) g(s, x(s)) ds, \tag{4.2}$$

where the Green's function $G(t,s)$ is defined by

$$G(t,s) = \begin{cases} t(1-s), & \text{if } 0 \leq t \leq s \leq 1, \\ s(1-t), & \text{if } 0 \leq s \leq t \leq 1. \end{cases}$$

We have the following properties of the Green's function $G$:

(a) $G(t,s) \geq 0$ for all $t, s \in [0,1]$;

(b) $\sup_{0 \leq t \leq 1} \int_0^1 G(t,s) ds = \dfrac{1}{8}$.

**Theorem 4.1.** *If the function '$g$' be such that $|g(s,a) - g(s,b)| \leq M|a-b|$ for all $s \in [0,1]$ and for all $a, b \in \mathbb{R}$ where $M$ is a non-negative real constant such that $M < 8$, then the boundary value problem* (4.1) *has a unique solution.*

*Proof.* Let us consider the set $S = C[0,1]$ of all real-valued continuous functions defined on $[0,1]$ equipped with the metrics $d_2, d_1$ defined by

$$d_2(x,y) = \sup_{0\leq t\leq 1} |x(t) - y(t)|$$

and

$$d_1(x,y) = \frac{d_2(x,y)}{1 + d_2(x,y)}$$

for all $x, y \in S$. Then we have

(i) $d_1(x,y) \leq d_2(x,y)$ for all $x, y \in S$;
(ii) $(S, d_1)$ is a complete metric space.

We now consider a mapping $T$ defined on $S$ by

$$Tx(t) = \int_0^1 G(t,s)g(s,x(s))ds \text{ for all } x \in S \text{ and for all } t \in [0,1].$$

Then, $T \in S^S$. Now, for all $x, y \in S$ and for all $t \in [0,1]$, we have

$$\begin{aligned} |Tx(t) - Ty(t)| &= \left| \int_0^1 G(t,s)g(s,x(s))ds - \int_0^1 G(t,s)g(s,y(s))ds \right| \\ &\leq \int_0^1 G(t,s)|g(s,x(s)) - g(s,y(s))|ds \\ &\leq M \int_0^1 G(t,s)|x(s) - y(s)|ds \\ &\leq M d_2(x,y) \int_0^1 G(t,s)ds \\ &\leq M d_2(x,y) \sup_{0\leq t\leq 1} \int_0^1 G(t,s)ds \\ &= \frac{M}{8} d_2(x,y). \end{aligned}$$

Taking supremum over all $t \in [0,1]$, we get

$$\begin{aligned} \sup_{0\leq t\leq 1} |Tx(t) - Ty(t)| &\leq \frac{M}{8} d_2(x,y) \\ \Longrightarrow d_2(Tx,Ty) &\leq \frac{M}{8} d_2(x,y) \\ \Longrightarrow d_2(Tx,Ty) &\leq \frac{M}{8} \max\{d_2(x,y), d_2(x,Ty), d_2(y,Tx)\}. \end{aligned} \tag{4.3}$$

Thus

$$d_2(Tx,Ty) \leq f(d_2(x,y), d_2(x,Ty), d_2(y,Tx)),$$

where $f \in \mathcal{A}'$ is defined by $f(u,v,w) = \alpha \max\{u,v,w\}$ with $\alpha = \frac{M}{8} < 1$. This shows that

(iii) $T$ is an $\mathscr{A}'$-contraction with respect to $d_2$.

Again, from (4.3), we have

(iv) $T$ is continuous with respect to $d_1$.

Hence it follows from Theorem 2.2 that $T$ has a unique fixed point, which is the unique solution of the boundary value problem (4.1) also.

## 5. Conclusions

We introduced two contractions using implicit relation of mappings and proved fixed point theorems of Maia type. As a result, we don't need to remember many more results of Maia type at once. Also, we constructed suitable examples to authenticate our established results. Finally, we give an application to one of our proven results.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

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