AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(9): 22319–22337.
DOI: 10.3934/math.20231138
Received: 07 May 2023
Revised: 02 July 2023
Accepted: 10 July 2023
Published: 13 July 2023

Research article

Complete integral convergence for weighted sums of negatively dependent random variables under sub-linear expectations

Lunyi Liu and Qunying Wu*

College of Science, Guilin University of Technology, Guilin 541006, China

* **Correspondence:** Email: wqy666@glut.edu.cn; Tel: +8615877031406.

Abstract: In the paper, the complete convergence and complete integral convergence for weighted sums of negatively dependent random variables under the sub-linear expectations are established. The results in the paper extend some complete moment convergence theorems from the classical probability space to the situation of sub-linear expectation space.

Keywords: negatively dependent; complete convergence; complete integral convergence; weighted sums; sub-linear expectation
Mathematics Subject Classification: 60F15

1. Introduction

Probability limit theory is an important research topic in mathematical statistics that has found extensive application in the fields of mathematics, statistics, and finance. However, the limitations of classical limit theory have become increasingly apparent with the application of limit theory in finance, risk measurement and other areas. In situations where the mathematical model is characterized by uncertainty, the analysis and computation of sub-linearity becomes feasible. To address this issue, academician Peng [1–3] put forward the concept of sub-linear expectation space, constructed the complete theoretical system of sub-linear expectation space and effectively solved the limitation of traditional probability space theory in statistics, economics, and other fields. In recent years, an increasing number of scholars have conducted extensive research in this field, yielding numerous relevant findings. Notably, Peng [1–3] and Zhang [4–6] have derived a series of significant conclusions, including the law of large numbers of strong numbers, the exponential inequality and Rosenthal's inequality under sub-linear expectations. These findings have established a solid groundwork for investigating of the limit theory of sub-linear expectation spaces. The results obtained by Peng and Zhang have greatly contributed to the advancement of our understanding of the sub-linear expectation space theorem.

The concepts of complete convergence and complete moment convergence hold significant importance in the probability limit theory. The theory of complete convergence was initially introduced by Hsu and Robbins [7]. Chow [8] introduced the concept of complete convergence of independent random variables, which has since been expanded upon. As a result of complete convergence, complete moment convergence is more accurate, prompting a further investigation by scholars. Qiu and Chen [9] established the complete moment convergence for independent and identically distributed random variables, while Yang and Hu [10] demonstrated the complete moment convergence for pairwise NQD random variables. Song and Zhu [11] derived the complete convergence theorem for extended negatively dependent random variables. Notably, in the sub-linear expectation space, the complete moment convergence is equivalent to the complete integral convergence. In recent years, an increasing number of scholars have conducted research on the topics of complete convergence and complete integral convergence within the context of sub-linear expectations, thereby significantly augmenting the associated theoretical frameworks. For example, Li and Wu [12] conducted a study on the convergence of complete integrals for arrays of row-wise extended negatively dependent random variables. Similarly, Lu and Weng [13] examined the complete and complete integral convergence of arrays consisting of row-wise widely negative dependent random variables. Additionally, Chen and Wu [14] investigated the complete convergence and complete integral convergence of partial sums for the moving average process. It is noteworthy that complete convergence and complete integral convergence with maxima under sub-linear expectation spaces are only valid when the sequences are independent or negatively dependent. For example, Feng and Zeng [15] proved a complete convergence theorem of the maximum of partial sums under the sub-linear expectations. Xu and Kong [16,17] discussed complete convergence and complete integral convergence under negatively dependent sequences. The aforementioned findings suggest a need for further development in the field of complete integral convergence. The objective of this research is to extend the complete moment convergence characteristic, as established by Wu and Wang [18], to sub-linear space through a probabilistic approach and, subsequently, derive relevant outcomes.

The present article is structured as follows: Section 2 provides an introduction to basic notations, concepts and related properties within the context of sub-linear expectations, along with the presentation of several lemmas. Section 3 establishes complete convergence and complete integral convergence for weighted sums of negatively dependent random variables under sub-linear expectations. Finally, in Section 4, the aforementioned lemmas are utilized to demonstrate the major findings of this study. The symbol c denotes an arbitrary constant and is independent of n. The $\ln x$ is denoted as $\log_2 x$ in the paper and $I(\cdot)$ denotes an indicator function.

2. Preliminaries

We use the framework and notions of Peng [1–3] and Zhang [6]. Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, X_2, \ldots, X_n \in \mathcal{H}$, then $\varphi(X_1, \ldots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}_n)$, where $\varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}_n)$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$|\varphi(x) - \varphi(y)| \le c\,(1 + |x|^m + |y|^m)\,|x - y|, \quad \forall x, y \in \mathbb{R}_n,$$

for some $c > 0, m \in \mathbb{N}$ depending on φ. $\mathcal{H}$ is considered as a space of random variable. In this case we denote $X \in \mathcal{H}$.

Definition 2.1. A sub-linear expectationis $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a function $\hat{\mathbb{E}} : \mathcal{H} \to [-\infty, +\infty]$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have
(a) Monotonicity: if $X \geq Y$, then $\hat{\mathbb{E}}(X) \geq \hat{\mathbb{E}}(Y)$;
(b) Constant preserving: $\hat{\mathbb{E}}(c) = c$;
(c) Sub-additivity: $\hat{\mathbb{E}}(X + Y) \leq \hat{\mathbb{E}}(X) + \hat{\mathbb{E}}(Y)$;
(d) Positive homogeneity: $\hat{\mathbb{E}}(\lambda X) = \lambda\hat{\mathbb{E}}(X), \lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sub-linear expectation space.

Given a sub-linear expectation $\hat{\mathbb{E}}$, let us denote the conjugate expectation $\hat{\varepsilon}$ of $\hat{\mathbb{E}}$ b

$$\hat{\varepsilon}(X) := -\hat{\mathbb{E}}(-X), \forall X \in \mathcal{H}.$$

Form the definition, it is easily shown that for all $X, Y \in \mathcal{H}$,

$$\begin{aligned} \hat{\varepsilon}(X) &\leq \hat{\mathbb{E}}(X), \\ \hat{\mathbb{E}}(X - Y) &\geq \hat{\mathbb{E}}(X) - \hat{\mathbb{E}}(Y), \\ \hat{\mathbb{E}}(X + c) &= \hat{\mathbb{E}}(X) + c, \qquad (2.1) \\ |\hat{\mathbb{E}}(X - Y)| &\leq \hat{\mathbb{E}}|X - Y|. \qquad (2.2) \end{aligned}$$

Definition 2.2. Let $\mathcal{G} \subset \mathcal{F}$, a function $V : \mathcal{G} \to [0, 1]$ is called a capacity, if

$$V(\emptyset) = 0, V(\Omega) = 1 \text{ and } V(A) \leq V(B) \text{ for } A \subset B, A, B \in \mathcal{G}.$$

It is called to be sub-additive if $V(A \cup B) \leq V(A) + V(B)$ for all $A, B \in \mathcal{G}$. In the sub-linear space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, we denote a pair $(\mathbb{V}, \mathcal{V})$ of capacities by

$$\mathbb{V}(A) := \inf\left\{\hat{\mathbb{E}}[\xi] : I_A \leq \xi, \xi \in \mathcal{H}\right\}, \mathcal{V}(A) := 1 - \mathbb{V}(A^c), A \in \mathcal{F},$$

where A^c is the complement set of A. It is obvious that $\mathbb{V}$ is sub-additive, and

$$\hat{\mathbb{E}}f \leq \mathbb{V}(A) \leq \hat{\mathbb{E}}g,\ \hat{\varepsilon}f \leq \mathcal{V}(A) \leq \hat{\varepsilon}g, \text{ if} f \leq I(A) \leq g, f,\ \text{g} \in \mathcal{H}.$$

This implies Markov inequality:

$$\mathbb{V}(|X| \geq x) \leq \hat{\mathbb{E}}|X|^P/x^p, \forall x > 0, p > 0.$$

Form $I(|X| \geq x) \leq |X|^P/x^P \in \mathcal{H}$, by Lemma 4.1 in Zhang [5], we have Hölder inequality: $\forall X, Y \in \mathcal{H}, p, q > 1$ satisfying $p^{-1} + q^{-1} = 1$,

$$\hat{\mathbb{E}}(|XY|) \leq \left(\hat{\mathbb{E}}\left(|X|^p\right)\right)^{1/p}\left(\hat{\mathbb{E}}\left(|X|^q\right)\right)^{1/q},$$

particularly, Jensen inequality: $\forall X \in \mathcal{H}$,

$$\left(\hat{\mathbb{E}}\left(|X|^r\right)\right)^{1/r} \leq \left(\hat{\mathbb{E}}\left(|X|^s\right)\right)^{1/s} \text{ for } 0 < r \leq s.$$

Definition 2.3. We define the Choquet integrals/expectations $(C_{\mathbb{V}}, C_{\mathcal{V}})$ by

$$C_V(X) = \int_0^\infty V(X \geq t)\mathrm{d}t + \int_{-\infty}^0 [V(X \geq t) - 1]\mathrm{d}t,$$

with V being replaced by $\mathbb{V}$ and $\mathcal{V}$ respectively.

Definition 2.4. (Identical distribution) Let X_1 and X_2 be two n-dimensional random vectors defined, respectively, in sub-linear expectation spaces $\left(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1\right)$ and $\left(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2\right)$. They are called identically distributed if

$$\hat{\mathbb{E}}_1\left(\varphi\left(X_1\right)\right) = \hat{\mathbb{E}}_2\left(\varphi\left(X_2\right)\right), \forall \varphi \in C_{l,\,\mathrm{Lip}}\left(\mathbb{R}_n\right),$$

whenever the sub-expectations are finite. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be identically distributed if, for each $i \geq 1$, X_i and X_1 are identically distributed.

Definition 2.5. (Negative dependence) In a sub-linear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, a random vector $Y = (Y_1, \cdots, Y_n)\,(Y_i \in \mathcal{H})$ is said to be negatively dependent (ND) to another random vector $X = (X_1, \cdots, X_m)\,(X_i \in \mathcal{H})$ under $\hat{\mathbb{E}}$ if for each pair of test functions $\varphi_1 \in C_{l,\,\mathrm{Lip}}\left(\mathbb{R}_m\right)$ and $\varphi_2 \in C_{l,\,\mathrm{Lip}}\left(\mathbb{R}_n\right)$, we have

$$\hat{\mathbb{E}}\left[\varphi_1(X)\varphi_2(Y)\right] \leq \hat{\mathbb{E}}\left[\varphi_1(X)\right]\hat{\mathbb{E}}\left[\varphi_2(Y)\right],$$

whenever $\varphi_1(X) \geq 0, \hat{\mathbb{E}}\left[\varphi_2(Y)\right] \geq 0, \hat{\mathbb{E}}\left[|\varphi_1(X)\varphi_2(Y)|\right] < \infty, \hat{\mathbb{E}}\left[|\varphi_1(X)|\right] < \infty, \hat{\mathbb{E}}\left[|\varphi_2(Y)|\right] < \infty$ and either φ_1 and φ_2 are coordinatewise non-increasing or coordinatewise non-decreasing.

A sequence of random variables $\{X_n, n \geq 1\}$ is said to be negatively dependent if X_{i+1} is negatively dependent to $(X_1, ..., X_i)$ for each $i \geq 1$.

It is obvious that, if $\{X_n, n \geq 1\}$ is a sequence of negatively dependent random variables and functions $f_1(x), f_2(x), ... \in C_{l,Lip}(\mathbb{R})$ are all non-decreasing (resp. all non-increasing), then $\{f_n\left(X_n\right), n \geq 1\}$ is also a sequence of negatively dependent random variables.

Definition 2.6. A sub-linear expectation $\hat{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$ is called to be countably sub-additive, if

$$\hat{\mathbb{E}}(X) \leq \sum_{n=1}^\infty \hat{\mathbb{E}}\left(X_n\right), \quad \text{where} \quad X \leq \sum_{n=1}^\infty X_n, \quad X, X_n \in \mathcal{H}, \quad X \geq 0, X_n \geq 0, \quad n \geq 1.$$

We need the following lemmas to prove the main results.

Lemma 2.1. *(Zhang [5]) Suppose that X_k is negatively dependent to $(X_{k+1}, \cdots, X_n)$ for each $k = 1, \ldots, n-1$ and $\hat{\mathbb{E}}(X_k) \leq 0$, then for $q \geq 2$,*

$$\hat{\mathbb{E}}\left[\max_{k\leq n} |S_k|^q\right] \leq c_q \left\{\sum_{k=1}^n \hat{\mathbb{E}}\left[|X_k|^q\right] + \left(\sum_{k=1}^n \hat{\mathbb{E}}\left[|X_k|^2\right]\right)^{\frac{q}{2}} + \left(\sum_{k=1}^n \left[(\hat{\varepsilon}X_k)^- + \left(\hat{\mathbb{E}}X_k\right)^+\right]\right)^q\right\}, \tag{2.3}$$

where c_q is a positive constant depending only on q.

Lemma 2.2. *Suppose $X \in \mathcal{H}$, $\gamma > 0$, $0 < \alpha \leq 2$ and $b_y = y^{1/\alpha} \ln^{1/\gamma} y$.*

(i) *Then, for any $c > 0$,*

$$C_{\mathbb{V}}\left(|X|^2 \ln^{1-2/\gamma} |X|\right) < \infty \Leftrightarrow \sum_{n=1}^\infty n^{2/\alpha - 1} \ln n \mathbb{V}\left(|X| > cb_n\right) < \infty. \tag{2.4}$$

(ii) *If* $C_{\mathbb{V}}\left(|X|^2\ln^{1-2/\gamma}|X|\right)<\infty$, *then for any* $\beta>1$ *and* $c>0$,

$$\sum_{k=1}^{\infty}\beta^{2k/\alpha}k\ln\beta\mathbb{V}\left(|X|>cb_{\beta^k}\right)<\infty. \tag{2.5}$$

Proof. (i) *Because*

$$C_{\mathbb{V}}\left(|X|^2\ln^{1-2/\gamma}|X|\right)<\infty\Leftrightarrow\int_1^{\infty}\mathbb{V}\left(|X|^2\ln^{1-2/\gamma}|X|>x\right)\mathrm{d}x<\infty. \tag{2.6}$$

Let $f(x)=x^2\ln^{1-2/\gamma}x, x>1$. *We define the inverse function of* $f(x)$ *as* $f^{-1}(x)$. *Then, we can get*

$$\int_1^{\infty}\mathbb{V}\left(|X|^2\ln^{1-2/\gamma}|X|>x\right)\mathrm{d}x=\int_1^{\infty}\mathbb{V}\left(|X|>f^{-1}(x)\right)\mathrm{d}x. \tag{2.7}$$

Let $f^{-1}(x)=cb_y=cy^{1/\alpha}\ln^{1/\gamma}y$, *for any* $c>0$, *we have*

$$x=f\left(cy^{1/\alpha}\ln^{1/\gamma}y\right)=cy^{2/\alpha}\ln^{2/\gamma}y\cdot\ln^{1-2/\gamma}\left(y^{1/\alpha}\ln^{1/\gamma}y\right).$$

Let

$$h(y):=\ln^{2/\gamma}y\cdot\ln^{1-2/\gamma}\left(y^{1/\alpha}\ln^{1/\gamma}y\right)=c_2\cdot\exp\left\{\int_{\mathrm{e}}^{y}\frac{g(u)}{u}\,\mathrm{d}u\right\},$$

where $c_2=\exp\left\{\left(1-\frac{2}{\gamma}\right)\ln\alpha\right\}$, $g(u)=\left(\frac{2}{\gamma}\frac{1}{\ln u}+\frac{1-\frac{2}{\gamma}}{\frac{1}{\alpha}\ln u+\frac{1}{\gamma}\ln\ln u}\left(\frac{1}{\alpha}+\frac{1}{\gamma\ln u}\right)\right)$ *and obviously* $g(u)\to 0, u\to\infty$.

Then, for any $c>0$, *we can get*

$$x'=\left(cy^{2/\alpha}h(y)\right)'=\frac{2c}{\alpha}\cdot y^{2/\alpha-1}\cdot h(y)+cy^{2/\alpha}\cdot h(y)\cdot\frac{g(y)}{y}\sim c\cdot y^{2/\alpha-1}\ln y.$$

Therefore, combining (2.7), for any $c>0$, *we have*

$$\int_1^{\infty}\mathbb{V}\left(|X|>f^{-1}(x)\right)\mathrm{d}x=\int_1^{\infty}\mathbb{V}\left(|X|>cb_y\right)x'\,\mathrm{d}y\sim c\int_1^{\infty}\mathbb{V}\left(|X|>cb_y\right)y^{2/\alpha-1}\ln y\,\mathrm{d}y. \tag{2.8}$$

Obviously, combining (2.6)–(2.8), we can get

$$C_{\mathbb{V}}\left(|X|^2\ln^{1-2/\gamma}|X|\right)<\infty\Leftrightarrow\sum_{n=1}^{\infty}n^{2/\alpha-1}\ln n\mathbb{V}\left(|X|>cb_n\right)<\infty,$$

hence, the proof of (i) *is established.*

(ii) *By the proof of* (i), *we can get (2.4), then for any* $\beta>1$ *and* $c>0$,

$$\begin{aligned}\infty&>\sum_{n=1}^{\infty}n^{2/\alpha-1}\ln n\mathbb{V}\left(|X|>cb_n\right)\\&\geq c\sum_{k=1}^{\infty}\sum_{\beta^{k-1}\leq n<\beta^k}\beta^{k(2/\alpha-1)}k\ln\beta\mathbb{V}\left(|X|>cb_{\beta^k}\right)\\&=c\sum_{k=1}^{\infty}\beta^{2k/\alpha}k\ln\beta\mathbb{V}\left(|X|>cb_{\beta^k}\right),\end{aligned}$$

hence, the proof of (ii) *is established.*

Lemma 2.3. *(Zhang [5]) If* $\hat{\mathbb{E}}$ *is countably sub-additive, then for* $X\in\mathcal{H}$,

$$\hat{\mathbb{E}}(|X|)\leq C_{\mathbb{V}}(|X|). \tag{2.9}$$

3. Main results

Theorem 3.1. *Assume that* $\{X, X_n, n \geq 1\}$ *is a sequence of negatively dependent and identically distributed random variables under sub-linear expectations. Suppose that* $\{a_{nk}, 1 \leq k \leq n, n \geq 1\}$ *is an array of positive real numbers and* $\hat{\mathbb{E}}$ *is countably sub-additive. Set* $b_n = n^{1/\alpha} \ln^{1/\gamma} n$*, where* $0 < \alpha \leq 2$*,* $0 < \gamma < 2$*, if*

$$C_{\mathbb{V}}\left(|X|^2 \ln^{1-2/\gamma} |X|\right) < \infty, \tag{3.1}$$

$$\sum_{k=1}^{n} a_{nk}^{\alpha} = O(n), \tag{3.2}$$

$$\hat{\mathbb{E}}X_k = \hat{\varepsilon}X_k = 0, \tag{3.3}$$

then for any $\varepsilon > 0$*,*

$$\sum_{n=1}^{\infty} n^{-1} \mathbb{V}\left(\max_{1\leq j\leq n}\left|\sum_{k=1}^{j} a_{nk}X_k\right| > b_n\varepsilon\right) < \infty. \tag{3.4}$$

Theorem 3.2. *Assume that the conditions of Theorem 3.1 are satisfied, then for* $0 < \theta < 2$ *and* $\varepsilon > 0$*,*

$$\sum_{n=1}^{\infty} n^{-1} C_{\mathbb{V}}\left\{b_n^{-1}\max_{1\leq j\leq n}\left|\sum_{k=1}^{j} a_{nk}X_k\right| - \varepsilon\right\}_+^{\theta} < \infty, \tag{3.5}$$

where $+$ *is the positive part.*

Remark 3.1. *Theorem 3.2 not only pushes the result of Wu and Wang[18] from probability space to sub-linear expectation space but also extends* $1 < \alpha \leq 2$ *to* $0 < \alpha \leq 2$*,* $0 < \gamma < \alpha$ *to* $0 < \gamma < 2$*,* $0 < \theta < \alpha$ *to* $0 < \theta < 2$*, extending the original range and enhancing the result.*

4. proof of main results

4.1. Proof of Theorem 3.1.

For fixed $n \geq 1$ and $1 \leq k \leq n$, denote

$$\begin{aligned} Y_{nk} &:= -b_n I(X_k < -b_n) + X_k I(|X_k| \leq b_n) + b_n I(X_k > b_n), \\ Z_{nk} &:= X_k - Y_{nk} = (X_k + b_n) I(X_k < -b_n) + (X_k - b_n) I(X_k > b_n). \end{aligned}$$

We can easily see that for any $\varepsilon > 0$,

$$\begin{aligned} \left\{\max_{1\leq j\leq n}\left|\sum_{k=1}^{j} a_{nk}X_k\right| > b_n\varepsilon\right\} &\subset \{\exists 1 \leq k \leq n, |X_k| > b_n\} \\ &\quad \cup \left\{\max_{1\leq j\leq n}\left|\sum_{k=1}^{j} a_{nk}X_k\right| > b_n\varepsilon, \forall 1 \leq k \leq n, |X_k| \leq b_n\right\} \\ &\subset \{\exists 1 \leq k \leq n, |X_k| > b_n\} \end{aligned}$$

$$\cup\left\{\max_{1\le j\le n}\left|\sum_{k=1}^{j}a_{nk}\left(Y_{nk}-\hat{\mathbb{E}}Y_{nk}\right)\right|>b_n\varepsilon-\max_{1\le j\le n}\left|\sum_{k=1}^{j}\hat{\mathbb{E}}(a_{nk}Y_{nk})\right|\right\}.$$

Then, we have

$$\begin{aligned}&\sum_{n=1}^{\infty}n^{-1}\mathbb{V}\left(\max_{1\le j\le n}\left|\sum_{k=1}^{j}a_{nk}X_k\right|>b_n\varepsilon\right)\\ \le&\sum_{n=1}^{\infty}n^{-1}\sum_{k=1}^{n}\mathbb{V}\left(|X_k|>b_n\right)+\sum_{n=1}^{\infty}n^{-1}\mathbb{V}\left(\max_{1\le j\le n}\left|\sum_{k=1}^{j}a_{nk}\left(Y_{nk}-\hat{\mathbb{E}}Y_{nk}\right)\right|>b_n\varepsilon-\max_{1\le j\le n}\left|\sum_{k=1}^{j}\hat{\mathbb{E}}(a_{nk}Y_{nk})\right|\right)\\ :=&I_1+I_2.\end{aligned}$$

In order to prove (3.4), we just need to prove

$$I_1<\infty, \tag{4.1}$$

$$I_2<\infty. \tag{4.2}$$

First of all, we prove (4.1). We know that in the probability space: $EI(|X|\le a)=P(|X|\le a)$ holds, nevertheless under the sub-linear expectation space, the expression $I(|x|\le a)$ not necessarily continuous. As a result, $EI(|X|\le a)$ does not necessarily exist. So, we need to modify the indicator function by functions in $C_{l,Lip}(\mathbb{R})$. We define the function $g(x)\in C_{l,Lip}(\mathbb{R})$ as follows.

For $2^{-1/\alpha}<\mu<1$, suppose that even function $g(x)\in C_{l,Lip}(\mathbb{R})$ and $g(x)$ is decreasing in $x\ge 0$, such that $0\le g(x)\le 1$ for all x and $g(x)=1$ if $|x|\le\mu$, $g(x)=0$ if $|x|>1$. Then

$$I(|x|\le\mu)\le g(|x|)\le I(|x|\le 1),\quad I(|x|>1)\le 1-g(|x|)\le I(|x|>\mu). \tag{4.3}$$

By (2.4), (3.1) and (4.3), we have

$$\begin{aligned}I_1&=\sum_{n=1}^{\infty}n^{-1}\sum_{k=1}^{n}\mathbb{V}\left(|X_k|>b_n\right)\\ &\le\sum_{n=1}^{\infty}n^{-1}\sum_{k=1}^{n}\hat{\mathbb{E}}\left(1-g\left(\frac{|X_k|}{b_n}\right)\right)\\ &=\sum_{n=1}^{\infty}\hat{\mathbb{E}}\left(1-g\left(\frac{|X|}{b_n}\right)\right)\\ &\le\sum_{n=1}^{\infty}\mathbb{V}\left(|X|>\mu b_n\right)\\ &<\infty,\end{aligned}$$

hence, the proof of (4.1) is established.

Next, we prove (4.2). First of all, we verify that

$$b_n^{-1}\max_{1\le j\le n}\left|\sum_{k=1}^{j}\hat{\mathbb{E}}\left(a_{nk}Y_{nk}\right)\right|\to 0,\ \text{as } n\to\infty.$$

Noting that

$$|Z_{nk}| = |X_k + b_n| I(X_k < -b_n) + |X_k - b_n| I(X_k > b_n) \le |X_k|\left(1 - g\left(\frac{|X_k|}{b_n}\right)\right). \quad (4.4)$$

According to (3.3) and a_{nk} is non-negative, we can get

$$\hat{\mathbb{E}}(a_{nk}X_k) = a_{nk}\hat{\mathbb{E}}X_k = 0. \quad (4.5)$$

Combining with (4.3), we have

$$|X|\left(1 - g\left(\frac{|X|}{b_n}\right)\right) \le |X| I(|X| > \mu b_n) \le \frac{|X|^2 \ln^{1-2/\gamma}|X|}{\mu b_n \ln^{1-2/\gamma}(\mu b_n)}. \quad (4.6)$$

When $1 \le \alpha \le 2$, we contact (2.2), (2.9), (3.1), (3.2), (4.4)–(4.6) and $\ln b_n \sim c \ln n$. It is easy to obtain that

$$\begin{aligned}
b_n^{-1}\max_{1\le j\le n}\left|\sum_{k=1}^{j}\hat{\mathbb{E}}(a_{nk}Y_{nk})\right| &\le b_n^{-1}\sum_{k=1}^{n}\left|\hat{\mathbb{E}}(a_{nk}Y_{nk})\right| \\
&= b_n^{-1}\sum_{k=1}^{n}\left|\hat{\mathbb{E}}(a_{nk}X_k) - \hat{\mathbb{E}}(a_{nk}Y_{nk})\right| \\
&\le b_n^{-1}\sum_{k=1}^{n}\hat{\mathbb{E}}|a_{nk}X_k - a_{nk}Y_{nk}| \\
&\le b_n^{-1}\sum_{k=1}^{n}a_{nk}\hat{\mathbb{E}}|Z_{nk}| \qquad (4.7)\\
&\le b_n^{-1}\left(\sum_{k=1}^{n}a_{nk}^{\alpha}\right)^{\frac{1}{\alpha}}\left(\sum_{k=1}^{n}1\right)^{1-\frac{1}{\alpha}}\hat{\mathbb{E}}\left(|X|\left(1 - g\left(\frac{|X|}{b_n}\right)\right)\right) \\
&\le nb_n^{-1}\left(\mu b_n \ln^{1-2/\gamma}(\mu b_n)\right)^{-1}\hat{\mathbb{E}}\left(|X|^2\ln^{1-2/\gamma}|X|\right) \\
&\le c\frac{n}{b_n^2 \ln^{1-2/\gamma} b_n} \\
&\le c\frac{1}{n^{2/\alpha-1}\ln n} \to 0, \text{ as } n \to \infty.
\end{aligned}$$

When $0 < \alpha < 1$, according to (2.9) and (3.1), we can get $\hat{\mathbb{E}}(|X|) \le C_{\mathbb{V}}\left(|X|^2\ln^{1-2/\gamma}|X|\right) < \infty$. Noting $|Y_{nk}| \le |X_k|$ and $\gamma > 0$, we have

$$\begin{aligned}
b_n^{-1}\max_{1\le j\le n}\left|\sum_{k=1}^{j}\hat{\mathbb{E}}(a_{nk}Y_{nk})\right| &\le b_n^{-1}\sum_{k=1}^{n}\left|\hat{\mathbb{E}}(a_{nk}Y_{nk})\right| \\
&= b_n^{-1}\sum_{k=1}^{n}a_{nk}\hat{\mathbb{E}}|Y_{nk}| \qquad (4.8)\\
&\le b_n^{-1}\sum_{k=1}^{n}a_{nk}\hat{\mathbb{E}}|X|
\end{aligned}$$

$$\leq n^{1/\alpha} b_n^{-1} \hat{\mathbb{E}}|X|$$
$$\leq c \frac{1}{\ln^{1/\gamma} n} \to 0, \text{ as } n \to \infty.$$

Thus, for $\varepsilon > 0$ and all n large enough, we have

$$\max_{1\leq j\leq n} \left| \sum_{k=1}^{j} \hat{\mathbb{E}}(a_{nk} Y_{nk}) \right| \leq \frac{b_n \varepsilon}{2}.$$

In order to prove (4.2), it suffices to show

$$I_3 = \sum_{n=1}^{\infty} n^{-1} \mathbb{V} \left(\max_{1\leq j\leq n} \left| \sum_{k=1}^{j} a_{nk} \left(Y_{nk} - \hat{\mathbb{E}} Y_{nk} \right) \right| > \frac{b_n \varepsilon}{2} \right) < \infty.$$

Noting that, for $p \geq 1$,

$$\hat{\mathbb{E}} \left| \left(Y_{nk} - \hat{\mathbb{E}} Y_{nk} \right) \right|^p \leq c_p \hat{\mathbb{E}} \left(|Y_{nk}|^p + \left| \hat{\mathbb{E}} Y_{nk} \right|^p \right) \leq 2c_p \hat{\mathbb{E}} |Y_{nk}|^p, \tag{4.9}$$

where c_p is a positive constant depending only on p.

We know a_{nk} is non-negative. By Definition 2.5, for fixed $n \geq 1$, $\left\{ a_{nk} \left(Y_{nk} - \hat{\mathbb{E}} Y_{nk} \right), 1 \leq k \leq n \right\}$ is still negatively dependent sequence of random variables. By (4.9), Markov inequality and (2.3) for $q = 2$, we can get

$$\begin{aligned}
I_3 \leq & c \sum_{n=1}^{\infty} n^{-1} b_n^{-2} \hat{\mathbb{E}} \left(\max_{1\leq j\leq n} \left| \sum_{k=1}^{j} a_{nk} \left(Y_{nk} - \hat{\mathbb{E}} Y_{nk} \right) \right|^2 \right) \\
\leq & c \sum_{n=1}^{\infty} n^{-1} b_n^{-2} \left(\sum_{k=1}^{n} \hat{\mathbb{E}} \left| a_{nk} \left(Y_{nk} - \hat{\mathbb{E}} Y_{nk} \right) \right|^2 \right) + \\
& c \sum_{n=1}^{\infty} n^{-1} b_n^{-2} \left(\sum_{k=1}^{n} \left[\left(\hat{\mathbb{E}} \left[a_{nk} \left(Y_{nk} - \hat{\mathbb{E}} Y_{nk} \right) \right] \right)^+ + \left(\hat{\varepsilon} \left[a_{nk} \left(Y_{nk} - \hat{\mathbb{E}} Y_{nk} \right) \right] \right)^- \right] \right)^2 \\
\leq & c \sum_{n=1}^{\infty} n^{-1} b_n^{-2} \sum_{k=1}^{n} \hat{\mathbb{E}} |a_{nk} Y_{nk}|^2 + \\
& c \sum_{n=1}^{\infty} n^{-1} b_n^{-2} \left(\sum_{k=1}^{n} \left[\left(\hat{\mathbb{E}} \left[a_{nk} \left(Y_{nk} - \hat{\mathbb{E}} Y_{nk} \right) \right] \right)^+ + \left(\hat{\varepsilon} \left[a_{nk} \left(Y_{nk} - \hat{\mathbb{E}} Y_{nk} \right) \right] \right)^- \right] \right)^2 \\
:= & I_4 + I_5.
\end{aligned}$$

By (4.3) and C_r inequality, for any $\lambda > 0$, we can obtain

$$|Y_{nk}|^\lambda \leq |X_k|^\lambda I(|X_k| \leq b_n) + b_n^\lambda I(|X_k| > b_n) \leq |X_k|^\lambda g\left(\frac{\mu |X_k|}{b_n} \right) + b_n^\lambda \left(1 - g\left(\frac{|X_k|}{b_n} \right) \right). \tag{4.10}$$

For I_4, combining (2.4), (3.1), (3.2) and (4.10), we have

$$I_4 = c \sum_{n=1}^{\infty} n^{-1} b_n^{-2} \sum_{k=1}^{n} \hat{\mathbb{E}} |a_{nk} Y_{nk}|^2$$

$$
\begin{aligned}
&= c\sum_{n=1}^{\infty} n^{-1}b_n^{-2}\sum_{k=1}^{n} a_{nk}^2\hat{\mathbb{E}}\,|Y_{nk}|^2 \\
&\le c\sum_{n=1}^{\infty} n^{2/\alpha-1}b_n^{-2}\left(\hat{\mathbb{E}}|X|^2 g\left(\frac{\mu|X|}{b_n}\right)+b_n^2\hat{\mathbb{E}}\left(1-g\left(\frac{|X|}{b_n}\right)\right)\right) \\
&\le c\sum_{n=1}^{\infty} n^{2/\alpha-1}b_n^{-2}\left(\hat{\mathbb{E}}|X|^2 g\left(\frac{\mu|X|}{b_n}\right)\right)+c\sum_{n=1}^{\infty} n^{2/\alpha-1}\mathbb{V}\left(|X|>\mu b_n\right) \\
&\le c\sum_{n=1}^{\infty} n^{2/\alpha-1}b_n^{-2}\left(\hat{\mathbb{E}}|X|^2 g\left(\frac{\mu|X|}{b_n}\right)\right).
\end{aligned}
$$

Let $g_j(x)\in C_{l,\,\mathrm{Lip}}(\mathbb{R})$, $j\ge 1$, such that $0\le g_j(x)\le 1$ for all x; $g_j\left(x/b_{2^j}\right)=1$ if $b_{2^{j-1}}<|x|\le b_{2^j}$ and $g_j\left(x/b_{2^j}\right)=0$ if $|x|\le \mu b_{2^{j-1}}$ or $|x|>(1+\mu)b_{2^j}$. Then for any $r>0$, we can obtain

$$
I\left(b_{2^{j-1}}<|X|\le b_{2^j}\right)\le g_j\left(\frac{|X|}{b_{2^j}}\right)\le I\left(\mu b_{2^{j-1}}<|X|\le (1+\mu)b_{2^j}\right), \tag{4.11}
$$

$$
|X|^r g\left(\frac{|X|}{b_{2^k}}\right)\le 1+\sum_{j=1}^{k}|X|^r g_j\left(\frac{|X|}{b_{2^j}}\right). \tag{4.12}
$$

It is noted that according to (2.5), (3.1), (4.11), (4.12), $0<\gamma<2$ and $g(x)$ is decreasing in $x\ge 0$. It is easy to prove that.

$$
\begin{aligned}
I_4 &\le c\sum_{n=1}^{\infty} n^{2/\alpha-1}b_n^{-2}\hat{\mathbb{E}}\left(|X|^2 g\left(\frac{\mu|X|}{b_n}\right)\right) \\
&\le c\sum_{k=1}^{\infty}\sum_{2^{k-1}\le n<2^k} 2^{k(2/\alpha-1)}b_{2^{k-1}}^{-2}\hat{\mathbb{E}}\left(|X|^2 g\left(\frac{\mu|X|}{b_{2^k}}\right)\right) \\
&\le c\sum_{k=1}^{\infty} 2^{2k/\alpha}b_{2^k}^{-2}\sum_{j=1}^{k}\hat{\mathbb{E}}\left(|X|^2 g_j\left(\frac{\mu|X|}{b_{2^j}}\right)\right) \\
&= c\sum_{j=1}^{\infty}\hat{\mathbb{E}}\left(|X|^2 g_j\left(\frac{\mu|X|}{b_{2^j}}\right)\right)\sum_{k=j}^{\infty} 2^{2k/\alpha}b_{2^k}^{-2} \\
&\le c\sum_{j=1}^{\infty}\hat{\mathbb{E}}\left(|X|^2 g_j\left(\frac{\mu|X|}{b_{2^j}}\right)\right)\sum_{k=j}^{\infty} k^{-2/\gamma} \\
&\le c\sum_{j=1}^{\infty} j^{1-2/\gamma}\cdot\hat{\mathbb{E}}\left(|X|^2 g_j\left(\frac{\mu|X|}{b_{2^j}}\right)\right) \\
&\le \sum_{j=1}^{\infty} j^{1-2/\gamma}\cdot b_{2^j}^2\mathbb{V}\left(|X|>b_{2^{j-1}}\right) \\
&= \sum_{j=1}^{\infty} 2^{2j/\alpha} j\mathbb{V}\left(|X|>cb_{2^j}\right) \\
&< \infty.
\end{aligned} \tag{4.13}
$$

Next, we estimate $I_5 < \infty$. According to (2.1), then we have $\hat{\mathbb{E}}\left(a_{nk}Y_{nk} - \hat{\mathbb{E}}\left(a_{nk}Y_{nk}\right)\right) = 0$. By Definition 2.1, we know $\hat{\mathbb{E}}(-X) = -\hat{\varepsilon}(X)$. Combining (2.2) and C_r inequality, we can obtain

$$\begin{aligned}
I_5 &= c\sum_{n=1}^{\infty} n^{-1}b_n^{-2}\left(\sum_{k=1}^{n}\left(\hat{\varepsilon}\left[a_{nk}\left(Y_{nk}-\hat{\mathbb{E}}Y_{nk}\right)\right]\right)^{-}\right)^2\\
&= c\sum_{n=1}^{\infty} n^{-1}b_n^{-2}\left(\sum_{k=1}^{n}\left(-\hat{\mathbb{E}}\left[-a_{nk}\left(Y_{nk}-\hat{\mathbb{E}}Y_{nk}\right)\right]\right)^{-}\right)^2\\
&\le c\sum_{n=1}^{\infty} n^{-1}b_n^{-2}\left(\sum_{k=1}^{n}\left|-\hat{\mathbb{E}}\left[-a_{nk}Y_{nk}+\hat{\mathbb{E}}a_{nk}Y_{nk}\right]\right|\right)^2\\
&= c\sum_{n=1}^{\infty} n^{-1}b_n^{-2}\left(\sum_{k=1}^{n}\left|\hat{\mathbb{E}}\left[-a_{nk}Y_{nk}\right]+\hat{\mathbb{E}}\left[a_{nk}Y_{nk}\right]\right|\right)^2\\
&\le c\sum_{n=1}^{\infty} n^{-1}b_n^{-2}\left(\sum_{k=1}^{n}\left(\left|\hat{\mathbb{E}}\left[-a_{nk}Y_{nk}\right]\right|+\left|\hat{\mathbb{E}}\left[a_{nk}Y_{nk}\right]\right|\right)\right)^2\\
&\le c\sum_{n=1}^{\infty} n^{-1}b_n^{-2}\left(\sum_{k=1}^{n}a_{nk}\left|\hat{\mathbb{E}}\left[-Y_{nk}\right]\right|\right)^2 + c\sum_{n=1}^{\infty} n^{-1}b_n^{-2}\left(\sum_{k=1}^{n}a_{nk}\left|\hat{\mathbb{E}}\left[Y_{nk}\right]\right|\right)^2\\
&= c\sum_{n=1}^{\infty} n^{-1}b_n^{-2}\left(\sum_{k=1}^{n}a_{nk}\left|\hat{\mathbb{E}}\left[-X_k\right]-\hat{\mathbb{E}}\left[-Y_{nk}\right]\right|\right)^2 +\\
&\quad c\sum_{n=1}^{\infty} n^{-1}b_n^{-2}\left(\sum_{k=1}^{n}a_{nk}\left|\hat{\mathbb{E}}\left[X_k\right]-\hat{\mathbb{E}}\left[Y_{nk}\right]\right|\right)^2\\
&\le c\sum_{n=1}^{\infty} n^{-1}b_n^{-2}\left(\sum_{k=1}^{n}a_{nk}\hat{\mathbb{E}}\left|-X_k-(-Y_{nk})\right|\right)^2 + c\sum_{n=1}^{\infty} n^{-1}b_n^{-2}\left(\sum_{k=1}^{n}a_{nk}\hat{\mathbb{E}}\left|X_k-Y_{nk}\right|\right)^2\\
&\le c\sum_{n=1}^{\infty} n^{-1}b_n^{-2}\left(\sum_{k=1}^{n}a_{nk}\hat{\mathbb{E}}\left|X_k-Y_{nk}\right|\right)^2.
\end{aligned}$$

Noting $2^{-1/\alpha} < \mu < 1$, we can get $\mu > b_{2^{k-1}}/b_{2^k}$. By (4.3), we have

$$1-g\left(\frac{|X|}{b_{2^k}}\right) \le I\left(\frac{|X|}{b_{2^k}} > \mu\right) \le I\left(|X| > b_{2^{k-1}}\right) = \sum_{j=k}^{\infty} I\left(b_{2^{j-1}} < |X| \le b_{2^j}\right) \le \sum_{j=k}^{\infty} g_j\left(\frac{|X|}{b_{2^j}}\right). \tag{4.14}$$

For I_5, we contact (2.5), (3.1), (3.2), (4.11), (4.14) and $\hat{\mathbb{E}}$ is countably sub-additive. We can get

$$\begin{aligned}
I_5 \le& c\sum_{n=1}^{\infty} n^{-1}b_n^{-2}\left(\sum_{k=1}^{n}a_{nk}\hat{\mathbb{E}}\left|X_k-Y_{nk}\right|\right)^2\\
\le& c\sum_{n=1}^{\infty} n^{-1}b_n^{-2}n^{\max(2,2/\alpha)}\hat{\mathbb{E}}^2\left(|X|\left(1-g\left(\frac{|X|}{b_n}\right)\right)\right)\\
\le& c\sum_{k=1}^{\infty}\sum_{2^{k-1}\le n<2^k} 2^{-k}b_{2^{k-1}}^{-2}2^{\max(2,2/\alpha)k}\hat{\mathbb{E}}^2\left(|X|\left(1-g\left(\frac{|X|}{b_{2^{k-1}}}\right)\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq c \sum_{k=1}^{\infty} b_{2^{k-1}}^{-2} 2^{\max(2,2/\alpha)k} \left(\sum_{j=k}^{\infty} \hat{\mathbb{E}}|X| g_j\left(\frac{|X|}{b_{2^j}}\right)\right)^2 \\
&\leq c \sum_{k=1}^{\infty} 2^{\max(2,2/\alpha)k} k^{2/\gamma-1} b_{2^k}^{-3} \cdot b_{2^k} k^{1-2/\gamma} \sum_{j=k}^{\infty} b_{2^j} \mathbb{V}\left(|X| > c b_{2^j}\right) \cdot \sum_{j=k}^{\infty} b_{2^j} \mathbb{V}\left(|X| > c b_{2^j}\right) \\
&\leq c \sum_{k=1}^{\infty} 2^{\max(2,2/\alpha)k} k^{2/\gamma-1} b_{2^k}^{-3} \sum_{j=k}^{\infty} b_{2^j}^{2} j^{1-2/\gamma} \mathbb{V}\left(|X| > c b_{2^j}\right) \cdot \sum_{j=k}^{\infty} b_{2^j} \mathbb{V}\left(|X| > c b_{2^j}\right) \\
&\leq c \sum_{k=1}^{\infty} 2^{\max(2,2/\alpha)k} k^{2/\gamma-1} b_{2^k}^{-3} \sum_{j=k}^{\infty} b_{2^j} \mathbb{V}\left(|X| > c b_{2^j}\right) \\
&= c \sum_{j=1}^{\infty} b_{2^j} \mathbb{V}\left(|X| > c b_{2^j}\right) \sum_{k=1}^{j} 2^{\max(2,2/\alpha)k} k^{2/\gamma-1} b_{2^k}^{-3} \\
&= c \sum_{j=1}^{\infty} b_{2^j} \mathbb{V}\left(|X| > c b_{2^j}\right) \sum_{k=1}^{j} 2^{\max(2-3/\alpha,-1/\alpha)k} k^{-1-1/\gamma} \\
&\leq c \begin{cases} \sum_{j=1}^{\infty} b_{2^j} \mathbb{V}\left(|X| > c b_{2^j}\right) & \text{if } 0 < \alpha \leq 3/2; \\ \sum_{j=1}^{\infty} b_{2^j} \mathbb{V}\left(|X| > c b_{2^j}\right) \sum_{k=1}^{j} 2^{k(2-3/\alpha)} k^{-1-1/\gamma} & \text{if } 3/2 < \alpha \leq 2; \end{cases} \\
&\leq c \begin{cases} \sum_{j=1}^{\infty} b_{2^j} \mathbb{V}\left(|X| > c b_{2^j}\right) & \text{if } 0 < \alpha \leq 3/2; \\ \sum_{j=1}^{\infty} 2^{j(2-3/\alpha)} j^{-1-1/\gamma} b_{2^j} \mathbb{V}\left(|X| > c b_{2^j}\right) & \text{if } 3/2 < \alpha \leq 2; \end{cases} \\
&\leq c \begin{cases} \sum_{j=1}^{\infty} 2^{j/\alpha} j^{1/\gamma} \mathbb{V}\left(|X| > c b_{2^j}\right) < \infty & \text{if } 0 < \alpha \leq 3/2. \\ \sum_{j=1}^{\infty} 2^{2j(1-1/\alpha)} j^{-1} \mathbb{V}\left(|X| > c b_{2^j}\right) < \infty & \text{if } 3/2 < \alpha \leq 2. \end{cases}
\end{aligned}
$$

4.2. Proof of Theorem 3.2.

For $\forall \varepsilon > 0$, we have by Theorem 3.1 that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1} C_{\mathbb{V}} \left\{ b_n^{-1} \max_{1 \leq j \leq n} \left| \sum_{k=1}^{j} a_{nk} X_k \right| - \varepsilon \right\}_{+}^{\theta} \\
= &\sum_{n=1}^{\infty} n^{-1} \int_0^{\infty} \mathbb{V}\left(b_n^{-1} \max_{1 \leq j \leq n} \left| \sum_{k=1}^{j} a_{nk} X_k \right| - \varepsilon > t^{1/\theta} \right) \mathrm{d}t \\
\leq &\sum_{n=1}^{\infty} n^{-1} \mathbb{V}\left(\max_{1 \leq j \leq n} \left| \sum_{k=1}^{j} a_{nk} X_k \right| > b_n \varepsilon \right) + \sum_{n=1}^{\infty} n^{-1} \int_1^{\infty} \mathbb{V}\left(\max_{1 \leq j \leq n} \left| \sum_{k=1}^{j} a_{nk} X_k \right| > b_n t^{1/\theta} \right) \mathrm{d}t \\
\leq &c \sum_{n=1}^{\infty} n^{-1} \int_1^{\infty} \mathbb{V}\left(\max_{1 \leq j \leq n} \left| \sum_{k=1}^{j} a_{nk} X_k \right| > b_n t^{1/\theta} \right) \mathrm{d}t.
\end{aligned}
$$

In order to prove (3.5), for $t \geq 1$, denoted as $1 \leq k \leq n$, we obtain

$$Y'_{nk} := -b_n t^{1/\theta} I\left(X_k < -b_n t^{1/\theta}\right) + X_k I\left(|X_k| \leq b_n t^{1/\theta}\right) + b_n t^{1/\theta} I\left(X_k > b_n t^{1/\theta}\right),$$
$$Z'_{nk} := X_k - Y'_{nk} = \left(X_k + b_n t^{1/\theta}\right) I\left(X_k < -b_n t^{1/\theta}\right) + \left(X_k - b_n t^{1/\theta}\right) I\left(X_k > b_n t^{1/\theta}\right).$$

We can easily see that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1} \int_1^{\infty} \mathbb{V}\left(\max_{1\leq j\leq n}\left|\sum_{k=1}^{j} a_{nk} X_k\right| > b_n t^{1/\theta}\right) \mathrm{d}t \\
\leq &\sum_{n=1}^{\infty} n^{-1} \sum_{k=1}^{n} \int_1^{\infty} \mathbb{V}\left(|X_k| > b_n t^{1/\theta}\right) \mathrm{d}t + \\
&\sum_{n=1}^{\infty} n^{-1} \int_1^{\infty} \mathbb{V}\left(\max_{1\leq j\leq n}\left|\sum_{k=1}^{j} a_{nk}\left(Y'_{nk} - \hat{\mathbb{E}} Y'_{nk}\right)\right| > b_n t^{1/\theta} - \max_{1\leq j\leq n}\left|\sum_{k=1}^{j} \hat{\mathbb{E}}\left(a_{nk} Y'_{nk}\right)\right|\right) \mathrm{d}t \\
:= &H_1 + H_2.
\end{aligned}$$

In order to prove (3.5), we just need to prove

$$H_1 < \infty, \tag{4.15}$$
$$H_2 < \infty. \tag{4.16}$$

First of all, we prove (4.15). For $t^{1/\theta} \geq 1$, since $g(x)$ is decreasing in $x \geq 0$, we have $1 - g\left(\frac{|X|}{b_n t^{1/\theta}}\right) \leq 1 - g\left(\frac{|X|}{b_n}\right)$. According to (2.5), (3.1), (4.11), (4.14), $0 < \theta < 2$ and $\hat{\mathbb{E}}$ is countably sub-additive. We can obtain

$$\begin{aligned}
H_1 &\leq \sum_{n=1}^{\infty} n^{-1} \int_1^{\infty} \sum_{k=1}^{n} \hat{\mathbb{E}}\left(1 - g\left(\frac{|X_k|}{b_n t^{1/\theta}}\right)\right) \mathrm{d}t \\
&\leq \sum_{n=1}^{\infty} \int_1^{\infty} \hat{\mathbb{E}} \frac{|X|^2 \ln^{1-2/\gamma-1/2}|X|}{(\mu b_n t^{1/\theta})^2 \ln^{1-2/\gamma-1/2}(\mu b_n t^{1/\theta})}\left(1 - g\left(\frac{|X|}{b_n t^{1/\theta}}\right)\right) \mathrm{d}t \\
&\leq c \sum_{n=1}^{\infty} \int_1^{\infty} \left(\mu b_n t^{1/\theta}\right)^{-2} \ln^{2/\gamma-1/2}\left(\mu b_n t^{1/\theta}\right) \hat{\mathbb{E}}\left(|X|^2 \ln^{1/2-2/\gamma}|X|\right)\left(1 - g\left(\frac{|X|}{b_n}\right)\right) \mathrm{d}t \\
&= c \sum_{n=1}^{\infty} \hat{\mathbb{E}}\left(|X|^2 \ln^{1/2-2/\gamma}|X|\right)\left(1 - g\left(\frac{|X|}{b_n}\right)\right) \int_{\mu b_n}^{\infty} y^{-2} \ln^{2/\gamma-1/2} y \cdot b_n^{-\theta} \mu^{-\theta} \theta y^{\theta-1} \mathrm{d}y \quad \left(\text{let } y = \mu b_n t^{1/\theta}\right) \\
&\leq c \sum_{n=1}^{\infty} \hat{\mathbb{E}}\left(|X|^2 \ln^{1/2-2/\gamma}|X|\right)\left(1 - g\left(\frac{|X|}{b_n}\right)\right) b_n^{-\theta} b_n^{\theta-2} \ln^{2/\gamma-1/2}(\mu b_n) \\
&\leq c \sum_{k=1}^{\infty} \sum_{2^{k-1}\leq n<2^k} b_{2^{k-1}}^{-2} \ln^{2/\gamma-1/2}(\mu b_{2^{k-1}}) \hat{\mathbb{E}}\left(|X|^2 \ln^{1/2-2/\gamma}|X|\right)\left(1 - g\left(\frac{|X|}{b_{2^{k-1}}}\right)\right) \\
&\leq c \sum_{k=1}^{\infty} 2^{k(1-2/\alpha)} k^{-1/2} \sum_{j=k}^{\infty} \hat{\mathbb{E}}\left(|X|^2 \ln^{1/2-2/\gamma}|X|\right) g_j\left(\frac{|X|}{b_{2^j}}\right)
\end{aligned}$$

$$= c\sum_{j=1}^{\infty}\hat{\mathbb{E}}\left(|X|^2\ln^{1/2-2/\gamma}|X|\right)g_j\left(\frac{|X|}{b_{2^j}}\right)\sum_{k=1}^{j}2^{k(1-2/\alpha)}k^{-1/2}$$

$$\le c\begin{cases}\sum\limits_{j=1}^{\infty} b_{2^j}^2 j^{1/2-2/\gamma}\mathbb{V}\left(|X|>cb_{2^j}\right) & \text{if } 0<\alpha<2;\\ \sum\limits_{j=1}^{\infty} j^{1/2}b_{2^j}^2 j^{1/2-2/\gamma}\mathbb{V}\left(|X|>cb_{2^j}\right) & \text{if } \alpha=2;\end{cases}$$

$$\le c\begin{cases}\sum\limits_{j=1}^{\infty} 2^{2j/\alpha}j^{-1/2}\mathbb{V}\left(|X|>cb_{2^j}\right)<\infty & \text{if } 0<\alpha<2.\\ \sum\limits_{j=1}^{\infty} 2^j j\mathbb{V}\left(|X|>cb_{2^j}\right)<\infty & \text{if } \alpha=2.\end{cases}$$

hence, the proof of (4.15) is established.

Next, we prove (4.16). Let's prove that

$$\sup_{t\ge1} b_n^{-1}t^{-1/\theta}\max_{1\le j\le n}\left|\sum_{k=1}^{j}\hat{\mathbb{E}}\left(a_{nk}Y'_{nk}\right)\right|\to 0,\ \text{as } n\to\infty.$$

When $1\le\alpha\le 2$, similar considerations to (4.7), we have

$$\begin{aligned}\sup_{t\ge1} b_n^{-1}t^{-1/\theta}\max_{1\le j\le n}\left|\sum_{k=1}^{j}\hat{\mathbb{E}}\left(a_{nk}Y'_{nk}\right)\right| &\le \sup_{t\ge1} b_n^{-1}t^{-1/\theta}\sum_{k=1}^{n}\left|\hat{\mathbb{E}}\left(a_{nk}Y'_{nk}\right)\right|\\ &\le \sup_{t\ge1} b_n^{-1}t^{-1/\theta}\sum_{k=1}^{n}\left|\hat{\mathbb{E}}\left(a_{nk}X_k\right)-\hat{\mathbb{E}}\left(a_{nk}Y'_{nk}\right)\right|\\ &\le \sup_{t\ge1} b_n^{-1}t^{-1/\theta}\sum_{k=1}^{n}a_{nk}\hat{\mathbb{E}}\left|Z'_{nk}\right|\\ &\le \sup_{t\ge1} b_n^{-1}t^{-1/\theta}\sum_{k=1}^{n}a_{nk}\hat{\mathbb{E}}\left(|X|\left(1-g\left(\frac{|X|}{b_nt^{1/\theta}}\right)\right)\right)\\ &\le \sup_{t\ge1} b_n^{-1}t^{-1/\theta}n\hat{\mathbb{E}}\left(|X|\left(1-g\left(\frac{|X|}{b_n}\right)\right)\right)\\ &\le nb_n^{-1}\hat{\mathbb{E}}\left(|X|\left(1-g\left(\frac{|X|}{b_n}\right)\right)\right)\\ &\le nb_n^{-1}\left(\mu b_n\ln^{1-2/\gamma}(\mu b_n)\right)^{-1}\hat{\mathbb{E}}\left(|X|^2\ln^{1-2/\gamma}|X|\right)\\ &\le c\frac{1}{n^{2/\alpha-1}\ln n}\to 0,\ \text{as } n\to\infty.\end{aligned}$$

When $0<\alpha<1$, noting $\left|Y'_{nk}\right|\le|X_k|$, similar to (4.8), we can get

$$\begin{aligned}\sup_{t\ge1} b_n^{-1}t^{-1/\theta}\max_{1\le j\le n}\left|\sum_{k=1}^{j}\hat{\mathbb{E}}\left(a_{nk}Y'_{nk}\right)\right| &\le \sup_{t\ge1} b_n^{-1}t^{-1/\theta}\sum_{k=1}^{n}\left|\hat{\mathbb{E}}\left(a_{nk}Y'_{nk}\right)\right|\\ &\le \sup_{t\ge1} b_n^{-1}t^{-1/\theta}\sum_{k=1}^{n}a_{nk}\hat{\mathbb{E}}\left|Y'_{nk}\right|\end{aligned}$$

$$\leq \sup_{t\geq 1} b_n^{-1} t^{-1/\theta} \sum_{k=1}^{n} a_{nk} \hat{\mathbb{E}} |X|$$
$$\leq b_n^{-1} \sum_{k=1}^{n} a_{nk} \hat{\mathbb{E}} |X|$$
$$\leq c \frac{1}{\ln^{1/\gamma} n} \to 0, \text{ as } n \to \infty.$$

Hence, for $t^{1/\theta} \geq 1$ and all n large enough, we can get

$$\max_{1\leq j\leq n} \left| \sum_{k=1}^{j} \hat{\mathbb{E}} (a_{nk} Y'_{nk}) \right| \leq \frac{b_n t^{1/\theta}}{2}.$$

In order to prove (4.14), it suffices to show

$$H_3 = \sum_{n=1}^{\infty} n^{-1} \int_1^{\infty} \mathbb{V} \left(\max_{1\leq j\leq n} \left| \sum_{k=1}^{j} a_{nk} \left(Y'_{nk} - \hat{\mathbb{E}} Y'_{nk} \right) \right| > \frac{b_n t^{1/\theta}}{2} \right) \mathrm{d}t < \infty.$$

We know a_{nk} is non-negative. By Definition 2.5, for fixed $n \geq 1$, $\left\{ a_{nk} \left(Y'_{nk} - \hat{\mathbb{E}} Y'_{nk} \right), 1 \leq k \leq n \right\}$ is still a negatively dependent sequence of random variables. By (4.9), Markov inequality and (2.3) for $q = 2$, we can get

$$\begin{aligned}
H_3 &= \sum_{n=1}^{\infty} n^{-1} \int_1^{\infty} \mathbb{V} \left(\max_{1\leq j\leq n} \left| \sum_{k=1}^{j} a_{nk} \left(Y'_{nk} - \hat{\mathbb{E}} Y'_{nk} \right) \right| > \frac{b_n t^{1/\theta}}{2} \right) \mathrm{d}t \\
&\leq c \sum_{n=1}^{\infty} n^{-1} \int_1^{\infty} \left(b_n t^{1/\theta} \right)^{-2} \hat{\mathbb{E}} \left(\max_{1\leq j\leq n} \left| \sum_{k=1}^{j} a_{nk} \left(Y'_{nk} - \hat{\mathbb{E}} Y'_{nk} \right) \right|^2 \right) \mathrm{d}t \\
&\leq c \sum_{n=1}^{\infty} n^{-1} \int_1^{\infty} \left(b_n t^{1/\theta} \right)^{-2} \sum_{k=1}^{n} \hat{\mathbb{E}} \left| a_{nk} \left(Y'_{nk} - \hat{\mathbb{E}} Y'_{nk} \right) \right|^2 \mathrm{d}t + \\
&\quad c \sum_{n=1}^{\infty} n^{-1} \int_1^{\infty} \left(b_n t^{1/\theta} \right)^{-2} \left(\sum_{k=1}^{n} \left[\left(\hat{\mathbb{E}} \left[a_{nk} \left(Y'_{nk} - \hat{\mathbb{E}} Y'_{nk} \right) \right] \right)^+ + \left(\hat{\varepsilon} \left[a_{nk} \left(Y'_{nk} - \hat{\mathbb{E}} Y'_{nk} \right) \right] \right)^- \right] \right)^2 \mathrm{d}t \\
&\leq c \sum_{n=1}^{\infty} n^{-1} \int_1^{\infty} \left(b_n t^{1/\theta} \right)^{-2} \sum_{k=1}^{n} \hat{\mathbb{E}} \left| a_{nk} Y'_{nk} \right|^2 \mathrm{d}t + \\
&\quad c \sum_{n=1}^{\infty} n^{-1} \int_1^{\infty} \left(b_n t^{1/\theta} \right)^{-2} \left(\sum_{k=1}^{n} \left[\left(\hat{\mathbb{E}} \left[a_{nk} \left(Y'_{nk} - \hat{\mathbb{E}} Y'_{nk} \right) \right] \right)^+ + \left(\hat{\varepsilon} \left[a_{nk} \left(Y'_{nk} - \hat{\mathbb{E}} Y'_{nk} \right) \right] \right)^- \right] \right)^2 \mathrm{d}t \\
&:= H_4 + H_5.
\end{aligned}$$

For H_4, similar considerations to (4.10), we can get for any $\lambda > 0$,

$$\left| Y'_{nk} \right|^{\lambda} \leq |X_k|^{\lambda} g \left(\frac{\mu |X_k|}{b_n t^{1/\theta}} \right) + \left(b_n t^{1/\theta} \right)^{\lambda} \left(1 - g \left(\frac{|X_k|}{b_n t^{1/\theta}} \right) \right). \tag{4.17}$$

Combining (3.2) and (4.17), we have

$$\begin{aligned}
H_4 &= \sum_{n=1}^{\infty} n^{-1} \int_1^{\infty} \left(b_n t^{1/\theta}\right)^{-2} \sum_{k=1}^{n} \hat{\mathbb{E}} \left|a_{nk} Y'_{nk}\right|^2 \, \mathrm{d}t \\
&= \sum_{n=1}^{\infty} n^{-1} \int_1^{\infty} \left(b_n t^{1/\theta}\right)^{-2} \sum_{k=1}^{n} a_{nk}^2 \hat{\mathbb{E}} \left|Y'_{nk}\right|^2 \, \mathrm{d}t \\
&\le \sum_{n=1}^{\infty} n^{-1} \int_1^{\infty} \left(b_n t^{1/\theta}\right)^{-2} \sum_{k=1}^{n} a_{nk}^2 \left(\hat{\mathbb{E}}|X|^2 g\left(\frac{\mu |X|}{b_n t^{1/\theta}}\right) + \left(b_n t^{1/\theta}\right)^2 \hat{\mathbb{E}}\left(1 - g\left(\frac{|X|}{b_n t^{1/\theta}}\right)\right)\right) \mathrm{d}t \\
&\le \sum_{n=1}^{\infty} n^{2/\alpha - 1} b_n^{-2} \int_1^{\infty} t^{-2/\theta} \hat{\mathbb{E}}|X|^2 g\left(\frac{\mu |X|}{b_n t^{1/\theta}}\right) \mathrm{d}t + c \sum_{n=1}^{\infty} n^{2/\alpha - 1} \int_1^{\infty} \hat{\mathbb{E}}\left(1 - g\left(\frac{|X|}{b_n t^{1/\theta}}\right)\right) \mathrm{d}t \\
&:= H_{41} + H_{42}.
\end{aligned}$$

For H_{42}, combining (2.5), (3.1), (4.11), (4.14), $0 < \gamma < 2$, $0 < \theta < 2$ and $\hat{\mathbb{E}}$ to countable sub-additivity and $g(x)$ is decreasing in $x \ge 0$. We have

$$\begin{aligned}
H_{42} &= \sum_{n=1}^{\infty} n^{2/\alpha - 1} \int_1^{\infty} \hat{\mathbb{E}}\left(1 - g\left(\frac{|X|}{b_n t^{1/\theta}}\right)\right) \mathrm{d}t \\
&\le \sum_{n=1}^{\infty} n^{2/\alpha - 1} \int_1^{\infty} \hat{\mathbb{E}} \frac{|X|^2 \ln^{1-2/\gamma-1/2} |X|}{(\mu b_n t^{1/\theta})^2 \ln^{1-2/\gamma-1/2} (\mu b_n t^{1/\theta})} \left(1 - g\left(\frac{|X|}{b_n t^{1/\theta}}\right)\right) \mathrm{d}t \\
&\le \sum_{n=1}^{\infty} n^{2/\alpha - 1} \int_1^{\infty} \left(\mu b_n t^{1/\theta}\right)^{-2} \ln^{2/\gamma - 1/2} \left(\mu b_n t^{1/\theta}\right) \hat{\mathbb{E}}\left(|X|^2 \ln^{1/2 - 2/\gamma} |X|\right) \left(1 - g\left(\frac{|X|}{b_n}\right)\right) \mathrm{d}t \\
&= \sum_{n=1}^{\infty} n^{2/\alpha - 1} \hat{\mathbb{E}}\left(|X|^2 \ln^{1/2 - 2/\gamma} |X|\right) \left(1 - g\left(\frac{|X|}{b_n}\right)\right) \int_{\mu b_n}^{\infty} y^{-2} \ln^{2/\gamma - 1/2} y \cdot b_n^{-\theta} \mu^{-\theta} \theta y^{\theta - 1} \mathrm{d}y \\
&\le c \sum_{n=1}^{\infty} n^{2/\alpha - 1} \hat{\mathbb{E}}\left(|X|^2 \ln^{1/2 - 2/\gamma} |X|\right) \left(1 - g\left(\frac{|X|}{b_n}\right)\right) b_n^{-\theta} b_n^{\theta - 2} \ln^{2/\gamma - 1/2} (\mu b_n) \\
&\le c \sum_{k=1}^{\infty} \sum_{2^{k-1} \le n < 2^k} 2^{k(2/\alpha - 1)} b_{2^{k-1}}^{-2} \ln^{2/\gamma - 1/2} (\mu b_{2^{k-1}}) \hat{\mathbb{E}}\left(|X|^2 \ln^{1/2 - 2/\gamma} |X|\right) \left(1 - g\left(\frac{|X|}{b_{2^{k-1}}}\right)\right) \\
&\le c \sum_{k=1}^{\infty} 2^{2k/\alpha} b_{2^k}^{-2} k^{2/\gamma - 1/2} \sum_{j=k}^{\infty} \hat{\mathbb{E}}\left(|X|^2 \ln^{1/2 - 2/\gamma} |X|\right) g_j\left(\frac{|X|}{b_{2^j}}\right) \qquad (4.18) \\
&\le c \sum_{j=1}^{\infty} \hat{\mathbb{E}}\left(|X|^2 \ln^{1/2 - 2/\gamma} |X|\right) g_j\left(\frac{|X|}{b_{2^j}}\right) \sum_{k=1}^{j} k^{-1/2} \\
&\le c \sum_{j=1}^{\infty} j^{1/2} b_{2^j}^2 j^{1/2 - 2/\gamma} \mathbb{V}\left(|X| > c b_{2^j}\right) \\
&= c \sum_{j=1}^{\infty} 2^{2j/\alpha} j \mathbb{V}\left(|X| > c b_{2^j}\right) \\
&< \infty.
\end{aligned}$$

Next, we prove $H_{41} < \infty$. Similar to the proof of (4.13) and (4.18), we can get

$$\begin{aligned}
H_{41} \leq & \sum_{n=1}^{\infty} n^{2/\alpha-1} b_n^{-2} \int_1^{\infty} t^{-2/\theta} \left(\hat{\mathbb{E}}|X|^2 \left(g\left(\frac{\mu|X|}{b_n t^{1/\theta}}\right) - g\left(\frac{\mu|X|}{b_n}\right) \right) \right) \mathrm{d}t + \\
& \sum_{n=1}^{\infty} n^{2/\alpha-1} b_n^{-2} \int_1^{\infty} t^{-2/\theta} \left(\hat{\mathbb{E}}|X|^2 g\left(\frac{\mu|X|}{b_n}\right) \right) \mathrm{d}t \\
\leq & \sum_{n=1}^{\infty} n^{2/\alpha-1} \int_1^{\infty} b_n^{-2} t^{-2/\theta} \ln^{2/\gamma-1/2} b_n \hat{\mathbb{E}} \left(|X|^2 \ln^{1/2-2/\gamma} |X| \left(1 - g\left(\frac{\mu|X|}{b_n}\right) \right) \right) \mathrm{d}t + \\
& c \sum_{n=1}^{\infty} n^{2/\alpha-1} b_n^{-2} \hat{\mathbb{E}} \left(|X|^2 g\left(\frac{\mu|X|}{b_n}\right) \right) \\
< & \infty.
\end{aligned}$$

Next, we estimate $H_5 < \infty$. Similar to the proof of I_5, we have

$$\begin{aligned}
H_5 &= c \sum_{n=1}^{\infty} n^{-1} \int_1^{\infty} \left(b_n t^{1/\theta}\right)^{-2} \left(\sum_{k=1}^{n} \left(\hat{\varepsilon}\left[a_{nk}\left(Y'_{nk} - \hat{\mathbb{E}} Y'_{nk}\right)\right]\right)^{-} \right)^2 \mathrm{d}t \\
&\leq c \sum_{n=1}^{\infty} n^{-1} \int_1^{\infty} \left(b_n t^{1/\theta}\right)^{-2} \left(\sum_{k=1}^{n} |a_{nk}| \hat{\mathbb{E}} \left| X_k - Y'_{nk} \right| \right)^2 \mathrm{d}t \\
&\leq c \sum_{n=1}^{\infty} n^{-1} b_n^{-2} \int_1^{\infty} t^{-2/\theta} \left(\sum_{k=1}^{n} |a_{nk}| \hat{\mathbb{E}} \left(|X| \left(1 - g\left(\frac{|X|}{b_n t^{1/\theta}}\right) \right) \right) \right)^2 \mathrm{d}t \\
&\leq c \sum_{n=1}^{\infty} n^{-1} b_n^{-2} \left(\sum_{k=1}^{n} |a_{nk}| \hat{\mathbb{E}} \left(|X| \left(1 - g\left(\frac{|X|}{b_n}\right) \right) \right) \right)^2 < \infty.
\end{aligned}$$

Hence, the proof of Theorem 3.2 is established.

5. Conclusions

This paper examines the concepts of complete convergence and complete integral convergence within sub-linear expectation space. The proof methodology employed differs from that utilized in probability space, as $\mathbb{V}$ and $\hat{\mathbb{E}}$ are not countably sub-additive in sub-linear expectation space. Additionally, the definition of identical distribution in sub-linear expectation is based on $\hat{\mathbb{E}}$ rather than $\mathbb{V}$.

Therefore, the use of suitable auxiliary tools is crucial for conducting a thorough investigation in the sub-linear expectation space. This study primarily relies on Zhang's [5] upper expectation inequality, which serves as a useful tool in our proof. Our findings indicate that the convergence integral convergence of maxima is more comprehensive than previous research results. In upcoming research endeavors, we aim to explore more intriguing outcomes.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This paper was supported by the National Natural Science Foundation of China (12061028) and Guangxi Colleges and Universities Key Laboratory of Applied Statistics.

Conflict of interest

In this article, all authors disclaim any conflict of interest.

References

1. S. G. Peng, G-expectation, G-Brownian motion and related stochastic calculus of Ito type, In: *Stochastic analysis and applications*, Berlin, Heidelberg: Springer, 2006, 541–567. http://doi.org/10.1007/978-3-540-70847-6_25
2. S. Peng, Multi-dimensional G-brownian motion and related stochastic calculus under gexpectation, *Stoch. Proc. Appl.*, **118** (2008), 2223–2253. https://doi.org/10.1016/j.spa.2007.10.015
3. S. G. Peng, Survey on normal distributions, central limit theorem, Brownian motion and the related stochastic calculus under sublinear expectations, *Sci. China Ser. A-Math.*, **52** (2009), 1391–1411. https://doi.org/10.1007/s11425-009-0121-8
4. L. X. Zhang, Strong limit theorems for extended independent random variables and extended negatively dependent random variables under sub-linear expectations, *Acta. Math. Sci.*, **42** (2016), 467–490. https://doi.org/10.1007/s10473-022-0203-z
5. L. X. Zhang, Rosenthal's inequalities for independent and negatively dependent random variables under sub-linear expectations with applications, *Sci. China Math.*, **59** (2016), 751–768. https://doi.org/10.1007/s11425-015-5105-2
6. L. X. Zhang, Exponential inequalities under the sub-linear expectations with applications to laws of the iterated logarithm, *Sci. China Math.*, **59** (2016), 2503–2526. https://doi.org/10.1007/s11425-016-0079-1
7. P. L. Hsu, H. Robbins, Complete convergence and the law of large numbers, *PNAS*, **33** (1947), 25–31. https://doi.org/10.1073/pnas.33.2.25
8. Y. S. Chow, On the rate of moment convergence of sample sums and extremes, *Bull. Inst. Math. Acad. Sin.*, **16** (1988), 177–201.
9. D. Qiu, P. Chen, Complete moment convergence for i.i.d.random variables, *Stat. Probabil. Lett.*, **91** (2014), 76–82. https://doi.org/10.1016/j.spl.2014.04.001
10. W. Yang, S. Hu, Complete moment convergence of pairwise NQD random variables, *Stochastics*, **87** (2015), 199–208. http://doi.org/10.1080/17442508.2014.939975
11. M. Song, Q. Zhu, Complete moment convergence of extended negatively dependent random variables, *J. Inequal. Appl.*, **2020** (2020), 150. https://doi.org/10.1186/s13660-020-02416-7
12. S. Li, Q. Wu, Complete integration convergence for arrays of rowwise extended negatively dependent random variables under the sub-linear expectations, *AIMS Mathematics*, **6** (2021), 12166–12181. https://doi.org/10.3934/math.2021706

13. D. Lu, Y. Meng, Complete and complete integral convergence for arrays of row wise widely negative dependent random variables under the sub-linear expectations, *Commun. Stat.-Theory. M.*, **51** (2022), 2994–3007. https://doi.org/10.1080/03610926.2020.1786585

14. X. Chen, Q. Wu. Complete convergence and complete integral convergence of partial sums for moving average process under sub-linear expectations, *AIMS Mathematics*, **7** (2022), 9694–9715. https://doi.org/10.3934/math.2022540

15. F. X. Feng, X. Zeng, A complete convergence theorem of the maximum of partial sums under the sub-linear expectations, *Filomat*, **36** (2022), 5725–5735. https://doi.org/10.2298/FIL2217725F

16. M. Xu, K. Cheng, W. Yu, Complete convergence for weighted sums of negatively dependent random variables under sub-linear expectations, *AMIS Mathematics*, **7** (2022), 19998–20019. https://doi.org/10.3934/math.20221094

17. M. Xu, K. Chng, Note on complete convergence and complete moment convergence for negatively dependent random variables under sub-linear expectations, *AMIS Mathematics*, **8** (2023), 8504–8521. https://doi.org/10.3934/math.2023428

18. Y. Wu, Y. Wang, On the complete moment convergence for weighted sums of weakly dependent random variables, *J. Math. Inequal.*, **15** (2021), 277–291. https://doi.org/10.7153/jmi-2021-15-21