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Research article

Boundedness of Marcinkiewicz integral operator of variable order in grand Herz-Morrey spaces

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Abstract: Let $\mathbb{S}^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure. Let $\Phi \in L^r(\mathbb{S}^{n-1})$ be a homogeneous function of degree zero. The variable Marcinkiewicz fractional integral operator is defined as

$$\mu_\Phi(f)(z_1) = \left(\int_0^\infty \left| \int_{|z_1 - z_2| \le s} \frac{\Phi(z_1 - z_2)}{|z_1 - z_2|^{n-1-\zeta(z_1)}} f(z_2) dz_2 \right|^2 \frac{ds}{s^3} \right)^{\frac{1}{2}}.$$

The Marcinkiewicz fractional operator of variable order $\zeta(z_1)$ is shown to be bounded from the grand Herz-Morrey spaces $M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,p(\cdot)}(\mathbb{R}^n)$ with variable exponent into the weighted space $M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,\rho,q(\cdot)}(\mathbb{R}^n)$ where

$$\rho = (1 + |z_1|)^{-\lambda}$$

and

$$\frac{1}{q(z_1)} = \frac{1}{p(z_1)} - \frac{\zeta(z_1)}{n}$$

when $p(z_1)$ is not necessarily constant at infinity.

Keywords: Lebesgue spaces; weighted estimates; Marcinkiewicz fractional; grand Herz-Morrey spaces
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1. Introduction

We consider the Riesz potential operator

$$I^{\zeta(z_1)}f(z_1) = \int_{\mathbb{R}^n} \frac{f(z_2)}{|z_1 - z_2|^{n-\zeta(z_1)}} dz_2, \qquad 0 < \zeta(z_1) < n. \tag{1.1}$$

Note that the $\zeta(z_1)$ is the order of the Riesz potential operator which is variable. Nowadays, there is a vast boom of research related to both the study of the Herz spaces themselves and the operator theory in these spaces. This is caused by the influence of some possible applications in modeling with non-standard local growth (in differential equations, fluid mechanics and elasticity theory, see for example [1, 2]). Boundedness of various operators has been intensively studied in the last years and one of the remarkable results was on the boundedness of the Hardy-Littlewood maximal operator in these spaces. For references we refer to the papers [3–7].

In [8], the authors considered variable potential operators $I^{\zeta(x)}$ to prove a Sobolev-type theorem for the potential operator from the Lebesgue space $L^{p(\cdot)}$ into the weighted Lebesgue space $L^{q(\cdot)}_w$ in $\mathbb{R}^n$ under the conditions that $p(x)$ is satisfying the logarithmic condition locally and at infinity. It was not supposed that $p(x)$ is constant at infinity but also assumed that $p(x)$ took its minimal value at infinity.

In [9], remarkable results were proven for boundedness on homogenous Herz spaces $K^{\zeta(\cdot),u}_{v(\cdot)}(\mathbb{R}^n)$ and non-homogenous Herz spaces $\dot{K}^{\zeta(\cdot),u}_{v(\cdot)}(\mathbb{R}^n)$ with variable exponents $\zeta(\cdot)$ and $v(\cdot)$. Meanwhile, they also proved boundedness for variable fractional integrals on Herz spaces with variable exponent. In [10], the authors considered the Herz Morrey spaces $M\dot{K}^{\zeta(\cdot),\lambda}_{u,v(\cdot)}(\mathbb{R}^n)$ with variable exponent and investigated mapping properties for the fractional Hardy type operators $\mathcal{H}_{\beta(\cdot)}$ and $\mathcal{H}^*_{\beta(\cdot)}$ in these spaces.

Sultan et al. [11] introduced the idea of grand variable Herz-Morrey spaces and proved boundedness for Riesz potential operator in these spaces. Inspired by the concept, in this article we will demonstrate the boundedness of Marcinkiewicz fractional integral operator of variable order from grand Herz-Morrey spaces to weighted space under some proper assumptions on weights.

We divided this article into different sections. Apart from introduction, a section is dedicated to basic lemmas and definitions. One section is for Sobolev type theorem for Marcinkiewicz fractional integral operator of variable order in grand Herz-Morrey spaces.

2. Preliminaries

For this section we refer to [12–18].

2.1. Lebesgue space with variable exponent

Definition 2.1. If H is a measurable set in $\mathbb{R}^n$ and $p(\cdot)\colon H \to [1, \infty)$ is a measurable function.

(a) Lebesgue space with variable exponent $L^{p(\cdot)}(H)$ can be defined as

$$L^{p(\cdot)}(H) = \left\{ f \text{ } measurable : \int_H \left(\frac{|f(y)|}{\gamma} \right)^{p(y)} dy < \infty \text{ } where\ \gamma\ is\ a\ constant \right\}.$$

Norm in $L^{p(\cdot)}(H)$ can be defined as

$$\|f\|_{L^{p(\cdot)}(H)} = \inf\left\{\gamma > 0 : \int_H \left(\frac{|f(y)|}{\gamma}\right)^{p(y)} dy \leq 1\right\}.$$

(b) The space $L^{p(\cdot)}_{\text{loc}}(H)$ can be defined as

$$L^{p(\cdot)}_{\text{loc}}(H) := \left\{f : f \in L^{p(\cdot)}(K)\, \textit{for all compact subsets } K \subset H\right\}.$$

We use these notations in this article:

(i) Let $f \in L^1_{\text{loc}}(H)$ be a locally integrable function, then the Hardy-Littlewood maximal operator M is defined as

$$Mf(y) := \sup_{s>0} s^{-n} \int\limits_{B(y,s)} |f(y)| dy, \quad (y \in H),$$

where

$$B(y, s) := \{x \in H : |y - x| < s\}.$$

(ii) The set $\mathfrak{P}(H)$ consists of all measuable functions $p(\cdot)$ satisfying

$$p_- := \operatorname*{ess\,inf}_{h \in H} p(h) > 1, \quad p_+ := \operatorname*{ess\,sup}_{h \in H} p(h) < \infty. \tag{2.1}$$

(iii) $\mathfrak{P}^{\log} = \mathfrak{P}^{\log}(H)$ consists of all functions $q \in \mathfrak{P}(H)$ satisfying (2.1) and log condition defined as,

$$|q(x) - q(y)| \leq \frac{C(q)}{-\ln|x - y|}, \quad |x - y| \leq \frac{1}{2}, \quad x, y \in H. \tag{2.2}$$

(iv) Let H unbounded. $\mathfrak{P}_\infty(H)$ and $\mathfrak{P}_{0,\infty}(H)$ are the subsets of $\mathfrak{P}(H)$ and values of these subsets lies in $[1, \infty)$. $\mathfrak{P}_\infty(H)$ and $\mathfrak{P}_{0,\infty}(H)$ satisfy following conditions:

$$|q(h) - q_\infty| \leq \frac{C}{\ln(e + |h|)}, \tag{2.3}$$

where $q_\infty \in (1, \infty)$.

$$|q(h) - q_0| \leq \frac{C}{\ln|h|}, |h| \leq \frac{1}{2}, \tag{2.4}$$

respectively.

(v)

$$\chi_l = \chi_{F_l},\ F_l = B_l \setminus B_{l-1},\ B_l = B(0, 2^l) = \{x \in \mathbb{R}^n : |x| < 2^l\}$$

for all $l \in \mathbb{Z}$.

C is a constant, its value varies from line to line and independent of main parameters involved.

We are assuming that order of Riesz potential operator $\zeta(x)$ is not continuous rather we are assuming that it is a measurable function in $\mathbb{R}^n$ satisfying the following conditions:

(1) $\zeta_0 := \inf_{x \in \mathbb{R}^n} \zeta(x) > 0$,

(2) $\operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(\infty)\zeta(x) < n$,

(3) $\operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x)\zeta(x) < n$,

where $p(\infty) = \lim_{|x|\to\infty} p(x)$.

The following proposition is the one of the main requirement to prove our main results. The given proposition was proved in [8] and commonly known as Sobolev theorem for Riesz potential operator in Lebesgue spaces under the some necessary assumptions on exponent.

Proposition 2.2. *Suppose that*

$$p(\cdot) \in \mathfrak{B}^{\log}(\mathbb{R}^n) \cap \mathfrak{B}_{0,\infty}(\mathbb{R}^n) \cap \mathfrak{B}(\mathbb{R}^n)$$

and assume

$$1 < p(\infty) \le p(x) \le p_+ < \infty.$$

Let $\zeta(x)$ satisfy the above conditions (1)–(3). Then, we have following weighted Sobolev-type estimate for the fractional operator $I^{\zeta(z)}$,

$$\|(1+|z|)^{-\lambda(z)} I^{\zeta(z)}(f)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where

$$\frac{1}{q(z)} = \frac{1}{p(z)} - \frac{\zeta(z)}{n}$$

is the Sobolev exponent.

$$\lambda(z) = C\zeta(z)\left(1 - \frac{\zeta(z)}{n}\right) \le C\frac{n}{4},$$

where C is denoting the Dini-Lipschitz constant from the inequality (2.3) *in which $q(\cdot)$ replaced by $p(\cdot)$.*

Remark 2.3. (i) If $\zeta(z)$ is satisfying the condition (2.3):

$$|\zeta(z) - \zeta_\infty| \le \frac{C_\infty}{\ln(e+|z|)}$$

for $x \in \mathbb{R}^n$. Then, $(1+|z|)^{-\lambda(z)}$ is equivalent to the weight $(1+|z|)^{-\lambda_\infty}$.
(ii) One can replace the variable order of Riesz potential operator $\zeta(x)$ by $\zeta(y)$ in the case of potentials over bounded domain, such potentials differ unessential if the function $\zeta(x)$ is satisfying the smoothness logarithmic condition as (2.2), since

$$C_1|z_1 - z_2|^{n-\zeta(z_2)}| \le |z_1 - z_2|^{n-\zeta(z_1)} \le C_2|z_1 - z_2|^{n-\zeta(z_2)}.$$

2.2. Variable exponent Herz spaces and Herz-Morrey spaces

In this subsection we will define variable exponent Herz spaces.

Definition 2.4. Let $u, v \in [1,\infty)$, $\zeta \in \mathbb{R}$, the classical versions of Herz spaces commonly known as homogenous and non-homogenous are defined by their norms such as

$$\|g\|_{K^{\zeta}_{u,v}(\mathbb{R}^n)} := \|g\|_{L^u(B(0,1))} + \left\{\sum_{l\in\mathbb{N}} 2^{l\zeta v}\left(\int\limits_{F_{2^{l-1},2^l}} |g(y)|^u dy\right)^{\frac{v}{u}}\right\}^{\frac{1}{v}}, \tag{2.5}$$

$$\|g\|_{\dot{K}^{\zeta}_{u,v}(\mathbb{R}^n)} := \left\{ \sum_{l\in\mathbb{Z}} 2^{l\zeta v} \left(\int_{F_{2^{l-1},2^l}} |g(y)|^u dy \right)^{\frac{v}{u}} \right\}^{\frac{1}{v}}, \tag{2.6}$$

respectively, such that $F_{t,\tau}$ is $F_{t,\tau} := B(0,\tau)\backslash B(0,t)$.

Definition 2.5. Let $u \in [1,\infty)$, $v(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$ and $\zeta \in \mathbb{R}$. $\dot{K}^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n)$ is the homogenous version of Herz space and its norm is given as

$$\dot{K}^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n) = \left\{ g \in L^{v(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{\dot{K}^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n)} < \infty \right\}, \tag{2.7}$$

where

$$\|g\|_{\dot{K}^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n)} = \left(\sum_{l=-\infty}^{l=\infty} \|2^{l\zeta} g\chi_l\|^u_{L^{v(\cdot)}} \right)^{\frac{1}{u}}.$$

Definition 2.6. Let $u \in [1,\infty)$, $\zeta \in \mathbb{R}$ and $v(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$. The non-homogenous Herz space $K^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n)$ can be defined as

$$K^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n) = \left\{ g \in L^{v(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{K^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n)} < \infty \right\}, \tag{2.8}$$

where

$$\|g\|_{K^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n)} = \|g\|_{L^{v(\cdot)}(B(0,1))} + \left(\sum_{k=-\infty}^{k=\infty} \|2^{k\zeta} g\chi_k\|^u_{L^{v(\cdot)}} \right)^{\frac{1}{u}}.$$

Now, we will define variable Herz-Morrey spaces.

Definition 2.7. For $a(\cdot)$: $\mathbb{R}^n \to \mathbb{R}$, $0 < u < \infty$, $v(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$ and $0 \le \beta < \infty$. A variable Herz-Morrey spaces $M\dot{K}^{a(\cdot),\beta}_{u,v(\cdot)}(\mathbb{R}^n)$ is defined by

$$M\dot{K}^{a(\cdot),\beta}_{u,v(\cdot)}(\mathbb{R}^n) = \left\{ g \in L^{v(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{M\dot{K}^{a(\cdot),\beta}_{u,v(\cdot)}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|g\|_{M\dot{K}^{a(\cdot),\beta}_{u,v(\cdot)}(\mathbb{R}^n)} = \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \left(\sum_{k=-\infty}^{k_0} 2^{ka(\cdot)u} \|g\chi_k\|^u_{L^{v(\cdot)}(\mathbb{R}^n)} \right)^{\frac{1}{u}}.$$

2.3. Grand variable Herz-Morrey spaces and Marcinkiewicz fractional integral operator

Definition 2.8. Let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $r \in [1,\infty)$, s: $\mathbb{R}^n \to [1,\infty)$, $\theta > 0$, $0 \le \beta < \infty$. We define the homogeneous grand variable Herz-Morrey spaces by the norm:

$$M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,s(\cdot)}(\mathbb{R}^n) = \left\{ g \in L^{s(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,s(\cdot)}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|g\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,s(\cdot)}(\mathbb{R}^n)} = \sup_{\psi>0} \sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta} \left(\psi^\theta \sum_{k=-\infty}^{l_0} 2^{k\alpha(\cdot)u(1+\psi)} \|g\chi_k\|^{u(1+\psi)}_{L^{s(\cdot)}(\mathbb{R}^n)} \right)^{\frac{1}{u(1+\psi)}}.$$

Non-homogeneous grand variable Herz-Morrey spaces can be defined in the similar way.

Let $\mathbb{S}^{n-1}$ is denoting the unit sphere in $\mathbb{R}^n$ with the normalized Lebesgue measure. Let $\Phi \in L^r(\mathbb{S}^{n-1})$ is a function of degree zero which is homogeneous such that

$$\int_{\mathbb{S}^{n-1}} \Phi(z_2')d\Phi(z_2') = 0, \tag{2.9}$$

where $z_2' = z_2/|z_2|$ and y is not zero. The Marcinkiewicz integral is introduced with Littlewood-Paley g-function as:

$$\mu_\Phi(f)(z_1) = \left(\int_0^\infty |F_{\Phi,s}(f)(z_1)|^2 \frac{ds}{s^3} \right)^{\frac{1}{2}},$$

where

$$F_{\Phi,s}(f)(z_1) = \int_{|z_1-z_2|\le s} \frac{\Phi(z_1-z_2)}{|z_1-z_2|^{n-1}} f(z_2)dz_2.$$

The variable Marcinkiewicz fractional integral can be defined as:

$$\mu_\Phi(f)(z_1) = \left(\int_0^\infty \left| \int_{|z_1-z_2|\le s} \frac{\Phi(z_1-z_2)}{|z_1-z_2|^{n-\zeta(z_1)-1}} f(z_2)dz_2 \right|^2 \frac{ds}{s^3} \right)^{\frac{1}{2}}.$$

2.4. *Important lemmas*

Following are the important lemmas to prove our main results.

Lemma 2.9. *[19] Let $B > 1$ and $q \in \mathfrak{P}_{0,\infty}(\mathbb{R}^n)$. Then,*

$$\frac{1}{t_0} s^{\frac{n}{q(0)}} \le \|\chi_{F_{s,B_s}}\|_{q(\cdot)} \le t_0 s^{\frac{n}{q(0)}}, \tag{2.10}$$

for $0 < s \le 1$ and

$$\frac{1}{t_\infty} s^{\frac{n}{q_\infty}} \le \|\chi_{F_{s,B_s}}\|_{q(\cdot)} \le t_\infty s^{\frac{n}{q_\infty}}, \tag{2.11}$$

for $s \ge 1$, respectively, where $t_0 \ge 1$ and $t_\infty \ge 1$ is depending on B but not depending on s.

Lemma 2.10. *[12] (Generalized Hölder's inequality) Consider a measurable subset H such that $H \subseteq \mathbb{R}^n$, and*

$$1 \le p_-(H) \le p_+(H) \le \infty.$$

Then,

$$\|fg\|_{L^{r(\cdot)}(H)} \le \|f\|_{L^{p(\cdot)}(H)} \|g\|_{L^{q(\cdot)}(H)}$$

holds where $f \in L^{p(\cdot)}(H)$, $g \in L^{q(\cdot)}(H)$ and

$$\frac{1}{r(z)} = \frac{1}{p(z)} + \frac{1}{q(z)}$$

for every $z \in H$.

Lemma 2.11. *[21] If $a > 0$, $s \in [1, \infty]$, $0 < d \le s$ and $-m + (m-1)ds < u < \infty$ then*

$$\left(\int\limits_{|z_2| \le a|z_1|} |z_2|^u |\Phi(z_1 - z_2)|^d dz_2 \right)^{1/d} \le |z_1|^{(u+m)/d} \|\Phi\|_{L^s(\mathbb{S}^{m-1})}.$$

3. Sobolev type theorem for grand Herz-Morrey spaces

The main purpose of this paper is to establish the boundedness of the Marcinkiewicz fractional integral operator of variable order on grand Herz-Morrey spaces by using some properties of exponent. As it can bee seen that grand Herz-Morrey spaces with variable exponent is the generalization of Herz-Morrey spaces with variable exponent, our main results hold for Herz-Morrey spaces for variable exponent. It is easy to see that our results also generalize the main theorems for [20].

Theorem 3.1. *Let $0 < v \le 1$, $\alpha(\cdot)$, $q_1(\cdot) \in \mathfrak{P}_{0,\infty}(\mathbb{R}^n)$ with $1 < q_1^- \le q_1^+ < \infty$, $1 \le u < \infty$. Let Φ be homogenous of degree zero and $\Phi \in L^s(\mathbb{S}^{n-1})$, $s > q_1'^-$. Let α be such that:*

(i)

$$-\frac{n}{q_1(0)} - v - \frac{n}{s} < \alpha(0) < \frac{n}{q_1'(0)} - v - \frac{n}{s},$$

(ii)

$$-\frac{n}{q_1(\infty)} - v - \frac{n}{s} < \alpha_\infty < \frac{n}{q_1'(\infty)} - v - \frac{n}{s},$$

$$\|(1+|z_1|)^{-\lambda(z_1)} \mu_\Phi(g)\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q_2(\cdot)}(\mathbb{R}^n)} \le C\|g\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q_1(\cdot)}(\mathbb{R}^n)}.$$

Proof. Let

$$g \in M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q_2(\cdot)}(\mathbb{R}^n)$$

and

$$g(z_1) = \sum_{l=-\infty}^{\infty} g(z_1)\chi_l(z_1) = \sum_{l=-\infty}^{\infty} g_l(z_1),$$

we have

$$\|(1+|z_1|)^{-\lambda(z_1)} \mu_\Phi(g)\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q_2(\cdot)}(\mathbb{R}^n)} = \sup_{\psi>0} \sup_{l_o \in \mathbb{Z}} 2^{-l_0\beta} \left(\psi^\theta \sum_{k=-\infty}^{l_0} 2^{k\alpha(\cdot)u(1+\psi)} \|\chi_k (1+|z_1|)^{-\lambda(z_1)} \mu_\Phi(g)\|^{u(1+\psi)}_{L^{q_2(\cdot)}} \right)^{\frac{1}{u(1+\psi)}}$$

$$\le \sup_{\psi>0} \sup_{l_o \in \mathbb{Z}} 2^{-l_0\beta} \left(\psi^\theta \sum_{k=-\infty}^{l_0} 2^{k\alpha(\cdot)u(1+\psi)} \left(\sum_{l=-\infty}^{\infty} \|\chi_k (1+|z_1|)^{-\lambda(z_1)} \mu_\Phi(g)(\chi_l)\|^{u(1+\psi)}_{L^{q_2(\cdot)}} \right) \right)^{\frac{1}{u(1+\psi)}}$$

$$\le \sup_{\psi>0} \sup_{l_o \in \mathbb{Z}} 2^{-l_0\beta} \left(\psi^\theta \sum_{k=-\infty}^{l_0} 2^{k\alpha(\cdot)u(1+\psi)} \left(\sum_{l=-\infty}^{k} \|\chi_k (1+|z_1|)^{-\lambda(z_1)} \mu_\Phi(g\chi_l)\|_{L^{q_2(\cdot)}} \right)^{u(1+\psi)} \right)^{\frac{1}{u(1+\psi)}}$$

$$+ \sup_{\psi>0} \sup_{l_o \in \mathbb{Z}} 2^{-l_0\beta} \left(\psi^\theta \sum_{k=-\infty}^{l_0} 2^{k\alpha(\cdot)u(1+\psi)} \left(\sum_{l=k+1}^{\infty} \|\chi_k (1+|z_1|)^{-\lambda(z_1)} \mu_\Phi(g\chi_l)\|_{L^{q_2(\cdot)}} \right)^{u(1+\psi)} \right)^{\frac{1}{u(1+\psi)}}$$

$$=: E_1 + E_2.$$

For E_1, splitting E_1 by using Minkowski's inequality we have

$$E_1 \leq \sup_{\psi>0}\sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta}\left(\psi^{\theta}\sum_{k=-\infty}^{-1}2^{k\alpha(\cdot)u(1+\psi)}\left(\sum_{l=-\infty}^{k}\|\chi_k(1+|z_1|)^{-\lambda(z_1)}\mu_{\Phi}(g\chi_l)\|_{L^{q_2(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$
$$+\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta}\left(\psi^{\theta}\sum_{k=0}^{l_0}2^{k\alpha(\cdot)u(1+\psi)}\left(\sum_{l=-\infty}^{k}\|\chi_k(1+|z_1|)^{-\lambda(z_1)}\mu_{\Phi}(g\chi_l)\|_{L^{q_2(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$
$$=: E_{11}+E_{12}.$$

For estimating E_{11}, we use the facts that, for each $k\in\mathbb{Z}$ and $l\leq k$ and a.e. $z_1\in F_k$, $z_2\in F_l$, we know that

$$|z_1-z_2|\approx|z_1|\approx 2^k,$$

$$|\mu_{\Phi}(g\chi_l)(z_1)| \leq \left(\int_o^{|z_1|}\left|\int_{|z_1-z_2|\leq t}\frac{\Phi(z_1-z_2)}{|z_1-z_2|^{n-1-\zeta(z_1)}}g\chi_l(z_2)dz_2\right|^2\frac{dt}{t^3}\right)^{1/2}$$
$$+\left(\int_{|z_1|}^{\infty}\left|\int_{|z_1-z_2|\leq t}\frac{\Phi(z_1-z_2)}{|z_1-z_2|^{n-1-\zeta(z_1)}}g\chi_l(z_2)dz_2\right|^2\frac{dt}{t^3}\right)^{1/2}$$
$$=: I_{11}+I_{12}.$$

By the virtue of the mean value theorem we obtain,

$$\left|\frac{1}{|z_1-z_2|^2}-\frac{1}{|z_1|^2}\right| \leq \frac{|z_2|}{|z_1-z_2|^3}. \tag{3.1}$$

For I_{11}, by using Minkowski inequaltiy, generalized Hölder's inequality and inequality (3.1) we have

$$I_{11} \leq \int_{\mathbb{R}^n}\frac{|\Phi(z_1-z_2)|}{|z_1-z_2|^{n-1-\zeta(z_1)}}|g\chi_l(z_2)|\left(\int_{|z_1-z_2|}^{|z_1|}\frac{dt}{t^3}\right)^{1/2}dz_2$$
$$\leq \int_{\mathbb{R}^n}\frac{|\Phi(z_1-z_2)|}{|z_1-z_2|^{n-1-\zeta(z_1)}}|g\chi_l(z_2)|\left|\frac{1}{|z_1-z_2|^2}-\frac{1}{|z_1|^2}\right|^{1/2}dz_2$$
$$\leq \int_{\mathbb{R}^n}\frac{|\Phi(z_1-z_2)|}{|z_1-z_2|^{n-1-\zeta(z_1)}}|g\chi_l(z_2)|\left|\frac{|z_2|}{|z_1-z_2|^3}\right|^{1/2}dz_2$$
$$\leq \frac{2^{l/2}}{|z_1|^{n+\frac{1}{2}}.|z_1|^{-\zeta(z_1)}}\int_{F_l}|\Phi(z_1-z_2)||g(z_2)|dz_2$$
$$\leq 2^{(l-k)/2}2^{-kn}|z_1|^{\zeta(z_1)}\|g\chi_l\|_{L^{q_1(\cdot)}}\|\Phi(z_1-\cdot)\chi_l(\cdot)\|_{L^{q_1'(\cdot)}}.$$

Similarly, now consider I_{12}, we have

$$
\begin{aligned}
I_{12} \leq & \int\limits_{\mathbb{R}^n} \frac{|\Phi(z_1 - z_2)|}{|z_1 - z_2|^{n-1-\zeta(z_1)}} |g\chi_l(z_2)| \left(\int\limits_{|z_1|}^{\infty} \frac{dt}{t^3} \right)^{1/2} dz_2 \\
\leq & \int\limits_{\mathbb{R}^n} \frac{|\Phi(z_1 - z_2)|}{|z_1 - z_2|^{n-1-\zeta(z_1)}} |g\chi_l(z_2)| dz_2 \\
\leq & |z_1|^{-n} |z_1|^{\zeta(z_1)} \int\limits_{F_l} |\Phi(z_1 - z_2)||g(z_2)| dz_2 \\
\leq & 2^{-kn} |z_1|^{\zeta(z_1)} \|g\chi_l(z_2)\|_{L^{q_1(\cdot)}} \|\Phi(z_1 - \cdot)\chi_l(\cdot)\|_{L^{q_1'(\cdot)}}.
\end{aligned}
$$

We define $q_1(\cdot)$ by the relation

$$
\frac{1}{q_1'(x)} = \frac{1}{q_1(x)} + \frac{1}{s}.
$$

By using Lemma 2.11 and generalized Hölder's inequality we have

$$
\begin{aligned}
\|\Phi(z_1 - \cdot)\chi_l(\cdot)\|_{L^{q_1'(\cdot)}} \leq & \|\Phi(z_1 - \cdot)\chi_l(\cdot)\|_{L^s} \|\chi_l(\cdot)\|_{L^{q_1(\cdot)}} \\
\leq & 2^{-lv} \left(\int\limits_{2^{l-1}<|z_2|<2^l} |\Phi(z_1 - z_2)|^s |z_2|^{sv} dz_2 \right)^{1/s} \|\chi_{B_l}\|_{L^{q_1(\cdot)}} \\
\leq & 2^{-lv} 2^{k(v+\frac{n}{s})} \|\Phi\|_{L^s(\mathbb{S}^{n-1})} \| \|\chi_{B_l}\|_{L^{q_1(\cdot)}}.
\end{aligned}
$$

It is known, see [10] that

$$
\begin{aligned}
I^{\zeta(\cdot)}(\chi_{B_k})(z_1) \geq & I^{\zeta(\cdot)}(\chi_{B_k})(z_1).(\chi_{B_k})(z_1) \\
= & \int_{B_k} \frac{1}{|z_1 - z_2|^{\zeta(z_1)-n}} dy.\chi_{B_k}(z_1) \\
\geq & C|z_1|^{\zeta(z_1)}.\chi_{B_k}(z_1) \\
\geq & C|z_1|^{\zeta(z_1)}.\chi_k(z_1).
\end{aligned}
$$

Consequently, by Proposition 2.2 we have

$$
\|\chi_k |z_1|^{\zeta(z_1)} (1 + |z_1|)^{-\lambda(z_1)}\|_{L^{q_2(\cdot)}} \leq \|(1 + |z_1|)^{-\lambda(z_1)} I^{\zeta(\cdot)}(\chi_{B_k})\|_{L^{q_2(\cdot)}} \leq \|\chi_{B_k}\|_{L^{q_1(\cdot)}}.
$$

Applying results to E_{11} we can get

$$
\begin{aligned}
E_{11} \leq & C \sup_{\psi>0} \sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta} \left[\psi^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(0)u(1+\psi)} \left(\sum_{l=-\infty}^{k} 2^{(l-k)(n/q_1'(0)-v-\frac{n}{s})} \|g\chi_l\|_{L^{q_1(\cdot)}} \right)^{u(1+\psi)} \right]^{\frac{1}{u(1+\psi)}} \\
\leq & C \sup_{\psi>0} \sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta} \left[\psi^\theta \sum_{k=-\infty}^{-1} \left(\sum_{l=-\infty}^{k} 2^{\alpha(0)l} \|g\chi_l\|_{L^{q_1(\cdot)}} 2^{b(l-k)} \right)^{u(1+\psi)} \right]^{\frac{1}{u(1+\psi)}}.
\end{aligned}
$$

Let

$$
b = \frac{n}{q_1'(0)} - v - \frac{n}{s} - \alpha(0) > 0.
$$

Applying Hölders inequality, $2^{-u(1+\psi)} < 2^{-u}$ and Fubini's theorem for series and we get

$$
\begin{aligned}
E_{11} &\le C \sup_{\psi>0} \sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta} \left[\psi^{\theta} \sum_{k=-\infty}^{-1} \left(\sum_{l=-\infty}^{k} 2^{\alpha(0)u(1+\psi)l} \|g\chi_l\|_{L^{q_1(\cdot)}}^{u(1+\psi)} 2^{bu(1+\psi)(l-k)/2} \times \sum_{l=-\infty}^{k} 2^{bu(1+\psi)'(l-k)/2} \right)^{\frac{u(1+\psi)}{u(1+\psi)'}} \right]^{\frac{1}{u(1+\psi)}} \\
&\le C \sup_{\psi>0} \sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta} \left(\psi^{\theta} \sum_{k=-\infty}^{-1} \sum_{l=-\infty}^{k} 2^{\alpha(0)u(1+\psi)l} \|g\chi_l\|_{L^{q_1(\cdot)}}^{u(1+\psi)} 2^{bu(1+\psi)(l-k)/2} \right)^{\frac{1}{u(1+\psi)}} \\
&\le C \sup_{\psi>0} \sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta} \left(\psi^{\theta} \sum_{l=-\infty}^{-1} 2^{\alpha(0)u(1+\psi)l} \|g\chi_l\|_{L^{q_1(\cdot)}}^{u(1+\psi)} \sum_{k=l}^{-1} 2^{bu(1+\psi)(l-k)/2} \right)^{\frac{1}{u(1+\psi)}} \\
&\le C \sup_{\psi>0} \sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta} \left(\psi^{\theta} \sum_{l=-\infty}^{-1} 2^{\alpha(0)u(1+\psi)l} \|g\chi_l\|_{L^{q_1(\cdot)}}^{u(1+\psi)} \sum_{k=l}^{-1} 2^{bp(l-k)/2} \right)^{\frac{1}{u(1+\psi)}} \\
&\le C \sup_{\psi>0} \sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta} \left(\psi^{\theta} \sum_{l=-\infty}^{-1} 2^{\alpha(0)u(1+\psi)l} \|g\chi_l\|_{L^{q_1(\cdot)}}^{u(1+\psi)} \right)^{\frac{1}{u(1+\psi)}} \\
&\le C \sup_{\psi>0} \sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta} \left(\psi^{\theta} \sum_{l=-\infty}^{l_0} 2^{\alpha(\cdot)u(1+\psi)l} \|g\chi_l\|_{L^{q_1(\cdot)}}^{u(1+\psi)} \right)^{\frac{1}{u(1+\psi)}} \\
&\le C \|g\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Now, for E_{12} using Minkowski's inequality we have

$$
\begin{aligned}
E_{12} &\le \sup_{\psi>0} \sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta} \left(\psi^{\theta} \sum_{k=0}^{l_0} 2^{k\alpha(\cdot)u(1+\psi)} \left(\sum_{l=-\infty}^{-1} \|\chi_k (1+|z_1|)^{-\lambda(z_1)} \mu_{\Phi}(g\chi_l)\|_{L^{q_2(\cdot)}} \right)^{u(1+\psi)} \right)^{\frac{1}{u(1+\psi)}} \\
&+ \sup_{\psi>0} \sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta} \left(\psi^{\theta} \sum_{k=0}^{l_0} 2^{k\alpha(\cdot)u(1+\psi)} \left(\sum_{l=0}^{k} \|\chi_k (1+|z_1|)^{-\lambda(z_1)} \mu_{\Phi}(g\chi_l)\|_{L^{q_2(\cdot)}} \right)^{u(1+\psi)} \right)^{\frac{1}{u(1+\psi)}} \\
&=: A_1 + A_2.
\end{aligned}
$$

The estimate for A_2 follows in a similar manner to E_{11} by replacing $q_1'(0)$ with $q_1'(\infty)$ and by the use of fact

$$b := \frac{n}{q_1'(\infty)} - v - \frac{n}{s} - \alpha_\infty > 0.$$

For A_1 we have

$$
\begin{aligned}
\|\chi_k (1+|z_1|)^{-\lambda(z_1)} \mu_{\Phi}(g\chi_l)\|_{L^{q_2(\cdot)}} &\le C 2^{-kn} 2^{-lv} 2^{k(v+\frac{n}{s})} \|\chi_{B_k}\|_{L^{q_1(\cdot)}} \|\chi_l\|_{L^{q_1(\cdot)}} \|g\chi_l\|_{L^{q_1(\cdot)}} \\
&\le C 2^{l(\frac{n}{q_1(0)}-v)} 2^{k(v+\frac{n}{s}-\frac{n}{q_1'(\infty)})} \|g\chi_l\|_{L^{q_1(\cdot)}}.
\end{aligned}
$$

Now, by using the fact

$$-\frac{n}{q_1'(\infty)} + v + \frac{n}{s} + \alpha_\infty < 0,$$

we have

$$A_1 \le \sup_{\psi>0}\sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta}\left(\psi^\theta \sum_{k=0}^{l_0} 2^{k\alpha_\infty u(1+\psi)}\left(\sum_{l=-\infty}^{-1} \|\chi_k(1+|z_1|)^{-\lambda(z_1)}\mu_\Phi(g\chi_l)\|_{L^{q_2(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$
$$\le C\sup_{\psi>0}\left[\psi^\theta \sum_{k=0}^{l_0} 2^{k\alpha_\infty u(1+\psi)}\left(\sum_{l=-\infty}^{-1} 2^{l(\frac{n}{q_1(0)}-v)}2^{k(v+\frac{n}{s}-\frac{n}{q_1'(\infty)})}\|g\chi_l\|_{L^{q_1(\cdot)}}\right)^{u(1+\psi)}\right]^{\frac{1}{u(1+\psi)}}$$
$$\le C\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta}\left[\psi^\theta \sum_{k=0}^{l_0} 2^{k(\alpha_\infty+v+\frac{n}{s}-\frac{n}{q_1'(\infty)})u(1+\psi)} \times\left(\sum_{l=-\infty}^{-1} 2^{l(\frac{n}{q_1(0)}-v)}\|g\chi_l\|_{L^{q_1(\cdot)}}\right)^{u(1+\psi)}\right]^{\frac{1}{u(1+\psi)}}$$
$$\le C\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta}\left(\psi^\theta\left(\sum_{l=-\infty}^{-1} 2^{l(\frac{n}{q_1(0)}-v)}\|g\chi_l\|_{L^{q_1(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$
$$\le C\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta}\left(\psi^\theta\left(\sum_{l=-\infty}^{-1} 2^{l\alpha(0)}\|g\chi_l\|_{L^{q_1(\cdot)}}2^{l(\frac{n}{q_1(0)}-v-\alpha(0))}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}.$$

Now, by applying Hölders inequality and using the fact that

$$\frac{n}{q_1'(0)}-\frac{n}{s}-v-\alpha(0)>0,$$

we have

$$A_1 \le C\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta}\left[\psi^\theta \sum_{l=-\infty}^{-1} 2^{\alpha(0)lu(1+\psi)}\|g\chi_l\|_{L^{q_1(\cdot)}}^{u(1+\psi)} \times\left(\sum_{l=-\infty}^{-1} 2^{l(\frac{n}{q_1'(0)}-v-\alpha(0))u(1+\psi)'}\right)^{\frac{u(1+\psi)}{u(1+\psi)'}}\right]^{\frac{1}{u(1+\psi)}}$$
$$\le C\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta}\left(\psi^\theta\left(\sum_{l=-\infty}^{l_0} 2^{\alpha(\cdot)lu(1+\psi)}\|g\chi_l\|_{L^{q_1(\cdot)}}^{u(1+\psi)}\right)\right)^{\frac{1}{u(1+\psi)}}$$
$$\le C\|g\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q_1(\cdot)}(\mathbb{R}^n)}.$$

Now, we estimate E_2 for each $k\in\mathbb{Z}$ and $l\ge k+1$ and a.e. $z_1\in F_k$, $z_2\in F_l$, we know that

$$|z_1-z_2|\approx|z_2|\approx 2^l,$$

we consider

$$|\mu_\Phi(g\chi_l)(z_1)| \le \left(\int_o^{|z_2|}\left|\int_{|z_1-z_2|\le t}\frac{\Phi(z_1-z_2)}{|z_1-z_2|^{n-1-\zeta(z_1)}}g\chi_l(z_2)dz_2\right|^2\frac{dt}{t^3}\right)^{1/2}$$
$$+\left(\int_{|z_2|}^{\infty}\left|\int_{|z_1-z_2|\le t}\frac{\Phi(z_1-z_2)}{|z_1-z_2|^{n-1-\zeta(z_1)}}g\chi_l(z_2)dz_2\right|^2\frac{dt}{t^3}\right)^{1/2}$$
$$=: I_{31}+I_{32}.$$

By using the similar arguments as used in I_{11}, we obtain

$$I_{31} \le 2^{(k-k)/2}2^{-ln}|z_1|^{\zeta(z_1)}\|g\chi_l\|_{L^{q_1(\cdot)}}\|\Phi(z_1-\cdot)\chi_l(\cdot)\|_{L^{q_1'(\cdot)}}.$$

By using same arguments of I_{12}, we obtain

$$I_{32} \leq 2^{-ln}|z_1|^{\zeta(z_1)}\|g\chi_l\|_{L^{q_1(\cdot)}}\|\Phi(z_1-\cdot)\chi_l(\cdot)\|_{L^{q_1'(\cdot)}}.$$

Consequently, by Proposition 2.2 we have

$$\|\chi_k|z_1|^{\zeta(z_1)}(1+|z_1|)^{-\lambda(z_1)}\|_{L^{q_2(\cdot)}} \geq \|(1+|z_1|)^{-\lambda(z_1)}I^{\zeta(\cdot)}(\chi_{B_k})\|_{L^{q_2(\cdot)}} \geq \|\chi_{B_k}\|_{L^{q_1(\cdot)}}.$$

Consequently, we will get

$$\begin{aligned}\|\chi_k(1+|z_1|)^{-\lambda(z_1)}\mu_\Phi(g\chi_l)\|_{L^{q_2(\cdot)}} &\leq C2^{-ln}\|g\chi_l\|_{L^{q_1(\cdot)}}\|\chi_{B_k}\|_{L^{q_1(\cdot)}}2^{-lv}2^{k(v+\frac{n}{s})}\|\chi_l\|_{L^{q_1(\cdot)}}\\ &\leq C2^{-ln}2^{-lv}2^{k(v+\frac{n}{s})}\|\chi_{B_k}\|_{L^{q_1(\cdot)}}\|\chi_{B_l}\|_{L^{q_1(\cdot)}}\|g\chi_l\|_{L^{q_1(\cdot)}}\\ &\leq C2^{-ln}2^{-lv}2^{k(v+\frac{n}{s})}2^{ln/q_1(\infty)}2^{kn/q_1(\infty)}\|g\chi_l\|_{L^{q_1(\cdot)}}\\ &\leq C2^{(k-l)(v+\frac{n}{s}+\frac{n}{q_1(\infty)})}\|g\chi_l\|_{L^{q_1(\cdot)}}.\end{aligned}$$

Now, splitting E_2 we have

$$\begin{aligned}E_2 \leq \max\Bigg\{&\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\beta}\left(\psi^\theta\sum_{k=-\infty}^{l_0}2^{k\alpha(\cdot)u(1+\psi)}\left(\sum_{l=k+1}^{\infty}\|\chi_k(1+|z_1|)^{-\lambda(z_1)}\mu_\Phi(g\chi_l)\|_{L^{q_2(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\ &\leq \sup_{\psi>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\beta}\left(\psi^\theta\sum_{k=-\infty}^{-1}2^{k\alpha(\cdot)u(1+\psi)}\left(\sum_{l=k+1}^{\infty}\|\chi_k(1+|z_1|)^{-\lambda(z_1)}\mu_\Phi(g\chi_l)\|_{L^{q_2(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\ &+\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\beta}\left(\psi^\theta\sum_{k=0}^{l_0}2^{k\alpha(\cdot)u(1+\psi)}\left(\sum_{l=k+1}^{\infty}\|\chi_k(1+|z_1|)^{-\lambda(z_1)}\mu_\Phi(g\chi_l)\|_{L^{q_2(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\ &=: E_{21}+E_{22}.\end{aligned}$$

For E_{22} we have

$$\begin{aligned}E_{22} &\leq C\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\beta}\left(\psi^\theta\sum_{k=0}^{l_0}2^{k\alpha_\infty u(1+\psi)}\left(\sum_{l=k+1}^{\infty}2^{(k-l)(v+\frac{n}{s}+\frac{n}{q_1(\infty)})}\|g\chi_l\|_{L^{q_1(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\ &\leq C\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\beta}\left(\psi^\theta\sum_{k=0}^{l_0}2^{k\alpha_\infty u(1+\psi)}\|g\chi_l\|^{u(1+\psi)}_{L^{q_1(\cdot)}}\left(\sum_{l=k+1}^{\infty}2^{(k-l)(v+\frac{n}{s}+\frac{n}{q_1(\infty)})}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\end{aligned}$$

where

$$d=\frac{n}{q_1(\infty)}+v+\frac{n}{s}+\alpha_\infty>0.$$

Then, we use Hölder's theorem for series and $2^{-u(1+\psi)}<2^{-u}$ to obtain

$$\begin{aligned}E_{22} &\leq C\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\beta}\left[\psi^\theta\sum_{k=0}^{l_0}\left(\sum_{l=k+1}^{\infty}2^{l\alpha_\infty u(1+\psi)}\|g\chi_l\|^{u(1+\psi)}_{L^{q_1(\cdot)}}2^{du(1+\psi)(k-l)/2}\right)\times\left(\sum_{l=k+1}^{\infty}2^{du(1+\psi)'(k-l)/2}\right)^{\frac{u(1+\psi)}{u(1+\psi)'}}\right]^{\frac{1}{u(1+\psi)}}\\ &\leq C\sup_{\psi>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\beta}\left(\psi^\theta\sum_{k=0}^{l_0}\sum_{l=k+1}^{\infty}2^{l\alpha_\infty u(1+\psi)}\|g\chi_l\|^{u(1+\psi)}_{L^{q_1(\cdot)}}2^{du(1+\psi)(k-l)/2}\right)^{\frac{1}{u(1+\psi)}}\end{aligned}$$

$$\leq C \sup_{\psi>0} \sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta}\left(\psi^{\theta}\sum_{l=0}^{\infty}2^{l\alpha_\infty u(1+\psi)}\|g\chi_l\|_{L^{q_1(\cdot)}}^{u(1+\psi)}\sum_{k=0}^{l-1}2^{du(1+\psi)(k-l)/2}\right)^{\frac{1}{u(1+\psi)}}$$

$$\leq C \sup_{\psi>0} \sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta}\left(\psi^{\theta}\sum_{l=0}^{\infty}\sum_{j=-\infty}^{l_0}2^{j\alpha_\infty u(1+\psi)}\|g\chi_j\|_{L^{q_1(\cdot)}}^{u(1+\psi)}\sum_{k=0}^{l-1}2^{du(1+\psi)(k-l)/2}\right)^{\frac{1}{u(1+\psi)}}$$

$$\leq C \sup_{\psi>0} \sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta}\left(\psi^{\theta}\sum_{l=0}^{\infty}\sum_{k=0}^{l-1}2^{du(1+\psi)(k-l)/2}\right)^{\frac{1}{u(1+\psi)}}\|g\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q_1(\cdot)}(\mathbb{R}^n)}$$

$$\leq C\|g\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q_1(\cdot)}(\mathbb{R}^n)}.$$

Now, for E_{21} using Minkowski's inequality we have

$$\begin{aligned} E_{21} &\leq \sup_{\psi>0} \sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta}\left(\psi^{\theta}\sum_{k=-\infty}^{-1}2^{k\alpha(\cdot)u(1+\psi)}\left(\sum_{l=k+1}^{-1}\|\chi_k(1+|z_1|)^{-\lambda(z_1)}\mu_\Phi(g\chi_l)\|_{L^{q_2(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}} \\ &+ \sup_{\psi>0} \sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta}\left(\psi^{\theta}\sum_{k=-\infty}^{-1}2^{k\alpha(\cdot)u(1+\psi)}\left(\sum_{l=0}^{\infty}\|\chi_k(1+|z_1|)^{-\lambda(z_1)}\mu_\Phi(g\chi_l)\|_{L^{q_2(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}} \\ &=: B_1 + B_2. \end{aligned}$$

The estimate for B_1 follows in a similar manner to E_{22} with $q_1(\infty)$ replaced by $q_1(0)$ and using the fact that

$$\frac{n}{q_1(0)} + v + \frac{n}{s} + \alpha(0) > 0.$$

For B_2 we have

$$\begin{aligned} \|\chi_k(1+|z_1|)^{-\lambda(z_1)}\mu_\Phi(g\chi_l)\|_{L^{q_2(\cdot)}} &\leq C2^{-ln}\|g\chi_l\|_{L^{q_1(\cdot)}}\|\chi_{B_k}\|_{L^{q_1(\cdot)}}2^{-lv}2^{k(v+\frac{n}{s})}\|\chi_l\|_{L^{q_1(\cdot)}} \\ &\leq C2^{-ln}2^{-lv}2^{k(v+\frac{n}{s})}\|\chi_{B_k}\|_{L^{q_1(\cdot)}}\|\chi_l\|_{L^{q_1(\cdot)}}\|g\chi_l\|_{L^{q_1(\cdot)}} \\ &\leq C2^{-ln}2^{-lv}2^{k(v+\frac{n}{s})}\|g\chi_l\|_{L^{q_1(\cdot)}}2^{ln/q_1(\infty)}2^{kn/q_1(0)} \\ &\leq C2^{-l(v+\frac{n}{s}+\frac{n}{q_1(\infty)})}2^{k(v+\frac{n}{q_1(0)}+\frac{n}{s})}\|g\chi_l\|_{L^{q_1(\cdot)}}. \end{aligned}$$

$$\begin{aligned} B_2 &\leq \sup_{\psi>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\left(\psi^{\theta}\sum_{k=-\infty}^{-1}2^{k\alpha(0)u(1+\psi)}\left(\sum_{l=0}^{\infty}\|\chi_k(1+|z_1|)^{-\lambda(z_1)}\mu_\Phi(g\chi_l)\|_{L^{q_1(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}} \\ &\leq C \sup_{\psi>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\left(\psi^{\theta}\sum_{k=-\infty}^{-1}2^{k\alpha(0)u(1+\psi)}\times\left(\sum_{l=0}^{\infty}2^{-l(v+\frac{n}{s}+\frac{n}{q_1(\infty)})}2^{k(v+\frac{n}{q_1(0)}+\frac{n}{s})}\|g\chi_l\|_{L^{q_1(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}} \\ &\leq C \sup_{\psi>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\left(\psi^{\theta}\sum_{k=-\infty}^{-1}2^{k(v+\frac{n}{q_1(0)}+\frac{n}{s}+\alpha(0))u(1+\psi)}\right)\times\left(\sum_{l=0}^{\infty}2^{-l(v+\frac{n}{s}+\frac{n}{q_1(\infty)})}\|g\chi_l\|_{L^{q_1(\cdot)}}\right)^{u(1+\psi)}\Bigg)^{\frac{1}{u(1+\psi)}} \\ &\leq C \sup_{\psi>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\left(\psi^{\theta}\left(\sum_{l=0}^{\infty}2^{-l(v+\frac{n}{s}+\frac{n}{q_1(\infty)})}\|g\chi_l\|_{L^{q_1(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}} \end{aligned}$$

$$\leq C \sup_{\psi>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \left(\psi^{\theta} \left(\sum_{l=0}^{\infty} 2^{\alpha_\infty l} \|g\chi_l\|_{L^{q_1(\cdot)}} 2^{-l(v+\frac{n}{s}+\frac{n}{q_1(\infty)}+\alpha_\infty)} \right)^{u(1+\psi)} \right)^{\frac{1}{u(1+\psi)}}$$

$$\leq C \sup_{\psi>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \left(\psi^{\theta} \left(\sum_{l=0}^{\infty} \sum_{j=-\infty}^{l} 2^{\alpha_\infty j} \|g\chi_j\|_{L^{q_1(\cdot)}} 2^{-l(v+\frac{n}{s}+\frac{n}{q_1(\infty)}+\alpha_\infty)} \right)^{u(1+\psi)} \right)^{\frac{1}{u(1+\psi)}}.$$

Now, by applying Hölders inequality and using the fact that

$$\frac{n}{q_1(\infty)} + v + \frac{n}{s} + \alpha(\infty) > 0,$$

we have

$$\begin{aligned} B_2 &\leq C \sup_{\psi>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \left(\psi^{\theta} \left(\sum_{l=0}^{\infty} 2^{-l(v+\frac{n}{s}+\frac{n}{q_1(\infty)}+\alpha_\infty)} \right)^{u(1+\psi)} \right)^{\frac{1}{u(1+\psi)}} \|g\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q_1(\cdot)}(\mathbb{R}^n)} \\ &\leq \|g\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q_1(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

Combining the estimates for E_1 and E_2 yields

$$\|(1+|z_1|)^{-\lambda(z_1)}\mu_\Phi(g)\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q_2(\cdot)}(\mathbb{R}^n)} \leq C\|g\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q_1(\cdot)}(\mathbb{R}^n)},$$

which ends the proof. □

4. Conclusions

In this paper we proved the boundedness of Marcinkiewicz fractional integral operator of variable order on grand Herz-Morrey spaces with variable exponent under some proper assumptions on exponent.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare no conflicts of interest.

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