Synthese (2023) 202:12
https://doi.org/10.1007/s11229-023-04196-1

ORIGINAL RESEARCH

The Frege–Hilbert controversy in context

Tabea Rohr[1]

Received: 5 January 2023 / Accepted: 16 May 2023

Abstract

This paper aims to show that Frege's and Hilbert's mutual disagreement results from different notions of *Anschauung* and their relation to axioms. In the first section of the paper, evidence is provided to support that Frege and Hilbert were influenced by the same developments of 19th-century geometry, in particular the work of Gauss, Plücker, and von Staudt. The second section of the paper shows that Frege and Hilbert take different approaches to deal with the problems that the developments in 19th-century geometry posed for the traditional Kantian philosophy of mathematics. Frege maintains that *Anschauung* is a source of knowledge by which we acknowledge geometrical axioms as true. For Hilbert, in contrast, axioms provide one of several correct "pictures" of reality. Hilbert's position is thereby deeply influenced by epistemological ideas from Hertz and Helmholtz, and, in turn, influenced the neo-Kantian Cassirer.

Keywords Frege · Hilbert · Geometry · Philosophy of mathematics · Epistemology

1 Introduction

In a couple of letters around 1900, Frege and Hilbert discuss the status of geometry and geometrical objects. After Hilbert terminated the correspondence, Frege published two series of papers, both titled "Über die Grundlagen der Geometrie", to make his concerns about Hilbert's attempt public.[1]

[1] Originally, Frege wanted to publish the correspondence between him and Hilbert instead. (He suggested this to Hilbert in a letter from January 6th, 1900.) However, Hilbert refused, so Frege decided to publish his concerns in a different way (Frege, 1903, p. 319).

✉ Tabea Rohr
tabea.rohr@gmail.com

[1] Laboratoire de Linguistique Formelle (CNRS and Université Paris Cité), 5 rue Thomas Mann, Case Postale 7031, 75205 Paris, France

Published online: 28 June 2023

The controversy is widely conceived as one between a conservative 19th-century mathematician—Frege—and a young man who represents a new age of mathematics—Hilbert.[2] Indeed, there is evidence that supports this picture. In his *Grundlagen der Geometrie*, Hilbert invented a completely new understanding of axiomatization in mathematics. He seemingly detaches geometrical axioms completely from intuition. As soon as 1899, he writes in a letter to Frege, "If in speaking of my points I think of some system of things, e.g., the system: love, law, chimney-sweep … and then assume all my axioms as relations between these things, then my propositions, e.g., Pythagoras' theorem, are also valid for these things."[3]

Frege, on the other hand, had trouble with this new picture. In an unpublished note taken after the conversation with Hilbert, Frege writes that only one set of axioms could be true: either that of Euclidean or that of non-Euclidean geometry. If Euclidean geometry is true, then non-Euclidean geometry is not scientific but has the same status as alchemy or astrology (Frege, 1979a, p. 169).

However, this paper, sheds new light on the relationship between Hilbert and Frege. In the first section of the paper, we show that both share the same heritage. Not only are they both deeply rooted in the geometry of the 19th century, but they both even react to the same authors, such as Gauss, von Staudt, and Plücker. We especially build on the pioneering work by Tappenden (1995a, 1995b, 2006) and Wilson (1992, 2005, 2010) on the geometrical background of Frege's philosophy.[4] It will become apparent that Frege's position on non-Euclidean geometry has been widely misconceived.[5] Further, we will see that Frege, in his mathematical writings, uses methods of his time from projective geometry to deal with extension elements (i.e., points in infinity and imaginary points)[6] as well as higher dimensions, even though that "contradicts our intuition."[7]

Hilbert, on the other hand, did not break as radically the connection between geometry and *Anschauung* (in his *Festschrift*) as some interpretations (among others, that

[2] For example, Blanchette writes, "Hilbert is clearly the winner in this debate, in the sense that roughly his conception of consistency is what one means today by 'consistency' in the context of formal theories, and a near relative of his methodology for consistency-proofs is now standard" Blanchette (2018).

[3] Letter to Frege from December 29th, 1899 (Frege, 1980, p. 40).

[4] More recent contributions have been made by Eder (2021), Schirn (2019), and Shipley (2015), to mention just a few.

[5] Freudental claims, "Frege, rebuking Hilbert like a schoolboy, also joins the Bœotians. (I have never understood why he is so highly esteemed today)" (Freudenthal, 1962, p. 618). As indicated earlier in Freudenthal's paper, "Bœotians" was used by Gauss to refer to the people who dismiss non-Euclidian geometry (Freudenthal, 1962, p. 613). Coffa writes, "Frege, for example, the most enlightened and penetrating among them [the philosophers, TR], argued that all non-Euclidean geometries were false and that they should therefore be placed together with astrology and alchemy in the category of pseudosciences. The geometers returned such advice with thanks and turned to the task of solving their own philosophical problems without outside help" (Coffa, 1986, p. 8). We show, however, that Frege does not dismiss non-Euclidean geometry from a mathematical point of view.

[6] We borrow the term "extension element" as a general term for both points in infinity and imaginary points from Wilson (2005, 2010).

[7] Frege uses this formulation in the opening passage of his thesis *On a Geometrical Representation of Imaginary Forms in the Plane* (Frege, 1984a, p. 1).

of Frege)[8] suggest.[9] He was deeply influenced by the discussions of late 19th-century geometry about the status of non-Euclidean geometry and projective geometry. In his lectures, he puts his own axiomatic approach in relation to the analytic and the synthetic–projective approach to geometry and calls his own approach an "analysis of *Anschauung*."[10]

Nonetheless, Frege and Hilbert undoubtedly have completely different philosophical convictions regarding geometry. In the second section of this paper, we show that, despite the fact that both Frege and Hilbert were influenced by Kant, these differences emerge from fundamental disagreements regarding the concept of axioms and the role of *Anschauung*.

Hilbert's perspective on axioms and *Anschauung* was heavily influenced by Hertz's picture theory of science, according to which scientific theories are not true statements about the world, but rather one among several possible "pictures" of the world. Hertz's picture theory later inspired the philosophy of symbolic forms of the neo-Kantian Cassirer, and it enabled Hilbert to take a very liberal approach toward the plurality of geometries, whereas Frege maintained the traditional Kantian position that only Euclidean geometry is "true" and *Anschauung* is the source of knowledge to recognize this truth.

2 The mathematical background of the Frege–Hilbert controversy

2.1 Non-Euclidean geometry and the relationship between geometry and arithmetic

Before Frege started to work on the *Begriffsschrift*, he studied mathematics in Jena and then in Göttingen, where he received his Ph.D. (and shortly afterwards, he finished his *Habilitation* back in Jena). His Ph.D. thesis from 1873 is titled *On a Geometrical Representation of Imaginary Forms in the Plane*.[11] Considering Frege's later project to show that logic is the foundation of arithmetic, it might be surprising that Frege's Ph.D. thesis was on a topic from geometry.

However, it is less surprising when one takes the academic environment in Göttingen at this time into account. In the early 1870s, the influence of Gauss and Riemann was still strong, and it was Gauss's authority that led to a wide acceptance of non-Euclidean geometry within the mathematical community (Klein, 1928, pp. 275–277). Riemann, in his famous *Habilitationsvortrag* from 1856, "Über die Hypothesen, welche der

[8] In a letter to Hilbert on January 6th, 1990, Frege writes, "It seems to me that you want to detach geometry entirely from spatial intuition and to turn it into a purely logical science like arithmetic" (Frege, 1980, p. 43).

[9] Freudenthal claims that Hilbert developed "a logically closed system of Euclidean geometry that avoids any *illegal appeal to intuition*" (Freudenthal, 1962, 616f., emphasis added).

[10] We discuss the meaning of this notion below.

[11] The original German title is *Über eine geometrische Darstellung der imaginären Gebilde in der Ebene.*

Geometrie zu Grunde liegen," laid out a totally new concept of geometry and of geometrical space. In this talk, Riemann introduced the notion of the curvature of space,[12] which enabled him to systematically distinguish different kinds of non-Euclidean geometries.[13] Furthermore, he introduced a new notion of dimension that is completely disconnected from intuition. According to this notion, a dimension is just an independent parameter that need not be either spatial or dense.[14]

Thus, Frege received his mathematical training in an environment where non-Euclidean geometry and geometries of higher dimensions were objects of serious mathematical investigation. This is, of course, no proof of Frege's positive attitude toward non-Euclidean geometry. In Göttingen, there was also a prominent opponent of non-Euclidean geometry—Lotze. Lotze was not a mathematician but a philosopher; however, in 19th-century Göttingen (and in many other German universities), mathematics belonged to the philosophical faculty, so Lotze interacted with the mathematicians. For instance, he was part of the panel for Riemann's *Habilitationsvortrag* (Gray, 2010, p. 198). Felix Klein mentions that Lotze's influence made it hard for him to discuss non-Euclidean geometry in 1870s Göttingen (Klein, 1926, p. 152). In his philosophical writings, Lotze even discusses the notion of a curved space, introduced by Riemann, arguing that only things in space, not space itself, can be curved.[15] In his *Metaphysics*, Lotze calls non-Euclidean geometry "one huge coherent error" ("ein einziger großer und zusammenhängender Irrthum") (Lotze, 1884, p. 209, § 122). So, does the passage from Frege's unpublished note, where he compares non-Euclidean geometry to alchemy, actually show that Frege took Lotze's side in the discussion about non-Euclidean geometry? A passage from *Grundlagen der Arithmetik* indicates something different. In § 14, Frege writes:

> Conceptual thought alone can after a fashion shake off this yoke, when it assumes, say, a space of four dimensions or positive curvature. To study such conceptions is not useless by any means; but it is to leave the ground of intuition entirely behind.

By mentioning the space of positive curvature, Frege, just like Lotze, refers to Riemann. However, unlike Lotze, he endorses the possibility of grasping such space conceptually.

[12] This idea is built on Gauss's *theorema egregium* (Gauß, 1828, p. 24). Roughly, this theorem says that the curvature of a surface is a property of its inner geometry. In other words, it is independent of the space it is embedded into.

[13] In particular, the distinction between elliptic geometries (with positive curvature) and hyperbolic geometries (with negative curvature) is made possible.

[14] A detailed study of the Riemannian influence on Frege can be found in Tappenden (2006).

[15] "Among the properties which our common apprehension believes most indispensable to space is the absolute homogeneousness of its infinite extension. The real elements which occupy it or move in it may, we think, have different densities of their aggregation and different rules for their relative positions at different points; space itself, on the other hand, as the impartial theater of all these events, cannot possess local differences of its own nature which might interfere with the liberty of everything that is or happens at one of its points to repeat itself without alteration at any other" (Lotze, 1884, p. 232, § 136). "It is clear to us what we are to think of as a spherical or pseudo-spherical surface, but not clear what can be meant by a spherical or pseudo-spherical space; designations which we meet with in the discussion of these subjects without any help being given to us in comprehending their meaning" (Lotze, 1884, p. 233, § 136).

Nevertheless, intuition can have a role in understanding alternative geometries, as Frege explains in the following sentence from § 14:

> If we do make use of intuition even here, as an aid, it is still the same old intuition of Euclidean space, the only one whose structures we can intuit. Only then the intuition is not taken at its face value, but as symbolic for something else; for example, we call straight or plane what we actually intuit as curved.

The aforementioned example refers to Euclidean models of non-Euclidean geometries, such as those by Beltrami and Poncelet.

As we will see in the next section, Frege does the very same thing with projective geometry in his Ph.D. thesis, where he uses real lines, in *Grundlagen*-terminology, as "symbols" for imaginary points.

In the next paragraph, Frege even argues that this possibility reveals the epistemic nature of geometry:

> For purposes of conceptual thought we can always assume the contrary of some one or other of the geometrical axioms, without involving ourselves in any self-contradictions when we proceed to our deductions, despite the conflict between our assumptions and our intuition. The fact that this is possible shows that the axioms of geometry are independent of one another and of the primitive laws of logic, and consequently are synthetic.

Thus, Frege claims that geometry is synthetic because it is logically possible to negate an axiom of Euclidean geometry.

A similar argument can be found in the work of Gauss and Riemann. According to Gauss, numbers are a "product of our mind" ("unseres Geistes Produkt"), whereas space has "a reality outside of our mind, whose laws we cannot prescribe completely a priori,"[16] because we are convinced of the necessity of the *Größenlehre* (arithmetic) but not of that of the *Raumlehre* (geometry).[17] Thus, for Gauss, geometry is an empirical science, just like physics, which can only be known a posteriori. Similarly, Riemann argues, "the propositions of geometry cannot be deduced from the general notion of number, but that the properties, by which space can be distinguished from other thinkable threefold extended magnitudes, can only be gained from experience."[18]

These philosophical positions were clearly influenced by the mathematical developments just outlined. From the fact that Euclidean geometry is non-contradictory, it follows that geometry cannot rest on logic alone. However, at that time, there was no real equivalent to non-Euclidean geometry in arithmetic—there was no alternative to the standard arithmetic. Hence, it should be no surprise that Frege was not the

[16] "außer unserem Geiste eine Realität hat, der wir ihre Gesetze a priori nicht vollständig vorschreiben können."

[17] Letter to Bessel. April 9th, 1830 Gauß (1880).

[18] "dass die Sätze der Geometrie sich nicht aus allgemeinen Größenbegriffen ableiten lassen, sondern dass diejenigen Eigenschaften, durch welche sich der Raum von anderen denkbaren dreifach ausgedehnten Größen unterscheidet, nur aus der Erfahrung entnommen werden können" (Riemann, 1868, p. 134).

only mathematician who claimed different sources of knowledge for geometry and arithmetic.[19]

The same argument for different sources of knowledge of geometry and arithmetic can be found in Hilbert's early work on geometry. In his "Lecture on Projective Geometry" from 1891, Hilbert writes:

> I need nothing beyond purely logical thinking when I occupy myself with number theory or algebra. With geometry it is totally different. I can just as little comprehend the properties of space through mere thinking as I can recognize the basic laws of mechanics, the law of gravity, or any other physical law. Space is not the product of my thinking, but is given to me by my senses.[20]

Here, Hilbert highlights the dissimilarities between geometry and arithmetic to argue that geometry has a different epistemic status. Just like Frege, he advocates the logical nature of arithmetic. However, he does not attempt to prove this himself; instead, he relies on Dedekind's work.[21] Hilbert maintained the conviction that arithmetic is a part of logic at the time he wrote his *Festschrift*. In a lecture he held in the year he wrote his *Festschrift*, he explicitly refers to Dedekind's work in *Was sind und was sollen die Zahlen?* as evidence (Hilbert, 2004b, p. 303).

2.2 Projective geometry

Projective-synthetic geometry developed from the end of the 18th century onward. Monge was the first who attempted to introduce a synthetic geometry that could catch up with the analytic geometry of that time.[22] "Analytic geometry" is the kind of geometry that originated with Descartes (in *La Géométrie*), in which one uses methods from arithmetic and algebra to express and prove geometrical statements. In synthetic geometry, these methods are banned. Poncelet, a student of Monge at the *École Polytechnique*, put this project forward and is often referred to as the founder of synthetic projective geometry (Klein, 1926, p. 80). Projective geometry differs from the usual affine geometry insofar as it considers only projective properties. A property is projective if it is preserved under projective transformation (i.e., central projection)

[19] There is less agreement regarding the question of whether geometry does or does not have the same source of knowledge as physics, or, to put it in a more Kantian framework (as Frege does), whether geometry is synthetic *a posteriori* or synthetic *a priori*. As we have seen above, Gauss argues that geometry is an empirical science, just like physics, and is, thus, synthetic a posteriori. Frege, on the other hand, maintains in *Grundlagen* that geometry is synthetic a priori, and he argues for a particular geometrical source of knowledge in "Erkenntnisquellen der Mathematik und Naturwissenschaften" Frege (1979b), which was written in the year before his death.

[20] "Ich brauche weiter nichts als rein logisches Denken, wenn ich mit Zahlentheorie oder Algebra mich beschäftige. Ganz anders verhält es sich mit der Geometrie. Ich kann die Eigenschaften des Raumes nimmer durch bloßes Nachdenken ergründen, so wenig, wie ich die Grundgesetze der Mechanik, das Gravitationsgesetz oder irgend ein anderes physikalisches Gesetz so erkennen kann. Es ist ja der Raum nicht ein Produkt meines Nachdenkens, sondern er ist mir durch meine Sinne gegeben" (Hilbert, 2004d, p. 22).

[21] In his lecture, Hilbert does not mention Dedekind explicitly, but he writes that one can gain numbers by "pure thought," e.g by "counting thoughts themselves." This idea was expressed by Dedekind in § 66 of his *Was sind und was sollen die Zahlen?*

[22] In *Géométrie descriptive* (Monge, 1799).

(Poncelet, 1822, p. 3). A non-projective property is called metric (Poncelet, 1822, p. 5). An important metric property is the distance ratio, on which coordinization in analytic geometry relies.

There is, however, a property that is not invariant under projection but that synthetic geometers want to consider nonetheless: the number of points of intersection. In order to classify the number of points of intersection as projective properties, Poncelet introduces new points to affine space: points at infinity and imaginary points. Their existence is demanded by Poncelet's "principle of continuity," according to which the number of points should be invariant under continuous movements of parts of a figure.[23] Another fruitful device for inferring geometrical sentences within synthetic projective geometry is the principle of reciprocity, which later became known by the name "principle of duality."[24] According to this principle, one can interchange the words "point" and "line" in the projective geometry of the plane while preserving truth and likewise the words "point" and "plane" in the geometry of space.[25]

Poncelet's principle of continuity is crucial for the strength of synthetic geometry, as it allows sentences of great generality to be deduced without employing analytic methods. However, these extension elements (i.e., points in infinity and imaginary points) are not based on intuition, unlike the usual points. Other mathematicians tried to equate extension elements with objects based on intuition. For example, von Staudt equates the expression "meeting at a point in infinity" with "being parallel" (von Staudt, 1847, § 5). According to him, a direction can be represented *(vertreten)* by a line out of the collection of parallel lines (which are usually said to "have this direction") (von Staudt, 1847, § 3). An imaginary point is an involution to which a certain "sense" is attached (von Staudt, 1856, § 7).[26]

It is noteworthy that the role of intuition is highly ambiguous when it comes to the principle of continuity. On the one hand, the introduction of extension elements is not motivated by intuition. In fact, they lead to results that seem to contradict our intuition at first glance. For example, it holds that every pair of circles has two points of intersection. However, one can easily imagine a pair of circles that do not intersect

23 Poncelet first mentions "the principle or law of continuity" ("le principe ou la loi de continuité") in his *Traité des propriétés projectives des figures* (Poncelet, 1822, p. xxiij).

24 The term "principle of duality" originated with Gergonne's "Philosophie mathématique. Considérations philosophiques sur les éléments de la science de l'étendue", where the term "duality" ("dualité") is used for the first time in this context (Gergonne, 1826, p. 210).

25 A very detailed study of the influence of the debate about the verification of duality in the early 19th century and its influence on Hilbert can be found in Eder and Schiemer (2018). They argue that metatheoretical reasoning emerged from this debate and that here we can find two different approaches: a "syntactic" (or "proof-theoretic") approache and a "semantic" (or "model-theoretic") approach, the former being attributed to Poncelet and the latter to Pasch (who relies on Plücker's work). They argue that Hilbert's writing reveals some ambiguity regarding whether Hilbert's metatheoretic investigations (i.e., his consistency and independence proofs) have to be understood in the syntactic or the semantic way (Eder & Schiemer, 2018, pp. 82–83).

26 Von Staudt already came up with the idea to identify imaginary points with involutions in von Staudt (1847); however, he struggled to find a way to bijectively map only one imaginary point to one involution. He finally solved this problem with the introduction of sense in von Staudt (1856). The most comprehensive article on this issue is by Nabonnand (2008).

in the intuitive understanding of "intersection."[27] On the other hand, the principle of continuity is central to the fruitfulness of synthetic projective geometry. For the pioneers of synthetic projective geometry, this geometry is an attempt to reinforce the connection of geometry to actual figures. Extension elements are introduced by synthetic geometers by finding substitutes that are based on intuition, i.e., parallel lines for points in infinity and involutions for imaginary points.

In his doctoral thesis, Frege draws the same conclusion. In the first paragraph, he remarks that "we seem well justified in questioning the sense of imaginary forms, since we attribute to them properties, which not infrequently contradict all our intuitions" (Frege, 1873, p. 3, 1984a, p. 1). A few sentences later, Frege mentions the possibility of taking points in infinity as just another expression for having the same direction (Frege, 1873, p. 3, 1984a, p. 1). In the next paragraph, he also mentions that "[i]maginary points can [...] be defined [...] by involution on a straight line" (Frege, 1873, p. 4, 1984a, 1f.). These solutions were, as we have seen, first suggested by von Staudt.

The aim of Frege's thesis, as the title already suggests, is to find "geometrical representations" of imaginary objects. Frege explains the notion of a "geometrical representation" as follows:

> [B]y geometrical representation of imaginary forms in the plane we understand accordingly a kind of correlation in virtue of which every real or imaginary element of the plane has a real, intuitive element corresponding to it. (Frege, 1873, 6f., 1984a, p. 7)

In the following paragraphs, Frege presents a new method to represent imaginary points.[28] According to this method, imaginary points in the plane are represented by lines in (three-dimensional) space. Therefore, Frege, introduces two planes parallel to each other: the "real plane" and the "imaginary plane." One finds the real line representing an imaginary point in the following way: An imaginary number can be thought of as a binary tuple of real numbers, with one of those numbers representing the real part and the other representing the imaginary part. A real point in the plane is analytically denoted by a binary tuple of real numbers—the coordinates. Analogously, an imaginary point in the plane is analytically denoted by a four-tuple of real numbers, two of which denote the real parts and two the imaginary parts. In Frege's representation, such an imaginary point is represented as a line going through one point of the real plane and one point of the imaginary plane. The point in the real plane is denoted by the two real numbers of the four-tuple denoting the real parts and the point in the imaginary plane is denoted by the two real numbers denoting the imaginary parts (see Fig. 1). A real point in the plane is represented by a real line in space, which goes through the origin of the imaginary plane (see line g in Fig. 1).

Thus, Frege indeed discusses the possibility of representing extension elements by different "real, intuitive elements." At first glance, this contrasts oddly with his later inability to understand Hilbert's method of reinterpreting geometrical signs differently for metatheoretical investigations. However, Frege's representations are not models in

[27] According to this common understanding of "intersection," which is based on the visual representations of geometrical figures, a pair of geometrical objects intersect if the lines of their representations cross.

[28] In the last paragraph of his thesis, Frege compares his representation to that of Gauss.

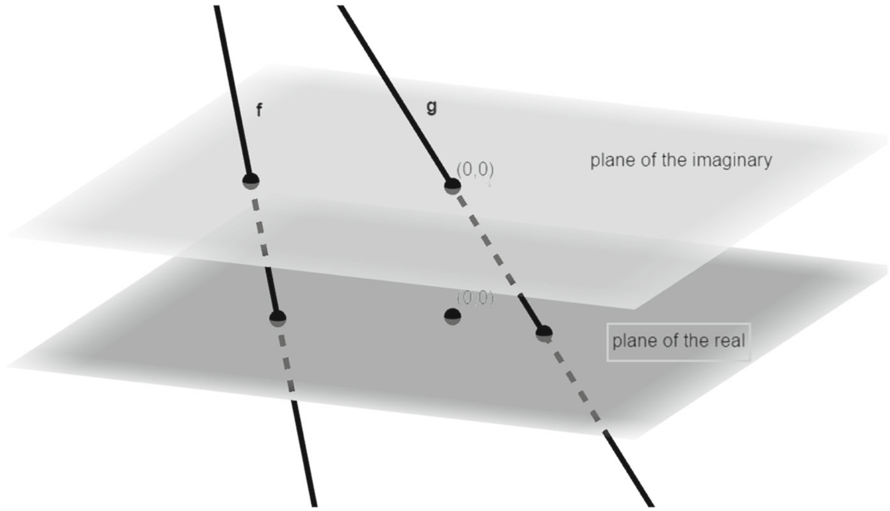


Fig. 1 Frege's representations of imaginary points

the modern sense of the term. There are no uninterpreted signs that are interpreted in a particular domain like in modern model theory;[29] instead, meaningful arithmetical expressions are represented by geometrical objects. The geometrical objects serve as "symbols," as Frege would later call them in *Grundlagen*.

At this time, Frege had not yet employed the strategy to let abstract objects such as sets define other objects, which he suggests in his *Grundlagen der Arithmetik*, since he basically did not yet have a notion of set, let alone his abstraction principle. Nevertheless, there is a continuity between Frege's way of dealing with extension elements from his thesis of 1873 until his *Grundlagen der Arithmetik*, i.e., the influence of von Staudt's approach. As already mentioned, Frege refers positively to von Staudt's equation of extension elements with more intuitive objects in his thesis. In his *Grundlagen der Arithmetik*, Frege states his conviction that one should define directions by presupposing the notion of parallel lines and not the other way round. This is a conviction he shares with von Staudt, who is always cautious about presupposing more intuitive concepts and defining less intuitive ones. In his *Geometrie der Lage*, von Staudt defines parallels as lines that have no point of intersection (von Staudt, 1847, §3 (31)). Line segments are parallels if they are part of parallel lines. They are unanimously (*einstimmig*) parallel if they are on the same side of a line connecting their starting points.

29 Eder (2019) comes to a similar conclusion: "Frege thus seems to identify the real projective plane with a particular model of the real projective plane. We can visualize or 'represent' this particular structure by means of other structures (say, by a sphere) or describe it by means of coordinates. But there is only one real projective plane, and so the question of associating different interpretations with words like 'point' or 'line' does not arise" (Eder, 2019, p. 5562).

Von Staudt does not define a direction as a set of parallel lines (or line segments),[30] but directions of rays can be "determined" ("bestimmt") by any line segment unanimously parallel to it, and this line segment can be "represented" ("vertreten") by any other line segment unanimously parallel to it in this context (von Staudt, 1847, § 3 (35)). We have to keep in mind that von Staudt lacked the concept of a set at this time. However, it is quite obvious that von Staudt inspired Frege.[31] This becomes even more apparent when considering that Frege, in the same paragraphs, mentions the notion of *Stellung* for what planes in space share if they are parallel.[32] This notion also originates with von Staudt (1847, § 3 (40)).

There is, however, a problem with the approach Frege presents in §§ 1–10 of his Ph.D. thesis. Since lines can be generally identified by two points, and points of the projective plane are presented by lines in the real space, imaginary lines are respectively represented by pairs of real lines, with each representing an imaginary point that lies on the imaginary line. However, as Frege mentions in § 11, this representation violates the principle of duality. Therefore, Frege presents an alternative identification, that preserves duality: imaginary lines, as well as imaginary points, are represented by two real lines. Frege does not use drawings to lay down his approach but continues purely analytically, using homogeneous tetradic Plücker coordinates.

These coordinates were introduced by Julius Plücker, who tried to capture the results of synthetic geometry within analytic geometry.[33] Therefore, his task is not to avoid metric notions for his coordinization, but to find a coordinization that allows the result of synthetic projective geometry to be deduced elegantly and efficiently. Thus, he developed an analytic foundation for the principle of reciprocity by introducing a coordinate system that is homogeneous,[34] i.e., a coordinate system in which coordinate tuples, those coordinates that have the same ratio, are equal. In such a coordinate system, a point in the plane is denoted by three coordinates and a point in three-dimensional space by four coordinates. In the projective plane, every line can be expressed by an equation of the form $ax + by + cz = 0$. In this equation, (x, y, z) and (a, b, c) are interchangeable. Thus, one could either, as usual, take a, b, and c as

[30] We thereby contradict Wilson, who indicates that von Staudt already used the definition of parallelism as a set (Wilson, 2005, p. 173). However, von Staut does not use the notion of a *set* In his paper from 2010, Wilson is much more cautious in his claims about von Staudt and no longer indicates that von Staudt defines directions as sets (Wilson, 2010, 397f.).

[31] Several authors have already remarked this similarity. We already mentioned (Wilson, 2005, p. 173). More recently, Eder (2021, p. 6531) has suggested that Frege had von Staudt in mind when writing the introduction of his thesis.

[32] This was already pointed out in Mancosu (2016, p. 62).

[33] As Jemma Lorenat points out, Plücker was, for that reason, sometimes even classified as a synthetic geometer. Plücker avoids all calculations that do not concern the final result. Cournot suggests that Plücker's analysis strongly resembles "la synththese." He employs the term "synthesis" to emphasize how Plücker's presentation works toward a known result, thus functioning well to prove known theorems (Lorenat, 2016, p. 432). Lorenat herself classifies Plücker as belonging to the "middle ground between the pure and analytic methods" (Lorenat, 2015a, p. 175). However, one must take into account that Plücker himself frequently distinguishes "analytic geometry" and the geometry that Poncelet introduced. Plücker thereby classifies himself as a member of the former approach. (See, e.g., Plücker, 1830)

[34] There are also other homogeneous coordinate systems, that were introduced at the beginning of the 19th century, e.g., that of Möbius. However, it was Plücker who had the idea to use his coordinate system to contribute to the discussion on the justification of the principle of duality. See also Boyer (2004, p. 294).

constants and $ax + by + cz = 0$ as a line equation with (x, y, z) as point variables, or one could take x, y, and z as constants and $ax + by + cz = 0$ as a point equation with (a, b, c) as point variables.[35] Thus, the projective plane's duality between points and lines becomes trivial (Pluecker, 1831, p. 2). In § 11 of his thesis, Frege uses Plücker coordinates for space, where a point is represented by four variables.[36]

Frege also builds on the ideas of Plücker later in his mathematical writings. For example, he uses Plücker coordinates extensively in his lecture "Analytische Geometrie nach neueren Methoden," which can be seen from the notes taken from Frege's first lecture in the winter semester of 1874/1875, a lecture he gave seven times between 1874 and 1895.[37] In this lecture, Frege provides an overview of three different coordinate systems: the "distance coordinate system" ("Abstandskoordinatensystem"), "distance ratio coordinate system" ("Abstandsverhältniscoordinatensystem"), and "general homogeneous coordinate system" ("Allgemeines homogenes Coordinatensystem"). He points out that these systems are based on three different properties: distance between two points, distance ratio, and cross-ratio. Moreover, he calls our usual coordinates a particular case of the "more general coordinate system"—the homogeneous coordinate system (Frege, 1874/1875, p. 363). Similarly, in his talk "Über Invarianten" from 1877, Frege identifies the properties that are preserved under the transformation of coordinate systems (for example, the transformation of classical coordinates preserves congruence) and explains how different coordinate systems differ in this respect (Frege, 1877, p. 378).

Frege also follows an approach invented by Plücker when it comes to understanding a fourth dimension. Plücker uses four line coordinates to describe a point in space (Plücker, 1868). Thus, we have a visual representation of an analytic expression that uses four dimensions. In his "Lecture on Geometry of Pairs of Points in the Plane," Frege develops a similar idea. He takes pairs of points as the basic elements of the plane, which means that four coordinates denote a pair of points in the plane. Frege appreciates that "[i]n this way we arrive at geometries of more than three dimensions without leaving the firm ground of intuition." Therefore, for Frege, these four coordinates do not refer to a point in four-dimensional space. Frege does not want to acknowledge such objects. Instead, the four coordinates refer to a pair of points in the plane, which is an unproblematic geometrical object for Frege.[38] This strategy is the same as the one used by Frege in his Ph.D. thesis, where he "represents" points in the imaginary plane by lines in three-dimensional real Euclidean space.

Therefore, we must conclude that Frege, in his mathematical writings, was heavily influenced by the projective geometry of his time, especially by the work of von Staudt

35 Tappenden (1995a, p. 445) presents a convincing argument that Frege's idea that a sentence might be decomposed into function and argument(s) in multiple ways is influenced by Plücker's proof of duality.

36 Frege got exposed to Plücker's work in a geometry lecture from Alfred Clebsch, which Frege attended in 1871 (Kreiser, 2001, p. 87). The lecture Frege attended appeared in print (Clebsch, 1876). Clebsch introduces Plücker's line coordinates very early in his lecture (section I.3 in the book).

37 In 1895, Frege wrote his first letter to Hilbert, just after they met at a conference in Lübeck (Frege, 1980, pp. 32–33).

38 As Tappenden (1995b, 327f.) shows, Frege and Plücker differ sharply in this respect from Riemann, for whom four-dimensional spaces are actually structures of four independent parameters.

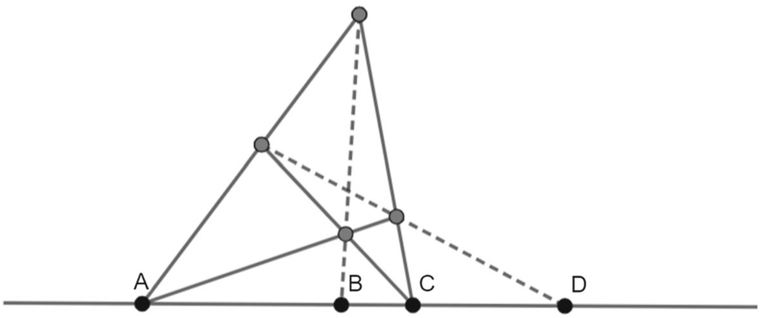


Fig. 2 Fourth harmonic point

and Plücker. Precisely these two authors were also essential to Hilbert's work in the 1890s.

According to Hilbert, the "Idol" ("Vorbild") for his "Lectures on Projective Geometry" from 1891 is von Staudt, who, according to Hilbert, solved the question of "whether it is possible to free projective geometry entirely from measuring and calculating."[39] In his "Lectures on the Foundations of Geometry" from 1894, Hilbert presents von Staudt's way of introducing coordinates without relying on metric notions. This method relies on the projective theorem of the fourth harmonic point, which von Staudt expresses in *Geometrie der Lage* as follows:

> If three points A, B, C are given on a straight line, and a quadrangle is constructed so that one diagonal passes through the second of the given points and at each of the two remaining points two opposite sides [of the quadrangle, TR] cut each other, then the other diagonal cuts the line at another point D, which is determined by the three given points. This point D is called the fourth harmonic point.[40] (See Fig. 2.)

One can now use this theorem to set up a metric-free coordinate system in the following way:[41] Let 0, α and ∞ be three colinear points and α a positive real number. We can assign the value 2α to the fourth harmonic point if we take 0 to be the point on the diagonal. Now we can proceed to assign the value 3α to the fourth harmonic point of α, 2α and ∞, where α is on the diagonal.

Similarly, we can assign the value $\frac{\alpha}{2}$ to the fourth harmonic point of 0, α and ∞, where ∞ is on the diagonal.

[39] "ob es möglich sei, die projektive Geometrie ganz vom Messen und Rechnen frei zu machen" (Hilbert, 2004d, p. 24).

[40] "Wenn in einer Geraden drei Punkte A, B, C gegeben sind, und alsdann ein Viereck so construiert wird, dass eine Diagonale durch den zweiten Punkt der gegebenen Punkte geht, in jedem der beiden übrigen aber zwei einander gegenüberliegende Seiten sich schneiden, so schneidet die andere Diagonale des Vierecks jene Gerade in einem weiteren Punkte D, welche durch die drei gegebenen Punkte bestimmt ist und zu denselben der vierte harmonische Punkt heißt" (von Staudt, 1847, §8 (93)).

[41] Hilbert presents this method of coordinization in a lecture "Die Grundlagen der Geometrie" (Hilbert, 2004a, pp. 85–93). The first presentation of the metric-free coordinization can be found in von Staudt (1856).

Finally, we can assign the value $\frac{1}{2n-1}$ to the fourth harmonic point of 0, 1 and $\frac{1}{n}$, where $\frac{1}{n}$ is between 0 and 1 while 1 lies on the diagonal.

Similarly, one can assign negative values to the line 0∞ (Hilbert, 2004a, p. 90). In this way, a number can be assigned to any point on the line.[42]

Giovannini (2016) showes that the discussion of von Staudt's coordinization among mathematicians of his time heavily influenced Hilbert's investigation of continuity axioms in his *Festschrift* (Giovannini, 2016, pp. 41–49).

Like Frege, Hilbert does not fully subscribe to synthetic geometry but sees analytic and synthetic methods as two interesting geometrical toolboxes that address different needs. In the closing passage of his lecture from 1891, he lists the advantages of both disciplines. According to him, in analytic geometry, "one arrives faster at propositions with highest generality,"[43] "while projective geometry [...] has the advantage of purity, closure and conceptual necessity of its methods."[44]

The distinction between "projective geometry" on the one hand and "analytic geometry" on the other is a bit odd, because projective geometry can also be done with analytic methods. A prominent example is the work of the previously mentioned German geometer Plücker.[45]

In fact, Hilbert was aware of Plücker's work. In his 1894 lecture, he introduces homogeneous Plücker coordinates as an alternative to von Staudt's metric-free coordinization (Hilbert, 2004a, 95ff.) and proves duality in Plücker's way (Hilbert, 2004a, p. 103), although he does not explicitly mention Plücker's name. Considering these two approaches to coordinization, Hilbert draws the following conclusion:

> At the same time one recognizes that projective and analytic geometry do not significantly differ at all, but rather only the starting points of the two branches of geometry are different, while their methods surely come to the same thing.[46]

This resembles Plücker's own evaluation of his results. In "Über ein neues Coordinatensystem," he writes:

> The general analytic method and the method which Mr. Poncelet developed in his *"Traité des propriétés projectives"* are based on the one hand on essentially different ideas; but on the other hand they coincide so completely in their results that the first method sometimes appears, admittedly astonishingly enough, to have been considered as a paraphrase, a plagiarism of the second. The question

42 Hilbert's proofs can be found in Hilbert (2004a, pp. 88–90).

43 "gelingt es rascher zu größter Allgemeinheit der Sätze zu gelangen" (Hilbert, 2004d, p. 55).

44 "Dagegen hat die projektive Geometrie [...] den Vorzug der Reinheit, Abgeschlossenheit und Denknotwendigkeit der Methoden" (Hilbert, 2004d, p. 55).

45 In France, Gergonne had a similar approach. Gergonne was the editor of the French journal *Annales des mathématiques pures et appliquées* and published early work of Plücker. An in-depth study of the different styles of Plücker and Gergonne (and Poncelet) can be found in Lorenat (2015a, 2015b).

46 "Zugleich erkennt man, dass projektive und analytische Geometrie sich garnicht wesentlich unterscheiden, vielmehr nur die Ausgangspunkte in beiden Zweigen der Geometrie verschieden sind, während sicherlich die Methoden auf das nämliche hinauslaufen" (Hilbert, 2004a, p. 103).

should instead have been answered of whether there is a necessary cause for this coincidence and where it is to be sought.[47]

Hilbert also builds on Plücker's work in his *Festschrift*. In § 9, he presents a consistency proof for his axiom system and provides an analytic interpretation of the geometrical notions that occur in his axioms, according to which points are identified with pairs of numbers (x, y) and lines with triples of numbers (u, v, w).[48] Hence, Hilbert here takes over Plücker's idea of line coordinates.[49]

We have thus seen that Frege and Hilbert share the same mathematical heritage. However, the question remains of where their profound misunderstanding comes from. In what follows, we will argue that both have different conceptions about the relationship between axioms and *Anschauung*, which make them evaluate the results of late 19th-century geometry, that they both acknowledge, differently.

3 Frege and Hilbert on axioms and intuition

The plurality of different geometries poses a problem for Kant's philosophy of mathematics. For Kant, sentences of Euclidean geometry are a priori truths, because Euclidean space is the *Anschauungsform* of outer things. The possibility of alternative geometries is discussed neither in Kant's philosophy nor in the mathematical writings of his contemporaries. However, with the rise of non-Euclidean geometry, the epistemic nature of Euclidean geometry came into question. As mentioned in Sect. 2, for Gauss, geometry is an empirical science. It is empirical insofar as one can ask which geometry describes our universe's form. According to Gauss, this can be investigated through physical experiments. Gauss builds his argument on a mathematical state of affairs. In non-Euclidean geometries, the inner angle sum is different from 180°. The bigger the triangles are, the greater the difference from 180°. Thus, Gauss concludes that measuring the angles of big triangles in physical space might eventually uncover the universe's geometry. The "big triangles" that he has in mind are triangles formed by light rays.

Frege and Hilbert (at the time he wrote his *Festschrift*).[50] both have a more Kantian position insofar as they both, in a way, hold that geometrical axioms are not empirical truths. According to Frege, we recognize the truth of geometrical axioms through a

[47] "Die allgemeine analytische Methode, und die Methode, die Herr Poncelet in seinem '*Traité des propriétés projectives*' entwickelt hat, beruhen auf der einen Seite auf ganz wesentlich verschiedenen Ideen, und stimmen doch auf der anderen Seite so sehr in den Resultaten überein, dass man, freilich sonderbar genug, die erste Methode als eine Periphrase, als ein Plagiat der zweiten hier und da betrachtet zu haben scheint, statt dass man sich ruhig die Frage beantworten sollte, ob nicht ein notwendiger Grund dieser Übereinstimmung vorhanden und wo derselbe zu suchen sei" (Plücker, 1830, p. 2).

[48] The numbers are algebraic numbers, which can be obtained starting from 1 by a finite application of the following operations: addition, subtraction, multiplication, and the operation $\sqrt{1+\omega^2}$.

[49] Here, Hilbert uses non-homogenous line coordinates.

[50] Hilbert's views changed over time. In the early 1890s, he seemed to agree with Gauss. As we will see in what follows, in 1894, Hilbert argued explicitly against Gauss's idea of an ultimate experimental proof for geometrical axioms. He still seemed to hold this view at the time he wrote his *Grundlagen* However, after learning about Einstein's theory of relativity, he again seemed to be more sympathetic to Gauss's view. These more empirical tendencies in this later period were already observed by Corry (2006, p. 168).

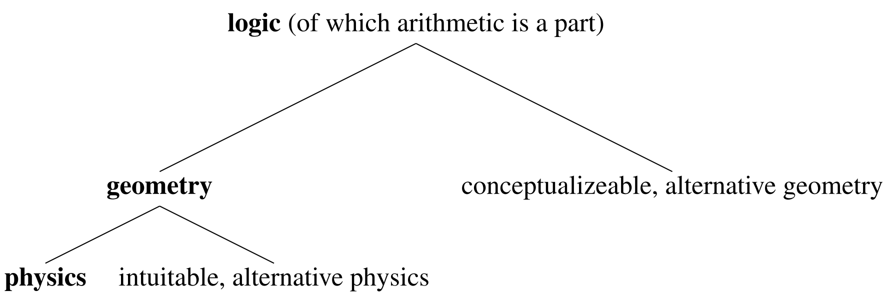
particular source of knowledge, which he sometimes calls *Anschauung*. According to Hilbert, on the other hand, geometrical axioms are not truths at all. However, we can ask which set of axioms is more suitable for describing the physical world. The answer to this question is not merely an empirical one. We will see that Hilbert is heavily influenced by Hertz and that his position resembles neo-Kantians such as Cassirer.

3.1 Frege's traditional notions of axioms and *Anschauung*

Two crucial convictions about axioms shape Frege's philosophical discussion of non-Euclidean geometry. Firstly, Frege thinks that the mathematical sciences have a hierarchical order, where the axioms of sciences that are lower in this hierarchy have to be consistent with axioms of sciences higher in the hierarchy but cannot be proven by them. Secondly, axioms express truths that rest on a specific source of knowledge, which is related to the axiom's position in the hierarchy.

Frege's hierarchical structure from top to bottom is as follows: (1) logic, which includes arithmetic; (2) geometry, the axioms of which have to be consistent with the axioms of logic but cannot be deduced from them (and that is the reason why alternative geometries can be grasped by conceptual thought); and (3) physics, the axioms of which must be consistent with the axioms of geometry but cannot be deduced from them (and that is why fictional situations, which do not obey the laws of physics, can nonetheless be imagined as long as they are intuitable).[51]

This hierarchy can be illustrated as follows:



Frege describes this hierarchical ordering in § 14 of his *Grundlagen*. We have already discussed the relationship between logic and geometry, which Frege explains in the passage from § 14 quoted on page 6. We have seen that according to Frege, Euclidean geometry is true, but non-Euclidean geometries are nonetheless perceivable by conceptual thought and, thus, by logical means because they obey the logical laws. One may recall Frege's conclusion that "the axioms of geometry are independent [...] of the primitive laws of logic, and consequently are synthetic." Thus, we need *Anschauung* in order to recognize the true geometry among logical possible geometries. Logic is not sufficient for that.

[51] Shipley (2015, p. 18) already mentioned that Frege "establishes a hierarchical structure of the mathematical discipline," by insisting that arithmetic is more general than geometry. However, Shipley does not discuss the position of physics in this hierarchy (which is not a "mathematical discipline" in the narrow sense).

Similarly, in the same paragraph Frege argues that alternatives to the true physical laws are intuitable:

> Empirical propositions hold good of what is physically or psychologically actual, the truths of geometry govern all that is spatially intuitable, whether actual or product of fantasy. The wildest vision of delirium, the boldest inventions of legend and poetry, where animals speak and stars stand still where men are turned to stone and trees turn into man, where the drowning haul themselves up out of swamps by their own topknots—all these remain, so long as they remain intuitable, still subject to the axioms of geometry.

In other words, the axioms of physics have to be consistent with the axioms of geometry but cannot be deduced from them; otherwise, it would be impossible to imagine a world where alternative physical axioms hold. Therefore, to determine the true axioms of physics among all the alternative axioms of physics that are intuitable, we need another source of knowledge, namely, sense experience.

According to this hierarchy of sources of knowledge, sense experience is not involved in the recognition of geometrical truths. Thus, a geometrical sentence cannot be refuted by a physical experiment. In particular, the Gaussean idea that physical experiments could challenge geometrical axioms makes no sense in Frege's conception of the sciences.

Thereby, Frege follows Kant, for whom the idea that experiments could challenge the geometrical axioms is similarly alien. However, unlike Kant, Frege does not have a profound philosophical explanation to justify this hierarchical order. Kant distinguishes between *Ding an sich* and *Ding der Erfahrung*. A *Ding an sich* cannot be cognized. Space is the form of our outer experience (*Erfahrung*). Thus, things are only spatial insofar as they are *Dinge der Erfahrung*. Geometry expresses truths about the property of space. From a modern perspective, one can still criticize Kant for not presenting an argument for the Euclidean nature of space (which is anachronistic, since non-Euclidean geometry was discovered after Kant's death). However, Kant's conception allows him to argue why experiments cannot refute geometrical sentences. Physics deals with *Dingen der Erfahrung*, space is the outer form of these *Dinge der Erfahrung*, and geometry is a science that determines the properties of space (Kant, 1929, A25/B40). Thus, the laws of physics have to obey the laws of geometry.

Frege never mentions this crucial Kantian conception.[52] He simply claims that there is a geometrical source of knowledge, which he sometimes calls *Anschauung*.[53] According to Frege, a source of knowledge is "what justifies the recognition of the truth, the judgment" (Frege, 1979b, p. 267). In the case of logic, Frege makes how this works quite explicit: the basic laws of logic must be acknowledged as true because we cannot think their negation (*Grundlagen*, § 14). Other logical truths (that, for Frege, include all sentences of arithmetic) can be acknowledged as true because they follow from the basic laws by simple rules, which are truth-preserving. However, with

[52] Dummett already pointed this out, concluding that "this leaves it obscure why we should treat our intuition as a ground of knowledge of geometry" (Dummett, 1982, p. 251).

[53] Frege later abandons the Kantian vocabulary he extensively uses in the *Grundlagen* In his unpublished essay "Erkenntnisquellen der Mathematik und Naturwissenschaft" (Frege, 1979b), for example, he just speaks about "the geometrical source of knowledge."

the discovery of the paradox, this procedure turns out to be fallacious.[54] The source of knowledge for geometrical sentences is *Anschauung*. Analogously, a geometrical axiom is true if it is not intuitable that its negation holds, i. e., if we cannot imagine a world, where they do not hold, not even in “the wildest vision of delirium” or “the boldest inventions of legend and poetry” (*Grundlagen* § 14). However, how is the *truth* of geometrical axioms justified by testing if its negation is not intuitable? This decision procedure only makes sense if geometry only governs our *Dinge der Erfahrung*;[55] otherwise, it could be possible that alternative geometrical axioms are true, even though we lack the capacity to imagine a world in which they do not hold. In § 26 of the *Grundlagen*, Frege himself indicates that there might be beings with a different spatial intuition (*Raumanschauung*),[56] Unfortunatly, Frege never discusses the distincion between *Ding an sich* and *Ding der Erfahrung*

Nonetheless, Frege does not doubt at all that the Euclidean axioms, and not the non-Euclidean ones, are intuitable and, thus, true. Even in his unpublished paper titled “Erkenntnisquellen der Mathematik und Naturwissenschaften” from 1924/1925, shortly before his death, Frege stresses that if we understand axioms “in the old Euclidean sense,” i.e., as true statements, “we need not fear that this source of knowledge will be contaminated” (Frege, 1979b, p. 273). Therefore, the geometrical source of knowledge is the most reliable source for Frege at this time (after the discovery of the Russell paradox).

When arguing for the truth of Euclidean geometry instead of non-Euclidean in his unpublished note “Über Euklidische Geometrie”, Frege gives a merely historical argument by mentioning that Euclid’s elements “have exercised unquestioned sway for 2000 years” (Frege, 1979a, p. 169). The infamous comparison of non-Euclidean geometry to astrology, which occurs in this note, serves as an expression of the conviction that non-Euclidean geometry is false (like astrology) since Euclidean geometry is true (like astronomy).

However, this does not demonstrate that Frege dismisses non-Euclidean geometry completely. We have already seen in Sect. 2 that Frege, unlike Lotze, acknowledges the possibility of conceptually grasping non-Euclidean geometry. Moreover, we have already established that this is crucial for Frege’s hierarchical understanding of the sciences.

For example, we have seen that Frege presented a way to visually represent the results of analytic projective geometry in his thesis. We can use arithmetic to express alternative geometries that are not in accordance with our *Anschauung*. This is possible because Frege has a much broader notion of the analytical than Kant. Unlike Kant,

[54] Frege addresses this issue explicitly in his unpublished paper “Erkenntnisquellen der Mathematik und Naturwissenschaft” from 1924/1925 (Frege, 1979b, pp. 269–273).

[55] Eder (2021, p. 6554) argues that “Frege does seem to understand the notion of intuition in the psychological sense,” and that intuition was “purely subjective.” However, if this is right, it is hard to understand how a geometrical proposition can be classified as objectively “true.” Something subjective can hardly be a foundation for objective truths.

[56] Schirn (2019, p. 957) tries to solve this puzzle by claiming that for Fege, geometrical axioms are objective, while spatial intuition is subjective. However, he admits that “I fail to see how the claimed dependence of the validity of the (objective) geometrical axioms on the nature of our faculty of intuition [...] could be reconciled with the subjectivity of our spatial intuitions, stressed as it is in *The Foundations* §26” (Schirn, 2019, p. 959).

Frege identifies being analytic with being deducible from logic. This resembles Kant's notion of analyticity insofar as Kant identifies analytic judgments with those that can be recognized by the principle of non-contradiction. Even so, Frege's notion of logic and analyticity goes well beyond Kant's. Most importantly, Frege can distinguish with his logic between concepts of different order and arity because he has quantifiers and relations.[57] These tools allow him to build very complex notions from logic alone, such as having a successor and being a property inherited in a series. These notions stand in much more complex inferential relations than those expressed by syllogistic logic.

Syllogistic propositions have the form "S is P". The predicates "S" and "P" are composed of simpler notions by mere conjunction. We may now ask if "P" is among the simpler notions of which "S" is composed by conjunction. If it is, "S is P" does not extend our knowledge because we only explicate what is already there (Kant, 1929, A7/B10f). Kant's famous example is the sentence "All bachelors are unmarried", where "unmarried" is already contained in the definition of "bachelor". Only this kind of sentences can be proven by the principle of non-contradiction. Frege argues in § 88 of *Grundlagen* that he can build complex notions, that are not just conjunctions of simpler notions and that, as a result, being logically deducible (thus, in his terminology, "analytic") and just explicating the subject term (thus, not extending our knowledge in Kant's terminology), do not coincide.

Therefore, for Frege, there are sentences that are analytic but nonetheless extend our knowledge.[58] This contrasts sharply with Kant's original conception of logic and analyticity,[59] which, according to Frege, is too narrow (Frege, 1999, § 88). It is only in the context of this broader notion of logic that it makes sense to call logic a source of knowledge. It is also essential for Frege's foundational project to deduce arithmetic from logic.

However, in order to acknowledge the truth of geometry, we still need *Anschauung*, according to Frege. Frege never makes his notion of *Anschauung* explicit, even though it is the source of knowledge of geometry. However, in the preceding paragraph of his *Grundlagen*, i.e., § 13, Frege provides a short explanation of what is particular about geometrical reasoning, which illuminates Frege's notion of *Anschauung*:

> One geometrical point, considered by itself, cannot be distinguished in any way from any other; the same applies to lines and planes. Only when several points, or lines or planes, are included together in a single intuition [*Anschauung*], do we distinguish them. In geometry, therefore, it is quite intelligible that general

[57] Michael Friedman points out that Kant does not have our means of polyadic logic to ensure that an axiom system has an infinite model (Friedman, 1990, 62f.) to express infinite indivisibility. As a result, conceptual knowledge is inadequate for geometry and intuition must play a role (Friedman, 1990, 70f.). Michael Wolff, on the other hand, claims that Frege's modern logic, would be classified by Kant as a particular logic (*Fachlogik*) of mathematics (Wolff, 1995, p. 220), because mathematical induction is a particular form of inference for ordered entities, which inherit a certain property—something that is particular for the natural numbers (Wolff, 1995, pp. 218–219).

[58] See Tappenden (1995a) and Rohr (2020) for a more detailed explanation of Frege's argument.

[59] However, the idea that logic is a science is not in accordance with Kant. In his *Critique of Pure Reason* Kant explicitly states that general, formal logic is "a *conditio sine qua non*, and is therefore the negative condition of all truth. But further than this logic cannot go. It has no touchstone for the discovery of such error as concerns not the form but the content" (Kant, 1929, p. 98, A59f/B84).

> propositions should be derived from intuition; the points or lines or planes which we intuit are not really particular at all, which is what enables them to stand as representatives of their kind.

This argument can be paraphrased as follows: what makes geometrical reasoning possible is that particular geometrical objects do not have specific properties besides their spatial relation; consequently, general sentences can be derived from single geometrical constructions. This idea resembles Kant's conception of mathematical reasoning.

According to this, it is crucial for mathematical proofs that they use constructions. In the chapter "Transcendentale Methodenlehre" of his *Kritik der reinen Vernunft*, Kant acknowledges that a figure as an empirical object is something particular. He argues that we can prove general statements through construction because we are simply considering the act of construction (*Handlung der Konstruktion*), thereby abstracting from all particularities, such as the absolute size of the figure. He thus sums up the difference between mathematical (on behalf of constructions) and philosophical (on behalf of logical deductions) reasoning in the following way: "Philosophical knowledge considers the particular only in the universal, mathematical knowledge the universal in the particular" (Kant, 1929, p. 577 (A714/B742)). This is exactly what Frege claims here regarding geometrical knowledge.

However, unlike Kant, Frege makes a sharp distinction between geometrical and arithmetical knowledge. He starts § 13 with the remark that "[w]e shall do well in general not to overestimate the extent to which arithmetic is akin to geometry." Then, he refers back to a remark from Leibniz on arithmetical objects, which Frege quotes and endorses in § 10 of *Grundlagen*:

> An even number can be divided into two equal parts, an odd number cannot; three and six are triangular numbers, four and nine are squares, eight is a cube, and so on.

In this respect, arithmetical objects such as numbers differ significantly from geometrical objects, which can only be distinguished "in a single intuition." Frege concludes that the deduction of a general proposition from a single intuition (i. e., construction) is not possible in arithmetic:

> To what extent a given particular number can represent all the others, and at what point its own special character comes into play, cannot be laid down generally in advance.

Hence, arithmetical knowledge allows no inference from the particular to the general—it is purely logical. Thus, for Frege, arithmetical reasoning is not like "mathematical reasoning" in the Kantian sense but rather like Kant's "philosophical knowledge," i.e., it is independent from *Anschauung*.

This coherent Fregean conception of *Anschauung* aims to clarify why geometry and arithmetic rest on different sources of knowledge. It fails, however, to provide evidence for the truth of Euclidean geometry in contrast to non-Euclidean geometry.

To summarize, Frege is aware that non-Euclidean geometries are without contradiction. Thus, his dismissive utterance about non-Euclidean geometry in "Über

Euklidische Geometrie" does not rest on mathematical ignorance. Philosophically, however, Frege does not present a good argument for the truth of Euclidean axioms and *Anschauung* as a source of knowledge for them. His notion of *Anschauung* clarifies why geometry and arithmetic rest on different sources of knowledge. However, it does not clarify why the axioms of Euclidean geometry are true and not those of non-Euclidean geometry. Hence, Frege's conviction that the Euclidean axioms are true and that *Anschauung* provides a source of knowledge of these truths is philosophically dogmatic.

3.2 Heinrich Hertz's picture theory and Hilbert's late 1890s notion of an axiom

Like Frege, Hilbert was convinced at the time of their correspondence that arithmetic is part of logic and that geometry is not. However, the agreement between the two authors basically ends there.

Hilbert opposes both of Frege's convictions about axioms when he wrote his *Festschrift*. Firstly, he holds that there is no hierarchical order of sciences (beyond the level of logic). In particular, physical experiments influence our choice of axioms for geometry. Secondly, for Hilbert at this time, axioms do not express truths but are merely a "picture" of reality. This is highly significant since different axiom systems (contradicting each other) can be pictures of reality.

Finally, in the mid-1890s, Hilbert moves away from the idea that *Anschauung* is a source of knowledge that can be presupposed as given and on which our axioms rest. Instead, axiomatization serves as a way to "analyze" *Anschauung*.

In what follows, we will examine these three theses and show how all three are closely related to each other.

The goal of his *Festschrift* is, according to its introduction, the "logical analysis of our spatial intuition" ("logische Analyse unserer räumlichen Anschauung") The word "Anschaunng" is of Kantian origin. The expression "logische Analyse unserer räumlichen Anschauung," however, is not. Nonetheless, this term is not explained in his *Festschrift*. However, we find an explanation of the similar expression "logical analysis of our intuitive faculty" ("logische Analyse unseres Anschauungsvermögens").[60] in the notes on Hilbert's "Lectures on Euclidean Geometry", from the same year[61]

> Utilizing an expression taken from Hertz (in the introduction to the *Principles of Mechanics*) we could formulate our main question as follows: to which necessary and sufficient and mutually independent conditions must a system of things be subordinated, so that to every property of these things corresponds a geometrical fact and vice versa, that is, so that these things are a complete and simple "picture" of geometrical reality?

60 Frege also uses the Kantian term "*Anschauungsvermögen*" in his thesis in a context where one would rather expect *Anschauung*. In the opening passage, he writes, "[W]e consider that the whole of geometry rests ultimately on axioms which derive their validity from the nature of our intuitive faculty" (Frege, 1984a, p. 1).

61 An almost identical passage can also be found in a lecture from 1902. Here, Hilbert again uses the term "intuitive faculty" ("Anschauungsvermögen") (Hilbert, 2004c, p. 541). It seems that he uses both expressions synonymously, unaware or ignorant of the different meanings they have in the context of Kant's philosophy, which they are borrowed from.

We can, finally, call our task a *logical analysis of our intuitive faculty*.[62]

First, we need to clarify some of Hilbert's other expressions here. As becomes clear from the context, the ("necessary" and "sufficient") "conditions" are the axioms.[63]

The word "system" ("System") can be found in Dedekind's work and roughly means set.[64] In his *Grundlagen*, Hilbert talks about "three systems of things," which he calls "points," "lines," and "planes." Thus, these mathematical objects are not part of what Hilbert calls "geometrical reality" but are a mere "picture" of it. Hence, there is no direct relationship between axioms and reality, but the "systems of things" serve as mediators.

As Hilbert indicates at the beginning of this passage, he took the picture notion from Hertz's introduction of the *Prinzipien der Mechanik*, a book published in 1894. Thus, to understand Hilbert's explanation, we must look closely at Hertz's picture theory.

Hertz sets up his picture theory in his *Prinzipien der Mechanik* to explain why there can be different physical theories explaining reality. In the introduction, Hertz writes, "We form for ourselves [picture]s or symbols of external objects, and the form which we give them is such that the necessary consequent of the [picture]s in thought are always the [picture]s of the necessary consequences in nature of the things pictured" (Hertz, 1899, p. 1).[65] In other words, pictures are what we would nowadays call scientific models; they allow us to predict future events.

It is crucial for Hertz's conception that our picture has no ontological commitment. He writes in the same passage, "The [picture]s which we here speak of are our conceptions of things. With the things themselves they are in conformity in *one* important respect, namely, in satisfying the above-mentioned requirement. For our purpose it is not necessary that they should be in conformity with the things in any other respect whatever" (Hertz, 1899, pp. 1–2). Hertz's own theory of mechanics, for example, contains so-called "hidden masses" (Hertz, 1899, pp. 25–26). For Hertz, this does not mean that there is necessarily such an entity in the outer world, but only that the hidden masses fulfill a task within the theory, namely, making good predictions about real-world events possible.

There is not only one picture of the world, but in order to be classified as a picture, a theory has to meet three criteria. First, it has to be logically permissible (*zulässig*),

62 "Mit Benutzung eines Ausdrucks von *Hertz* (In der Einleitung zu den *Prinzipien der Mechanik*) könnten wir unsere Hauptfrage so formulieren: Welches sind die notwendigen und hinreichenden und unter sich unabhängigen Bedingungen, denen man ein System von Dingen unterwerfen muss, damit jeder Eigenschaft dieser Dinge eine geometrische Tatsache entspreche und umgekehrt, damit also diese Dinge eine vollständiges und einfaches "Bild" der geometrischen Wirklichkeit seien?

Endlich können wir die Aufgabe als eine *logische Analyse unseres Anschauungsvermögens* bezeichnen" (Hilbert, 2004b, p. 303).

63 Hilbert introduces his lecture with the remark that it is necessary to uncover the "mutual relations" of the axioms and to "reduce their number" (Hilbert, 2004b, p. 302). Moreover, before the quoted passage, he asks which sentences must be added to the laws of logic to obtain Euclidean geometry (Hilbert, 2004b, p. 303). The quoted passage is meant to lead to an answer to this question, and, obviously, they are the axioms of geometry, which must be added.

64 Unlike a set, however, a system, which contains only one object, is identical to the object it contains (Dedekind, 1888, § 1).

65 The word "image" in the translation here and in what follows has been replaced by "picture," which is now generally accepted to be the better translation of "Bild" in this context.

i.e., consistent. Unlike Hilbert, Hertz did not have any formal tools to prove a theory's consistency.

Of course, being consistent is not sufficient for a theory to be a picture of reality. Therefore, the theory also has to be correct (*richtig*), i.e., in accordance with empirical observations. This is the most important criterion. It is also expressed in the aforementioned definition of a picture "that the necessary consequent of the [picture]s in thought are always the [picture]s of the necessary consequences in nature of the things pictured" (Hertz, 1899, p. 1).

Different, mutually inconsistent theories can meet this criterion. In fact, at the time, there were several theories of mechanics that all met this criterion. In his book, Hertz compares the classical Newtonian mechanics, the mechanics of Helmholtz, and his own with each other.

Consequently, there can be several theories that are all correct. To pick one of several equally correct theories, one must have a different criterion. This third criterion is called "appropriateness" ("Zweckmäßigkeit") (Hertz, 1899, p. 2). A theory is appropriate if it is more "distinct" (i.e., if it "pictures more of the essential relations of the object"). Of two equally distinct pictures, the simpler picture (i.e., the one which has "the smaller number of superfluous or empty relations") is more appropriate (Hertz, 1899, p. 2). Whether a particular picture is appropriate or not is, however, disputable. Unsurprisingly, Hertz argues in *Prinzipien der Mechanik* that his own theory for mechanics is the most appropriate one.

Utilizing Hertz's expression of "picture" in the context of geometry, Hilbert extends Hertz's picture theory beyond its original scope—physics—to geometry. Hilbert discusses all three criteria for being a picture when evaluating his axiom systems for geometry. Both Frege and Hilbert would agree on the first criterion: a geometrical axiom system must be consistent in order to be considered to be scientific at all.

The criterion of correctness, however, marks an important difference to Frege's convictions about sciences. For Frege, it is crucial that we want to express true thoughts with scientific sentences.[66] For Frege, there are also sentences which express thoughts that are neither true nor false, like "Odysseus was ashore at Ithaca sound asleep,"[67] but these sentences belong to poetry, not science, because "Odysseus" does not refer to a real human. The logical laws tell us that if a sentence is true, its negation has to be false. Therefore, if the axiom of parallels is true, its negation has to be false. However, if the axiom of parallels had no truth value at all, it would not be a scientific sentence but belong instead to poetry.

As Ulrich Majer points out, "Hertz was the first to notice that not all scientific sentences are either true or false, but some of them have a rather different and very peculiar relation to reality" (Majer, 1998, p. 235). These are the theoretical sentences of our theory. They are correct, not true, in the Fregean sense. The theoretical terms

66 Blanchette investigates the consequences of Frege's understanding of thoughts for his reception of Hilbert's consistency and independence proofs. She points out that, for Frege, consistency holds between thoughts, not merely sentences (Blanchette, 1996, p. 322). Thoughts can be expressed by different sentences. Thus, consistency and independence proofs, which Hilbert presents, are no proof for the consistency and independence of thoughts (Blanchette, 1996, p. 325).

67 Frege presents this example in "Über Sinn und Bedeutung" (Frege 1892, p. 30, 1984b, p. 163). Frege also mentions that it *is* true or false if "Odysseus" refers to a real person.

like "hidden mass" do not refer to a certain entity. Still, they fulfill the crucial task of enabling us to set up a theory where "the necessary consequents of the images in thought are always the images of the necessary consequences in nature of the things pictured" (Hertz, 1899, 1f.). Consequently, for Hertz and Hilbert, the fact that a sentence includes words that do not refer is not a sign of it being non-scientific.

In this context, we also have to read Hilbert's claim that one could substitute the words "point," "line," and "plane" with the words "love," "law," and "chimney sweep." Hilbert was not interested in the philosophy of language, and neither did he conduct any own research on logic at this time.[68] It is the picture theory of scientific theories that lies behind this claim.[69] For Hilbert, there are no such things as "real" points, planes, and lines, just as there are no real hidden masses. They are theoretical concepts that enable us to set up scientific theories that meet the correctness criteria. They have no existence outside of their role in the theory.

As we have seen above, appropriateness is a criterion that is only applicable if our theories are correct but not true, since different theories can be correct (while, for Frege, only one can be true). In his "Lectures on the Foundations of Geometry" from 1894, the year in which Hertz's *Prinzipien der Mechanik* was published,[70] Hilbert evaluates the Gaussian experience mentioned above, in which one measures the inner triangle sum of astronomical triangles in order to determine the geometrical form of physical space. After criticizing Lotze for neglecting the possibility (i.e., absence of contradictions) of non-Euclidean geometries, he discusses the Gaussian experiment in the following way:

> Of course, to come back to Lotze's prejudice, no experiment can force us to acknowledge hyperbolic geometry. Rather, even if the angle sum [of a triangle, TR] appears from experiment to be $< \pi$, it is always possible to get by with the usual Euclidean space. [...] Still, it would be simpler, more transparent, and would need fewer axioms, if under these circumstances we were to postulate the hyperbolic nature of our space.[71]

From this quote, we can conclude two things. Firstly, Hilbert acknowledges the possibility of classifying different geometries as correct, although one might be more appropriate. Thus, they are not "true" in the Fregean sense.

Secondly, Hilbert, at this time, disagrees with the Gaussian idea that one could determine the geometrical form of our universe by experiment. If we measure triangle sums smaller than π, it would be "easier, more transparent, and it would need fewer axioms" if we were to use hyperbolic, rather than Euclidean, geometry. This is a Hertzian idea applied to geometry. The hyperbolic geometry, as well as the Euclidean

[68] As mentioned before, Hilbert relied on Dedekind's work at the time he wrote the *Festschrift*.

[69] Of course, this picture theory entails certain claims on the philosophy of language.

[70] We know that Hilbert was already familiar with Hertz at this time because he mentioned Hertz's picture theory in this lecture (Hilbert, 2004a, p. 79).

[71] "Natürlich, um auf Lotze's Vorurteil zurückzukommen, kann uns kein Experiment zwingen, die hyperbolische Geometrie anzuerkennen. Vielmehr ist es, falls sich experimentell die Winkelsumme $< \pi$ herausstellen sollte, immer noch möglich, mit dem üblichen Euklidischen Raum auszukommen. [...] Nun, einfacher, durchsichtiger wird es und es bedarf weniger Axiome, wenn wir unter solchen Umständen die hyperbolische Natur unseres Raumes postulieren würden" (Hilbert, 2004a, p. 120).

geometry, could be part of a correct picture of space. However, claiming that space is hyperbolic would be the more "appropriate" choice since it is simpler.

Likewise, Hilbert rejects the idea that one could ultimately determine the physical laws of our universe. He claims that nobody could force us to assume the Copernican model since one could also choose the Ptolomean model. However, the description would be more complicated with the Ptolomean model.[72]

Thus, Hilbert agrees with Frege in the rejection of Gauss. Empirical data cannot falsify geometrical axioms. Even if we measure triangle sums smaller than π, we are not *forced* to accept hyperbolic geometry. However, unlike Frege, Hilbert admits that empirical data influence our choice of an axiom system for geometry.[73] He, therefore, rejects the Fregean idea of a hierarchical order of geometry and physics.

This shows that Hilbert and Frege have different convictions regarding not just geometry but axiomatic theories in general. As seen in Sect. 3.2, Frege claims that geometrical axioms cannot be refuted by physical experiments because they express truths about the knowledge source (*Anschauung*) on which physics also rests. For Hilbert, *Anschauung* is not a knowledge source for the axioms but an object of axiomatic analysis, which we have to presuppose to gain a "picture" of reality.

This idea can be illustrated with an example. Hilbert (1899) presents six groups of axioms in his *Festschrift*. In what follows, we will examine the meaning of "analysis of *Anschauung*" for one example—namely, the group of axioms of congruence. The group of the axioms of congruence consists of five axioms. The first three concern line segments, while the last two concern angles. These axioms demand that segments and angles can be moved in space without changing their size or shape. This is important because these axioms are not fulfilled in a space with non-constant curvature. As we have seen in the first section, this property was first identified by Riemann in his

[72] In fact, the Copernican revolution is a commonly used example to show that physical theories are not simply abandoned if experiments and mathematics show that another theory is simpler. In *The Copernican Revolution* Thomas Kuhn shows that the Copernican revolution was part of a modern shift of "man's relation to the universe and to god" (Kuhn, 1985, p. 2).

[73] In this respect, Hilbert's position also differs significantly from Poincaré's conventionalism. Poincaré also discusses the mathematical example of measuring a triangle in his article "Les Géométries Non Euclidiennes" (1891, p. 774). Just like Hertz and Hilbert, he stresses that geometrical axioms cannot be falsified by intuition. However, Poincaré attributes no role to empiricism when it comes to choosing the "most appropriate" ("plus commode") geometry. Instead, it is purely an inner mathematical choice, i. e., a "convention." Poincaré argues that Euclidean geometry is the simplest one from a mathematical point of view.

In the passage quoted above, Hilbert, in contrast, argues for the choice of hyperbolic geometry because this geometry is simpler and more transparent *under these circumstances* (i. e. from the empirical viewpoint). It is simpler because it allows us to set up a simpler physical theory that is in accordance with the empirical results.

In later writings, Hilbert stresses the role of the empirical in geometry in theoretical physics even more. For example, in his lecture "Grundsätzliche Fragen der modernen Physik" from 1923, Hilbert harshly criticizes Poincaré's conventionalism, which attributes no role to empiricism when it comes to deciding which geometry to take as a mathematical framework for physical theories: "According to Poincaré's analogy, it would be as if someone slipped a pancake recipe between the axioms and wanted thereby to establish the importance of the art of cooking for logic. Conventionalism has led to utter confusion: the ultimate consequence of Poincaré's point of view is that there are not any laws of nature at all" ("Nach Analogie von Poincaré wäre es, als wenn jemand ein Eierkuchenrezept zwischen die Axiome schieben und dann die Bedeutung der Kochkunst für die Logik damit begründen wollte. Der Konventionalismus hat heillose Konfusion angestiftet: Die letzte Konsequenz der Poincaréschen Ansicht ist, dass sich überhaupt keinerlei Naturgesetze finden lassen") (Hilbert, 2009, p. 430).

Habilitationsvortrag. Riemann points out that in a space of constant curvature, "every line can be measured by every other [line]" ("dass [...] jede Linie durch jede messbar sei"). This means precisely that one can take a segment and compare it in size to any other line anywhere in space.

One can further illustrate this property by saying that a segment should not change its form or size when moved in space. This was illustrated by Helmholtz in his paper "Über den Ursprung und die Bedeutung geometrischer Axiome," which uses the example of flat beings living on the surface of an egg. An egg has a higher curvature on the top than in the middle. Thus, a triangle with sides of a certain size would have a lower angle sum on the top than in the middle. Therefore, one cannot move a triangle on the surface of an egg without changing its shape (von Helmholtz, 1883, pp. 8–10). Thus, Hilbert's axioms do not hold for these fictional beings.

Hemholtz comments on his mathematical investigation in his paper in a way which is very interesting for our understanding of Hilbert:

> I offer these remarks, at first only to show what difficulties attend the complete *analysis* of the presuppositions we make, in employing the common *intuitive method* [*anschauliche Methode*]. We evade them when we apply, to the investigation of principles, the analytical method of modern algebraical geometry (Helmholtz, 1870, p. 7, emphasis added).[74]

The formulation "*analysis* of the presuppositions we make, in employing the common *intuitive method* [*Analyse der anschaulichen Methode*]" has an interesting similarity to Hilbert's notions of "analysis of *Anschauung*" and "analysis of our *Anschauungsvermögen*." Both authors want to further *analyze* what is traditionally (e. g., by Frege and Kant) taken to be the ultimate foundation on which our geometrical axioms rest, i. e., "Anschauung." For both authors, this means that we have to make explicit what distinguishes Euclidean geometry from other geometries. Therefore, it seems plausible that Hilbert borrowed this expression from Helmholtz.[75]

The thought experiment with the beings living on the surface of an egg shows that the property of a space that figures can be moved in all directions without changing their shape is not a logical necessity. Therefore, the property needs to be explicit in the axiomatization.

The analysis of *Anschauung* via axiomatizations aims for this kind of understanding of what distinguishes a correct picture of reality from an axiom set that does not fulfill this property of correctness. Hilbert explicitly puts this axiomatic approach to geometry in relation to the analytic and synthetic approach by claiming that "[i]n both disciplines these principle questions are not treated of" and supports his claim

[74] Translation from Atkinson with minor changes (von Helmholtz, 1893, pp. 33–34). "Ich führe diese Überlegungen hier zunächst nur an, um klar zu machen, auf welche Schwierigkeiten wir bei der vollständigen *Analyse* aller von uns gemachten Voraussetzungen nach der *Methode der Anschauung* stoßen. Ihnen entgehen wir, wenn wir die von der neueren rechnenden Geometrie ausgearbeitete analytische Methode auf die Untersuchung der Principien anwenden." (Emphasis added.) Atkinson translates "anschauliche Methode" as "constructive method." Thus, the striking similarity to Hilbert's notion of "Analyse der Anschauung" vanishes.

[75] Nevertheless, one should not overemphasize the parallel between Hilbert and Helmholtz. Hilbert seeks an axiomatic foundation of Euclidean geometry, and Helmholtz develops an analytic geometry. Furthermore, Helmholtz seems much more critical about the intuitive methods (*anschauliche Methode*) than Hilbert.

by pointing out that "[i]n analytic geometry one starts with the introduction of number." As seen in Sect. 2.2, this was criticized by synthetic geometers, who developed a geometry that does not use metric notions. We have also seen that Hilbert broadly followed von Staudt in his lectures in the early 1890s. After adopting his own geometrical approach, Hilbert criticizes synthetic geometry thus: "In projective geometry one appeals to intuition from the start, whereas *we want to analyze the intuition, in order, so to speak, to reconstruct it from its individual components*."[76] (Emphasis added.) "Analysis" (of *Anschauung*), thus, has to be understood literally as "breaking up" (which is the meaning of the Greek word "ανάλυσις") into particles, in order to identify the "necessary and sufficient conditions" to obtain (mediated by the system of things) a "picture of geometrical reality." It is crucial for this project that axioms are not perceived as true sentences; otherwise, they could not be understood as particles, which are the result of our analysis and which may form entire different geometries when combined with other particles. Hilbert thereby goes beyond von Staudt, his former idol, and also Frege, who both simply presuppose *Anschauung* as given.

Despite Hilbert's use of Kantian terms like *Anschauung* and his argument against Gauss's anti-Kantian philosophy of geometry, Hilbert is obviously not a full-blooded Kantian. As we pointed out earlier, for Kant, it would make no sense to challenge geometrical claims through physical experiments. However, Hilbert—and Hertz—influenced the neo-Kantianism of Cassirer with the idea of different "pictures" of the world. As early as 1910 in *Substanzbegriff und Funktionsbegriff*, but also later in *Einsteins Relativitätstheorie* (1921) and in the third volume of *Philosophie der symbolischen Form*, *Phänomenologie der Erkenntnis* (1928), Cassirer praises Hertz and Hilbert for recognizing the *creative* character of theoretical concepts in physics and mathematics. Mathematical concepts are "the intellectual establishment of a *constructive* connection"; they cannot be gained from "physically present bodies" by "simple 'abstraction" (Cassirer, 1923, p. 12). This idea "receives its clearest expression" in Hilbert's axiomatization (Cassirer, 1923, p. 93). Later in his *Philosophy of Symbolic Forms*, he extends this Hertzian–Hilbertian idea to the scope of human culture, including art, religion, and myth. They "[a]ll live in particular [picture]-worlds [Bildwelten], which do not merely reflect the empirically given, but which rather produce it in accordance with an independent principle" (Cassirer, 1980, p. 78).[77] Symbols are "created by the intellect itself;" therefore, they are no "passive images" ("passive Abbilder")[78] (Cassirer, 1980, p. 75) and differ from the Kantian notion of *Anschauungsform* insofar as we can create different symbols.

What makes this approach neo-*Kantian* is that the object of experience depends on the subject of experience. We find this idea in Hilbert and Hertz, who acknowledge

76 "In beiden Disciplinen werden die prinzipiellen Fragen nicht behandelt; in der analytischen Geometrie beginnt man mit der Einführung der Zahl, [...]; in der projektiven Geometrie appeliert man von vornherein an die Anschauung, wogegen wir ja die Anschauung analysieren wollen, um sie dann sozusagen aus ihren einzelnen Bestandteilen wieder aufzubauen" (Hilbert, 2004b, p. 303, p. 3 in the manuscript).

77 Again, "image" is changed to "picture" in the translation.

78 Both German words "Bild" and "Abbild" are translated as "images." This translation is misleading since both German words are opposed. We create "Bilder," while the "Abbilder" are mere copies of what is there. Therefore, here "picture" would have been once again the more suitable translation for the German word "Bild."

that our theory/axioms/picture frames our knowledge of the outer world, but not in Frege, who is less Kantian in this respect. What makes this *neo*-Kantian, i.e., breaking with Kantian conviction, is that the subject can deliberately change the form of the experience, something Cassirer takes from Hertz's picture theory and its reception in Hilbert. In this respect, Frege is the more conservative Kantian.

4 Conclusion

We have seen in this paper that Frege and Hilbert came from similar mathematical backgrounds. Their controversy cannot simply be explained by Frege's ignorance, because Frege was well aware of the developments in the mathematics of the time and did not dismiss them. The Frege–Hilbert controversy is rooted in a disagreement that goes well beyond the subject matter of geometry and logic. It concerns the general question of how our theoretical terms can relate to the real world and how we can generate theoretical knowledge.

Similarly, the controversy cannot be explained as a disagreement on issues of logic or philosophy of language. Instead, Hilbert developed his non-referential understanding of axioms motivated by epistemological concerns, which were influenced by Hertz's picture theory of physics. Thus, Hilbert's model-theoretic logic emerged from this epistemological perspective, not the other way around. Keeping this in mind might help avoid an anachronistic reading of the Frege–Hilbert controversy.

In this paper, we focused on the Hilbert of the 1890s. As Corry (2006) already pointed out, Hilbert's reference to *Anschauung* shifted after he started working on the general theory of relativity in a more empiricist direction. The question of why and how precisely Hilbert's philosophical convictions changed in that period is an interesting one. Its answer would help to get a unified picture of Hilbert's rich and diverse work. However, it is a research question that goes beyond the scope of this paper.

Acknowledgements This article benefited from the helpful comments of Andrew Arana, Adrien Champougny, Richard Lawrance and two anonymous reviewers.

Funding The research leading to these results received funding from *Deutsche Forschungsgemeinschaft* (DFG) under Grant Agreement No. 426133767 and from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (Grant Agreement No. 850539).

References

Blanchette, P. A. (1996). Frege and Hilbert on consistency. *The Journal of Philosophy, 93*(7), 317–336.

Blanchette, P. (2018). The Frege-Hilbert controversy. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy* (Fall 2018 Ed.). Metaphysics Research Lab, Stanford University.

Boyer, C. (2004). *History of analytic geometry*. Dover Publications.

Cassirer, E. (1923). Substance and function. In *Substance and function and Einstein's theory of relativity* (pp. 1–346). The Open Court Publishing Company.

Cassirer, E. (1980). *The philosophy of symbolic forms* (R. Manheim, Trans.). Yale University Press.

Clebsch, A. (1876). *Vorlesungen über Geometrie* (Vol. 1 (Geometrie der Ebene)) (F. Lindemann, Ed.). Teubner.

Coffa, A. (1986). Geometry and generality in Frege's philosophy of arithmetic. In R. G. Colodny (Ed.), *From quarks to quasars: Philosophical problems of modern physics* (pp. 3–70).

Corry, L. (2006). Axiomatics, empiricism, and anschauung in Hilbert's conception of geometry: Between arithmetic and general relativity. In J. F. Domínguez & J. Gray (Eds.), *The architecture of modern mathematics: Essays in history and philosophy* (pp. 133–156). Oxford University Press.

Dedekind, R. (1888). *Was sind und was sollen die Zahlen?* Vieweg.

Dummett, M. (1982). Frege and Kant on geometry. *Inquiry, 25*(2), 233–254.

Eder, G. (2019). Frege and the origins of model theory in nineteenth century geometry. *Synthese, 198*(6), 5547–5575.

Eder, G. (2021). Frege on intuition and objecthood in projective geometry. *Synthese, 199*(3–4), 6523–6561.

Eder, G., & Schiemer, G. (2018). Hilbert, duality, and the geometrical roots of model theory. *The Review of Symbolic Logic, 11*(1), 48–86.

Frege, G. (1873). Über eine Geometrische Darstellung der imaginären Gebilde in der Ebene. A. N. Jena

Frege, G. (1874/1875). Analytische Geometrie nach neueren Methoden. In H. Hermes, F. Kambartel, & F. Kaulbach (Eds.), *Nachgelassene Schriften* (pp. 347–364). Meiner.

Frege, G. (1877). Über Invarianten. In H. Hermes, F. Kambartel, & F. Kaulbach (Eds.), *Nachgelassene Schriften* (pp. 378–380). Meiner.

Frege, G. (1892). Über Sinn und Bedeutung. *Zeitschrift für Philosophie und philosophische Kritik* 25–50.

Frege, G. (1903). Über die Grundlagen der Geometrie. *Jahres-Bericht der deutschen Mathematiker-Vereinigung 12*, 319–324.

Frege, G. (1979a). On Euclidean geometry. In H. Hermes, F. Kambartel, & F. Kaulbach (Eds.), *Posthumous writings* (pp. 167–169). University of Chicago Press.

Frege, G. (1979b). Sources of knowledge of mathematics and the mathematical natural sciences. In H. Hermes, F. Kambartel, & F. Kaulbach (Eds.), *Posthumous writings* (pp. 267–274). University of Chicago Press.

Frege, G. (1980). Philosophical and mathematical correspondence. In H. Hermes, F. Kambartel, & F. Kaulbach (Eds.). University of Chicago Press.

Frege, G. (1984a). On a geometrical representation of imaginary forms in the plane. In B. McGuinness (Ed.), *Collected papers on mathematics, logic and philosophy* (pp. 1–55). Basil Blackwell.

Frege, G. (1984b). On sense and meaning. In B. McGuinness (Ed.), *Collected papers on mathematics, logic and philosophy* (pp. 157–177). Basil Blackwell.

Frege, G. (1999). *The foundations of arithmetic?: A logico-mathematical inquiry into the concept of number*. Northwestern University Press.

Freudenthal, H. (1962). The main trends in the foundations of geometry in the 19th century. In E. Nagel, P. Suppes, & A. Tarski (Eds.), *Logic, methodology, and philosophy of science. Proceedings of 1960 international congress* (pp. 613–621). Stanford University Press.

Friedman, M. (1990). *Kant and the exact sciences*. Harvard University Press.

Gauß, C. F. (1828). Disquisitiones generales circa superficies. Göttingen Typis Dieterichianis.

Gauß, C. F. (1880). Gauß an Bessel. Göttingen den 9. April 1830. In Briefwechsel zwischen Gauß und Bessell (pp. 495–499). Verlag von Wilhelm Engelmann. Edited on behalf of the königlich preußischen Akademie der Wissenschaften.

Gergonne, J. (1826). Philosophie mathématique. Considérations philosophiques sur les élémens de la science de l'étendue. *Annales de mathématiques pures et appliquées, 16*, 209–231.

Giovannini, E. (2016). Bridging the gap between analytic and synthetic geometry: Hilbert's axiomatic approach. *Synthese, 193*(1), 31–70.

Gray, J. (2010). *Worlds out of nothing: A course in the history of geometry in the 19th century*. Springer.

Hertz, H. (1899). *The principles of mechanics*. Macmillan and Co.
Hilbert, D. (2004a). Die Grundlagen der Geometrie (1894). In M. Hallett & U. Majer (Eds.), David Hilbert's lectures on the foundations of geometry 1891–1902 (pp. 72–127). Springer.
Hilbert, D. (2004b). Elemente der eukldischen Geometrie (1898–99). In M. Hallett & U. Majer (Eds.), David Hilbert's lectures on the foundations of geometry 1891–1902 (pp. 302–395). Springer.
Hilbert, D. (2004c). Grundlagen der Geometrie (1902). In M. Hallett & U. Majer (Eds.), David Hilbert's lectures on the foundations of geometry 1891–1902 (pp. 540–606). Springer.
Hilbert, D. (2004d). Projective Geometrie (1891). In M. Hallett & U. Majer (Eds.), David Hilbert's lectures on the foundations of geometry 1891–1902 (pp. 21–62). Springer.
Hilbert, D. (2009). Grundsätzliche fragen der modernen physik (1923). In T. Sauer & U. Majer (Eds.), David Hilbert's lectures on the foundations of physics 1915–1927 (pp. 396–432). Springer.
Kant, I. (1929). *Critique of pure reason* (N. K. Smith, Trans). Palgrave Macmillan.
Klein, F. (1926). Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert. Springer.
Klein, F. (1928). Vorlesungen über nicht-euklidische Geometrie. Springer.
Kreiser. (2001). Gottlob frege: Leben-Zeit-Werk. Meiner.
Kuhn, T. (1985). *The Copernican revolution. Planetary astronomy in the development of western thought*. Harvard University Press.
Lorenat, J. (2015a). Figures real, imagined, and missing in Poncelet, Plücker, and Gergonne. *Historia Mathematica, 42*(2), 155–192.
Lorenat, J. (2015b). Polemics in public: Poncelet, Gergonne, Plücker, and the duality controversy. *Science in Context, 28*(4), 545–585.
Lorenat, J. (2016). Synthetic and analytic geometries in the publications of Jakob Steiner and Julius Plücker (1827–1829). *Archive for History of Exact Sciences, 70*(4), 413–462.
Lotze, H. (1884). *Metaphysics in three books: Ontology, cosmology and psychology*. The Clarendon Press.
Majer, U. (1998). Heinrich hertz's picture-conception of theories: Its elaboration by Hilbert, Weyl, and Ramsey. In D. Baird, R. Hughes, & A. Nordmann (Eds.), *Heinrich hertz: Classical physicist, modem philosopher* (pp. 225–242). Springer.
Mancosu, P. (2016). *Abstraction and infinity*. Oxford University Press.
Monge, G. (1799). *Géométrie descriptive. leçons données aux écoles normales*. Baudouin.
Nabonnand, P. (2008). La théorie des "würfe" de von staudt - une irruption de l'algèbre dans la géométrie pure. *Archive for History of Exact Sciences, 62*(3), 201–241.
Plücker, J. (1830). *Über ein neues Coordinatensystem., 5*(1), 1–36.
Plücker, J. (1831). *Analytisch-geometrische Entwicklungen* (Vol. 2). Baedeker.
Plücker, J. (1868). *Neue geometrie des raumes: gegründet auf die betrachtung der geraden linie als raumelement*. Teubner.
Poncelet, J. V. (1822). *Traité des propriétés projectives des figures*. Bachelier.
Riemann, B. (1868). Über die Hypothesen, welche der Geometrie zu Grunde liegen. Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen, 13, 133–150.
Rohr, T. (2020). Freges Begriff der Logik. Mentis.
Schirn, M. (2019). Frege's philosophy of geometry. *Synthese, 196*(3), 929–971.
Shipley, J. (2015). Frege on the foundation of geometry in intuition. *Journal for the History of Analytical Philosophy, 3*(6), 1–23.
Tappenden, J. (1995a). Extending knowledge and "fruitful concepts": Fregean themes in the foundations of mathematics. *Noûs, 29*(4), 427–467.
Tappenden, J. (1995b). Geometry and generality in Frege's philosophy of arithmetic. *Synthese, 102*(3), 319–361.
Tappenden, J. (2006). The Riemannian background to Frege's philosophy. In J. Ferreirós & J. J. Gray (Eds.), *The architecture of modern mathematics: Essays in history and philosophy* (pp. 107–150). Oxford University Press.
von Helmholtz, H. (1883). Über den Ursprung und die Bedeutung der geometrischen Axiome : Vortrag gehalten im Dozentenverein zu Heidelberg 1870. In Helmholtz, Herrmann: Vorträge und Reden (Vol. 2, pp. 1–31). Vieweg.
von Helmholtz, H. (1893). On the origin and significance of geometrical axioms. In *Popular lectures on scientific subjects* (E. Atkinson, Trans.) (Vol. 2, pp. 27–72). Longmans, Greens and co.
von Staudt, K. G. C. (1847). *Geometrie der Lage*. Friedrich Kon'sche Buchhandlung.
von Staudt, K. G. C. (1856). *Beiträge zur Geometrie der Lage*. Friedrich Kon'sche Buchhandlung.
Wilson, M. (1992). Frege: The royal road from geometry. *Noûs, 39*(2), 149–180.

Wilson, M. (2005). Ghost world: A context for Frege's context principle. In E. Reck & M. Beaney (Eds.), *Frege: Critical assessments of leading philosophers.* (Vol. 3). Routledge.

Wilson, M. (2010). Frege's mathematical setting. In M. Potter & T. Ricketts (Eds.), *The Cambridge companion to Frege* (pp. 379–412). Cambridge University Press.

Wolff, M. (1995). *Die Vollständigkeit der kantischen Urteilstafel: mit einem Essay über Freges Begriffsschrift*. Klostermann.