FINITE INDEX SUBGROUPS IN CHEVALLEY GROUPS ARE BOUNDED: AN ADDENDUM TO "ON BI-INVARIANT WORD METRICS"

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ABSTRACT. We prove that subgroups commensurable with S-arithmetic Chevalley groups are bounded.

1. Introduction

A group G is called *bounded* if every conjugation invariant norm on G has finite diameter. Examples of bounded groups include $\mathrm{SL}_n(\mathbf{Z})$ for $n \geq 3$, $\mathrm{Diff}_0(M)$, where M is a manifold of dimension different from 2 and 4, the commutator subgroup of Thompson's group F and many others. A finite index subgroup of a bounded group does not have to be bounded. The simplest example is the infinite cyclic subgroup of the infinite dihedral group. The purpose of this note is to prove the following result.

Theorem. *Let G be a Chevalley group over the ring of S-integers in a number field $\mathbf{k}$ constructed from a root system whose irreducible components all have rank at least 2. If H is commensurable with G then it is bounded.*

The above theorem generalises the main result of the paper [7]. The proof is similar with an additional ingredient being an explicit form of bounded generation of a finite index subgroup of a boundedly generated group. We also correct a couple of mistakes from the original proof. First, the reduction to rank two sublattices in the proof of [7, Theorem 1.1] needs an extra argument if α is not contained in a rank two root subsystem isomorphic to A_2. This is done in Lemma 1 of the current paper. Second, the same proof in [7] erroneously assumed that $(\mathcal{O}_S, +)$ is always a finitely generated abelian group. Lemma 2 of the current paper fixes the resulting gap in the proof.

Motivation and context. The study of general conjugation-invariant norms have several sources [5]. In finite groups there is a well studied notion of a covering number [6]. Moreover, generation by conjugacy classes of finite simple groups has been extensively investigated [10, 11]. In symplectic geometry, there is a natural conjugation-invariant norm, called the Hofer norm [14], on the Hamiltonian transformations of a symplectic manifold. General conjugation-invariant norms can be used in understanding Hamiltonian group actions on symplectic manifolds. For example, in [9] it is shown that certain bounded groups don't admit Hamiltonian actions on symplectic manifolds. In differential topology diffeomorphisms of manifolds have *fragmentation* property [2]. That is, they can be expressed as composition of diffeomorphisms supported in balls, for example. Investigation how complicated such decompositions are can be done in the framework of conjugation invariant norms [4].

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2. Definitions and known facts

2.1. **Norm.** A *conjugation invariant norm* of a group G is a nonnegative function $\nu\colon G \to \mathbf{R}$ such that the following conditions

(1) $\nu(g) = 1$ if and only if $g = 1_G$,

(2) $\nu\left(g^{-1}\right) = \nu(g)$,

(3) $\nu(gh) \leq \nu(g) + \nu(h)$,

(4) $\nu\left(h^{-1}gh\right) = \nu(g)$

hold for every $g, h \in G$.

2.2. **Bounded group.** A group G is called *bounded* if the diameter

$$\operatorname{diam}(G, \nu) = \sup\{\nu(g) \mid g \in G\} < \infty$$

for every conjugation invariant norm ν. If G is generated by finitely many conjugacy classes then its boundedness is equivalent to the boundedness of any conjugation invariant *word* norm [7, Section 2.C].

2.3. **Arithmetic group.** Let $\mathbf{k}$ be a number field (i.e. a finite extension of $\mathbf{Q}$) and let $V_{\mathbf{k}}$ denote the set of equivalence classes of valuations of $\mathbf{k}$. Let $S \subset V_{\mathbf{k}}$ be a finite set containing the set of all Archimedean valuations S_∞. The ring of S-integers is defined by

$$\mathcal{O}_S = \{x \in \mathbf{k} \mid v(x) \geq 0 \text{ for all } v \notin S\}.$$

In case S is precisely the set S_∞, we write $\mathcal{O}$ instead of $\mathcal{O}_{S_\infty}$. Let $\mathbf{G}$ be a connected algebraic group defined over $\mathbf{k}$ with a fixed $\mathbf{k}$-embedding $\mathbf{G} \to \mathrm{GL}_r(\mathbf{k})$. A subgroup of $\mathbf{G}$ that is commensurable with $\mathbf{G}(\mathcal{O}_S) = \mathbf{G} \cap \mathrm{GL}_r(\mathcal{O}_S)$ is called an *S-arithmetic group* [12, page 61]. More generally, an S-arithmetic group can be defined over any global field.

2.4. **Chevalley group** [1]**.** Let $\mathfrak{g}$ be a semi-simple complex Lie algebra over $\mathbf{C}$ with Cartan subalgebra $\mathfrak{h}$. Let Φ be a root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ with simple roots $\{\alpha_1, \dots, \alpha_k\} \subset \Phi$. Let

$$\{H_{\alpha_i}(1 \leq i \leq k); X_\alpha(\alpha \in \Phi)\}$$

be a Chevalley basis of the algebra $\mathfrak{g}$, and let $\mathfrak{g}_{\mathbf{Z}}$ be its linear envelope over $\mathbf{Z}$. Let $\varphi\colon \mathfrak{g} \to \mathfrak{gl}(r, \mathbf{C})$ be a faithful representation. There is a lattice $L \subset \mathbf{C}^r$ which is invariant with respect to all operators of the form $\varphi(X_\alpha)^m/m!$, where $m \in \mathbf{N}$. If $\mathbf{k}$ is an arbitrary field then homomorphisms $x_\alpha\colon (\mathbf{k}, +) \to \mathrm{GL}(L \otimes \mathbf{k})$ of the additive group of $\mathbf{k}$ into $\mathrm{GL}(L \otimes \mathbf{k})$ are defined and given by the formulas

$$x_\alpha(t) = \sum_{m=0}^{\infty} t^m \frac{\varphi(X_\alpha)^m}{m!}.$$

The subgroup $G(\Phi, \mathbf{k}) \subset \mathrm{GL}(L \otimes \mathbf{k})$ generated by $\{x_\alpha(t) : \alpha \in \Phi, t \in \mathbf{k}\}$, is called the *adjoint Chevalley group* associated with the root system Φ, the representation φ and the field $\mathbf{k}$. We will follow the custom to call those groups Chevalley group, for short. We make a remark about other Chevalley groups at the end of the paper.

2.5. **Chevalley's commutator formula.** The root elements of the Chevalley group $G(\Phi, \mathbf{k})$ satisfy the following relations:

$$\begin{aligned} x_\alpha(s)x_\alpha(t) &= x_\alpha(s+t) \\ [x_\alpha(s), x_\beta(t)] &= \prod_{i,j>0} x_{i\alpha+j\beta}(C_{i,j}t^i s^j) \end{aligned}$$

where the product is taken in the increasing order of $i + j > 0$ and $C_{i,j} \in \{\pm 1, \pm 2, \pm 3\}$ [8, Lemma 32.5, Propositions 33.3–5].

2.6. **Semisimple elements of Chevalley groups.** Besides the root elements one also has semisimple elements of $G(\Phi, \mathbf{k})$ which are defined as follows. Let $\alpha \in \Phi$ be a root, $0 \neq t \in \mathbf{k}$ and set

$$h_\alpha(t) := x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)x_\alpha(-1)x_{-\alpha}(1)x_\alpha(-1).$$

They are related to the root elements as follows. Let $\alpha, \beta \in \Phi$ be roots and let $\langle \beta, \alpha \rangle := 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}$ be a corresponding Cartan integer. Then the following equation holds

$$h_\alpha(t)x_\beta(u)h_\alpha(t)^{-1} = x_\beta(t^{\langle \beta,\alpha \rangle}u).$$

where $u, t \in \mathbf{k}$ and $t \neq 0$ [15, Lemma 20(c)].

2.7. **S-arithmetic Chevalley groups.** Let $G(\Phi, \mathbf{k})$ be a Chevalley group over a number field $\mathbf{k}$. We consider the S-arithmetic group $G(\Phi, \mathcal{O}_S)$ over the ring of S-integers $\mathcal{O}_S \subset \mathbf{k}$. Let $E(\Phi, \mathcal{O}_S) \subset G(\Phi, \mathcal{O}_S)$ be the subgroup generated by the root elements $x_\alpha(t)$, where $\alpha \in \Phi$ and $t \in \mathcal{O}_S$. This subgroup is called the *elementary S-arithmetic Chevalley Group* (or the *elementary subgroup of* $G(\Phi, \mathcal{O}_S)$). It is known that $E(\Phi, \mathcal{O}_S) = G(\Phi, \mathcal{O}_S)$ holds in the case of the rings of S-algebraic integers $\mathcal{O}_S$ if all irreducible components of Φ have rank at least 2 [13, Corollaire 4.6],[3, Theorem 3.6].

2.8. **Tavgen's theorem.** Let Φ be an irreducible root system of rank at least 2. If $G = E(\Phi, \mathcal{O}_S)$ is an elementary S-arithmetic Chevalley group then there exists a number $m \in \mathbf{N}$ and roots $\alpha_1, \ldots, \alpha_m \in \Phi$ such that every element $g \in G$ can be written as $g = x_{\alpha_1}(t_1) \cdots x_{\alpha_m}(t_m)$, for some $t_1, \ldots, t_m \in \mathcal{O}_S$ [16, Theorem A]. We say that G has *bounded generation with respect to root elements.*

3. Proof

First note that if H contains a finite index bounded subgroup H', then H itself is bounded, because for any conjugation-invariant norm ν defined on H one has for a given finite set of representatives T of left cosets of H' in H that

$$\operatorname{diam}(H, \nu) \leq \operatorname{diam}(H', \nu) + \max\{\nu(t) | t \in T\} < \infty.$$

This implies that it suffices to prove the statement for finite index subgroups of $G = E(\Phi, \mathcal{O}_S)$. Next observe that each finite index subgroup H of G contains a normal subgroup H' of finite index. Thus it suffices to consider only the case H of finite index and normal in G.

Lemma 1. *Let $G = E(\Phi, \mathcal{O})$ be the elementary arithmetic Chevalley group of rank at least 2 constructed from the irreducible root system Φ and let H be a finite index normal subgroup of G. Further let α be a root and ν a conjugation-invariant norm on H. Then the group $\{x_\alpha(a) | a \in \mathcal{O}\} \cap H$ is bounded with respect to ν.*

Proof. Let $\xi_0 = 1, \xi_1, \ldots, \xi_r \in \mathcal{O}$ be a basis of $\mathcal{O}$ over $\mathbf{Z}$, i.e. $\mathcal{O}$ splits as a direct product $\bigoplus_{i=0}^{r} \mathbf{Z}\xi_i$. Let $p \in \mathbf{N}$ be the smallest positive integer such that the elements $x_\delta(p\xi_l) \in H$ for all $\delta \in \Phi$ and $0 \leq l \leq r$. Now for $a \in \mathcal{O}$ observe that there are integers $m_0, \ldots, m_r$ such that $a = m_0 + m_1\xi_1 + \cdots + m_r\xi_r$. Now using division with remainder we can find integers n_l and r_l with $0 \leq r_l < p$ such that $m_l = pn_l + r_l$ for $0 \leq l \leq r$. But then we get

$$x_\alpha(a) = (x_\alpha(p\xi_0)^{n_0} \cdots x_\alpha(p\xi_r)^{n_r})x_\alpha(r_0\xi_0 + \cdots + r_r\xi_r).$$

Now observe that for the second factor there are only finitely many possibilities and thus it suffices to show that the cyclic subgroup generated by $x_\alpha(p\xi_l)$ for $0 \leq l \leq r$ is bounded with respect to ν. Using the same division with remainder trick for these subgroups again, it actually suffices to find a non-zero multiple v of p such that the cyclic subgroup generated by $x_\alpha(v\xi_l)$ for $0 \leq l \leq r$ is bounded with respect to ν. In the following let ξ be one of the ξ_l.

Now there exists a subsystem $\Psi \subset \Phi$ isomorphic to one of $\mathbf{A}_2$, $\mathbf{B}_2$ or $\mathbf{G}_2$ such that $\alpha \in \Psi$. Moreover, if $\alpha \in \mathbf{A}_2$ there exist $\beta, \gamma \in \mathbf{A}_2$ such that $\alpha = \beta + \gamma$ and that no other positive combination of β and γ is a root. The same holds if $\alpha \in \mathbf{B}_2$ or $\alpha \in \mathbf{G}_2$ is a long root. It then follows from (2.5) that

$$x_\alpha(p\xi)^{Cpn} = [x_\beta(p\xi), x_\gamma(pn)]$$

for some fixed $C \in \{\pm 1, \pm 2, \pm 3\}$ and any $n \in \mathbf{Z}$. In particular, we have that

$$\nu(x_\alpha(Cp^2\xi)^n) \leq 2\nu(x_\beta(p\xi))$$

which implies that $\nu(x_\alpha(p\xi)^n)$ is bounded independently of n if α is either a root contained in $\mathbf{A}_2$ or a long root in $\mathbf{B}_2$ or $\mathbf{G}_2$. Note in particular that this implies already that the set $\{x_\alpha(pa)^n | n \in \mathbf{Z}\}$ is bounded with respect to ν for all $a \in \mathcal{O}$ if α is a long root in $\mathbf{G}_2$ or $\mathbf{B}_2$.

If $\alpha \in \mathbf{B}_2$ is a short root then there exist $\beta, \gamma \in \mathbf{B}_2$ such that $\alpha = \beta + \gamma$ and $\beta + 2\gamma$ is a long root and no other positive combination of β and γ is a root. Applying (2.5) again, we get that

$$[x_\beta(p\xi), x_\gamma(pn)] = x_\alpha(p^2\xi Cn)x_{\beta+2\gamma}(p^3\xi C'n^2),$$
$$x_\alpha(p\xi)^{Cpn} = [x_\beta(p\xi), x_\gamma(pn)]\, x_{\beta+2\gamma}(-p^2\xi)^{C'pn^2}$$

for some fixed $C, C' \in \{\pm 1, \pm 2, \pm 3\}$ and any $n \in \mathbf{Z}$. This implies that

$$\nu(x_\alpha(p^2C\xi)^n) \leq 2\nu(x_\beta(p\xi)) + \nu(x_{\beta+2\gamma}(p\xi)^{-C'p^2n^2}).$$

However note that $\beta + 2\gamma$ is a long root and hence we already know that $\nu(x_{\beta+2\gamma}(p\xi)^{-C'p^2n})$ is bounded independently from n.

Now if $\alpha \in \mathbf{G}_2$ is a short root, then there are $\beta, \gamma \in \mathbf{G}_2$ such that $\beta + \gamma = \alpha$ and further $\beta + 2\gamma$ and $2\beta + \gamma$ are both long roots and no other positive combination of β and γ are roots. Applying (2.5) again, we get for any $n \in \mathbf{Z}$ that

$$[x_\beta(p\xi), x_\gamma(pn)] = x_\alpha(p^2\xi C n) x_{\beta+2\gamma}(p^3\xi C' n^2) x_{2\beta+\gamma}(p^3\xi^2 C'' n),$$
$$x_\alpha(p\xi)^{Cpn} = [x_\beta(p\xi), x_\gamma(pn)]\, x_{\beta+2\gamma}(-p^3\xi C' n^2) x_{2\beta+\gamma}(-p^3\xi^2 C'' n)$$

for some fixed $C, C', C'' \in \{\pm 1, \pm 2, \pm 3\}$, which implies that

$$\nu(x_\alpha(p^2 C\xi)^n) \leq 2\nu(x_\beta(p\xi)) + \nu(x_{\beta+2\gamma}(p\xi)^{-C'p^2n^2}) + \nu(x_{2\beta+\gamma}(p\xi^2)^{-p^2C''n}).$$

Now the terms $\nu(x_{\beta+2\gamma}(p\xi)^{-C'p^2n^2})$ and $\nu(x_{2\beta+\gamma}(p\xi^2)^{-p^2C''n})$ are bounded independently of n as $\beta + 2\gamma$ and $2\beta + \gamma$ are long roots in $\mathbf{G}_2$. This concludes the proof. □

Lemma 2. *Let $G = E(\Phi, \mathcal{O}_S)$ be an S-arithmetic Chevalley group of rank at least 2 constructed from the irreducible root system Φ and H a finite index normal subgroup of G. Further let ν be a conjugation-invariant norm on H. Then the set*

$$\{y x_\alpha(a) y^{-1} | a \in \mathcal{O}_S, y \in G, \alpha \in \Phi\} \cap H$$

is bounded with respect to ν.

Proof. We start with the observation that there exists an element $u \in \mathcal{O}^*$ such that $\mathcal{O}_S = \mathcal{O}[u^{-1}]$. Indeed, let $p_i \subseteq \mathcal{O}$ be the ideal corresponding to a non-archimedean valuation $v_i \in S$. Let $U = \prod_i p_i$. Since the class number of $\mathcal{O}$ is finite, there exists $n \in \mathbf{N}$ and $u \in \mathcal{O}$ such that $U^n = (u)$. Then for any $x \in \mathcal{O}_S$ we have that $u^k x \in \mathcal{O}$ for some $k \in \mathbf{N}$ which implies the claim.

Since there are only finitely many elements in Φ, it suffices to show the boundedness of the set

$$\{y x_\alpha(a) y^{-1} | a \in \mathcal{O}_S, y \in G\} \cap H$$

for a given root α. Since $\mathcal{O}_S = \mathcal{O}[u^{-1}]$, for each element $a \in \mathcal{O}_S$ there exists $k \in \mathbf{N}$ and $b \in \mathcal{O}$ such that $a = u^{-k}b$. Thus one gets by $\langle \alpha, \alpha \rangle = 2$ and (2.6) that $h_\alpha(u^{-k}) x_\alpha(u^k b) h_\alpha(u^k) = x_\alpha(u^{-k}b)$ holds. But H is normal and so it suffices to show boundedness for the set

$$\{y x_\alpha(a) y^{-1} | a \in \mathcal{O}, y \in G\} \cap H.$$

Let $T \subset G$ be a finite set of representatives of right cosets of H in G. Thus each $y \in G$ can be written as $y = ht$ for some $h \in H$ and $t \in T$. Next observe that if $y x_\alpha(a) y^{-1} \in H$ for $a \in \mathcal{O}$ then

$$\nu(y x_\alpha(a) y^{-1}) = \nu(h(t x_\alpha(a) t^{-1}) h^{-1}) = \nu(t x_\alpha(a) t^{-1}).$$

Since the set T is finite, it suffices to show that $\{tx_\alpha(a)t^{-1}|a \in \mathcal{O}\} \cap H$ is bounded with respect to ν for any given $t \in T$. As H is normal, the function

$$\nu_t\colon H \cap E(\Phi, \mathcal{O}) \to \mathbf{R}_{\geq 0}, x \mapsto \nu(txt^{-1})$$

defines a conjugation invariant norm on the finite index, normal subgroup $H \cap E(\Phi, \mathcal{O})$ of $E(\Phi, \mathcal{O})$. Thus Lemma 1 yields that

$$\{tx_\alpha(a)t^{-1}|a \in \mathcal{O}\} \cap H$$

is indeed bounded with respect to ν. $\square$

Proof of Theorem. We first prove the special case for G constructed from an irreducible root system Φ.

For each $\alpha \in \Phi$ let $E_\alpha := \{x_\alpha(a)|a \in \mathcal{O}_S\}$ be given. Now $H \cap E_\alpha$ has finite index in E_α, so let T_α be a finite set of representatives of right cosets of $H \cap E_\alpha$ in E_α. Now let $h \in H$ be given. By [16, Theorem A] there is an $m \in \mathbf{N}, \alpha_1, \ldots, \alpha_m \in \Phi$ (independent from h) and $e_i \in E_{\alpha_i}$ for $i = 1, \ldots, m$ (dependent on h) such that $h = e_1 \cdots e_m$. Now for each $i \leq m$ let $h_i \in E_{\alpha_i} \cap H$ and $t_i \in T_{\alpha_i}$ be given such that $e_i = h_i t_i$. Now note that

$$\begin{aligned} h = e_1 \cdots e_m &= \prod_{i=1}^m h_i t_i \\ &= \left(\prod_{i=1}^m (t_1 \cdots t_{i-1})\, h_i\, (t_1 \cdots t_{i-1})^{-1}\right) \cdot (t_1 \cdots t_m) \end{aligned}$$

Now by Lemma 2 all of the elements $(t_1 \cdots t_{i-1})\, h_i\, (t_1 \cdots t_{i-1})^{-1}$ are bounded independently from h and thus as m is also independent of h, the element

$$\prod_{i=1}^m (t_1 \cdots t_{i-1}) h_i (t_1 \cdots t_{i-1})^{-1}$$

is also bounded independently from h. Furthermore Φ is finite, all the sets T_α are finite, and thus there are only finitely many possibilities for the product $t_1 \cdots t_m$ (independently of h) and hence h has a bound not depending on itself. This finishes the proof for the irreducible case.

We now proceed with the general case. If $\Phi_1, \ldots, \Phi_m$ are the irreducible components of Φ, then G decomposes as the direct product $\prod_{i=1}^m G(\Phi_i, \mathcal{O}_S)$. Set $G_i := G(\Phi_i, \mathcal{O}_S)$ for $i = 1, \ldots, m$. But now consider the subgroups $H_i := H \cap G_i$ for $1 \leq i \leq m$ and $H' := \prod_{i=1}^m H_i$. Observe that the H_i have finite index in the G_i and thus H' has finite index in H. Hence it suffices to show the statement in the case that H is a product of finite index subgroups in the corresponding $G(\Phi_i, \mathcal{O}_S)$. But then applying [7, Proposition 3.9] inductively on the series of short exact sequences given by the direct product decomposition of H', it suffices to show the statement for $G = G(\Phi, \mathcal{O}_S)$ and H of finite index in G. Thus we are done. $\square$

Remark.

(1) The assumption on the components of Φ to have rank at least 2 is necessary. Namely the group $SL_2(\mathbf{Z})$ is known to be unbounded.

(2) Theorem 1 is proven for adjoint Chevalley groups, but is equally valid for other types of Chevalley groups like the simply connected ones. Without going into too much details, a possible proof strategy could be as follows: Any other type $G'(\Phi, \mathcal{O}_S)$ of Chevalley groups arise from a finite central extension $\pi : G'(\Phi, \mathcal{O}_S) \to G(\Phi, \mathcal{O}_S)$ of the adjoint type. Hence given a bi-invariant norm ν on $G'(\Phi, \mathcal{O}_S)$, one can define

$$\pi_*\nu(A) := \max\{\nu(X) \mid X \in \pi^{-1}(A)\}$$

for $A \in G(\Phi, \mathcal{O}_S)$ to obtain a bi-invariant norm $\pi_*\nu$ on $G(\Phi, \mathcal{O}_S)$ with the property that $\pi_*\nu(\pi(X)) \geq \nu(X)$ holds for all $X \in G'(\Phi, \mathcal{O}_S)$. But $G(\Phi, \mathcal{O}_S)$ is bounded and so $\pi_*\nu$ has bounded diameter and so ν has bounded diameter, too. The finite index cases work similarly.

(3) Theorem 1 is stated for arithmetic Chevalley groups arising from rings of S-integers $\mathcal{O}_S$ in number fields. But the theorem is also valid for arithmetic Chevalley groups arising from rings of S-algebraic integers in global function fields. The proofs work essentially the same barring some complications that arise in the case that the global function field K has characteristic 2 and the fact that Tavgen's Theorem 2.8 has to be replaced by [17, Theorem 1.3].

References

[1] Chevalley group. *Encyclopedia of Mathematics*. Available at `https://www.encyclopediaofmath.org/index.php/Chevalley_group`.

[2] Augustin Banyaga. *The structure of classical diffeomorphism groups*, volume 400 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1997.

[3] H. Bass, J. Milnor, and J.-P. Serre. Solution of the congruence subgroup problem for $\mathrm{SL}_n\,(n \geq 3)$ and $\mathrm{Sp}_{2n}\,(n \geq 2)$. *Inst. Hautes Études Sci. Publ. Math.*, (33):59–137, 1967.

[4] Michael Brandenbursky and Jarek Kędra. Fragmentation norm and relative quasimorphisms. *Procedings of the AMS, to appear*, 2022. Available at https://arxiv.org/pdf/1804.06592.

[5] Dmitri Burago, Sergei Ivanov, and Leonid Polterovich. Conjugation-invariant norms on groups of geometric origin. In *Groups of diffeomorphisms*, volume 52 of *Adv. Stud. Pure Math.*, pages 221–250. Math. Soc. Japan, Tokyo, 2008.

[6] Erich W. Ellers, Nikolai Gordeev, and Marcel Herzog. Covering numbers for Chevalley groups. *Israel J. Math.*, 111:339–372, 1999.

[7] Światosław R. Gal and Jarek Kędra. On bi-invariant word metrics. *J. Topol. Anal.*, 3(2):161–175, 2011.

[8] James E. Humphreys. *Linear algebraic groups, corrected fifth printing.* Springer-Verlag, New York-Heidelberg, 1975. Graduate Texts in Mathematics, No. 21.

[9] Jarek Kędra, Assaf Libman, and Ben Martin. On boundedness properties of groups. *Journal of Topology and Analysis*, 2022.

[10] R. Lawther and Martin W. Liebeck. On the diameter of a Cayley graph of a simple group of Lie type based on a conjugacy class. *J. Combin. Theory Ser. A*, 83(1):118–137, 1998.

[11] Martin W. Liebeck and Aner Shalev. Diameters of finite simple groups: sharp bounds and applications. *Ann. of Math. (2)*, 154(2):383–406, 2001.

[12] G. A. Margulis. *Discrete subgroups of semisimple Lie groups*, volume 17 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1991.

[13] Hideya Matsumoto. Sur les sous-groupes arithmétiques des groupes semi-simples déployés. *Ann. Sci. École Norm. Sup. (4)*, 2:1–62, 1969.

[14] Dusa McDuff and Dietmar Salamon. *Introduction to symplectic topology.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1998.

[15] Robert Steinberg. *Lectures on Chevalley groups*, volume 66 of *University Lecture Series.* American Mathematical Society, Providence, RI, 2016. Notes prepared by John Faulkner and Robert Wilson, Revised and corrected edition of the 1968 original [MR0466335], With a foreword by Robert R. Snapp.

[16] O. I. Tavgen′. Bounded generability of Chevalley groups over rings of S-integer algebraic numbers. *Izv. Akad. Nauk SSSR Ser. Mat.*, 54(1):97–122, 221–222, 1990.

[17] Alexander A. Trost. Bounded generation by root elements for chevalley groups defined over rings of integers of function fields with an application in strong boundedness, 2021.

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