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OPTIMAL DISTRIBUTED GLOBALLY BOUNDED CONTROL FOR PARABOLIC – HYPERBOLIC EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS AND A LINEAR QUALITY CRITERION

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Abstract. For the problem of optimal control of a parabolic-hyperbolic process with nonlocal point boundary conditions, an explicit form of the solution is obtained in the form of formal series according to the system of eigenfunctions, which are generated by the spatial differential operator and boundary conditions. At the same time, the unequivocal solvability of the intermediate problems is established for each iteration. In addition, sufficient conditions for the convergence of the series are established, which determine the obtained formal solution of the optimal control problem, which justifies its correctness

Key words: Optimal control problem, parabolic-hyperbolic equations, Fourier coefficients, approximation.

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1. Introduction

It is known [1] that problems of optimal globally bounded distributed control for linear stationary parabolic or hyperbolic equations with local boundary conditions and a linear quality criterion have unique solutions in the form convergent series in a complete system of eigenfunctions generated by a spatial differential operator and boundary conditions. Problems of construction of approximate controls for of nonlinear parabolic equations were considered in [2, 3] and for wave equations were studied in [4]

In this paper, we substantiate the possibility of generalizing the above approach to non-self-adjoint optimal control problems related to parabolic-hyperbolic equations with non-local boundary conditions and a linear quality criterion. A

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formal solution of the problem is constructed. The unique solvability of intermediate problems at each iteration is established. Sufficient conditions for the convergence of the series defining found a formal solution.

Some types of optimal control problems for such systems were considered in [5]. The properties of solutions such problems that had no analogues in the self-adjoint case were set there.

When implementing this generalization, a number of features should be borne in mind. First, under a fixed control, we consider the classical solution of the corresponding boundary value problem with absolutely continuous control. Secondly, as a control norm, which is globally bounded, we consider a special equivalent norm generated by some positive definite operator. Thirdly, at intermediate iterations, the Fourier coefficients of controls are determined at different time intervals.

2. Statement of the problem and its preliminary analysis

Let the controlled process be described by function $y(x,t)$, which satisfies the equation

$$L\,y(x,t) = u(x,t),\ \ (x,t) \in D, \tag{2.1}$$

initial condition

$$y(x,-\alpha) = \varphi(x) \tag{2.2}$$

and boundary conditions

$$y(0,t) = 0, \quad y'(0,t) = y'(1,t), \quad -\alpha \le t \le T, \tag{2.3}$$

where $D = \{(x,t):\ 0 < x < 1,\ -\alpha < t \le T,\ \ \alpha, T > 0\}$, control u and function φ will be assumed to be given, and their smoothness properties will be refined below,

$$L\,y \ = \ \begin{cases} y_t - y_{xx}, & t \ > \ 0, \\ y_{tt} - y_{xx}, & t \ < \ 0. \end{cases}$$

The formal solution of the problem (2.1) – (2.3) can be represented [6] as

$$y(x,t) = X_0(x)y_0(t) + \sum_{k=1}^{\infty}(X_{2k-1}(x)y_{2k-1}(t) + X_{2k}(x)y_{2k}(t)), \tag{2.4}$$

where the functions $y_i(t)$ are defined as solutions to the Cauchy problems

$$\begin{gathered} \frac{dy_0(t)}{dt} = u_0(t),\ \ t > 0, \\ \frac{d^2y_0(t)}{dt^2} = u_0(t),\ \ t < 0, \\ y_0(-\alpha) = \varphi_0; \end{gathered} \tag{2.5}$$

$$\frac{dy_{2k-1}(t)}{dt} + \lambda_k^2 y_{2k-1}(t) = u_{2k-1}(t),\ \ t > 0$$

$$\frac{d^2 y_{2k-1}(t)}{dt^2} + \lambda_k^2 y_{2k-1}(t) = u_{2k-1}(t),\ \ t < 0, \tag{2.6}$$

$$y_{2k-1}(-\alpha) = \varphi_{2k-1},\ \ \lambda_k = 2k\pi;$$

$$\frac{dy_{2k}(t)}{dt} + \lambda_k^2 y_{2k}(t) = -2\lambda_k y_{2k-1}(t) + u_{2k}(t),\ \ t > 0,$$

$$\frac{d^2 y_{2k}(t)}{dt^2} + \lambda_k^2 y_{2k}(t) = -2\lambda_k y_{2k-1}(t) + u_{2k}(t),\ \ t < 0, \tag{2.7}$$

$$y_{2k}(-\alpha) = \varphi_{2k},\ \ k = 1, 2, ...,$$

and here $y_i(t) = (y(\cdot, t), Y_i(\cdot))_{L_2(0,1)} \in C^1(-\alpha, T),\ \ u_i(t) = (u(\cdot, t), Y_i(\cdot))_{L_2(0,1)}$, $i \geq 0$; functions $X_i(x)$ and $Y_i(x)$ belong to the Riesz bases

$$W_0 = \{X_0(x) = x,\ \ X_{2k-1}(x) = x\cos(2\pi kx),\ \ X_{2k}(x) = \sin(2\pi kx),\ \ k = 1, \ldots\},$$

$$\begin{aligned} R_0 \ &= \ \{Y_0(x) = 2,\ Y_{2k-1}(x) = 4\cos(2\pi kx), \\ &\qquad Y_{2k}(x) = 4(1-x)\sin(2\pi kx),\ k = 1, \ldots\}; \end{aligned}$$

$$(X_i, Y_j)_{L_2(0,1)} \ = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j,\ i,j = 0, 1, ...; \end{cases}$$

the sequence of numbers $\{\varphi_k\}$ is taken from the representation functions $\varphi(x)$ in the Riesz basis W_0.

For any function $\phi(x) \in L_2(0,1)$ fair assessment [7]

$$r||\phi||^2_{L_2(0,1)} \leq \sum_{k=0}^{\infty} \phi_k^2 \leq R||\phi||^2_{L_2(0,1)}, \tag{2.8}$$

где $r = 3/4,\ \ R = 16,\ \ \phi_k = (\phi, Y_k)_{L_2(0,1)}$.

Moreover, in [8] it is proved that in the space $L_2(0,1)$ we can introduce an equivalent norm according to the rule

$$||\phi||^2_D = (D\phi, \phi) = \sum_{k=0}^{\infty} \phi_k^2, \tag{2.9}$$

where $D : L_2(0,1) \to L_2(0,1)$ – some positive definite operator.

In [6] the problem (2.1) – (2.3) was studied for $u(x,t) = 0$ and in [9] for $u(x,t) \neq 0$ it is proved that series (2.4) is the unique solution to the problem (2.1) – (2.3) and $y(x,t) \in C^1(\bar{D}) \cap C^2(D_-) \cap C^{2,1}(D_+)$ if α is a rational number, $\varphi \in C(0,1),\ \ u \in C(D)$ and the conditions

$$\sum_{k=1}^{\infty} \lambda_k^2 \Big(|\varphi_{2k-1}| + |\varphi_{2k}|\Big) < \infty, \tag{2.10}$$

$$\sum_{k=1}^{\infty}\lambda_k\Big(||u_{2k-1}||_{C(-\alpha,T)}+||u_{2k}||_{C(-\alpha,T)}\Big)<\infty, \tag{2.11}$$

where $D_- = \{(x,t):\ 0<x<1,\ -\alpha<t\le 0\}$, $D_+ = \{(x,t):\ 0<x<1,\ 0<t\le T\}$.

In the paper [9] for a fixed continuous control an integral representation for solution of problems (2.5) – (2.7) is obtained

$$y_0(t) = \Phi^0_{0,+}(t)\varphi_0 + \int_{-\alpha}^{0} V^0_{0,+}(t,\tau)u_0(\tau)d\tau$$
$$+ U^0_{0,+}(t)u_0(0) + \int_0^t \mathcal{U}^0_{0,+}(t,\tau)u_0(\tau)d\tau, \quad t > 0,$$

$$y_0(t) = \Phi^0_{0,-}(t)\varphi_0 + \int_{-\alpha}^{0} V^0_{0,-}(t,\tau)u_0(\tau)d\tau$$
$$+ U^0_{0,-}(t)u_0(0) + \int_{-\alpha}^t \mathcal{V}^0_{0,-}(t,\tau)u_0(\tau)d\tau, \quad t < 0, \tag{2.12}$$

where

$$\Phi^0_{0,+}(t)=1,\quad V^0_{0,+}(t,\tau)=-(\alpha+\tau),\quad U^0_{0,+}(t)=\alpha,\qquad \mathcal{U}^0_{0,+}(t,\tau)=1,$$
$$\Phi^0_{0,-}(t)=1,\quad V^0_{0,-}(t,\tau)=-(\alpha+t),\quad U^0_{0,-}(t)=\alpha+t,\quad \mathcal{V}^0_{0,-}(t,\tau)=t-\tau;$$

$$y_{2k-1}(t) = \Phi^{2k-1}_{2k-1,+}(t)\varphi_{2k-1} + \int_{-\alpha}^{0} V^{2k-1}_{2k-1,+}(t,\tau)v_{2k-1}(\tau)d\tau$$
$$+ U^{2k-1}_{2k-1,+}(t)u_{2k-1}(0) + \int_0^t \mathcal{U}^{2k-1}_{2k-1,+}(t,\tau)u_{2k-1}(\tau)d\tau, \quad t>0;$$

$$y_{2k-1}(t) = \Phi^{2k-1}_{2k-1,-}(t)\varphi_{2k-1} + \int_{-\alpha}^{0} V^{2k-1}_{2k-1,-}(t,\tau)u_{2k-1}(\tau)d\tau$$
$$+ U^{2k-1}_{2k-1,-}(t)u_{2k-1}(0) + \int_{-\alpha}^t \mathcal{V}^{2k-1}_{2k-1,-}(t,\tau)u_{2k-1}(\tau)d\tau, \quad t<0, \tag{2.13}$$

here the following notations are used

$$\Phi^{2k-1}_{2k-1,+}(t)=\frac{\exp(-\lambda_k^2 t)}{\delta_k(\alpha)},\qquad V^{2k-1}_{2k-1,+}(t,\tau)=-\frac{\exp(-\lambda_k^2 t)}{\delta_k(\alpha)\lambda_k}\sin\lambda_k(\alpha+\tau),$$
$$U^{2k-1}_{2k-1,+}(t)=\frac{\exp(-\lambda_k^2 t)\sin(\lambda_k\alpha)}{\delta_k(\alpha)\lambda_k},\quad \mathcal{U}^{2k-1}_{2k-1,+}(t,\tau)=\exp(-\lambda_k^2(t-\tau)),$$
$$\Phi^{2k-1}_{2k-1,-}(t)=\frac{\delta_k(|t|)}{\delta_k(\alpha)},\qquad V^{2k-1}_{2k-1,-}(t,\tau)=-\frac{\sin\lambda_k(t+\alpha)}{\lambda_k\delta_k(\alpha)}\delta_k(|\tau|),$$
$$U^{2k-1}_{2k-1,-}(t)=\frac{\sin\lambda_k(t+\alpha)}{\lambda_k\delta_k(\alpha)},\qquad \mathcal{V}^{2k-1}_{2k-1,-}(t,\tau)=\frac{1}{\lambda_k}\sin\lambda_k(t-\tau),$$

$$
\begin{aligned}
y_{2k}(t) = &\Phi^{2k}_{2k-1,+}(t)\varphi_{2k-1} + \Phi^{2k}_{2k,+}(t)\varphi_{2k} \\
&+ \int_{-\alpha}^{0} \Big(V^{2k}_{2k-1,+}(t,\tau)u_{2k-1}(\tau) + V^{2k}_{2k,+}(t,\tau)u_{2k}(\tau)\Big)\,d\tau \\
&+ U^{2k}_{2k-1,+}(t)u_{2k-1}(0) + U^{2k}_{2k,+}(t)u_{2k}(0) \\
&+ \int_{0}^{t} \Big(\mathcal{U}^{2k}_{2k-1,+}(t,\tau)u_{2k-1}(\tau) + \mathcal{U}^{2k}_{2k,+}(t,\tau)u_{2k}(\tau)\Big)\,d\tau, \quad t>0,
\end{aligned}
$$

$$
\begin{aligned}
y_{2k}(t) = &\Phi^{2k}_{2k-1,-}(t)\varphi_{2k-1} + \Phi^{2k}_{2k,-}(t)\varphi_{2k} \\
&+ \int_{-\alpha}^{0} \Big(V^{2k}_{2k-1,-}(t,\tau)u_{2k-1}(\tau) + V^{2k}_{2k,-}(t,\tau)u_{2k}(\tau)\Big)\,d\tau \\
&+ U^{2k}_{2k-1,-}(t)u_{2k-1}(0) + U^{2k}_{2k,-}(t)u_{2k}(0) \\
&+ \int_{-\alpha}^{t} \Big(\mathcal{V}^{2k}_{2k-1,-}(t,\tau)u_{2k-1}(\tau) + \mathcal{V}^{2k}_{2k,-}(t,\tau)u_{2k}(\tau)\Big)\,d\tau, \quad t<0; \qquad (2.14)
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi^{2k}_{2k-1,+}(t) &= \frac{\exp(-\lambda_k^2 t)}{\delta_k^2(\alpha)}\Big((\alpha-1)\sin\lambda_k\alpha - \alpha\lambda_k\cos\lambda_k\alpha - 2\lambda_k t\delta_k(\alpha)\Big), \\
\Phi^{2k}_{2k,+}(t) &= \frac{\exp(-\lambda_k^2 t)}{\delta_k(\alpha)}, \\
V^{2k}_{2k-1,+}(t,\tau) &= \frac{\exp(-\lambda_k^2 t)}{\lambda_k\delta_k(\alpha)}\Bigg(\frac{\cos\lambda_k\alpha\sin\lambda_k\tau}{\lambda_k} - \tau\cos\lambda_k(\alpha+\tau) \\
&\quad - \frac{\delta_k(|\tau|)}{\delta_k(\alpha)}\Big(\alpha - \frac{\sin 2\lambda_k\alpha}{2\lambda_k}\Big) + 2\sin\lambda_k(\alpha+\tau)\Big(\frac{\sin\lambda_k\alpha}{\delta_k(\alpha)} + \lambda_k t\Big)\Bigg), \\
V^{2k}_{2k,+}(t,\tau) &= -\frac{\exp(-\lambda_k^2 t)}{\lambda_k\delta_k(\alpha)}\sin\lambda_k(\tau+\alpha), \\
U^{2k}_{2k-1,+}(t) &= \frac{\exp(-\lambda_k^2 t)}{\lambda_k\delta_k^2(\alpha)}\Bigg(-2\sin^2\lambda_k\alpha + \alpha - \frac{\sin 2\lambda_k\alpha}{2\lambda_k} - 2t\lambda_k\sin\lambda_k\alpha\delta_k(\alpha)\Bigg), \\
U^{2k}_{2k,+}(t) &= \frac{\exp(-\lambda_k^2 t)}{\delta_k(\alpha)}\frac{\sin\lambda_k\alpha}{\lambda_k}, \\
\mathcal{U}^{2k}_{2k-1,+}(t,\tau) &= -2\lambda_k(t-\tau)\exp(-\lambda_k^2(t-\tau)), \\
\mathcal{U}^{2k}_{2k,+}(t,\tau) &= \exp(-\lambda_k^2(t-\tau)), \\
\Phi^{2k}_{2k-1,-}(t) &= \frac{1}{\delta_k^2(\alpha)}\Big(-2\sin\lambda_k(t+\alpha) + \delta_k(|t|)(\sin\lambda_k\alpha - \lambda_k\alpha\cos\lambda_k\alpha) \\
&\quad + \delta_k(\alpha)(\sin\lambda_k t - \lambda_k t\cos\lambda_k t) + \alpha\delta_k(|t|)\sin\lambda_k\alpha - t\delta_k(\alpha)\sin\lambda_k t\Big),
\end{aligned}
$$

$$\Phi^{2k}_{2k,-}(t) = \frac{\delta_k(|t|)}{\delta_k(\alpha)},$$

$$V^{2k}_{2k-1,-}(t,\tau) = -\frac{\sin\lambda_k(t+\alpha)}{\delta_k(\alpha)}\left(\frac{(1-\tau)\sin\lambda_k\tau}{\lambda_k} - \tau\cos\lambda_k\tau - \frac{2\sin\lambda_k(\alpha+\tau)}{\lambda_k\delta_k(\alpha)}\right)$$
$$+ \frac{\cos\lambda_k\alpha\ \delta_k(|\tau|)}{\lambda_k}\left(\frac{\delta_k(|t|)}{\delta_k^2(\alpha)}\Big(\frac{\sin\lambda_k\alpha}{\lambda_k} - \alpha\cos\lambda_k\alpha\Big) + \frac{1}{\delta_k^2(\alpha)}\Big(\frac{\sin\lambda_k t}{\lambda_k}\right.$$
$$\left. - t\cos\lambda_k t\Big)\right) - \frac{\sin\lambda_k\alpha\delta_k(|\tau|)}{\lambda_k\delta_k^2(\alpha)}\Big(\alpha\delta_k(|t|)\sin\lambda_k\alpha - t\delta_k(\alpha)\sin\lambda_k t\Big),$$

$$V^{2k}_{2k,-}(t,\tau) = -\frac{\sin\lambda_k(t+\alpha)\delta_k(|\tau|)}{\lambda_k\delta_k(\alpha)},$$

$$U^{2k}_{2k-1,-}(t) = -\frac{2\sin\lambda_k(t+\alpha)\sin\lambda_k\alpha}{\lambda_k\delta_k^2(\alpha)} - \frac{\cos\lambda_k\alpha}{\lambda_k}\left(\frac{\delta_k(|t|)}{\delta_k^2(\alpha)}\Big(\frac{\sin\lambda_k\alpha}{\lambda_k}\right.$$
$$\left. - \alpha\cos\lambda_k\alpha\Big) + \frac{1}{\delta_k(\alpha)}\Big(\frac{\sin\lambda_k t}{\lambda_k} - t\cos\lambda_k t\Big)\right)$$
$$+ \frac{\sin\lambda_k\alpha}{\lambda_k\delta_k^2(\alpha)}\Big(\alpha\delta_k(|t|)\sin\lambda_k\alpha - t\delta_k(\alpha)\sin\lambda_k t\Big),$$

$$U^{2k}_{2k,-}(t) = \frac{\sin\lambda_k(t+\alpha)}{\lambda_k\delta_k(\alpha)},$$

$$\mathcal{V}^{2k}_{2k-1,-}(t,\tau) = \frac{1}{\lambda_k}\left((t-\tau)\cos\lambda_k(t-\tau) - \frac{\sin\lambda_k(t-\tau)}{\lambda_k}\right),$$

$$\mathcal{V}^{2k}_{2k,-}(t,\tau) = \frac{\sin\lambda_k(t-\tau)}{\lambda_k}\ .$$

The above integral representation of the solution to the problem (2.5) – (2.7) contains a function

$$\delta_k(\alpha) = \cos\lambda_k\alpha + \lambda_k\sin\lambda_k\alpha \tag{2.15}$$

in the denominator. For sufficiently large k and rational positive α, it satisfies the estimates

$$|\delta_k(\alpha)| = \sqrt{1+\lambda_k^2}\ \Big|\sin(\lambda_k\alpha+\gamma_k)\Big| \le \sqrt{1+\lambda_k^2},$$
$$|\delta_k(\alpha)| = \sqrt{1+\lambda_k^2}\ \Big|\sin(\lambda_k\alpha+\gamma_k)\Big| \ge \sqrt{1+\lambda_k^2}\sin\frac{\pi}{2q}\ , \tag{2.16}$$

here $\gamma_k = \arcsin\left(\sqrt{1+\lambda_k^2}\right)^{-1}$, $\alpha = p/q$, $2kp = sq+r$, p,q are positive integers, r is a non-negative integer not exceeding $q-1$, s is a non-negative integer.

Consider the following optimal control problem: we need to find the function $u(x,t)$, which returns an extreme value to the quality criterion

$$J(u) = \int_{-\alpha}^{T}\int_0^1 q(x,t)y(x,t)dxdt, \tag{2.17}$$

when restrictions are met

$$\sum_{i=0}^{\infty}\left(\hat{u}_i^2(-\alpha)+\int_{-\alpha}^{T}v_i^2(t)dt\right)\le\nu^2, \tag{2.18}$$

$$y(x,T)=\psi(x), \tag{2.19}$$

where $q(x,t)\in L_2(D)$, $\psi(x)\in L_2(0,1)$ are fixed functions whose properties will be specified below.

$$u_i(t)=\hat{u}_i(-\alpha)+\int_{-\alpha}^{t}v_i(\tau)d\tau,\quad t\ge-\alpha,\ i=0,1,\dots. \tag{2.20}$$

If the condition (2.19) is satisfied, then the solution to the problem (2.17) – (2.20) exists and is reached on boundary of the domain (2.18): the linear functional reaches extreme values in control when the last is located on the boundary of a closed convex region.

3. Formal solution of an extremal problem

The functions $q(x,t)$, $\psi(x)$ have series expansions

$$q(x,t)=Y_0(x)q_0(t)+\sum_{k=1}^{\infty}\Big(Y_{2k-1}(x)q_{2k-1}(t)+Y_{2k}(x)q_{2k}(t)\Big), \tag{3.1}$$

$$\psi(x)=X_0(x)\psi_0+\sum_{k=1}^{\infty}\Big(X_{2k-1}(x)\psi_{2k-1}+X_{2k}(x)\psi_{2k}\Big), \tag{3.2}$$

where $q_i(t)=\big(q(\cdot,t),X_i(\cdot)\big)_{L_2(0,1)}$, $\psi_i=\big(\psi(\cdot),Y_i(\cdot)\big)_{L_2(0,1)}$, $i=0,1,\dots.$

Then the problem (2.17) – (2.19) becomes

$$J(u)=\sum_{i=0}^{\infty}\left(\int_{-\alpha}^{T}q_i(t)y_i(t)dt\right)\to\ extr, \tag{3.3}$$

if

$$\sum_{i=0}^{\infty}\left(\hat{u}_i^2(-\alpha)+\int_{-\alpha}^{T}v_i^2(t)dt\right)=\nu^2, \tag{3.4}$$

$$y_i(T)=\psi_i(x),\ \ i=0,1,\dots. \tag{3.5}$$

The Lagrange functional for the problem (3.3) – (3.5) will have the form

$$\begin{aligned} L=&\sum_{i=0}^{\infty}\left(\int_{-\alpha}^{T}q_i(t)y_i(t)dt\right)+\sum_{i=0}^{\infty}\mu_i\big(y_i(T)-\psi_i(x)\big)\\ &+\ \frac{m}{2}\left(\nu^2-\sum_{i=0}^{\infty}\left(\hat{u}_i^2(-\alpha)+\int_{-\alpha}^{T}v_i^2(t)dt\right)\right), \end{aligned} \tag{3.6}$$

where $m,\ \mu_i,\ i=0,1,\ldots$ are Lagrange multipliers.

The Lagrange functional (3.6) can be represented as

$$L = L_0 + \sum_{k=1}^{\infty} L_{2k-1,2k} + \frac{m}{2}\,\nu^2, \tag{3.7}$$

where

$$L_0 = \int_{-\alpha}^{T} q_0(t)y_0(t)dt + \mu_0\big(y_0(T) - \psi_0(x)\big) - \frac{m}{2}\left(\hat{u}_0^2(-\alpha) + \int_{-\alpha}^{T} v_0^2(t)dt\right), \tag{3.8}$$

$$\begin{aligned} L_{2k-1,2k} = \sum_{j=2k-1}^{2k} \Bigg(&\int_{-\alpha}^{T} q_j(t)y_j(t) + \mu_j(y_j(T) - \psi_j(x)) \\ &- \frac{m}{2}\left(\hat{u}_j^2(-\alpha) + \int_{-\alpha}^{T} v_j^2(t)dt\right)\Bigg). \end{aligned} \tag{3.9}$$

Let us obtain optimality conditions for the above problems, taking into account integral representations (2.12) – (2.14), (2.20). For the problem (3.8) we get

$$\frac{\partial L_0}{\partial \hat{u}_0(-\alpha)} = a_0^0 + \mu_0 b_0^0 - m\hat{u}_0(-\alpha) = 0,$$

$$0 = \frac{\partial L_0}{\partial v_0}\bigg|(t) = \begin{cases} a_{0,-}^0(t) + \mu_0 b_{0,-}^0(t) - mv_{0,-}(t), & t \in (-\alpha, 0), \\ a_{0,+}^0(t) + \mu_0 b_{0,+}^0(t) - mv_{0,+}(t), & t \in [0, T], \end{cases} \tag{3.10}$$

where

$$\begin{aligned} a_0^0 = &\int_{-\alpha}^{0}\left(q_0(t)\left(\int_{-\alpha}^{0} V_{0,-}^0(t,\tau)d\tau + U_{0,-}^0(t)\right) + \int_t^0 q_0(\tau)\mathcal{V}_{0,-}^0(\tau,t)d\tau\right)dt \\ &+ \int_0^T\left(q_0(t)\left(\int_{-\alpha}^{0} V_{0,+}^0(t,\tau)d\tau + U_{0,+}^0(t)\right) + \int_t^T q_0(\tau)\mathcal{U}_{0,+}^0(\tau,t)d\tau\right)dt, \end{aligned}$$

$$b_0^0 = \int_{-\alpha}^{0} V_{0,+}^0(T,\tau)d\tau + U_{0,+}^0(T) + \int_0^T \mathcal{U}_{0,+}^0(T,\tau)d\tau,$$

$$\begin{aligned} a_{0,-}^0(t) = &\int_t^0\left(\int_{-\alpha}^{0} q_0(\xi)V_{0,-}^0(\xi,\tau)d\xi\right)d\tau + \int_{-\alpha}^{0} q_0(\xi)U_{0,-}^0(\xi)d\xi \\ &+ \int_t^0 q_0(\xi)\left(\int_t^{\xi} \mathcal{V}_{0,-}^0(\xi,\tau)d\tau\right)d\xi + \int_t^0\left(\int_0^T q_0(\xi)V_{0,+}^0(\xi,\tau)d\xi\right)d\tau \\ &+ \int_0^T q_0(\xi)U_{0,+}^0(\xi)d\xi + \int_0^T\int_{\tau}^{T} q_0(\xi)\mathcal{U}_{0,+}^0(\xi,\tau)d\xi d\tau, \end{aligned}$$

$$b_{0,-}^0(t) = \int_t^0 V_{0,+}^0(T,\xi)d\xi + U_{0,+}^0(T) + \int_0^T \mathcal{U}_{0,+}^0(T,\tau)d\tau,$$

$$a^0_{0,+}(t) = \int_t^T q_0(\xi)\left(\int_t^\xi \mathcal{U}^0_{0,+}(\xi,\tau)d\tau\right)d\xi, \qquad b^0_{0,+}(t) = \int_t^T \mathcal{U}^0_{0,+}(T,\tau)d\tau,$$

$\left.\dfrac{\partial\, L_0}{\partial v_0}\right|(t)$ is the Frechet derivative of the functional L_0.

The parameter μ_0 in the equations (3.10) is determined from the moment equality (3.5) for $i = 0$ and has the form

$$\mu_0 = \frac{m\Big(\psi_0 - \Phi^0_{0,+}(T)\varphi_0\Big) - r_0}{R_0}\ , \tag{3.11}$$

where

$$r_0 = a^0_0 b^0_0 + \int_{-\alpha}^0 a^0_{0,-}(t)b^0_{0,-}(t)dt + \int_0^T a^0_{0,+}(t)b^0_{0,+}(t)dt,$$
$$R_0 = (b^0_0)^2 + \int_{-\alpha}^0 (b^0_{0,-}(t))^2dt + \int_0^T (b^0_{0,+}(t))^2dt.$$

In (3.11) $R_0 > 0$. Indeed, let $R_0 = 0$. Then

$$b^0_0 = 0, \qquad b^0_{0,-}(t) = 0,\ t \in [-\alpha, 0], \qquad b^0_{0,+}(t) = 0,\ t \in (0, T]. \tag{3.12}$$

From the last equation of the system (3.12) it follows

$$\mathcal{U}^0_{0,+}(T,t) = 0, \quad t \in (0,T].$$

Since $b^0_{0,-}(-\alpha) = b^0_0$, then subtracting from the first equation of the system (3.12) second one, we get

$$V^0_{0,+}(T,t) = 0, \quad t \in [-\alpha, 0].$$

Taking into account the two equalities obtained above, it follows from the first equation of the system (3.12) that $U^0_{0,+}(T) = 0$. The resulting equalities contradict the definition of the functions $\mathcal{U}^0_{0,+}(t,\tau)$, $V^0_{0,+}(t,\tau)$, $U^0_{0,+}(t)$, i.e. $R_0 > 0$.

Then from (3.10) – (3.11) we find

$$\hat{u}_0(-\alpha) = \frac{1}{m}\mathcal{A}^0_0 + \mathcal{B}^0_0, \qquad v_{0,\mp}(t) = \frac{1}{m}\mathcal{A}^0_{0,\mp}(t) + \mathcal{B}^0_{0,\mp}(t), \tag{3.13}$$

where

$$\mathcal{A}^0_0 = a^0_0 - \frac{b^0_0 r_0}{R_0}\ , \qquad \mathcal{B}^0_0 = \frac{b^0_0\Big(\psi_0 - \Phi_{0,+}(T)\varphi_0\Big)}{R_0}\ ,$$
$$\mathcal{A}^0_{0,\mp}(t) = a^0_{0,\mp}(t) - \frac{b^0_{0,\mp}(t) r_0}{R_0}\ , \qquad \mathcal{B}^0_{0,\mp}(t) = \frac{b^0_{0,\mp}(t)\Big(\psi_0 - \Phi_{0,+}(T)\varphi_0\Big)}{R_0}\ .$$

For the problem (3.9) we get

$$\frac{\partial L_{2k-1,2k}}{\partial \hat{u}_{2k-1}(-\alpha)} = a^{2k-1}_{2k-1} + \sum_{j=2k-1}^{2k} \mu_j b^{2k-1}_j - m\hat{u}_{2k-1}(-\alpha) = 0,$$

$$\frac{\partial L_{2k-1,2k}}{\partial \hat{u}_{2k}(-\alpha)} = a^{2k}_{2k} + \sum_{j=2k-1}^{2k} \mu_j b^{2k}_j - m\hat{u}_{2k}(-\alpha) = 0,$$

$$0 = \frac{\partial L_{2k-1,2k}}{\partial v_{2k-1}}\Big|(t) = \begin{cases} a^{2k-1}_{2k-1,-}(t) + \sum_{j=2k-1}^{2k} \mu_j b^{2k-1}_{j,-}(t) - mv_{2k-1,-}(t), & t<0, \\ a^{2k-1}_{2k-1,+}(t) + \sum_{j=2k-1}^{2k} \mu_j b^{2k-1}_{j,+}(t) - mv_{2k-1,+}(t), & t>0, \end{cases}$$

$$0 = \frac{\partial L_{2k-1,2k}}{\partial v_{2k}}\Big|(t) = \begin{cases} a^{2k}_{2k,-}(t) + \sum_{j=2k-1}^{2k} \mu_j b^{2k}_{j,-}(t) - mv_{2k,-}(t), & t<0, \\ a^{2k}_{2k,+}(t) + \sum_{j=2k-1}^{2k} \mu_j b^{2k}_{j,+}(t) - m\, v_{2k,+}(t), & t>0, \end{cases} \tag{3.14}$$

where

$$\begin{aligned} a^{2k-1}_{2k-1} = &\int_{-\alpha}^{0} \Bigg(q_{2k-1}(t)\Big(\int_{-\alpha}^{0} V^{2k-1}_{2k-1,-}(t,\tau)d\tau + U^{2k-1}_{2k-1,-}(t)\Big) \\ &+ \int_t^0 q_{2k-1}(\tau)\mathcal{V}^{2k-1}_{2k-1,-}(\tau,t)d\tau \Bigg) dt \\ &+ \int_0^T \Bigg(q_{2k-1}(t)\Big(\int_{-\alpha}^{0} V^{2k-1}_{2k-1,+}(t,\tau)d\tau + U^{2k-1}_{2k-1,+}(t)\Big) \\ &+ \int_t^T q_{2k-1}(\tau)\mathcal{U}^{2k-1}_{2k-1,+}(\tau,t)d\tau \Bigg) dt \\ &+ \int_{-\alpha}^{0} \Bigg(q_{2k}(t)\Big(\int_{-\alpha}^{0} V^{2k}_{2k-1,-}(t,\tau)d\tau + U^{2k}_{2k-1,-}(t)\Big) \\ &+ \int_t^0 q_{2k}(\tau)\mathcal{V}^{2k}_{2k-1,-}(\tau,t)d\tau \Bigg) dt \\ &+ \int_0^T \Bigg(q_{2k}(t)\Big(\int_{-\alpha}^{0} V^{2k}_{2k-1,+}(t,\tau)d\tau + U^{2k}_{2k-1,+}(t)\Big) \\ &+ \int_t^T q_{2k}(\tau)\mathcal{U}^{2k}_{2k-1,+}(\tau,t)d\tau \Bigg) dt, \\ b^{2k-1}_{2k-1} = &\int_{-\alpha}^{0} V^{2k-1}_{2k-1,+}(T,\tau)d\tau + U^{2k-1}_{2k-1,+}(T) + \int_0^T \mathcal{U}^{2k-1}_{2k-1,+}(T,\tau)d\tau, \\ b^{2k-1}_{2k} = &\int_{-\alpha}^{0} V^{2k}_{2k-1,+}(T,\tau)d\tau + U^{2k}_{2k-1,+}(T) + \int_0^T \mathcal{U}^{2k}_{2k-1,+}(T,\tau)d\tau, \end{aligned}$$

$$a_{2k}^{2k} = \int_{-\alpha}^{0} \left(q_{2k}(t)\Big(\int_{-\alpha}^{0} V_{2k,-}^{2k}(t,\tau)d\tau + U_{2k,-}^{2k}(t)\Big) + \int_{t}^{0} q_{2k}(\tau)\mathcal{V}_{2k,-}^{2k}(\tau,t)d\tau \right) dt$$
$$+ \int_{0}^{T} \left(q_{2k}(t)\Big(\int_{-\alpha}^{0} V_{2k,+}^{2k}(t,\tau)d\tau + U_{2k,+}^{2k}(t)\Big) + \int_{t}^{T} q_{2k}(\tau)\mathcal{U}_{2k,+}^{2k}(\tau,t)d\tau \right) dt$$

$$b_{2k-1}^{2k} = 0, \qquad b_{2k}^{2k} = \int_{-\alpha}^{0} V_{2k,+}^{2k}(T,\tau)d\tau + U_{2k,+}^{2k}(T) + \int_{0}^{T} \mathcal{U}_{2k,+}^{2k}(T,\tau)d\tau,$$

$$a_{2k-1,-}^{2k-1}(t) = \int_{t}^{0} \left(\int_{-\alpha}^{0} q_{2k-1}(\xi)V_{2k-1,-}^{2k-1}(\xi,\tau)d\xi \right) d\tau + \int_{-\alpha}^{0} q_{2k-1}(\xi)U_{2k-1,-}^{2k-1}(\xi)d\xi$$
$$+ \int_{t}^{0} q_{2k-1}(\xi) \left(\int_{t}^{\xi} \mathcal{V}_{2k-1,-}^{2k-1}(\xi,\tau)d\tau \right) d\xi + \int_{t}^{0} \left(\int_{0}^{T} q_{2k-1}(\xi)V_{2k-1,+}^{2k-1}(\xi,\tau)d\xi \right) d\tau$$
$$+ \int_{0}^{T} q_{2k-1}(\xi)U_{2k-1,+}^{2k-1}(\xi)d\xi + \int_{0}^{T} \int_{\tau}^{T} q_{2k-1}(\xi)\mathcal{U}_{2k-1,+}^{2k-1}(\xi,\tau)d\xi d\tau$$
$$\int_{t}^{0} \left(\int_{-\alpha}^{0} q_{2k}(\xi)V_{2k-1,-}^{2k}(\xi,\tau)d\xi \right) d\tau + \int_{-\alpha}^{0} q_{2k}(\xi)U_{2k-1,-}^{2k}(\xi)d\xi$$
$$+ \int_{t}^{0} q_{2k}(\xi) \left(\int_{t}^{\xi} \mathcal{V}_{2k-1,-}^{2k}(\xi,\tau)d\tau \right) d\xi + \int_{t}^{0} \left(\int_{0}^{T} q_{2k}(\xi)V_{2k-1,+}^{2k}(\xi,\tau)d\xi \right) d\tau$$
$$+ \int_{0}^{T} q_{2k}(\xi)U_{2k-1,+}^{2k}(\xi)d\xi + \int_{0}^{T} \int_{\tau}^{T} q_{2k}(\xi)\mathcal{U}_{2k-1,+}^{2k}(\xi,\tau)d\xi d\tau,$$

$$b_{2k-1,-}^{2k-1}(t) = \int_{t}^{0} V_{2k-1,+}^{2k-1}(T,\xi)d\xi + U_{2k-1,+}^{2k-1}(T) + \int_{0}^{T} \mathcal{U}_{2k-1,+}^{2k-1}(T,\tau)d\tau,$$

$$b_{2k,-}^{2k-1}(t) = \int_{t}^{0} V_{2k-1,+}^{2k}(T,\xi)d\xi + U_{2k-1,+}^{2k}(T) + \int_{0}^{T} \mathcal{U}_{2k-1,+}^{2k}(T,\tau)d\tau,$$

$$a_{2k-1,+}^{2k-1}(t) = \int_{t}^{T} q_{2k-1}(\xi) \left(\int_{t}^{\xi} \mathcal{U}_{2k-1,+}^{2k-1}(\xi,\tau)d\tau \right) d\xi$$
$$+ \int_{t}^{T} q_{2k}(\xi) \left(\int_{t}^{\xi} \mathcal{U}_{2k-1,+}^{2k}(\xi,\tau)d\tau \right) d\xi,$$

$$b_{2k-1,+}^{2k-1}(t) = \int_{t}^{T} \mathcal{U}_{2k-1,+}^{2k-1}(T,\tau)d\tau, \qquad b_{2k,+}^{2k-1}(t) = \int_{t}^{T} \mathcal{U}_{2k-1,+}^{2k}(T,\tau)d\tau,$$

$$a_{2k,-}^{2k}(t) = \int_{t}^{0} \left(\int_{-\alpha}^{0} q_{2k}(\xi)V_{2k,-}^{2k}(\xi,\tau)d\xi \right) d\tau + \int_{-\alpha}^{0} q_{2k}(\xi)U_{2k,-}^{2k}(\xi)d\xi$$
$$+ \int_{t}^{0} q_{2k}(\xi) \left(\int_{t}^{\xi} \mathcal{V}_{2k,-}^{2k}(\xi,\tau)d\tau \right) d\xi + \int_{t}^{0} \left(\int_{0}^{T} q_{2k}(\xi)V_{2k,+}^{2k}(\xi,\tau)d\xi \right) d\tau$$
$$+ \int_{0}^{T} q_{2k}(\xi)U_{2k,+}^{2k}(\xi)d\xi + \int_{0}^{T} \int_{\tau}^{T} q_{2k}(\xi)\mathcal{U}_{2k,+}^{2k}(\xi,\tau)d\xi d\tau,$$

$$b^{2k}_{2k-1,-}(t) = 0, \quad b^{2k}_{2k,-}(t) = \int_t^0 V^{2k}_{2k,+}(T,\xi)d\xi + U^{2k}_{2k,+}(T) + \int_0^T \mathcal{U}^{2k}_{2k,+}(T,\tau)d\tau,$$

$$a^{2k}_{2k,+}(t) = \int_t^T q_{2k}(\xi)\int_t^\xi \mathcal{U}^{2k}_{2k,+}(\xi,\tau)d\tau d\xi,$$

$$b^{2k}_{2k-1,+}(t) = 0, \quad b^{2k}_{2k,+}(t) = \int_t^T \mathcal{U}^{2k}_{2k,+}(T,\tau)d\tau.$$

The parameters $\mu_{2k-1}, \mu_{2k},\ k = 1, 2, \dots$ for each k are determined from the moment equalities (3.5) ($i = \overline{2k-1, 2k}$) that generate the system

$$\begin{aligned} R^{2k-1}_{2k-1}\mu_{2k-1} + R^{2k-1}_{2k}\mu_{2k} &= m\left(\psi_{2k-1} - \Phi^{2k-1}_{2k-1,+}(T)\varphi_{2k-1}\right) - r^{2k-1}_{2k-1}, \\ R^{2k}_{2k-1}\mu_{2k-1} + R^{2k}_{2k}\mu_{2k} & \\ &= m\left(\psi_{2k} - \Phi^{2k}_{2k-1,+}(T)\varphi_{2k-1} - \Phi^{2k}_{2k,+}(T)\varphi_{2k}\right) - r^{2k}_{2k}, \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} r^{2k-1}_{2k-1} &= a^{2k-1}_{2k-1}b^{2k-1}_{2k-1} + \int_{-\alpha}^0 a^{2k-1}_{2k-1,-}(t)b^{2k-1}_{2k-1,-}(t)dt + \int_0^T a^{2k-1}_{2k-1,+}(t)b^{2k-1}_{2k-1,+}(t)dt, \\ R^{2k-1}_{2k-1} &= \left(b^{2k-1}_{2k-1}\right)^2 + \int_{-\alpha}^0 \left(b^{2k-1}_{2k-1,-}(t)\right)^2 dt + \int_0^T \left(b^{2k-1}_{2k-1,+}(t)\right)^2 dt, \\ R^{2k-1}_{2k} &= b^{2k-1}_{2k-1}b^{2k-1}_{2k} + \int_{-\alpha}^0 b^{2k-1}_{2k-1,-}(t)b^{2k-1}_{2k,-}(t)dt + \int_0^T b^{2k-1}_{2k-1,+}(t)b^{2k-1}_{2k,+}(t)dt, \\ r^{2k}_{2k} &= a^{2k-1}_{2k-1}b^{2k-1}_{2k} + \int_{-\alpha}^0 a^{2k-1}_{2k-1,-}(t)b^{2k-1}_{2k,-}(t)dt + \int_0^T a^{2k-1}_{2k-1,+}(t)b^{2k-1}_{2k,+}(t)dt \\ &\quad + a^{2k}_{2k}b^{2k}_{2k} + \int_{-\alpha}^0 a^{2k}_{2k,-}(t)b^{2k}_{2k,-}(t)dt + \int_0^T a^{2k}_{2k,+}(t)b^{2k}_{2k,+}(t)dt, \\ R^{2k}_{2k-1} &= b^{2k-1}_{2k-1}b^{2k-1}_{2k} + \int_{-\alpha}^0 b^{2k-1}_{2k-1,-}(t)b^{2k-1}_{2k,-}(t)dt + \int_0^T b^{2k-1}_{2k-1,+}(t)b^{2k-1}_{2k,+}(t)dt, \\ R^{2k}_{2k} &= \left(b^{2k-1}_{2k}\right)^2 + \int_{-\alpha}^0 \left(b^{2k-1}_{2k,-}(t)\right)^2 dt + \int_0^T \left(b^{2k-1}_{2k,+}(t)\right)^2 dt \\ &\quad + \left(b^{2k}_{2k}\right)^2 + \int_{-\alpha}^0 \left(b^{2k}_{2k,-}(t)\right)^2 dt + \int_0^T \left(b^{2k}_{2k,+}(t)\right)^2 dt. \end{aligned}$$

Consider the solvability of the system (3.15). Its determinant has the form

$$\Delta_{2k-1,2k} = R^{2k-1}_{2k-1}R^{2k}_{2k} - \left(R^{2k-1}_{2k}\right)^2, \tag{3.16}$$

because $R^{2k-1}_{2k} = R^{2k}_{2k-1}$.

Let us show that for all $k : \Delta_{2k-1,2k} > 0$. To do this, we use the Hilbert space $\mathcal{H} = R^1 \times L_2(-\alpha, 0) \times L_2(0, T)$ with scalar product

$$\left(\hat{a}, \hat{b}\right)_{\mathcal{H}} = ab + \int_{-\alpha}^{0} a_-(t) b_-(t) dt + \int_{0}^{T} a_+(t) b_+(t) dt, \quad \hat{a}, \hat{b} \in \mathcal{H}. \tag{3.17}$$

Then the determinant (3.16) in terms of the characteristics of the space $\mathcal{H}$ can be represented in the form

$$\Delta_{2k-1,2k} = \left\|\hat{b}^{2k-1}_{2k-1}\right\|^2_{\mathcal{H}} \left(\left\|\hat{b}^{2k-1}_{2k}\right\|^2_{\mathcal{H}} + \left\|\hat{b}^{2k}_{2k}\right\|^2_{\mathcal{H}} \right) - \left(\hat{b}^{2k-1}_{2k-1}, \hat{b}^{2k-1}_{2k}\right)^2_{\mathcal{H}}, \tag{3.18}$$

where $\left(\hat{b}^{2k-1}_{2k-1}\right)' = \left(b^{2k-1}_{2k-1},\ b^{2k-1}_{2k-1,-}(t),\ b^{2k-1}_{2k-1,+}(t)\right)$, $\left(\hat{b}^{2k-1}_{2k}\right)' = \left(b^{2k-1}_{2k},\ b^{2k-1}_{2k,-}(t),\ b^{2k-1}_{2k,+}(t)\right)$, $\left(\hat{b}^{2k}_{2k}\right)' = \left(b^{2k}_{2k},\ b^{2k}_{2k,-}(t),\ b^{2k}_{2k,+}(t)\right)$.

Let us estimate from below the value of the determinant $\Delta_{2k-1,2k}$, using the Cauchy-Bunyakovsky inequality for the scalar product,

$$\begin{aligned} \Delta_{2k-1,2k} &\geq \|\hat{b}^{2k-1}_{2k-1}\|^2_{\mathcal{H}} \left(\|\hat{b}^{2k-1}_{2k}\|^2_{\mathcal{H}} + \|\hat{b}^{2k}_{2k}\|^2_{\mathcal{H}} \right) - \|\hat{b}^{2k-1}_{2k-1}\|^2_{\mathcal{H}} \|\hat{b}^{2k-1}_{2k}\|^2_{\mathcal{H}} \\ &= \|\hat{b}^{2k-1}_{2k-1}\|^2_{\mathcal{H}} \|\hat{b}^{2k}_{2k}\|^2_{\mathcal{H}} = \|\hat{b}^{2k-1}_{2k-1}\|^4_{\mathcal{H}}, \end{aligned} \tag{3.19}$$

since $\hat{b}^{2k}_{2k} = \hat{b}^{2k-1}_{2k-1}$ due to the integral representation (2.13) – (2.14) of the original boundary value problem solution.

Let us show that $\Delta_{2k-1,2k} > 0$ for any k.

Indeed, suppose that $\hat{b}^{2k-1}_{2k-1} = 0$. Then the system of equations

$$\begin{aligned} b^{2k-1}_{2k-1} &= \int_{-\alpha}^{0} V^{2k-1}_{2k-1,+}(T, \tau) d\tau + U^{2k-1}_{2k-1,+}(T) + \int_{0}^{T} \mathcal{U}^{2k-1}_{2k-1,+}(T, \tau) d\tau = 0, \\ b^{2k-1}_{2k-1,-}(t) &= \int_{t}^{0} V^{2k-1}_{2k-1,+}(T, \xi) d\xi + U^{2k-1}_{2k-1,+}(T) \\ &\quad + \int_{0}^{T} \mathcal{U}^{2k-1}_{2k-1,+}(T, \tau) d\tau = 0, \quad t \in [-\alpha, 0), \\ b^{2k-1}_{2k-1,+}(t) &= \int_{t}^{T} \mathcal{U}^{2k-1}_{2k-1,+}(T, \tau) d\tau = 0, \quad t \in [0, T]. \end{aligned}$$

The written system will take place provided that

$$\mathcal{U}^{2k-1}_{2k-1,+}(T, \tau) = V^{2k-1}_{2k-1,+}(T, \tau) = U^{2k-1}_{2k-1,+}(T) = 0$$

(see the analysis of the $R_0 = 0$ case above). This contradicts the definition of functions $\mathcal{U}^{2k-1}_{2k-1,+}(T, \tau)$, $V^{2k-1}_{2k-1,+}(T, \tau)$, $U^{2k-1}_{2k-1,+}(T)$.

Thus,

$$\mu_{2k-1} = \Delta^{-1}_{2k-1,2k}\Bigg[m\Bigg(\Big(\psi_{2k-1} - \Phi^{2k-1}_{2k-1,+}(T)\varphi_{2k-1}\Big)R^{2k}_{2k} - \Big(\psi_{2k}$$
$$- \Phi^{2k}_{2k-1,+}(T)\varphi_{2k-1} - \Phi^{2k}_{2k,+}(T)\varphi_{2k}\Big)R^{2k-1}_{2k}\Bigg) + r^{2k}_{2k}R^{2k-1}_{2k} - r^{2k-1}_{2k-1}R^{2k}_{2k}\Bigg],$$
$$\mu_{2k} = \Delta^{-1}_{2k-1,2k}\Bigg[\Bigg(m\Big(\psi_{2k} - \Phi^{2k}_{2k-1,+}(T)\varphi_{2k-1} - \Phi^{2k}_{2k,+}(T)\varphi_{2k}\Big)\ R^{2k-1}_{2k-1}$$
$$- \Big(\psi_{2k-1} - \Phi^{2k-1}_{2k-1,+}(T)\varphi_{2k-1}\Big)R^{2k}_{2k-1}\Bigg) + r^{2k-1}_{2k-1}R^{2k}_{2k-1} - r^{2k}_{2k}R^{2k-1}_{2k-1}\Bigg]. \tag{3.20}$$

Then from (3.14) and (3.20) we find

$$\hat{u}_j(-\alpha) = \frac{1}{m}\mathcal{A}^j_j + \mathcal{B}^j_j,$$
$$v_{j,\mp}(t) = \frac{1}{m}\mathcal{A}^j_{j,\mp}(t) + \mathcal{B}^j_{j,\mp}(t), \quad j = \overline{2k-1, 2k},\ k = 1, 2, \ldots, \tag{3.21}$$

where

$$\mathcal{A}^{2k-1}_{2k-1} = a^{2k-1}_{2k-1} + \frac{1}{\Delta_{2k-1,2k}}\Bigg[b^{2k-1}_{2k-1}\Big(r^{2k}_{2k}R^{2k-1}_{2k} - r^{2k-1}_{2k-1}R^{2k}_{2k}\Big)$$
$$+ b^{2k-1}_{2k}\Big(r^{2k-1}_{2k-1}R^{2k-1}_{2k} - r^{2k}_{2k}R^{2k-1}_{2k-1}\Big)\Bigg],$$
$$\mathcal{B}^{2k-1}_{2k-1} = \frac{1}{\Delta_{2k-1,2k}}\Bigg[b^{2k-1}_{2k-1}\Bigg(\Big(\psi_{2k-1} - \Phi^{2k-1}_{2k-1,+}(T)\varphi_{2k-1}\Big)R^{2k}_{2k}$$
$$- \Big(\psi_{2k} - \Phi^{2k}_{2k-1,+}(T)\varphi_{2k-1} - \Phi^{2k}_{2k,+}(T)\varphi_{2k}\Big)R^{2k-1}_{2k}\Bigg)$$
$$+ b^{2k-1}_{2k}\Bigg(\Big(\psi_{2k} - \Phi^{2k}_{2k-1,+}(T)\varphi_{2k-1} - \Phi^{2k}_{2k,+}(T)\varphi_{2k}\Big)R^{2k-1}_{2k-1}$$
$$- \Big(\psi_{2k-1} - \Phi^{2k-1}_{2k-1,+}(T)\varphi_{2k-1}\Big)R^{2k}_{2k-1}\Bigg)\Bigg],$$
$$\mathcal{A}^{2k-1}_{2k-1,\mp}(t) = a^{2k-1}_{2k-1,\mp}(t) + \frac{1}{\Delta_{2k-1,2k}}\Bigg[b^{2k-1}_{2k-1,\mp}(t)\Big(r^{2k}_{2k}R^{2k-1}_{2k} - r^{2k-1}_{2k-1}R^{2k}_{2k}\Big)$$
$$+ b^{2k-1}_{2k,\mp}(t)\Big(r^{2k-1}_{2k-1}R^{2k-1}_{2k} - r^{2k}_{2k}R^{2k-1}_{2k-1}\Big)\Bigg],$$

$$\mathcal{B}^{2k-1}_{2k-1,\mp}(t) = \frac{1}{\Delta_{2k-1,2k}}\bigg[b^{2k-1}_{2k-1,\mp}(t)\Big(\Big(\psi_{2k-1} - \Phi^{2k-1}_{2k-1,+}(T)\varphi_{2k-1}\Big)R^{2k}_{2k}$$
$$- \Big(\psi_{2k} - \Phi^{2k}_{2k-1,+}(T)\varphi_{2k-1} - \Phi^{2k}_{2k,+}(T)\varphi_{2k}\Big)R^{2k-1}_{2k}\Big)$$
$$+ b^{2k-1}_{2k,\mp}(t)\Big(\Big(\psi_{2k} - \Phi^{2k}_{2k-1,+}(T)\varphi_{2k-1} - \Phi^{2k}_{2k,+}(T)\varphi_{2k}\Big)R^{2k-1}_{2k-1}$$
$$- \Big(\psi_{2k-1} - \Phi^{2k-1}_{2k-1,+}(T)\varphi_{2k-1}\Big)R^{2k}_{2k-1}\Big)\bigg];$$

$$\mathcal{A}^{2k}_{2k} = a^{2k}_{2k} + \frac{1}{\Delta_{2k-1,2k}}\bigg[b^{2k}_{2k-1}\Big(r^{2k}_{2k}R^{2k-1}_{2k} - r^{2k-1}_{2k-1}R^{2k}_{2k}\Big)$$
$$+ b^{2k}_{2k}\Big(r^{2k-1}_{2k-1}R^{2k-1}_{2k} - r^{2k}_{2k}R^{2k-1}_{2k-1}\Big)\bigg],$$

$$\mathcal{B}^{2k}_{2k} = \frac{1}{\Delta_{2k-1,2k}}\bigg[b^{2k}_{2k-1}\Big(\Big(\psi_{2k-1} - \Phi^{2k-1}_{2k-1,+}(T)\varphi_{2k-1}\Big)R^{2k}_{2k}$$
$$- \Big(\psi_{2k} - \Phi^{2k}_{2k-1,+}(T)\varphi_{2k-1} - \Phi^{2k}_{2k,+}(T)\varphi_{2k}\Big)R^{2k-1}_{2k}\Big)$$
$$+ b^{2k}_{2k}\Big(\Big(\psi_{2k} - \Phi^{2k}_{2k-1,+}(T)\varphi_{2k-1} - \Phi^{2k}_{2k,+}(T)\varphi_{2k}\Big)R^{2k-1}_{2k-1}$$
$$- \Big(\psi_{2k-1} - \Phi^{2k-1}_{2k-1,+}(T)\varphi_{2k-1}\Big)R^{2k}_{2k-1}\Big)\bigg],$$

$$\mathcal{A}^{2k}_{2k,\mp}(t) = a^{2k}_{2k,\mp}(t) + \frac{1}{\Delta_{2k-1,2k}}\bigg[b^{2k}_{2k-1,\mp}(t)\Big(r^{2k}_{2k}R^{2k-1}_{2k} - r^{2k-1}_{2k-1}R^{2k}_{2k}\Big)$$
$$+ b^{2k}_{2k,\mp}(t)\Big(r^{2k-1}_{2k-1}R^{2k-1}_{2k} - r^{2k}_{2k}R^{2k-1}_{2k-1}\Big)\bigg],$$

$$\mathcal{B}^{2k}_{2k,\mp}(t) = \frac{1}{\Delta_{2k-1,2k}}\bigg[b^{2k}_{2k-1,\mp}(t)\Big(\Big(\psi_{2k-1} - \Phi^{2k-1}_{2k-1,+}(T)\varphi_{2k-1}\Big)R^{2k}_{2k}$$
$$- \Big(\psi_{2k} - \Phi^{2k}_{2k-1,+}(T)\varphi_{2k-1} - \Phi^{2k}_{2k,+}(T)\varphi_{2k}\Big)R^{2k-1}_{2k}\Big)$$
$$+ b^{2k}_{2k,\mp}(t)\Big(\Big(\psi_{2k} - \Phi^{2k}_{2k-1,+}(T)\varphi_{2k-1} - \Phi^{2k}_{2k,+}(T)\varphi_{2k}\Big)R^{2k-1}_{2k-1}$$
$$- \Big(\psi_{2k-1} - \Phi^{2k-1}_{2k-1,+}(T)\varphi_{2k-1}\Big)R^{2k}_{2k-1}\Big)\bigg].$$

Lagrange multiplier m, according to condition (3.4) and formulas (3.13), (3.21), determined from the equation

$$c\left(\frac{1}{m}\right)^2+2d\left(\frac{1}{m}\right)+e=0, \tag{3.22}$$

where

$$\begin{aligned} c&=\sum_{i=0}^{\infty}\left[\left(\mathcal{A}_i^i\right)^2+\int_{-\alpha}^{0}\left(\mathcal{A}_{i,-}^i(t)\right)^2dt+\int_0^T\left(\mathcal{A}_{i,+}^i(t)\right)^2dt\right],\\ d&=\sum_{i=0}^{\infty}\left[\mathcal{A}_i^i\mathcal{B}_i^i+\int_{-\alpha}^{0}\mathcal{A}_{i,-}^i(t)\mathcal{B}_{i,-}^i(t)dt+\int_0^T\mathcal{A}_{i,+}^i(t)\mathcal{B}_{i,+}^i(t)dt\right],\\ e&=-\nu^2+\sum_{i=0}^{\infty}\left[\left(\mathcal{B}_i^i\right)^2+\int_{-\alpha}^{0}\left(\mathcal{B}_{i,-}^i(t)\right)^2dt+\int_0^T\left(\mathcal{B}_{i,+}^i(t)\right)^2dt\right]. \end{aligned}$$

Let the series defining the coefficients c, d, e of the equation (3.22) converge. Then

$$\left(\frac{1}{m}\right)_{1,2}=\frac{-d\pm\sqrt{D}}{c}, \tag{3.23}$$

where $D=d^2-ce$ is the discriminant of the equation (3.22).

The number ν in the constraint (2.18) must be such that the inequality

$$\begin{aligned} D=&\left(\sum_{i=0}^{\infty}\left[\mathcal{A}_i^i\mathcal{B}_i^i+\int_{-\alpha}^{0}\mathcal{A}_{i,-}^i(t)\mathcal{B}_{i,-}^i(t)dt+\int_0^T\mathcal{A}_{i,+}^i(t)\mathcal{B}_{i,+}^i(t)dt\right]\right)^2\\ &+\left(\sum_{i=0}^{\infty}\left[\left(\mathcal{A}_i^i\right)^2+\int_{-\alpha}^{0}\left(\mathcal{A}_{i,-}^i(t)\right)^2dt+\int_0^T\left(\mathcal{A}_{i,+}^i(t)\right)^2dt\right]\right)\\ &\times\left(\nu^2-\sum_{i=0}^{\infty}\left[\left(\mathcal{B}_i^i\right)^2+\int_{-\alpha}^{0}\left(\mathcal{B}_{i,-}^i(t)\right)^2dt+\int_0^T\left(\mathcal{B}_{i,+}^i(t)\right)^2dt\right]\right)\geq 0. \end{aligned} \tag{3.24}$$

Then the formulas (3.13), (3.21) are a formal solution to the problem (2.17) – (2.19).

4. Substantiation of the solution of an extremal problem

Let us find the conditions on the data of the original problem under which the above formal results hold.

Let us first estimate the right-hand side of the inequality (3.19) from below.

$$\|\hat{b}^{2k-1}_{2k-1}\|^4_{\mathcal{H}} =\Bigg(\Big(\int_{-\alpha}^{0} V^{2k-1}_{2k-1,+}(T,\tau)d\tau + U^{2k-1}_{2k-1,+}(T) + \int_0^T \mathcal{U}^{2k-1}_{2k-1,+}(T,\tau)d\tau\Big)^2$$
$$+\int_{-\alpha}^{0} \Big(\int_t^0 V^{2k-1}_{2k-1,+}(T,\xi)d\xi + U^{2k-1}_{2k-1,+}(T) + \int_0^T \mathcal{U}^{2k-1}_{2k-1,+}(T,\tau)d\tau\Big)^2 dt$$
$$+\int_0^T \Big(\int_t^T \mathcal{U}^{2k-1}_{2k-1,+}(T,\tau)d\tau\Big)^2 dt\Bigg)^2. \tag{4.1}$$

Applying to the right side of equality (4.1) the inequality about the relationship between the arithmetic mean and the geometric mean, we get

$$\|\hat{b}^{2k-1}_{2k-1}\|^4_{\mathcal{H}} \geq 9\Bigg[\Big(\int_{-\alpha}^{0} V^{2k-1}_{2k-1,+}(T,\tau)d\tau + U^{2k-1}_{2k-1,+}(T) + \int_0^T \mathcal{U}^{2k-1}_{2k-1,+}(T,\tau)d\tau\Big)^2$$
$$\times\int_{-\alpha}^{0} \Big(\int_t^0 V^{2k-1}_{2k-1,+}(T,\xi)d\xi + U^{2k-1}_{2k-1,+}(T) + \int_0^T \mathcal{U}^{2k-1}_{2k-1,+}(T,\tau)d\tau\Big)^2 dt$$
$$\times\int_0^T \Big(\int_t^T \mathcal{U}^{2k-1}_{2k-1,+}(T,\tau)d\tau\Big)^2 dt\Bigg]^{\frac{2}{3}}. \tag{4.2}$$

Let us estimate the factors of the right-hand side of the inequality (4.2) from below.

$$\Big(\int_{-\alpha}^{0} V^{2k-1}_{2k-1,+}(T,\tau)d\tau + U^{2k-1}_{2k-1,+}(T) + \int_0^T \mathcal{U}^{2k-1}_{2k-1,+}(T,\tau)d\tau\Big)^2$$
$$=\Big(-\frac{\exp(-\lambda_k^2T)}{\delta_k(\alpha)\lambda_k^2}\Big(1-\cos\lambda_k\alpha\Big) + \frac{\exp(-\lambda_k^2T)}{\delta_k(\alpha)\lambda_k}\sin(\lambda_k\alpha)$$
$$+\frac{1}{\lambda_k^2}\Big(1-\exp(-\lambda_k^2T)\Big)\Big)^2 \geq \frac{1}{\lambda_k^4}\Big(1-\frac{\exp(-\lambda_1^2T)}{\lambda_1}\Big)^2;$$

$$\int_{-\alpha}^{0}\Big(\int_t^0 V^{2k-1}_{2k-1,+}(T,\xi)d\xi + U^{2k-1}_{2k-1,+}(T) + \int_0^T \mathcal{U}^{2k-1}_{2k-1,+}(T,\tau)d\tau\Big)^2 dt$$
$$=\int_{-\alpha}^{0}\Big(\frac{\exp(-\lambda_k^2T)}{\delta_k(\alpha)\lambda_k^2}\Big(\cos\lambda_k\alpha - \cos\lambda_k(\alpha+t)\Big) + \frac{\exp(-\lambda_k^2T)}{\delta_k(\alpha)\lambda_k}\sin(\lambda_k\alpha)$$
$$+\frac{1}{\lambda_k^2}\Big(1-\exp(-\lambda_k^2T)\Big)\Big)^2 dt \geq \frac{\alpha}{\lambda_k^4}\Big(1-\frac{\exp(-\lambda_1^2T)}{\lambda_1}\Big)^2;$$

$$\int_0^T\Big(\int_t^T \mathcal{U}^{2k-1}_{2k-1,+}(T,\tau)d\tau\Big)^2 dt = \frac{1}{\lambda_k^4}\int_0^T\Big(1-\exp\Big(-\lambda_k^2(T-t)\Big)\Big)^2 dt$$
$$> \frac{1}{\lambda_k^4}\int_0^T\Big(1-\exp\Big(-\lambda_1^2(T-t)\Big)\Big)^2 dt.$$

The above estimates imply the inequality

$$\Delta_{2k-1,2k} > \frac{C}{\lambda_k^8}\ . \tag{4.3}$$

The optimal control formally found in the previous subsection must be an absolutely continuous function, i.e., the series must converge

$$U=\sum_{i=0}^{\infty}\Big(\hat{u}_i^2(-\alpha)+\int_{-\alpha}^T v_i^2(t)dt\Big)\le C\Bigg[\frac{1}{\hat{m}^2}\sum_{i=0}^{\infty}\Big(\big(\mathcal{A}_i^i\big)^2+\int_{-\alpha}^0\big(\mathcal{A}^i_{i,-}(t)\big)^2dt\ +$$
$$+\int_0^T\big(\mathcal{A}^i_{i,+}(t)\big)^2dt\Big)+\sum_{i=0}^{\infty}\Big(\big(\mathcal{B}_i^i\big)^2+\int_{-\alpha}^0\big(\mathcal{B}^i_{i,-}(t)\big)^2dt+\int_0^T\big(\mathcal{B}^i_{i,+}(t)\big)^2dt\Big)\Bigg], \tag{4.4}$$

where

$$\frac{1}{\hat{m}} < \frac{|d|+\sqrt{D}}{c}\ .$$

Due to the representation for the constant c,, the estimate for the quantity U will be a fair

$$U < C\Bigg[\frac{d^2+D}{c_0}\ +\sum_{i=0}^{\infty}\Big(\big(\mathcal{B}_i^i\big)^2+\int_0^T\big(\mathcal{B}^i_{i,-}(t)\big)^2dt+\int_{-\alpha}^0\big(\mathcal{B}^i_{i,+}(t)\big)^2dt\Big)\Bigg], \tag{4.5}$$

where

$$c_0=\big(\mathcal{A}_0^0\big)^2+\int_{-\alpha}^0\big(\mathcal{A}^0_{0,-}(t)\big)^2dt+\int_0^T\big(\mathcal{A}^0_{0,+}(t)\big)^2dt\neq 0.$$

Further, two variants of sufficient conditions on the initial data of the problem under consideration are possible, under which the above series will converge, defining the formal solution of the problem.

Case 1. Let the input data be $\varphi(x),\ \psi(x)\in C_0^\infty(0,1)$, $q(x.t)\in C_0^{0,\infty}(D)$, i.e. by spatial coordinate they are infinitely differentiable and finite on the boundary.

Consider, for example, how it will look for the function $\varphi(x)$. The above series will converge if the series is convergent

$$\sum_{k=0}^{\infty}\lambda_k^l\Big(\varphi_{2k-1}^2+\varphi_{2k}^2\Big), \tag{4.6}$$

where l – some positive integer and let here $\varphi_{-1}=0$. Really,

$$\varphi_{2k-1}=\int_0^1\varphi(x)Y_{2k-1}(x)dx,\quad \varphi_{2k}=\int_0^1\varphi(x)Y_{2k}(x)dx.$$

Integrating by parts, for the coefficient φ_{2k-1} we obtain

$$\begin{aligned}
\varphi_{2k-1} &= \int_0^1 \varphi(x)Y_{2k-1}(x)dx = 4\int_0^1 \varphi(x)\cos(\lambda_k x)dx \\
&= \frac{4}{\lambda_k}\,\varphi(x)\sin(\lambda_k x)\Big|_0^1 - \frac{4}{\lambda_k}\int_0^1 \frac{d\varphi(x)}{dx}\sin(\lambda_k x)dx \\
&= \frac{4}{\lambda_k}\,\varphi(x)\sin(\lambda_k x)\Big|_0^1 + \frac{4}{\lambda_k^2}\,\frac{d\varphi(x)}{dx}\cos(\lambda_k x)\Big|_0^1 - \frac{4}{\lambda_k^2}\int_0^1 \frac{d^2\varphi(x)}{dx^2}\cos(\lambda_k x)dx \\
&= \frac{4}{\lambda_k}\,\varphi(x)\sin(\lambda_k x)\Big|_0^1 + \frac{4}{\lambda_k^2}\,\frac{d\varphi(x)}{dx}\cos(\lambda_k x)\Big|_0^1 - \frac{4}{\lambda_k^3}\,\frac{d^2\varphi(x)}{dx^2}\sin(\lambda_k x)\Big|_0^1 \\
&+ \frac{4}{\lambda_k^3}\int_0^1 \frac{d^3\varphi(x)}{dx^3}\sin(\lambda_k x)dx = \frac{4}{\lambda_k}\varphi(x)\sin(\lambda_k x)\Big|_0^1 + \frac{4}{\lambda_k^2}\,\frac{d\varphi(x)}{dx}\cos(\lambda_k x)\Big|_0^1 \\
&- \frac{4}{\lambda_k^3}\frac{d^2\varphi(x)}{dx^2}\sin(\lambda_k x)\Big|_0^1 - \frac{4}{\lambda_k^4}\frac{d^3\varphi(x)}{dx^3}\cos(\lambda_k x)\Big|_0^1 \\
&+ \frac{4}{\lambda_k^4}\int_0^1 \frac{d^4\varphi(x)}{dx^4}\cos(\lambda_k x)dx = \ldots = 4\sum_{i=1}^{m}\frac{(-1)^{\mu_i+1}}{\lambda_k^i}\frac{d^{i-1}\varphi(x)}{dx^{i-1}}\varrho_i(\lambda_k,x)\Big|_0^1 \\
&+ (-1)^{\mu_m}\frac{4}{\lambda_k^m}\int_0^1 \frac{d^m\varphi(x)}{dx^m}\varrho_m(\lambda_k,x)dx, \qquad (4.7)
\end{aligned}$$

where

$$\mu_m = \begin{cases} j, & m = 2j-1, \quad j = 1,2,\ldots, \\ n, & m = 2n, \qquad n = 1,2,\ldots\ , \end{cases}$$

$$\varrho_m(\lambda_k,x) = \begin{cases} \sin(\lambda_k x), & m - \text{odd}, \\ \cos(\lambda_k\ x), & m - \text{even}. \end{cases}$$

Since the function $\varphi(x)$ is finite, the equality (4.7) takes the form

$$\varphi_{2k-1} = (-1)^{\mu_m}\frac{4}{\lambda_k^m}\int_0^1 \frac{d^m\varphi(x)}{dx^m}\varrho_m(\lambda_k,x)dx. \qquad (4.8)$$

Similarly, setting $\omega(x) = \varphi(x)(1-x)$, for the coefficient φ_{2k} we have

$$\begin{aligned}
\varphi_{2k} &= \int_0^1 \varphi(x)Y_{2k}(x)dx = 4\int_0^1 \omega(x)\sin(\lambda_k x)dx \\
&= -\frac{4}{\lambda_k}\,\omega(x)\cos(\lambda_k x)\Big|_0^1 + \frac{4}{\lambda_k}\int_0^1 \frac{d\omega(x)}{dx}\cos(\lambda_k x)dx \\
&= -\frac{4}{\lambda_k}\,\omega(x)\cos(\lambda_k x)\Big|_0^1 + \frac{4}{\lambda_k^2}\,\frac{d\omega(x)}{dx}\sin(\lambda_k x)\Big|_0^1 - \frac{4}{\lambda_k^2}\int_0^1 \frac{d^2\omega(x)}{dx^2}\sin(\lambda_k x)dx \\
&= -\frac{4}{\lambda_k}\,\omega(x)\cos(\lambda_k x)\Big|_0^1 + \frac{4}{\lambda_k^2}\,\frac{d\omega(x)}{dx}\sin(\lambda_k x)\Big|_0^1 + \frac{4}{\lambda_k^3}\,\frac{d^2\omega(x)}{dx^2}\cos(\lambda_k x)\Big|_0^1 \\
&- \frac{4}{\lambda_k^3}\int_0^1 \frac{d^3\omega(x)}{dx^3}\cos(\lambda_k x)dx = -\frac{4}{\lambda_k}\,\omega(x)\cos(\lambda_k x)\Big|_0^1
\end{aligned}$$

$$+\frac{4}{\lambda_k^2}\frac{d\omega(x)}{dx}\sin(\lambda_k x)\Big|_0^1+\frac{4}{\lambda_k^3}\frac{d^2\omega(x)}{dx^2}\cos(\lambda_k x)\Big|_0^1-\frac{4}{\lambda_k^4}\frac{d^3\omega(x)}{dx^3}\sin(\lambda_k x)\Big|_0^1$$
$$+\frac{4}{\lambda_k^4}\int_0^1\frac{d^4\omega(x)}{dx^4}\sin(\lambda_k x)dx=\dots$$
$$=-\frac{4}{\lambda_k}\omega(x)\cos(\lambda_k x)\Big|_0^1+4\sum_{i=1}^{m-1}\frac{(-1)^{\mu_i+1}}{\lambda_k^{i+1}}\frac{d^i\omega(x)}{dx^i}\varrho_i(\lambda_k,x)\Big|_0^1+$$
$$+(-1)^m\frac{4}{\lambda_k^m}\int_0^1\frac{d^m\omega(x)}{dx^m}\varrho_{m+1}(\lambda_k,x)dx. \tag{4.9}$$

Taking into account that the function $\varphi(x)$, is finite, the equality (4.9) takes the form

$$\varphi_{2k}=(-1)^m\frac{4}{\lambda_k^m}\int_0^1\frac{d^m\omega(x)}{dx^m}\varrho_{m+1}(\lambda_k,x)\,dx. \tag{4.10}$$

Due to the fact that

$$\frac{d^n\omega(x)}{dx^n}=(1-x)\frac{d^n\varphi(x)}{dx^n}-n\frac{d^{n-1}\varphi(x)}{dx^{n-1}},$$

from the formulas (4.8), (4.10) we get the estimate

$$|\varphi_j|\le\frac{C}{\lambda_k^m}\sum_{\nu=0}^m\max_{x\in[0,1]}\left|\frac{d^\nu\varphi(x)}{dx^\nu}\right|,\quad j=\overline{2k-1,2k}. \tag{4.11}$$

Taking into account (4.11), for the series (4.6), we obtain the estimate

$$\sum_{k=0}^\infty\lambda_k^l\left(\varphi_{2k-1}^2+\varphi_{2k}^2\right)\le C\sum_{\nu=0}^m\max_{x\in[0,1]}\left|\frac{d^\nu\varphi(x)}{dx^\nu}\right|^2\sum_{k=0}^\infty\frac{1}{\lambda_k^{2m-l}}. \tag{4.12}$$

We choose the number m from the condition

$$2m-l\ge 2. \tag{4.13}$$

Then the series (4.6) will converge.

Case 2. The class of original functions can be extended for the convergence of the above series, defining the formal solution of the problem. Indeed, let a priori estimates of the type (4.5) result in series of the form (4.6). Then due to (4.7), (4.9), we set $\varphi(x)\in C^m(0,1)$ и

$$\frac{d^{2k-1}\varphi(0)}{dx^{2k-1}}=\frac{d^{2k-1}\varphi(1)}{dx^{2k-1}},\quad 2k-1>0,$$
$$\frac{d^{2k}\varphi(0)}{dx^{2k}}=0,\quad k=\overline{0,m}.$$

If the conditions (4.13) are met, the series (4.6) will converge.

For other inputs, the analog result will take place. The general case is not presented here due to the cumbersomeness of a priori estimates of the form (4.5).

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