JOURNAL OF OPTIMIZATION, DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS (JODEA)
Volume **31**, Issue **1**, June **2023**, pp. 125–142, DOI 10.15421/142307

ISSN (print) 2617–0108
ISSN (on-line) 2663–6824

ON THE ASYMPTOTIC EQUIVALENCE OF ORDINARY AND FUNCTIONAL STOCHASTIC DIFFERENTIAL EQUATIONS

Olexandr M. Stanzhytskyi*, Grygoriy O. Petryna†, Maryna V. Hrysenko‡

Abstract. This paper studies the asymptotic behavior of solutions of linear stochastic functional-differential equations. This behavior is investigated using the method of asymptotic equivalence, according to which an ordinary system of linear differential equations is constructed based on the initial stochastic system, and the asymptotic behavior of the solutions of this system is analogous to the behavior of the solutions of the initial system.

Key words: asymptotic behavior, delay, Cauchy matrix, probability.

2010 Mathematics Subject Classification: 60H15, 60H30, 35R60.

Communicated by Prof. P. Kogut

1. Introduction

The work is dedicated to the study of the asymptotic behavior of solutions in linear systems of stochastic functional-differential equations. Functional-differential equations model evolutionary processes in which the future depends not only on the current state but also on the system's past state (delay effect). The presence of delays significantly influences the qualitative behavior of the system. The right-hand side of such mathematical models is a functional of a segment of the solution, which complicates the research object and requires the development and application of methods of infinite-dimensional analysis. The wide application of such models has led to a rapid development of the theory of functional-differential equations. Its foundations for deterministic functional-differential equations in the finite-dimensional case are thoroughly presented in the monograph [1], and for deterministic equations in the infinite-dimensional case in the monograph [2]. As for stochastic functional-differential equations in finite-dimensional spaces, the monograph [3] provides a detailed bibliography and presents elements of the asymptotic and qualitative theory of such equations. Regarding stochastic functional-differential equations in infinite-dimensional spaces, the monograph [4] is noteworthy. The existence of invariant measures in shift spaces for stochastic functional-differential equations with partial derivatives is addressed in works [5–8]. In this

*Department of General Mathematics, Taras Shevchenko National University of Kyiv, 4e, Akademika Hlushkova av., Kyiv, 03127, Ukraine, ostanzh@gmail.com
†Department of General Mathematics, Taras Shevchenko National University of Kyiv, 4e, Akademika Hlushkova av., Kyiv, 03127, Ukraine, grpetryna@gmail.com
‡Department of General Mathematics, Taras Shevchenko National University of Kyiv, 4e, Akademika Hlushkova av., Kyiv, 03127, Ukraine, magryss@gmail.com

work, the asymptotic behavior of solutions at infinity is investigated using a well-known method in the theory of differential equations called the method of asymptotic equivalence. According to this method, a simpler system is constructed based on the original system, and the behavior of solutions at infinity of the simpler system is equivalent to the behavior of solutions of the original system. The classical result by Levinson [9] is relevant in the linear case. For stochastic systems without delay, this approach is further developed in works [10, 11]. The article is structured as follows: Section 2 introduces the notation and formulates the main results. Section 3 is devoted to proving the main results of the study. Finally, an illustrative example is provided at the end of the work.

2. Preliminaries

For $h > 0$ we define a function space $C_h = C([-h,0];\mathbf{R}^d)$ of continuous functions with a norm $||\phi||_C = \sup_{\theta\in[-h,0]} |\phi(\theta)|$. We denote the norm of a vector in $\mathbf{R}^d$ space using the symbol $|\cdot|$ and the norm of a $(d\times d)$ matrix, consistent with a vector norm, using $||\cdot||$ throughout this paper. Consider the system of ordinary differential equations (ODE) in the following form

$$dx = Ax\,dt, \tag{2.1}$$

with the initial conditions $x(t_0) = x_0,\ \ t \geq t_0 \geq 0,\ \ x \in \mathbf{R}^d$, and A be a constant deterministic matrix. Along with system (2.1), we consider the system of functional stochastic differential equations (FSDE)

$$dy = \left(Ay + \int_{-h}^{0} B(t,\theta)y(t+\theta)\,d\theta\right)dt + \left(\int_{-h}^{0} D(t,\theta)y(t+\theta)d\theta\right)dW(t), \tag{2.2}$$

where $B(t,\theta), D(t,\theta)$ are continuous deterministic matrices for $t \geq 0,\ \theta \in [-h,0]$, integrable with respect to θ. $W(t)$ is a Wiener process on a probability space $(\Omega,\mathbf{F},P)$ with filtration $\{\mathcal{F}_t, t\geq 0\}\subset \mathbf{F}$, and there exist such $b(t)$ and $d(t)$

$$\left\|\int_{-h}^{0} B(t,\theta)\phi(\theta)\,d\theta\right\| \leq b(t)\|\phi\|_C, \tag{2.3}$$

$$\left\|\int_{-h}^{0} D(t,\theta)\phi(\theta)\,d\theta\right\| \leq d(t)\|\phi\|_C. \tag{2.4}$$

We introduce the definition of asymptotic equivalence, which is a generalization of the classical definition of asymptotic equivalence for systems of ordinary differential equations to the stochastic case.

Definition 2.1. If for each solution $y(t)$ of system (2.2) there corresponds a solution $x(t)$ of (2.1) such that

$$\lim_{t\to\infty} \mathbf{E}|x(t)-y(t)|^2 = 0,$$

then system (2.2) is called asymptotically mean square equivalent to system (2.1). In case when for each solution $y(t)$ of system (2.2) there corresponds a solution $x(t)$ of system (2.1) such that

$$P\left\{\lim_{t\to\infty} |x(t) - y(t)| = 0\right\} = 1,$$

then system 2.2 is called asymptotically equivalent to system 2.1 with probability 1.

Now let us formulate the main result of our work.

Theorem 2.1. *Let all solutions of system (2.1) be bounded on $t \in [0, \infty)$. If*

$$\int_0^\infty |b(t)|\, dt \le K_1 < \infty, \tag{2.5}$$

$$\int_0^\infty |d(t)|^2\, dt \le K_1 < \infty, \tag{2.6}$$

Then system (2.2) is asymptotically equivalent to the system (2.1) in the mean square sense. Also, if we change (2.6) on

$$\int_0^\infty t d(t)^2\, dt \le K_1, \tag{2.7}$$

then, (2.2) is asymptotically equivalent to the system (2.1) with the probability 1.

3. Proof of the main result

Proof. This theorem consists of two parts the following proof will deal with them sequentially. We will start with the first part.
By our conditions, the solutions of system (2.1) are bounded, hence the eigen values $\lambda(A)$ of the matrix A satisfy the inequality $Re\lambda(A)$, also the values which real part equals to zero have simple elementary divisors. We can assume that matrix A has a quasi-diagonal form,

$$A = \mathrm{diag}(A_1, A_2), \tag{3.1}$$

where A_1 and A_2 are $(p \times p)$ and $(q \times q)$ - matrices, $p + q = d$, such that

$$Re\lambda(A_1) \le -\alpha < 0, \quad Re\lambda(A_2) = 0. \tag{3.2}$$

Let

$$X(t) = \mathrm{diag}(e^{tA_1}, e^{tA_2}), \tag{3.3}$$

be a fundamental matrix of system (2.1), normalized in zero, $X(0) = E_d$, and

$$I_1 = \mathrm{diag}(E_p, 0), \quad I_2 = \mathrm{diag}(0, E_q),$$

where E_p and E_q are the identity matrices of order p and q, and $I_1 + I_2 = E_d$. Let's define

$$\begin{aligned} X(t) &= X_1(t) + X_2(t) = X(t)I_1 + X(t)I_2 \\ &= \mathrm{diag}(e^{tA_2}, 0) + \mathrm{diag}(0, e^{tA_2}). \end{aligned} \tag{3.4}$$

Therefore, the Cauchy matrix can be written in the following way

$$\begin{aligned} \tilde{X} &= X(t)X^{-1}(\tau) = X(t-\tau) \\ &= X_1(t-\tau) + X_2(t-\tau). \end{aligned} \tag{3.5}$$

Using our previous estimates we get

$$\|X_1(t)\| = \left\|e^{tA_1}\right\| \le a_1 e^{-\alpha t}, \quad t \ge t_0 \ge 0, \tag{3.6}$$

$$\|X_2(t)\| = \left\|e^{tA_2}\right\| \le a_2, \quad t \in \mathbb{R}. \tag{3.7}$$

Where a_1, a_2, α are some positive constants. Let us write a solution of system (2.2) with the initial conditions $y(t_0) = y_0$ in terms of a Cauchy matrix for the deterministic differential system (2.1).

$$\begin{aligned} y(t) = X(t-t_0)y(t_0) &+ \int_{t_0}^{t} X_1(t-\tau) \int_{-h}^{0} [B(\tau,\theta)y(\tau+\theta)]\, d\theta\, d\tau \\ &+ \int_{t_0}^{t} X_2(t-\tau) \int_{-h}^{0} [B(\tau,\theta)y(\tau+\theta)]\, d\theta\, d\tau \\ &+ \int_{t_0}^{t} X_1(t-\tau) \int_{-h}^{0} [D(\tau,\theta)y(\tau+\theta)]\, d\theta\, dW(\tau) \\ &+ \int_{t_0}^{t} X_2(t-\tau) \int_{-h}^{0} [D(\tau,\theta)y(\tau+\theta)]\, d\theta\, dW(\tau), \end{aligned} \tag{3.8}$$

for $t \ge t_0 \ge 0$ and $\theta \in [-h, 0]$. Using the evolution properties of the matriciant

$$X_2(t-\tau) = X(t-\tau)I_2 = X(t-t_0)X(t_0-\tau)I_2 = X(t-t_0)X_2(t_0-\tau), \tag{3.9}$$

we can rewrite (3.8) in the following way:

$$\begin{aligned} y(t) = X(t-t_0)\{y(t_0) &+ \int_{t_0}^{\infty} X_2(t_0-\tau) \int_{-h}^{0} [B(\tau,\theta)y(\tau+\theta)]\, d\theta\, d\tau \\ &+ \int_{t_0}^{\infty} X_2(t_0-\tau) \int_{-h}^{0} [D(\tau,\theta)y(\tau+\theta)]\, d\theta\, dW(\tau)\} \\ &+ \int_{t_0}^{t} X_1(t-\tau) \int_{-h}^{0} [B(\tau,\theta)y(\tau+\theta)]\, d\theta\, d\tau \\ &+ \int_{t_0}^{t} X_1(t-\tau) \int_{-h}^{0} [D(\tau,\theta)y(\tau+\theta)]\, d\theta\, dW(\tau) \\ &- \int_{t}^{\infty} X_2(t-\tau) \int_{-h}^{0} [B(\tau,\theta)y(\tau+\theta)]\, d\theta\, d\tau \\ &- \int_{t}^{\infty} X_2(t-\tau) \int_{-h}^{0} [D(\tau,\theta)y(\tau+\theta)]\, d\theta\, dW(\tau). \end{aligned} \tag{3.10}$$

Let $y(t) \equiv y(t,\omega)$ be a solution of system (2.2) with the initial condition $y(t_0+\theta) = \phi(\theta),\ \theta \in [-h,0]$, which correspond to a solution $x(t)$ of system (2.1) with the initial condition

$$\begin{aligned} x(t_0) = y(t_0) + \int_{t_0}^{\infty} X_2(t_0-\tau) \int_{-h}^{0} [B(\tau,\theta)y(\tau+\theta)]\, d\theta\, d\tau \\ + \int_{t_0}^{\infty} X_2(t_0-\tau) \int_{-h}^{0} [D(\tau,\theta)y(\tau+\theta)]\, d\theta\, dW(\tau). \end{aligned} \tag{3.11}$$

For every solution of system (2.2) with the initial condition $y(t_0+\theta) = \phi(\theta)$ by formula (3.11) we define correspondence between the set of solutions $\{y(t) \equiv y(t,\omega)\}$ of system (2.2) and the set of solutions $\{x(t)\}$ of system (2.1)
Now we can start proving our first statement. We know that

$$\begin{aligned} y(t) = X(t-t_0)y(t_0) + \int_{t_0}^{t} X(t-\tau) \int_{-h}^{0} [B(\tau,\theta)y(\tau+\theta)]\, d\theta\, d\tau \\ + \int_{t_0}^{t} X(t-\tau) \int_{-h}^{0} [D(\tau,\theta)y(\tau+\theta)]\, d\theta\, dW(\tau), \end{aligned} \tag{3.12}$$

Hence, using the stochastic integral properties and the above equality we obtain

$$\begin{aligned} \mathbf{E}|y(t)|^2 \leq 3||X(t-t_0)||^2\mathbf{E}|y(t_0)|^2 \\ + 3\mathbf{E}\left|\int_{t_0}^{t} X(t-\tau) \int_{-h}^{0} [B(\tau,\theta)y(\tau+\theta)]\, d\theta\, d\tau\right|^2 \\ + 3\mathbf{E}\left|\int_{t_0}^{t} X(t-\tau) \int_{-h}^{0} [D(\tau,\theta)y(\tau+\theta)]\, d\theta\, dW(\tau)\right|^2, \end{aligned} \tag{3.13}$$

For simplicity, let us explicitly consider each term in the above inequality:

$$\|X(t-t_0)\|^2\mathbf{E}|y(t_0)|^2 \leq \max(a_1^2,a_2^2)\mathbf{E}|y(t_0)|, \tag{3.14}$$

$$\begin{aligned} \mathbf{E}\Big|\int_{t_0}^{t} X(t_0-\tau) \int_{-h}^{0} [B(\tau,\theta)y(\tau+\theta)]\, d\theta\, d\tau\Big|^2 \\ \leq \mathbf{E}\left(\int_{t_0}^{t} \sqrt{||X(t_0-\tau)||}\sqrt{\|X(t_0-\tau)\|}\sqrt{b(\tau)}\sqrt{b(\tau)}\|y_\tau\|_C\right)^2 \\ \leq \int_{t_0}^{t} \|X(t-\tau)\|b(\tau)\mathbf{E}\|y_\tau\|_C^2\, d\tau \int_{t_0}^{t} ||X(t-\tau)||b(\tau)\, d\tau \\ \leq \max(a_1^2,a_2^2) \int_{0}^{\infty} b(\tau)\, d\tau \int_{t_0}^{t} b(\tau)E\|y_\tau\|_C^2\, d\tau, \end{aligned} \tag{3.15}$$

$$\begin{aligned}
&\mathbf{E}\left|\int_{t_0}^{t} X(t_0-\tau)\int_{-h}^{0}[D(\tau,\theta)y(\tau+\theta)]\,d\theta\,dW(\tau)\right|^2\\
&\le \int_{t_0}^{t} E\left|X(t-\tau)\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\right|^2\,d\tau\\
&\le \int_{t_0}^{t}\|X(t-\tau)\|^2\mathbf{E}\left\|\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\right\|^2\,d\tau\\
&\le \int_{t_0}^{t}\|X(t-\tau)\|^2 d^2(\tau)\mathbf{E}\|y_\tau\|_C^2\,d\tau\\
&\le \max(a_1^2,a_2^2)\int_{t_0}^{t} d(\tau)^2\mathbf{E}\|y_\tau\|_C^2\,d\tau.
\end{aligned} \tag{3.16}$$

Now we can substitute our estimates into the (3.13).

$$\begin{aligned}
\mathbf{E}|y(t)|^2 \le 3\max(a_1^2,a_2^2)(\mathbf{E}|y(t_0)|^2 + \int_0^{\infty} b(\tau)\,d\tau\int_{t_0}^{t} b(\tau)\mathbf{E}\|y_\tau\|_C^2\,d\tau\\
+\int_{t_0}^{t} d^2(\tau)\mathbf{E}\|y_\tau\|_C^2\,d\tau),
\end{aligned} \tag{3.17}$$

It is clear that

$$\max_{s\in[0,t]}\mathbf{E}||y_s||_C^2 \le \max_{s\in[-h,0]}\mathbf{E}|\phi(s)|^2 + \max_{s\in[0,t]}\mathbf{E}|y(s)|^2, \tag{3.18}$$

hence,

$$\begin{aligned}
\max_{s\in[0,t]}\mathbf{E}|y(s)|^2 \le 3\max(a_1^2,a_2^2)\Big(\mathbf{E}\|\phi(\theta)\|_C^2\\
+\int_0^{\infty} b(\tau)\,d\tau\int_{t_0}^{t} b(\tau)\max_{s\in[0,\tau]}\mathbf{E}|y(s)|^2\,d\tau\\
+\int_{t_0}^{t} d^2(\tau)\max_{s\in[0,\tau]}\mathbf{E}|y(s)|^2\,d\tau\Big),
\end{aligned} \tag{3.19}$$

Using the Gronwall-Bellman inequality we get:

$$\begin{aligned}
\max_{s\in[0,t]}\mathbf{E}|y(s)|^2 &\le 3\max(a_1^2,a_2^2)\mathbf{E}\|\phi(\theta)\|_C^2\\
&\quad\times\exp\left(3\max(a_1^2,a_2^2)\int_{t_0}^{t}(K_1 b(\tau)+d^2(\tau))\,d\tau\right)\\
&\le 3\max(a_1^2,a_2^2)\mathbf{E}\|\phi(\theta)\|_C^2\\
&\quad\times\exp\left(3\max(a_1^2,a_2^2)\int_0^{\infty}(K_1 b(\tau)+d^2(\tau))\,d\tau\right)\\
&\le \tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2,\\
\tilde{K} &= 3\max(a_1^2,a_2^2)\exp\left(3\max(a_1^2,a_2^2)\int_0^{\infty}(K_1 b(\tau)+d^2(\tau))\,d\tau\right).
\end{aligned} \tag{3.20}$$

The latter inequality indicates that the integrals in equation (3.11) exhibit mean square convergence. Next, we will assess the expected difference in square norms between the respective solutions $x(t)$ and $y(t)$. Since

$$x(t) = X(t - t_0)x(t_0), \tag{3.21}$$

where $x(t_0)$ is defined in (3.11), using (3.10) we obtain

$$\begin{aligned}
\mathbf{E}|x(t) - y(t)|^2 = \mathbf{E}\Big| & \int_{t_0}^{t} X_1(t-\tau)\left[\int_{-h}^{0} B(\tau,\theta)y(\tau+\theta)\,d\theta\right] d\tau \\
& + \int_{t_0}^{t} X_1(t-\tau)\left[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\right] dW(\tau) \\
& - \int_{t}^{\infty} X_2(t-\tau)\left[\int_{-h}^{0} B(\tau,\theta)y(\tau+\theta)\,d\theta\right] d\tau \\
& - \int_{t}^{\infty} X_2(t-\tau)[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta]\,dW(\tau)|^2 \\
\le 4\mathbf{E} & \left|\int_{t_0}^{t} X_1(t-\tau)\left[\int_{-h}^{0} B(\tau,\theta)y(\tau+\theta)\,d\theta\right] d\tau\right|^2 \\
& + 4\mathbf{E}\left|\int_{t_0}^{t} X_1(t-\tau)\left[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\right] dW(\tau)\right|^2 \\
& + 4\mathbf{E}|\int_{t}^{\infty} X_2(t-\tau)\left[\int_{-h}^{0} B(\tau,\theta)y(\tau+\theta)\,d\theta\right] d\tau|^2 \\
& + 4\mathbf{E}|\int_{t}^{\infty} X_2(t-\tau)\left[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\right] dW(\tau)|^2.
\end{aligned} \tag{3.22}$$

Using (3.20), let's estimate each term of the last inequality:

$$\begin{aligned}
& \mathbf{E}\left|\int_{t_0}^{t} X_1(t-\tau)\left[\int_{-h}^{0} B(\tau,\theta)y(\tau+\theta)\,d\theta\right] d\tau\right|^2 \\
& \le \mathbf{E}\left(\int_{t_0}^{t} \|X_1(t-\tau)\| \left\|\int_{-h}^{0} B(\tau,\theta)y(\tau+\theta)\,d\theta\right\| d\tau\right)^2 \\
& \le \mathbf{E}\left(\int_{t_0}^{t} \|X_1(t-\tau)\|b(\tau)\|y_\tau\|_C\,d\tau\right)^2 \\
& = \mathbf{E}\left(\int_{t_0}^{t} \sqrt{\|X_1(t-\tau)\|b(\tau)}\sqrt{\|X_1(t-\tau)\|b(\tau)}\|y_\tau\|_C\,d\tau\right)^2 \\
& \le \mathbf{E}\left(\int_{t_0}^{t} \|X_1(t-\tau)\|b(\tau)d\tau \int_{t_0}^{t} ||X_1(t-\tau)||b(\tau)\|y_\tau\|_C^2\,d\tau\right)
\end{aligned}$$

$$\begin{aligned} &\leq \tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2 \left(\int_{t_0}^t \|X_1(t-\tau)\| b(\tau)\, d\tau\right)^2 \\ &\leq \tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2 \left(\int_{t_0}^t a_1 e^{-\alpha(t-\tau)} b(\tau)\, d\tau\right)^2, \end{aligned} \tag{3.23}$$

Since $b(t)$ is absolutely integrable for $t \geq 2t_0$:

$$\begin{aligned} \int_{t_0}^t e^{-\alpha(t-\tau)} b(\tau)\, d\tau &= \int_{t_0}^{\frac{t}{2}} e^{-\alpha(t-\tau)} b(\tau)\, d\tau + \int_{\frac{t}{2}}^t e^{-\alpha(t-\tau)} b(\tau)\, d\tau \\ &\leq e^{-\frac{\alpha t}{2}} \int_{t_0}^{\frac{t}{2}} b(\tau)\, d\tau + \int_{\frac{t}{2}}^t b(\tau)\, d\tau \\ &\leq e^{-\frac{\alpha t}{2}} \int_0^{\infty} b(\tau)\, d\tau + \int_{\frac{t}{2}}^t b(\tau)\, d\tau, \end{aligned}$$

From the last inequality, it becomes evident that the first term on the right-hand side of (3.22) tends to 0, as $t \to \infty$.

$$\begin{aligned} &\mathbf{E}\Big|\int_{t_0}^t X_1(t-\tau)\left[\int_{-h}^0 D(\tau,\theta) y(\tau+\theta)\, d\theta\right] dW(\tau)\Big|^2 \\ &\qquad \leq \int_{t_0}^t \mathbf{E}|X_1(t-\tau)\left[\int_{-h}^0 D(\tau,\theta) y(\tau+\theta)\, d\theta\right]|^2\, d\tau \\ &\qquad \leq \int_{t_0}^t \|X_1(t-\tau)\|^2 d^2(\tau) \mathbf{E}\|y_\tau\|_C^2 d\tau \\ &\qquad \leq \tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2 \int_{t_0}^t a_1^2 e^{-2\alpha(t-\tau)} d^2(\tau) d\tau, \end{aligned} \tag{3.24}$$

From $d^2(\tau)$ integrability and the previous term we can conclude that the second term in (3.22) also tends to 0 as $t \to \infty$.

$$\begin{aligned} &\mathbf{E}\Big|\int_t^{\infty} X_2(t-\tau)\left[\int_{-h}^0 B(\tau,\theta) y(\tau+\theta)\, d\theta\right] d\tau\Big|^2 \\ &\qquad \leq \mathbf{E}\left(\int_t^{\infty} \|X_2(t-\tau)\| \left\|\int_{-h}^0 B(\tau,\theta) y(\tau+\theta)\, d\theta\right\| d\tau\right)^2 \\ &\qquad \leq \mathbf{E}\left(\int_t^{\infty} \|X_2(t-\tau)\| b(\tau) \|y_\tau\|_C\, d\tau\right)^2 \\ &\qquad \leq \int_t^{\infty} \|X_2(t-\tau)\| b(\tau) E\|y_\tau\|_C^2\, d\tau \int_t^{\infty} \|X_2(t-\tau)\| b(\tau)\, d\tau \\ &\qquad \leq \tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2 \left(\int_t^{\infty} \|X_2(t-\tau)\| b(\tau)\, d\tau\right)^2 \\ &\qquad \leq \tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2 a_2^2 \left(\int_t^{\infty} b(\tau)\, d\tau\right)^2, \end{aligned} \tag{3.25}$$

$$\mathbf{E}\Big|\int_t^\infty X_2(t-\tau)\left[\int_{-h}^0 D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\Big|^2$$
$$\leq \mathbf{E}\int_t^\infty \|X_2(t-\tau)\|^2 d^2(\tau)\mathbf{E}\|y\|_C^2\,d\tau \tag{3.26}$$
$$\leq \tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2 a_2^2\int_t^\infty d^2(\tau)\,d\tau.$$

Both (3.25, 3.26) tend to 0 as $t\to\infty$. These results prove the first part of our theorem,

$$\lim_{t\to\infty}\mathbf{E}|x(t)-y(t)|^2=0.$$

Next, we will proceed with proving the second part of the theorem. We start with introducing a sequence denoted as $\{n_k|k\geq 1\}$, satisfying the condition $n_k>k$, $k\geq 1$, such that

$$\int_{n_k}^\infty b(\tau)\,d\tau\leq\frac{1}{2^k},\quad k\geq 1,$$

and a sequence $m_k|k\geq 1$, where $m_k>k,\ k\geq 1$ and,

$$\int_{m_k}^\infty \tau d^2(\tau)\,d\tau\leq\frac{1}{2^k},\quad k\geq 1.$$

Now we use these sequences in order to construct l_k:

$$l_k=2\max\{n_k,m_k\},\quad k\geq 1.$$

Using (3.20), where $x(t_0)$ is defined in (3.11), from (3.10) we know that arbitrary solutions $x(t)$ and $y(t)$ satisfy the following:

$$\begin{aligned}
&P\Big\{\sup_{t\geq l_k}|x(t)-y(t)|\geq\frac{1}{k}\Big\}\\
&\quad=P\Big\{\sup_{t\geq l_k}\int_{t_0}^t X_1(t-\tau)\left[\int_{-h}^0 B(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,d\tau\\
&\qquad+\int_{t_0}^t X_1(t-\tau)\left[\int_{-h}^0 D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)|\\
&\qquad-\int_t^\infty X_2(t-\tau)\left[\int_{-h}^0 B(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,d\tau\\
&\qquad-\int_t^\infty X_2(t-\tau)\left[\int_{-h}^0 D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)|\geq\frac{1}{k}\Big\}\\
&\quad\leq P\left\{\sup_{t\geq l_k}\left|\int_{t_0}^t X_1(t-\tau)\left[\int_{-h}^0 B(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,d\tau\right|\geq\frac{1}{4k}\right\}
\end{aligned}$$

$$
\begin{aligned}
&+ P\left\{\sup_{t\ge l_k}\left|\int_{t_0}^{t} X_1(t-\tau)\left[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\right|\ge\frac{1}{4k}\right\}\\
&+ P\left\{\sup_{t\ge l_k}\left|\int_{t}^{\infty} X_2(t-\tau)\left[\int_{-h}^{0} B(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,d\tau\right|\ge\frac{1}{4k}\right\}\\
&+ P\left\{\sup_{t\ge l_k}\left|\int_{t}^{\infty} X_2(t-\tau)\left[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\right|\ge\frac{1}{4k}\right\},
\end{aligned}
\tag{3.27}
$$

$k\in\mathbf{N}$.

Similar to the approach taken in the first part of the theorem, we shall now proceed to estimate each term present on the right-hand side of the aforementioned inequality. We start from the first term. By applying Chebyshev's inequality, we obtain the following expression:

$$
\begin{aligned}
&P\Big\{\sup_{t\ge l_k}\left|\int_{t_0}^{t} X_1(t-\tau)\left[\int_{-h}^{0} B(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,d\tau\right|\ge\frac{1}{4k}\Big\}\\
&\qquad\le 4k\mathbf{E}\sup_{t\ge l_k}\left|\int_{t_0}^{t} X_1(t-\tau)\left[\int_{-h}^{0} B(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,d\tau\right|\\
&\qquad\le 4k\mathbf{E}\sup_{t\ge l_k}\int_{t_0}^{t}\|X_1(t-\tau)\|\,\left\|\int_{-h}^{0} B(\tau,\theta)y(\tau+\theta)\,d\theta\right\|\,d\tau\\
&\qquad\le 4k\mathbf{E}\sup_{t\ge l_k}\int_{t_0}^{t}\|X_1(t-\tau)\|b(\tau)\|y_\tau\|_C\,d\tau\\
&\qquad\le 4k\mathbf{E}\sup_{t\ge l_k}\int_{t_0}^{t} a_1e^{-\alpha(t-\tau)}b(\tau)\|y_\tau\|_C\,d\tau\\
&\qquad= 4ka_1\mathbf{E}\sup_{t\ge l_k}\left(\int_{t_0}^{\frac{t}{2}} e^{-\alpha(t-\tau)}b(\tau)\|y_\tau\|_C\,d\tau+\int_{\frac{t}{2}}^{t} e^{-\alpha(t-\tau)}b(\tau)\|y_\tau\|_C\,d\tau\right)\\
&\qquad\le 4ka_1\sqrt{\tilde K\mathbf{E}\|\phi(\theta)\|_C^2}(e^{\frac{-\alpha l_k}{2}}\int_{t_0}^{\infty} b(\tau)\,d\tau+\int_{\frac{l_k}{2}}^{\infty} b(\tau)\,d\tau)\\
&\qquad\le 4ka_1\sqrt{\tilde K\mathbf{E}\|\phi(\theta)\|_C^2}(e^{\frac{-\alpha l_k}{2}}K_1+\frac{1}{2^k})=:I_k^{(1)}.
\end{aligned}
\tag{3.28}
$$

In order to estimate the second term on the right-hand side of the inequality (3.27), let's consider the sequence of random events

$$
A_N=\{\omega|\sup_{l_k\le t\le N}|\int_{t_0}^{t} X_1(t-\tau)[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)d\theta]\,dW(\tau)|\ge\frac{1}{4k}\}.
$$

For an arbitrary $K_1\le K_2$ we have $A_{K_1}\subset A_{K_2}$. Therefore A_N is a monotone sequence of sets, and

$$A = \lim_{N\to\infty} A_N = \bigcup_{N=0}^{\infty} A_N$$
$$= \{\omega | \sup_{l_k\le t} | \int_{t_0}^{t} X_1(t-\tau)[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta]\,dW(\tau)| \ge \frac{1}{4k}\},$$

so that

$$P\{A\} = \lim_{N\to\infty} P\{A_N\}.$$

Hence, for $N \ge l_k$,

$$\begin{aligned}
&\sup_{l_k\le t\le N} \left|\int_{t_0}^{t} X_1(t-\tau)[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta]\,dW(\tau)\right| \\
&\quad\le \sup_{l_k\le t\le N} \left|\int_{t_0}^{l_k} X_1(t-\tau)[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta]\,dW(\tau)\right| \\
&\quad+ \sup_{l_k\le t\le N} \left|\int_{l_k}^{t} X_1(t-\tau)[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta]\,dW(\tau)\right|.
\end{aligned} \tag{3.29}$$

Thus, we have

$$\begin{aligned}
&P\Big\{\sup_{l_k\le t\le N} \left|\int_{t_0}^{t} X_1(t-\tau)\left[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\right| \ge \frac{1}{4k}\Big\} \\
&\le P\left\{\sup_{l_k\le t\le N} \left|\int_{t_0}^{l_k} X_1(t-\tau)\left[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\right| \ge \frac{1}{8k}\right\} \\
&\quad + P\left\{\sup_{l_k\le t\le N} \left|\int_{l_k}^{t} X_1(t-\tau)\left[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\right| \ge \frac{1}{8k}\right\}.
\end{aligned} \tag{3.30}$$

Let us start with the first term of the last inequality

$$\begin{aligned}
&P\Big\{\sup_{l_k\le t\le N} \left|\int_{t_0}^{l_k} X_1(t-\tau)\left[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\right| \ge \frac{1}{8k}\Big\} \\
&\le 64k^2\mathbf{E}\left(\sup_{l_k\le t\le N} \left|\int_{t_0}^{l_k} X_1(t-\tau)\left[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\right|^2\right) \\
&\le 64k^2\tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2\left(e^{-\alpha l_k}\int_{t_0}^{\frac{l_k}{2}} d^2(\tau)\,d\tau + \int_{\frac{l_k}{2}}^{l_k} d^2(\tau)\,d\tau\right) \\
&\le 64k^2\tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2\left(e^{-\alpha l_k}\int_{t_0}^{\infty} d^2(\tau)\,d\tau + \int_{\frac{l_k}{2}}^{\infty} d^2(\tau)\,d\tau\right) \\
&\le 64k^2\tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2\left(e^{-\alpha l_k}K_1 + \frac{1}{2^k}\right) =: I_k^{(2)}.
\end{aligned} \tag{3.31}$$

Now we can move to the second term on the right-hand side of the inequality (3.30)

$$
\begin{aligned}
&P\left\{\sup_{l_k\le t\le N}\left|\int_{l_k}^{t} X_1(t-\tau)\left[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\right|\ge\frac{1}{8k}\right\}\\
&\quad= P\Big\{\sup_{l_k\le t\le N}\Big|\int_{l_k}^{t} X_1(t-\tau)+X_1(t-l_k)\\
&\qquad - X_1(t-l_k)\left[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\Big|\ge\frac{1}{8k}\Big\}\\
&\quad\le P\Big\{\sup_{l_k\le t\le N}\Big|\int_{l_k}^{t} X_1(t-\tau)-X_1(t-l_k)\\
&\qquad\times\left[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\Big|\ge\frac{1}{16k}\Big\}\\
&\qquad+P\left\{\sup_{l_k\le t\le N}\left|\int_{l_k}^{t} X_1(t-l_k)\left[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\right|\ge\frac{1}{16k}\right\},
\end{aligned}
\tag{3.32}
$$

Next, we can estimate each of the terms in the above inequality. Let us start with the second term.

$$
\begin{aligned}
&P\left\{\sup_{l_k\le t\le N}\left|\int_{l_k}^{t} X_1(t-l_k)\left[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\right|\ge\frac{1}{16k}\right\}\\
&\quad\le P\left\{\sup_{l_k\le t\le N}\|X_1(t-l_k)\|\sup_{l_k\le t\le N}\left|\int_{l_k}^{t}\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\,dW(\tau)\right|\ge\frac{1}{16k}\right\}\\
&\quad\le 256k^2a_1^2\mathbf{E}\left(\sup_{l_k\le t\le N}(|\int_{l_k}^{t}\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\,dW(\tau)|^2)\right)\\
&\quad\le 1024k^2a_1^2\int_{l_k}^{N} d^2(\tau)\tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2\,d\tau\\
&\quad\le 1024k^2a_1^2\tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2\int_{l_k}^{\infty} d^2(\tau)\tau\,d\tau\\
&\quad\le 1024k^2a_1^2\tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2\frac{1}{2^k}.
\end{aligned}
\tag{3.33}
$$

The following estimations are essential in order to deal with the first term of the inequality we are considering at this step.

$$
\begin{aligned}
P\Big\{\sup_{l_k\le t\le N}\Big|\int_{l_k}^{t} X_1(t-\tau)+X_1(t-l_k)-X_1(t-l_k)\\
\times\left[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\Big|\ge\frac{1}{8k}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_{l_k}^{t}\Big(\int_{l_k}^{\tau} X_1(t-s)A\Big[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\Big]\,dW(\tau)\Big)\\
&= -\int_{l_k}^{t}\Big(\int_{l_k}^{t} X_1(t-s)AI_{\{s\le\tau\}}\Big[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\Big]\,dW(\tau)\Big) \qquad (3.34)\\
&= -\int_{l_k}^{t} X_1(t-s)A\Big(\int_{l_k}^{t} I_{\{s\le\tau\}}\Big[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\Big]\,dW(\tau)\Big)\,ds.
\end{aligned}
$$

From this, we obtain

$$
\begin{aligned}
&P\Big\{\sup_{l_k\le t\le N}\Big|\int_{l_k}^{t} X_1(t-\tau)-X_1(t-l_k)\\
&\qquad\times\Big[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\Big]\,dW(\tau)\Big|\ge\frac{1}{16k}\Big\}\\
&\le P\Big\{\sup_{l_k\le t\le N}\Big|\int_{l_k}^{t} X_1(t-s)A\Big(\int_{l_k}^{t} I_{\{s\le\tau\}}\\
&\qquad\times\Big[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\Big]\,dW(\tau)\Big)\,ds\Big|\ge\frac{1}{16k}\Big\}\\
&\le 256k^2\mathbf{E}\Big(\sup_{l_k\le t\le N}\Big|\int_{l_k}^{t} X_1(t-s)A\Big(\int_{l_k}^{t} I_{\{s\le\tau\}}\\
&\qquad\times\Big[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\Big]\,dW(\tau)\Big)\,ds\Big|\Big)^2\\
&\le 256k^2\mathbf{E}\Big(\sup_{l_k\le t\le N}\Big(\int_{l_k}^{t} a_1e^{-\alpha(t-s)}\|A\|\\
&\qquad\times\Big|\int_{l_k}^{t} I_{\{s\le\tau\}}\Big[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\Big]\,dW(\tau)\Big)\Big|\,ds\Big)^2\\
&\le 256k^2E\Big(\sup_{l_k\le t\le N}(\int_{l_k}^{t} a_1^2e^{-2\alpha(t-s)}\|A\|^2\\
&\qquad\times\int_{l_k}^{t}\Big|\int_{l_k}^{t} I_{\{s\le\tau\}}\Big[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)d\theta\Big]\,dW(\tau))\Big|^2\,ds\Big)\\
&\le \frac{256k^2a_1^2\|A\|^2}{2\alpha}\mathbf{E}\Big(\sup_{l_k\le t\le N}\int_{l_k}^{t}\Big|\int_{l_k}^{t} I_{\{s\le\tau\}}\\
&\qquad\times\Big[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\Big]\,dW(\tau)\Big|^2\,ds\Big)\\
&\le \frac{256k^2a_1^2\|A\|^2}{2\alpha}\int_{l_k}^{N}\mathbf{E}\sup_{l_k\le t\le N}\Big|\int_{l_k}^{t} I_{\{s\le\tau\}}\\
&\qquad\times\Big[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\Big]\,dW(\tau)\Big|^2\,ds\\
&\le \frac{1024^2a_1^2\|A\|^2}{2\alpha}\int_{l_k}^{N}\Big(\int_{l_k}^{N} d^2(\tau)\mathbf{E}\|y_\tau\|_C^2\,d\tau\,ds\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1024^2 a_1^2 \|A\|^2 \tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2 k^2}{2\alpha} \int_{l_k}^{N} \left(\int_{l_k}^{N} d^2(\tau)\, d\tau \right) ds \\
&\le \frac{1024^2 a_1^2 \|A\|^2 \tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2 k^2}{2\alpha} \int_{l_k}^{N} \left(d^2(\tau) \int_{l_k}^{\tau} ds \right) d\tau \\
&\le \frac{1024^2 a_1^2 \|A\|^2 \tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2 k^2}{2\alpha} \int_{l_k}^{N} \tau d^2(\tau)\, d\tau \\
&\le \frac{1024^2 a_1^2 \|A\|^2 \tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2 k^2}{2\alpha 2^k}.
\end{aligned} \tag{3.35}
$$

According to (3.33) and (3.35)

$$
\begin{aligned}
P\Big\{ \sup_{l_k \le t \le N} \Big| \int_{l_k}^{t} X_1(t-\tau)[\int_{-h}^{0} D(\tau,\theta) y(\tau+\theta)\, d\theta]\, dW(\tau) \Big| \ge \frac{1}{8k} \Big\} \\
\le 256 \tilde{K}\mathbf{E}||\phi(\theta)||_C^2 (1 + 4a_1^2 \|A\|^2) 2^{-k} =: I_k^{(3)}.
\end{aligned}
$$

Let us now estimate the third term on the right-hand side of the inequality (3.27).

$$
\begin{aligned}
&P\Big\{ \sup_{t \ge l_k} \Big| \int_{t}^{\infty} X_2(t-\tau) \left[\int_{-h}^{0} B(\tau,\theta) y(\tau+\theta)\, d\theta \right] d\tau \Big| \ge \frac{1}{4k} \Big\} \\
&\quad \le P\left\{ \sup_{t \ge l_k} \left| \int_{t}^{\infty} \|X_2(t-\tau)\| \left\| \int_{-h}^{0} B(\tau,\theta) y(\tau+\theta)\, d\theta \right\| d\tau \right| \ge \frac{1}{4k} \right\} \\
&\quad \le P\left\{ \sup_{t \ge l_k} \left| \int_{l_k}^{\infty} \|X_2(t-\tau)\| \left\| \int_{-h}^{0} B(\tau,\theta) y(\tau+\theta)\, d\theta \right\| d\tau \right| \ge \frac{1}{4k} \right\} \\
&\quad \le 16k^2 \sup_{t \ge l_k} \left| \int_{l_k}^{\infty} \|X_2(t-\tau)\| b(\tau) \|y_\tau\|_C \, d\tau \right| \\
&\quad \le 16k^2 a_2^2 \sqrt{\tilde{K}\mathbf{E}\|\phi(\theta)\|^2} \int_{l_k}^{\infty} b(\tau)\, d\tau \\
&\quad \le 6k^2 a_2^2 \sqrt{\tilde{K}\mathbf{E}\|\phi(\theta)\|^2} \frac{1}{2^k} =: I_k^{(4)}.
\end{aligned} \tag{3.36}
$$

We shall now proceed to estimate the final term located on the right-hand side of inequality (3.27). Let us consider the random events sequence

$$
A_N = \left\{ \omega \Big| \sup_{l_k \le t \le N} \left| \int_{t}^{\infty} X_2(t-\tau) \left[\int_{-h}^{0} D(\tau,\theta) y(\tau+\theta)\, d\theta \right] dW(\tau) \right| \ge \frac{1}{4k} \right\}.
$$

By definition, A_N is a monotone sequence of sets, therefore

$$
\begin{aligned}
A &= \lim_{N \to \infty} A_N = \bigcup_{N=0}^{\infty} A_N \\
&= \left\{ \omega \Big| \sup_{l_k \le t} \left| \int_{t}^{\infty} X_2(t-\tau) \left[\int_{-h}^{0} D(\tau,\theta) y(\tau+\theta)\, d\theta \right] dW(\tau) \right| \ge \frac{1}{4k} \right\}.
\end{aligned}
$$

so that

$$P\{A\} = \lim_{N\to\infty} P\{A_N\}.$$

Since $l_k \le t$, the following inequality holds

$$\begin{aligned} &P\left\{\sup_{l_k\le t\le N}\left|\int_t^\infty X_2(t-\tau)\left[\int_{-h}^0 D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\right| \ge \frac{1}{4k}\right\} \\ &\quad\le P\left\{\sup_{l_k\le t\le N}\left|\int_{l_k}^\infty X_2(t-\tau)\left[\int_{-h}^0 D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\right| \ge \frac{1}{8k}\right\} \\ &\qquad+ P\left\{\sup_{l_k\le t\le N}\left|\int_{l_k}^t X_2(t-\tau)\left[\int_{-h}^0 D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\right| \ge \frac{1}{8k}\right\}. \end{aligned} \tag{3.37}$$

Let's proceed with each term of the latter inequality

$$\begin{aligned} &P\left\{\sup_{l_k\le t\le N}\left|\int_{l_k}^\infty X_2(t-\tau)\left[\int_{-h}^0 D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\right| \ge \frac{1}{8k}\right\} \\ &\le P\left\{\sup_{l_k\le t\le N}\|X_2(\tau)\|\,\left|\int_{l_k}^\infty X_2^{-1}(t)\left[\int_{-h}^0 D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\right| \ge \frac{1}{8k}\right\} \\ &\le 64k^2a_2^4\tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2\frac{1}{2^k} =: I_k^{(5)}. \end{aligned} \tag{3.38}$$

Now we move to the second term,

$$\begin{aligned} P\Big\{\sup_{l_k\le t\le N}&\left|\int_{l_k}^t X_2(t-\tau)\left[\int_{-h}^0 D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\right| \ge \frac{1}{8k}\Big\} \\ &\le 256k^2a_2^4\tilde{K}\mathbf{E}||\phi(\theta)||_C^2\frac{1}{2^k} =: I_k^{(6)}. \end{aligned} \tag{3.39}$$

By taking the limit $N\to\infty$ in (3.37), we get

$$P\left\{\sup_{t\ge l_k}\left|\int_t^\infty X_2(t-\tau)\left[\int_{-h}^0 B(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,d\tau\right| \ge \frac{1}{4k}\right\} \le I_k^{(5)} + I_k^{(6)}.$$

Finally

$$P\left\{\sup_{t\ge l_k}|x(t)-y(t)| \ge \frac{1}{k}\right\} \le I_k^{(1)} + I_k^{(2)} + I_k^{(3)} + I_k^{(4)} + I_k^{(5)} + I_k^{(6)} = I_k.$$

The convergence of the series $\sum_{k=1}^\infty I_k$ is evident and according to the Borel-Cantelli lemma there exists a positive integer $M = M(\omega)$ such that, for arbitrary $k \ge M(\omega)$ for arbitrary $k \ge M(\omega)$

$$\sup_{t\ge l_k}|x(t)-y(t)| \ge \frac{1}{k},$$

with probability 1. Hence, for almost all ω and arbitrary $\epsilon > 0$ there exists $T = T(\epsilon, \omega) = l_{k_0}$, where $k_0 = \max\{[\frac{1}{\epsilon}], M(\omega)\}$, such that the following inequality holds for all $t \geq T$:

$$|x(t) - y(t)| \leq \sup_{t \geq T} |x(t) - y(t)| \geq \frac{1}{k_0} \leq \epsilon.$$

Theorem is proved. □

4. Example

Let us give an application of our theorem.

Example 4.1. We consider a system of ordinary differential equations,

$$d \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} dt, \tag{4.1}$$

Together with the system (4.1), consider the following system of functional stochastic differential equations

$$\begin{aligned} d \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} dt + \int_{-h}^{0} B(t, \theta) \begin{bmatrix} y_1(t+\theta) \\ y_2(t+\theta) \end{bmatrix} d\theta \, dt \\ + \int_{-h}^{0} D(t, \theta) \begin{bmatrix} y_1(t+\theta) \\ y_2(t+\theta) \end{bmatrix} d\theta \, dW(t), \end{aligned} \tag{4.2}$$

where

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \tag{4.3}$$

$$B(t, \theta) = \begin{bmatrix} \frac{b_1(\theta)}{(t+1)^3} & 0 \\ 0 & \frac{b_1(\theta)}{(t+1)^3} \end{bmatrix}, \tag{4.4}$$

$$D(t, \theta) = \begin{bmatrix} 0 & \frac{d_1(\theta)}{(t+1)^2} \\ \frac{d_1(\theta)}{(t+1)^2} & 0 \end{bmatrix}, \tag{4.5}$$

here $b_1(\theta), d_1(\theta)$ are continuous functions on $[-h, 0]$. Then

$$\begin{aligned} \left\| \int_{-h}^{0} B(t, \theta) \phi(\theta) \, d\theta \right\| &\leq \int_{-h}^{0} \|B(t, \theta)\| \, d\theta \|\phi\|_C \\ &= \frac{\sqrt{2}}{(t+1)^2} \int_{-h}^{0} b_1(\theta) \, d\theta \|\phi\|_C. \end{aligned} \tag{4.6}$$

Therefore, $b(t) = \frac{\sqrt{2}}{(t+1)^2} \int_{-h}^{0} b_1(\theta) \, d\theta$ and $\int_0^\infty b(t) \, dt < \infty$

$$\begin{aligned} \left\| \int_{-h}^{0} D(t, \theta) \phi(\theta) \, d\theta \right\| &\leq \int_{-h}^{0} \|D(t, \theta)\| \, d\theta \|\phi\|_C \\ &= \frac{\sqrt{2}}{(t+1)^2} \int_{-h}^{0} d_1(\theta) \, d\theta \|\phi\|_C. \end{aligned} \tag{4.7}$$

Therefore, $d(t) = \frac{\sqrt{2}}{(t+1)^2}\int_{-h}^{0} d_1(\theta)\,d\theta$ and $\int_0^\infty td^2(t)\,dt < \infty$. From this follows that system (4.2) is asymptotically equivalent to the system (4.1) in the mean square sense and with probability 1.

5. Conclusions

This work proposes a new method for studying the asymptotic behavior at infinity of solutions to linear stochastic functional-differential equations. According to this method, the problem is reduced to investigating a much simpler object: a system of ordinary linear equations with constant coefficients. This system is constructed in such a way that for every solution of the original system, a corresponding solution of the deterministic system is assigned, and the difference between them tends to zero at infinity, either in mean square or with probability one. Naturally, some smallness of the stochastic perturbation at infinity is required in terms of the convergence of integrals of the noise intensity.

6. Acknowledgments

The research of Oleksandr Stanzhytskyi was partially supported by the National Research Foundation of Ukraine No. F81/41743 and by the Ukrainian Government Scientific Research Grant No. 210BF38-01. ‘

References

1. J.K. Hale, *Theory of functional differential equations*, Springer-Verlag, New York, Heidelberg, 1977.
2. Jian. Wu, *Theory and Applications of Partial Functional Differential Equations*, Applied Mathematical Sciences,**119**, Springer-Verlag, New York, Inc., 1996.
3. E.F. Carkov, *Random perturbations of functional-differential equations*, Zinatne, Riga, 1989.
4. S.-E.A. Mohammed, *Stochastic functional differential equations*, Pitman (Advanced Publishing Program), Boston, MA, 1984.
5. A.M. Samoilenko, N.I. Mahmudov, A.N. Stanzhitskii, *Existence, uniqueness, and controllability results for neutral FSDEs in Hilbert spaces*,Dynam. Systems & Appl., **17** (1), 2008, 53–70.
6. O M. Stanzhytskyi, V.V. Mogilova, A.O. Tsukanova, *On Comparison Results for Neutral Stochastic Differential Equations of Reaction-Diffusion Type in* $L^2(\mathbb{R}^d)$, in Modern Mathematics and Mechanics, Underst. Complex Syst., Springer, Cham, 2019, 351–395.
7. A.O. Stanzhytskyi, *On Weak and Strong Solutions of Paired Stochastic Functional Differential Equations in Infinite-Dimensional Spaces*, Journal of Optimization, Differential Equations and Their Applications (JODEA), **29** (2), 2021, 48–76.
8. A. Stanzhytsky, O. Misiats, O. Stanzhytskyi, *Invariant Measure for Neutral Stochastic Functional Differential Equations with Non-Lipchitz Coefficients*, Evolution Equations and Control Theory, **11** (6), 2022, 1929–1953.
9. N. Levinson, *The Asymptotic Behavior of Systems of Linear Differential Equations*, Amer. J. Math., **68**, 1946, 1–6.

10. O.M. Stanzhytskyi, *Investigation of Exponential Dichotomy of Ito Stochastic Systems by Using Quadratic Forms*, Ukr. Mat. J., **53** (11), 2001, 1882–1894.
11. O.M. Stanzhytskyi, A.P. Krenevich, I.G. Novak, *Asymptotic Equivalence of Linear Stochastic Ito Systems and Oscillation of Solutions of Linear Second-Order Equations*, Differ. Equ., **47** (6), 2011, 799–813.

Received 21.05.2023