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STABILITY W.R.T. DISTURBANCES FOR THE GLOBAL ATTRACTOR OF MULTI-VALUED SEMIFLOW GENERATED BY NONLINEAR WAVE EQUATION

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Abstract. The paper investigates the issue of stability with respect to external disturbances for the global attractor of the wave equation under conditions that do not ensure the uniqueness of the solution to the initial problem. Under general conditions for nonlinear terms, it is proved that the global attractor of the undisturbed problem is locally stable in the sense of ISS and has the AG property with respect to disturbances.

Key words: global attractor, multi-valued semiflow, local input-to-state stability, asymptotic gain, wave equation.

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1. Introduction

Properties of global attractors of nonlinear wave equations with dissipation under different assumptions on the interaction functions have been under investigation in many papers (see [1, 2] and references therein). With the appearance of the works [3, 4], it became possible to study invariant uniformly attracting sets of infinite-dimensional dynamical systems without uniqueness of the solution of the initial problem, considering instead of a classical semigroup its multivalued counterpart called an m-semiflow. In particular, for the wave equation with nonsmooth nonlinear term f the existence and properties of the global attractor of the corresponding m-semiflow were investigated in [5].

In the presence of external disturbances, the problem becomes non-autonomous and its dynamics can be described in terms of uniform attractors of semi-processes [6–9]. It turned out, that this theory also allows us to solve the problem of estimating the deviation of the solution of the disturbed equation from the global attractor of the undisturbed system. In the case of a trivial attractor consisting

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of a single asymptotically stable equilibrium point, for the simplest partial differential equation of the reaction-diffusion type, such results first appeared in [10]. The technique of this work was based on the classical ISS approach of Lyapunov functions [11–13] and could not be applied to systems with non-trivial attractors. The corresponding technique was developed in the works of [14, 15] and applied to the wave equation with a smooth interaction function f and disturbances of the type $h(x)d(t)$ in the work [16]. The extension of this theory to the case of non-uniqueness of solution of the initial problem was carried out in [17], where the local ISS property of the attractor was established for the reaction-diffusion system.

In the present paper, we consider a wave equation with a non-smooth nonlinearity $f(y)$ and a $g(y)d(t)$-type disturbance with a non-smooth function g. Local ISS and AG stability properties with respect to disturbances are established for the global attractor of the undisturbed problem ($d \equiv 0$).

2. Setting of the problem

In a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$ we consider the following boundary-value problem

$$\begin{cases} \frac{\partial^2 y(t,x)}{\partial t^2} + \alpha \frac{\partial y(t,x)}{\partial t} - \triangle y(t,x) + f(y(t,x)) = g(y(t,x))d(t), \ t > 0, \\ y(t,x)|_{x\in\partial\Omega} = 0, \end{cases} \tag{2.1}$$

where $\alpha > 0$, $f, g \in C(\mathbb{R})$ are given, $d \in L^\infty(\mathbb{R}_+)$ is a disturbance parameter.

We prove (see Lemma 3.1) that under rather general assumptions on f, g the problem (2.1) is globally resolvable (in weak sense) in the phase space $X = H_0^1(\Omega) \times L^2(\Omega)$. The uniqueness of solutions is not guaranteed.

Let us consider a multi-valued map $S_d : \mathbb{R}_+ \times X \mapsto 2^X$,

$$S_d(t, z_0) = \{z(t) \mid z = \begin{pmatrix} y \\ y_t \end{pmatrix} \text{ is a solution of (2.1), } z(0) = z_0\}. \tag{2.2}$$

For $d \equiv 0$ (undisturbed problem) the multi-valued map $S_0 : \mathbb{R}_+ \times X \mapsto 2^X$ is a multi-valued semigroup (m-semiflow), which possesses a global attractor $\Theta \subset X$, i.e., there exists a compact set $\Theta \subset X$ such that

$$\Theta = S_0(t, \Theta) \ \forall t \geq 0,$$

$$\forall r > 0 \quad \sup_{\|z_0\| \leq r} \operatorname{dist}(S_0(t, z_0), \Theta) \to 0, t \to \infty.$$

Here and after we use denotations:

$$\operatorname{dist}(A, B) = \sup_{\xi \in A} \inf_{\eta \in B} \|\xi - \eta\|_X, \ \|A\|_\Theta = \operatorname{dist}(A, \Theta).$$

Thus, in the undisturbed case, all trajectories (2.1) eventually end up in an arbitrarily small neighborhood of Θ. The paper investigates the issue of estimating the deviation of the trajectory of the disturbed problem (2.1) from the set Θ depending on the value of $\|d\|_\infty = \operatorname{ess\,sup}_{t\in(0,+\infty)} |d(t)|$.

This question in terms of Input-to-State Stability (ISS) theory can be solved by setting the estimate (ISS property): $\forall t \geq 0$

$$\|S_d(t, z_0)\|_\Theta \leq \beta(\|z_0\|_\Theta, t) + \gamma(\|d\|_\infty). \tag{2.3}$$

Here $\gamma : [0, +\infty) \mapsto [0, +\infty)$ is a continuous strictly increasing function with $\gamma(0) = 0$ ($\gamma \in \mathcal{K}$), $\beta : [0, +\infty) \times [0, +\infty) \mapsto [0, +\infty)$ is a continuous function, $\forall t \geq 0$ $\beta(, t) \in \mathcal{K}$, $\forall s \geq 0$ $\beta(s,)$ decreases to 0 ($\beta \in \mathcal{KL}$).

The main results of this work are a local variant of (2.3) (local ISS) (see Theorem 4.1) and Asymptotic Gain (AG) property: $\forall z_0 \in X$

$$\overline{\lim_{t\to\infty}} \|S_d(t, z_0)\|_\Theta \leq \gamma(\|d\|_\infty). \tag{2.4}$$

3. Existence, a priori estimates, and regularity of solutions.

Assume that there exist positive constants m, c_1, c_2, c_3, c_4 such that $\forall s \in \mathbb{R}$

$$|f(s)| \leq c_1(1 + |s|^{\frac{n}{n-2}}), \tag{3.1}$$

$$F(s) \geq -as^2 - c_2,\ f(s)s - F(s) + as^2 \geq -c_3, \tag{3.2}$$

$$|g(s)| \leq c_4, \tag{3.3}$$

where $a < \frac{\lambda_1}{2}$, λ_1 is the first eigenvalue of $-\triangle$ in $H_0^1(\Omega)$, $F(s) := \int_0^s f(t)\,dt$.

Remark 3.1. In all further arguments in the case $n = 2$ we can assume that in (3.1) f has arbitrary power growth because of embedding $H_0^1(\Omega) \subset L^p(\Omega)$, $\forall\ p \geq 1$, and in the case $n = 1$ assumption (3.1) is not needed because of embedding $H_0^1(\Omega) \subset C(\overline{\Omega})$.

A solution of (2.1) we will understand in a weak sense, i.e., a pair of functions $z() = \begin{pmatrix} y() \\ y_t() \end{pmatrix} \in L^\infty(0, T; X)$ is called a solution of (2.1) on $(0, T)$ if $\forall \psi \in H_0^1(\Omega)$, $\forall \eta \in C_0^\infty(0, T)$ the following equality holds

$$-\int_0^T (y_t, \psi)\eta_t + \int_0^T \Big(\alpha(y_t, \psi) + (y, \psi)_{H_0^1} + (f(y), \psi) - (g(y), \psi)d(t)\Big)\,\eta = 0, \tag{3.4}$$

where by $\|\|$ and $(,)$ we denote the norm and scalar product in $L^2(\Omega)$.

If $z \in L^\infty_{loc}(\mathbb{R}_+; X)$ satisfies (3.4) $\forall T > 0$, then z is called a global solution (a solution for short) of (2.1).

Lemma 3.1. *Under assumptions* (3.1)-(3.3) $\forall z_0 \in X$, $\forall d \in L^2_{loc}(\mathbb{R}_+)$ *there exists at least one solution of* (2.1) *with* $z|_{t=0} = z_0$.

Proof. First, it should be noted that due to embedding $H_0^1(\Omega) \subset L^{\frac{2n}{n-2}}(\Omega)$, $n \geq 3$, from conditions (3.1), (3.3) we deduce that for $y \in L^\infty(0,T;H_0^1(\Omega))$

$$f(y) \in L^2(0,T;L^2(\Omega)),\ g(y)d(t) \in L^2(0,T;L^2(\Omega)).$$

So, results of [1] allow us to claim that for every solution of (2.1) and $\forall T > 0$

$$z = \begin{pmatrix} y \\ y_t \end{pmatrix} \in C([0,T];X).$$

In particular, the initial condition $z|_{t=0} = z_0$ makes sense.

We prove an existence of solution of (2.1) by Galerkin method [1]. Let $z_0 = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \in X, T > 0$ be given. For every $m \geq 1$ we consider an approximation function

$$y_m(t) = \sum_{i=1}^m g_{im}(t)\omega_i,$$

where $\{\omega_i\}_{i\geq 1}$ are eigenfunctions of $-\triangle$ in $H_0^1(\Omega)$, and $\{g_{im}()\}$ are solutions of ODE system

$$\frac{d^2}{dt^2}(y_m,\omega_j) + \alpha\frac{d}{dt}(y_m,\omega_j) + (y_m,\omega_j)_{H_0^1} + (f(y_m),\omega_j) - (g(y_m),\omega_j)d(t) = 0,\ j = \overline{1,m} \quad (3.5)$$

$$y_m|_{t=0} = y_m(0) \to y_0 \text{ in } H_0^1(\Omega),\ y_m'|_{t=0} = y_m'(0) \to y_1 \text{ in } L^2(\Omega).$$

Due to *Carathéodory's* theorem we have a solution of (3.5) on $[0,T_m]$. Let us derive a priori estimates which would imply that $T_m = T$. For this purpose, we introduce a function

$$Y_m(t) = \frac{1}{2}\|y_m'(t)\|^2 + \frac{1}{2}\|y_m(t)\|^2_{H_0^1} + (F(y_m(t)),1) + \delta(y_m'(t),y_m(t)),$$

where $\delta \in (0,\alpha)$ we will choose later.

Due to (3.5) we get:

$$\begin{aligned}
\frac{dY_m}{dt} &= -(\alpha-\delta)\|y_m'(t)\|^2 - \delta\|y_m(t)\|^2_{H_0^1} - \alpha\delta(y_m',y_m) \\
&\quad - \delta(f(y_m),y_m) + (y_m',g(y_m))d(t) - \delta(y_m,g(y_m))d(t) \\
&= -\delta Y_m(t) + \left(-\alpha + \frac{3\delta}{2}\right)\|y_m'(t)\|^2 - \frac{\delta}{2}\|y_m(t)\|^2_{H_0^1} \\
&\quad + \delta\left((F(y_m),1) - (f(y_m),y_m)\right) - \alpha\delta(y_m',y_m) + \delta^2(y_m',y_m) \\
&\quad + (y_m',g(y_m))d(t) - \delta(y_m,g(y_m))d(t) \\
&\leq -\delta Y_m(t) + \left(-\alpha + \frac{3\delta}{2}\right)\|y_m'(t)\|^2 - \frac{\delta}{2}\|y_m(t)\|^2_{H_0^1} \\
&\quad + \delta m\|y_m\|^2 - \delta c_3 - \delta(\alpha-\delta)(y_m',y_m) \\
&\quad + (y_m',g(y_m))d(t) - \delta(y_m,g(y_m))d(t).
\end{aligned}$$

Taking into account the *Poincaré* inequality $\|y_m\|^2_{H^1_0} \geq \lambda_1 \|y_m\|^2$ and assumption

$$\lambda_1 - 2a > 0,$$

we derive that for sufficiently small $\delta \in (0, \alpha)$ there exists a constant $c_5 > 0$ such that

$$\frac{d}{dt} Y_m(t) \leq -\delta Y_m(t) + c_5 (1 + \|d\|^2). \tag{3.6}$$

Using estimate (3.6) and assumption (3.2) we get

$$\begin{aligned}
&\frac{1}{2}\|y'_m\|^2 + \left(\frac{1}{2} - \frac{a}{\lambda_1}\right)\|y_m\|^2_{H^1_0} + \delta(y'_m, y_m) - c_2|\Omega| \\
&\quad \leq \left(\frac{1}{2}\|y'_m(0)\|^2 + \frac{1}{2}\|y_m(0)\|^2_{H^1_0} + (F(y_m(0)), 1)\right) e^{-\delta t} \\
&\qquad + \delta(y'_m(0), y_m(0)) e^{-\delta t} + c_5 \left(\frac{1}{\alpha} + \int_0^t \|d(s)\|^2 e^{-\delta(t-s)}\, ds\right).
\end{aligned}$$

Thus, there exists a constant $c_6 > 0$ such that for sufficiently small $\delta > 0$ and for every $m \geq 1$ the following estimate holds:

$$\begin{aligned}
\|y'_m(t)\|^2 + \|y_m(t)\|^2_{H^1_0} \leq c_6 \Big((\|y'_m(0)\|^2 + \|y_m(0)\|^2_{H^1_0} \\
+ \|y_m(t)\|^{\frac{2n-2}{n-2}}_{H^1_0}) e^{-\delta t} + 1 + \int_0^t \|d(s)\|^2 e^{-\delta(t-s)}\, ds \Big). \quad (3.7)
\end{aligned}$$

This estimate allows us to claim that solutions y_m exist on $[0, T]$ and for some function $z = \begin{pmatrix} y \\ y_t \end{pmatrix} \in L^\infty(0, T; X)$ up to subsequence

$$\begin{aligned}
&y_m \to y \text{ weak-* in } L^\infty(0, T; H^1_0(\Omega)), \\
&y'_m \to y_t \text{ weak-* in } L^\infty(0, T; L^2(\Omega)).
\end{aligned} \tag{3.8}$$

So, due to the Compactness Lemma [18]

$$y_m \to y \text{ in } L^2(0, T; L^2(\Omega)) \text{ and almost everywhere (a.e.) on } (0, T) \times \Omega. \tag{3.9}$$

Then

$$f(y_m) \to f(y),\ g(y_m) \to g(y) \text{ weakly in } L^2(0, T; L^2(\Omega)). \tag{3.10}$$

Passing to the limit in (3.5), we get that the function $z = \begin{pmatrix} y \\ y_t \end{pmatrix}$ satisfies (3.4) with $z(0) = z_0$. Therefore, z is the required solution of (2.1), and estimate (3.7) takes place. Lemma is proved. □

Remark 3.2. Since for the solution $z = \begin{pmatrix} y \\ y_t \end{pmatrix}$

$$f(y),\ g(y)d(t) \in L^2(0,T;L^2(\Omega)),$$

then from [1] it follows that functions

$$t \mapsto \|y_t(t)\|^2 + \|y(t)\|^2_{H_0^1},\ t \mapsto (F(y(t)),1)\,,\ t \mapsto (y_t(t),y(t))$$

are absolutely continuous. Therefore, for the function

$$Y(t) = \frac{1}{2}\|y_t(t)\|^2 + \frac{1}{2}\|y(t)\|^2_{H_0^1} + (F(y(t)),1) + \delta(y_t(t),y(t))$$

we can repeat all arguments (3.6), (3.7) and obtain that every solution of (2.1) satisfies (3.7).

Moreover, if $d \in L^\infty(\mathbb{R}_+)$, then from (3.7) we deduce that every solution of (2.1) $z = \begin{pmatrix} y \\ y_t \end{pmatrix}$ satisfies the following estimate: $\forall t \geq 0$

$$\|y_t(t)\|^2 + \|y(t)\|^2_{H_0^1} \leq c_6 \Big((\|y_t(0)\|^2 + \|y(0)\|^2_{H_0^1} + \|y(0)\|^{\frac{2n-2}{n-2}}_{H_0^1})e^{-\delta t} + 1 + \frac{1}{\delta}\|d\|^2_\infty \Big). \quad (3.11)$$

Remark 3.3. For $n = 1,2$ in estimates (3.7), (3.11) the term with degree $\frac{2n-2}{n-2}$ is absent.

Lemma 3.2. *Let* $\{z_n = \begin{pmatrix} y \\ y_{n_t} \end{pmatrix}\}$ *be solutions of* (2.1) *on* $(0,T)$ *with disturbances* $\{d_n\} \subset L^2(0,T)$, *initial conditions* $\{z_n^0\} \subset X$, *and* $t_n \to t_0$. *If*

$$z_n^0 \to z^0 \text{ weakly in } X,\ d_n \to d \text{ weakly in } L^2(0,T), \quad (3.12)$$

then there exists a solution of (2.1) $z = \begin{pmatrix} y \\ y_t \end{pmatrix}$ *on* $(0,T)$ *such that* $z(0) = z_0$ *and up to subsequence*

$$z_n(t_n) \to z(t_0) \text{ weakly in } X. \quad (3.13)$$

If convergence in (3.12) *is strong, then*

$$z_n(t_n) \to z(t_0) \text{ in } X.$$

Proof. Assume that (3.12) are fulfilled. Using estimate (3.7) and the Compactness Lemma we can repeat arguments (3.9) and claim that z_n converges to z in the sense of (3.8), (3.9). Moreover,

$$y_n(t_n) \to y(t_0) \text{ in } L^2(\Omega),\ y_{n_t}(t_n) \to y_t(t_0) \text{ in } H^{-1}(\Omega). \quad (3.14)$$

Due to (3.9) and *Lebesgue's* dominated convergence theorem we get

$$(g(y_n),\psi) \to (g(y),\psi) \text{ in } L^2(0,T). \quad (3.15)$$

So, we can pass to the limit in (3.4) and obtain that $z = \begin{pmatrix} y \\ y_t \end{pmatrix}$ is a solution of (2.1), $z(0) = z_0$.

Estimate (3.7), convergence (3.14) and compact embedding $H_0^1(\Omega) \subset L^2(\Omega)$ guarantee that (3.13) is fulfilled.

Let convergence in (3.12) be strong. Taking into account Remark 3.2, for the absolutely continuous function

$$V_n(t) = \frac{1}{2}\|y_{n_t}(t)\|^2 + \frac{1}{2}\|y_n(t)\|^2_{H_0^1} + (F(y_n(t)), 1)$$

we have the following equality: for almost all $t \in (0, T)$

$$\frac{d}{dt}V_n(t) = -\alpha\|y_{n_t}(t)\|^2 + (g(y_n(t)), y_{n_t}(t))\, d_n(t).$$

So, for all $t \in [0, T]$, in particular, for $t = t_n$, we deduce:

$$\frac{1}{2}\left(\|y_{n_t}(t_n)\|^2 + \|y_n(t_n)\|^2_{H_0^1}\right) + \alpha\int_0^{t_n}\|y_{n_t}(s)\|^2\, ds$$
$$= V_n(0) - (F(y_n(t_n)), 1) + \int_0^{t_n}(g(y_n(s)), y_{n_t}(s))\, d_n(s)\, ds. \quad (3.16)$$

Let us justify the limit transition in the right-hand part of (3.16). It is clear that $V_n(0) \to V(0)$. Due to (3.14)

$$F(y_n(t_n, x)) \to F(y(t_0, x)) \text{ for a.a. } x \in \Omega.$$

Additionally, due to the compact embedding $H_0^1(\Omega) \subset L^{\frac{2n-2}{n-2}}(\Omega)$ we have

$$y_n(t_n) \to y(t_0) \text{ in } L^{\frac{2n-2}{n-2}}(\Omega).$$

Since from (3.1) we get the estimate

$$|F(s)| \le c_7\left(1 + |s|^{\frac{2n-2}{n-2}}\right),$$

so due to *Lebesgue's* dominated convergence theorem

$$(F(y_n(t_n)), 1) \to (F(y(t_0)), 1)\,. \quad (3.17)$$

From the same reasons

$$g(y_n) \to g(y) \text{ in } L^2(0, T; L^2(\Omega)).$$

Thus, from (3.8) and strong convergence $d_n \to d$ in $L^2(0, T)$ we derive:

$$\int_0^T (g(y_n(\tau)), y_{n_t}(\tau))\, d_n(\tau)\, d\tau \to \int_0^T (g(y(\tau)), y_t(\tau))\, d(\tau)\, d\tau. \quad (3.18)$$

Estimate (3.7) implies

$$\int_{t_0}^{t_n} |\,(g(y_n(s)), y_{n_t}(s))\, d_n(s)|\, ds \le c \int_{t_0}^{t_n} |d_n(s)|\, ds \to 0 \ as\ n \to \infty.$$

Therefore, from (3.18) we can justify the limit transition in the last term of equality (3.16). Then (3.16) yields

$$\begin{aligned} \frac{1}{2} \varliminf_{n\to\infty} \left(\|y_{n_t}(t_n)\|^2 + \|y_n(t_n)\|^2_{H_0^1} \right) + \alpha \int_0^{t_0} \|y_t(s)\|^2\, ds \\ \le V(0) - (F(y(t_0)), 1) + \int_0^{t_0} (g(y(s)), y_t(s))\, d(s)\, ds \\ = \frac{1}{2} \left(\|y_t(t_0)\|^2 + \|y(t_0)\|^2_{H_0^1} \right) + \alpha \int_0^{t_0} \|y_t(s)\|^2\, ds. \quad (3.19) \end{aligned}$$

From (3.19) we deduce that $\varliminf_{n\to\infty} \|z_n(t_n)\|_X \le \|z(t_0)\|_X$, which means that $z_n(t_n)$ converges to $z(t_0)$ strongly in X. Lemma is proved. □

4. Local ISS property for the attractor.

We consider the undisturbed problem

$$\begin{cases} \frac{\partial^2 y(t,x)}{\partial t^2} + \alpha \frac{\partial y(t,x)}{\partial t} - \triangle y(t,x) + f(y(t,x)) = 0, \ t > 0, \ x \in \Omega, \\ y(t,x)|_{x\in\partial\Omega} = 0. \end{cases} \quad (4.1)$$

Under assumptions (3.1), (3.2) it is known [5], that the m-semiflow

$$S_0(t, z_0) = \{z(t) \mid z = \begin{pmatrix} y \\ y_t \end{pmatrix} \text{ is a solution of } (4.1), z(0) = z_0\} \quad (4.2)$$

possesses global attractor Θ in the phase space $X = H_0^1(\Omega) \times L^2(\Omega)$.

Lemma 3.2 and estimate (3.11) guarantee the following properties of S_0:

$$\forall t_n \to t_0 \ge 0, \ \forall z_0^n \to z_0, \ \forall \xi_n \in S_0(t_n, z_0^n)$$

$$\text{up to subsequence } \xi_n \to \xi_0 \in S_0(t_0, z_0), \quad (4.3)$$

$$\forall r > 0 \text{ the set } \{S_0(t, z_0) \mid t \ge 0, \ \|z_0\|_X \le r\} \text{ is bounded in } X. \quad (4.4)$$

Properties (4.3), (4.4) imply stability of Θ in the following sense [17]:

$$\exists \beta \in \mathcal{KL} \ \forall z_0 \in X, \ \forall t \ge 0 \ \|S_0(t, z_0)\|_\Theta \le \beta(\|z_0\|_\Theta, t). \quad (4.5)$$

Let us consider the family of maps $\{S_d\}_{d\in U}$ defined in (2.2). Here $U = L^\infty(\mathbb{R}_+)$ describes the set of disturbances in (2.1).

In addition to conditions (3.1)-(3.3), we will make an additional assumption:

$$f \in C^1(\mathbb{R}) \text{ and } \exists c_8 > 0\ \forall s \in \mathbb{R}\ |f'(s)| \le c_8(1+|s|^r),\ r < \frac{n}{n-2}. \tag{4.6}$$

It is known [6], that assumption (4.6) ensures the uniqueness of solution in (4.1), i.e., the map S_0 defined by (4.2) is single-valued and generates a classical semigroup. It should be noted that the function g can be non-smooth, so we cannot expect uniqueness for the disturbed problem (2.1).

Theorem 4.1. *Assume that conditions* (3.1)-(3.3), (4.6) *are fulfilled. Then the family*

$$\{S_d\}_{d\in U},\ U = L^\infty(\mathbb{R}_+)$$

possesses local ISS property for the global attractor Θ, *i.e.,*

$$\exists r > 0,\ \exists \beta \in \mathcal{KL},\ \exists \gamma \in \mathcal{K} \text{ such that}$$

$$\forall \|z_0\|_\Theta \le r,\ \forall \|d\|_\infty \le r,\ \forall t \ge 0$$
$$\|S_d(t,z_0)\|_\Theta \le \beta(\|z_0\|_\Theta, t) + \gamma(\|d\|_\infty). \tag{4.7}$$

Proof. According to [17], it is enough to verify the following properties:

$$\forall r > 0 \text{ the set } \{S_d(t,z_0) \mid t \ge 0,\ \|d\|_\infty \le r,\ \|z_0\|_X \le r\} \text{ is bounded in } X, \tag{4.8}$$

$$\forall r > 0\ \exists c(r) > 0\ \forall \|z_0^{(1)}\|_X \le r,\ \|z_0^{(2)}\|_X \le r,\ \forall t \ge 0$$
$$\|S_0(t,z_0^{(1)}) - S_0(t,z_0^{(2)})\|_X \le e^{c(r)t}\|z_0^{(1)} - z_0^{(2)}\|_X, \tag{4.9}$$

$$\exists\ \kappa \in \mathcal{K},\ \exists\ \eta : \mathbb{R}_+^2 \mapsto \mathbb{R}_+ \text{ such that } \forall r > 0$$
$$\overline{\lim_{t\to 0+}} \frac{\eta(r,t)}{t} < \infty \text{ and } \forall t \ge 0,\ \forall \|z_0\|_X \le r,\ \forall \|d\|_\infty \le r$$
$$dist(S_d(t,z_0), S_0(t,z_0)) \le \eta(r,t)\kappa(\|d\|_\infty). \tag{4.10}$$

Property (4.8) is a consequence of estimate (3.11). Property (4.9) can be derived from the following arguments [16]: for $\|y_1\|_{H_0^1} \le r$, $\|y_2\|_{H_0^1} \le r$ from (4.6), $H\ddot{o}lder's$ inequality and embedding $H_0^1(\Omega) \subset L^{\frac{2n}{n-2}}(\Omega)$ we get

$$\int_\Omega |f(y_1) - f(y_2)|^2\,dx \le$$
$$c\left(1 + \|y_1\|^{\frac{n}{n-2}}_{L^{\frac{2n}{n-2}}} + \|y_2\|^{\frac{n}{n-2}}_{L^{\frac{2n}{n-2}}}\right)\|y_1 - y_2\|^2_{L^{\frac{2n}{n-2}}} \le c(r)\|y_1 - y_2\|^2_{H_0^1}. \tag{4.11}$$

Let $z^{(1)} = \begin{pmatrix} y^{(1)} \\ y_t^{(1)} \end{pmatrix}$, $z^{(2)} = \begin{pmatrix} y^{(2)} \\ y_t^{(2)} \end{pmatrix}$ be solutions of (4.1), and $\|z^{(1)}(0)\|_X \le r$, $\|z^{(2)}(0)\|_X \le r$. Then from (4.11) for the function $\omega(t) = y^{(1)}(t) - y^{(2)}(t)$, we deduce:

$$\frac{1}{2}\frac{d}{dt}\left(\|\omega_t\|^2 + \|\omega\|^2_{H_0^1}\right) + \alpha\|\omega_t\|^2 \le c^{\frac{1}{2}}(r)\|\omega\|_{H_0^1}\|\omega_t\|,$$

$$\frac{d}{dt}\left(\|\omega_t\|^2+\|\omega\|^2_{H_0^1}\right)\le c^{\frac{1}{2}}(r)\left(\|\omega_t\|^2+\|\omega\|^2_{H_0^1}\right).$$

After applying *Grönwall's* lemma we obtain (4.9).

For proving (4.10) we consider arbitrary solution $z^{(1)}=\begin{pmatrix} y^{(1)} \\ y_t^{(1)} \end{pmatrix}$ of (2.1) with disturbance d, $\|d\|_\infty\le r$ and initial data z_0. Let $z^{(2)}=\begin{pmatrix} y^{(2)} \\ y_t^{(2)} \end{pmatrix}$ be a unique solution of (4.1) with initial data z_0, $\|z_0\|_X\le r$. Then for the function $\omega(t)=y^{(1)}(t)-y^{(2)}(t)$ we have the following estimate: for a.a. $t\in(0,T)$

$$\frac{d}{dt}\left(\|\omega_t\|^2+\|\omega\|^2_{H_0^1}\right)\le c^{\frac{1}{2}}(r)\left(\|\omega_t\|^2+\|\omega\|^2_{H_0^1}\right) + c_4|\Omega|^{\frac{1}{2}}\|d\|_\infty\sup_{t\in[0,T]}\left(\|\omega_t\|+\|\omega\|_{H_0^1}\right). \quad (4.12)$$

Integrating over $[0,t]$, we get: $\forall\ t\in(0,T)$

$$\|\omega_t(t)\|^2+\|\omega(t)\|^2_{H_0^1}\le c^{\frac{1}{2}}(r)\int_0^t\left(\|\omega_t(s)\|^2+\|\omega(s)\|^2_{H_0^1}\right)ds + c_4T|\Omega|^{\frac{1}{2}}\|d\|_\infty\sup_{t\in[0,T]}\left(\|\omega_t\|+\|\omega\|_{H_0^1}\right). \quad (4.13)$$

After applying *Grönwall's* lemma from (4.13) we derive the existence of $c>0$, $\eta(r)>0$ such that

$$\sup_{t\in[0,T]}\|z^{(1)}(t)-z^{(2)}(t)\|_X\le c\|d\|_\infty Te^{\eta(r)T}.$$

So, we have (4.10). Theorem is proved. □

5. AG property for the attractor.

In this part of the work we show that under assumptions (3.1)-(3.3) for sufficiently wide class of disturbances $U_1\subset L^\infty(\mathbb{R}_+)$ the global attractor Θ of the m-semiflow S_0 is globally stable in the AG sense, i.e., robust estimate (2.4) takes place.

Assume that the set of disturbances U_1 consists of all functions $d\in L^\infty(\mathbb{R}_+)$ with

$$\sup_{t\ge0}\int_t^{t+1}|d(s+\tau)-d(s)|^2\,ds\le\psi(|l|), \quad (5.1)$$

where ψ may depend on d and $\psi(p)\to0,\ p\to0+$.

Property (5.1) is true for absolutely continuous functions $d\in L^\infty(\mathbb{R}_+)$ with $d'\in L^\infty(\mathbb{R}_+)$.

It is clear that the set U_1 is translation-invariant, i.e.,

$$\forall d() \in U_1,\ \forall h \geq 0\ d(+h) \in U_1.$$

Moreover, it is known [6] that for every $d \in U_1$ the set

$$\Sigma(d) := cl_{L^2_{loc}}\{d(+h) \mid h \geq 0\}$$

is a translation-invariant compact subset of $L^2_{loc}(\mathbb{R}_+)$, $d \in \Sigma(d)$, $\Sigma(0) = \{0\}$ i $\forall \sigma \in \Sigma(d)$

$$\sup_{t\geq 0} \int_t^{t+1} |\sigma(s)|^2\, ds \leq \sup_{t\geq 0} \int_t^{t+1} |d(s)|^2\, ds \leq \|d\|^2_\infty. \tag{5.2}$$

Theorem 5.1. *Assume that conditions* (3.1)-(3.3), (5.1) *are fulfilled. Then the family* $\{S_d\}_{d\in U_1}$ *possesses* AG *property for the global attractor* Θ, *i.e.,*

$$\exists \gamma \in \mathcal{K}\ \forall d \in U_1,\ \forall z_0 \in X$$

$$\overline{\lim_{t\to\infty}} \|S_d(t, z_0)\|_\Theta \leq \gamma(\|d\|_\infty). \tag{5.3}$$

Proof. As $\forall t \geq 0,\ \forall d \in U_1,\ \forall \sigma \in \Sigma(d)$ due to (5.2)

$$\int_0^t |\sigma(s)|^2 e^{-\sigma(t-s)}\, ds \leq \frac{1}{\sigma}\|d\|^2_\infty, \tag{5.4}$$

so from (3.7) we derive: $\exists c > 0\ \forall r > 0\ \exists T(r)\ \forall t \geq T(r),\ \forall \|z_0\|_X \leq r$ and for arbitrary solution $z()$ of (2.1) with $z(0) = z_0$ and disturbance $\sigma \in \Sigma(d)$ the following estimate holds

$$\|z(t)\|_X \leq c(1 + \|d\|_\infty). \tag{5.5}$$

Taking into account dissipative property (5.5), compactness of $\Sigma(d)$, estimate (5.2), and abstract results from [16], we conclude that for proving robust estimate (5.3) it is sufficient to verify the following properties:

$$\sigma_n \to \sigma \text{ in } L^2_{loc}(\mathbb{R}_+),\ z_0^n \to z_0 \text{ in } X,\ \xi_n \in S_{\sigma_n}(t, z_0^n),\ \xi_n \to \xi \text{ in } X \Rightarrow$$
$$\Rightarrow \xi \in S_\sigma(t, z_0), \tag{5.6}$$

$\{\sigma_n\} \subset \Sigma(d)$, $d \in U_1$ (or $\sigma_n \in \Sigma(d_n)$, $\|d_n\|_\infty \to 0$), $z_0^n \to z_0$ weakly in X, $t_n \nearrow \infty$,

$$\xi_n \in S_{\sigma_n}(t_n, z_0^n) \Rightarrow \{\xi_n\} \text{ is precompact in } X. \tag{5.7}$$

Property (5.6) is a direct consequence of Lemma 3.2.

Let us prove (5.7). We put $\xi_n = z_n(t_n)$, where $z_n()$ is a solution of (2.1) with $d = \sigma_n,\ z_n(0) = z_0^n$.

From estimates (3.7),(5.2) and assumption (5.7) we derive that the sequence $\{\xi_n\}$ is bounded in X. So, up to subsequence

$$\xi_n \to \xi \text{ weakly in } X. \tag{5.8}$$

We can extract a subsequence such that $\forall M \geq 1$

$$z_n(t_n - M) \to \xi_M \text{ weakly in } X.$$

Moreover, $\forall t \geq 0$ for sufficiently large n we have from the cocycle property:

$$z_n(t_n - M + t) \in S_{\sigma_n(+t_n - M)}(t, 0, z_n(t_n - M)).$$

Let us put $\bar{\sigma_n}(t) := \sigma_n(t + t_n - M)$. Assumption (5.7) allows us to claim that for some $\bar{\sigma}$ we have that

$$\bar{\sigma_n} \to \bar{\sigma} \text{ in } L^2_{loc}(\mathbb{R}_+). \tag{5.9}$$

Therefore, from Lemma 3.2 for $\bar{z_n}(t) = z_n(t + t_n - M)$ we have that $\forall t \geq 0$

$$\bar{z_n}(t) \to \bar{z}(t) \text{ weakly in } X,$$

$$\bar{z}(t) \in S_{\bar{\sigma}}(t, 0, \xi_M).$$

In particular,

$$\bar{z_n}(M) = \xi_n \to \bar{z}(M) = \xi \text{ weakly in } X.$$

It is known [5] that every solution $z()$ of (2.1) with disturbance $d()$ satisfies the equality

$$\frac{d}{dt} I(z(t)) + \alpha I(z(t)) = H_d(t, z(t)), \tag{5.10}$$

where

$$\begin{aligned} I(z) &= \frac{1}{2}\|y_t\|^2 + \frac{1}{2}\|y\|^2_{H^1_0} + (F(y), 1) + \frac{\alpha}{2}(y_t, y), \\ H_d(t, z) &= \alpha(F(y(t)), 1) - \frac{\alpha}{2}(f(y(t)), y(t)) \\ &\quad + \frac{\alpha}{2}(g(y(t)), y(t))d(t) + (g(y(t)), y_t(t))d(t). \end{aligned}$$

We write (5.10) for $\bar{z_n}$ and after integrating over $[0, M]$ we get:

$$I(\xi_n) = I(z_n(t_n - M))e^{-\alpha M} + \int_0^M e^{\alpha(p-M)} H_{\bar{\sigma_n}}(p, \bar{z_n}(p))\, dp. \tag{5.11}$$

Applying to $\{\bar{z}_n\}$ arguments (3.17),(3.18), and taking into account strong convergence (5.9), we deduce that $\forall M \geq 0$

$$\int_0^M e^{\alpha(p-M)} H_{\bar{\sigma_n}}(p, \bar{z_n}(p))\, dp \to \int_0^M e^{\alpha(p-M)} H_{\bar{\sigma}}(p, \bar{z}(p))\, dp \text{ as } n \to \infty.$$

From estimate (3.7) $\exists c > 0\ \forall t \geq 0,\ \forall n \geq 1$

$$|I(z_n(t))| \leq c, \tag{5.12}$$

where c does not depend on M.

Then from (5.11), (5.12) we conclude that

$$\overline{\lim_{n\to\infty}} I(\xi_n) \le ce^{-\alpha M} + \int_0^M e^{-\alpha(p-M)} H_{\bar{\sigma}}(\bar{z}(p))\,dp$$
$$= ce^{-\alpha M} + I(\xi) - I(\xi_M)e^{-\alpha M} \le 2ce^{-\alpha M} + I(\xi).$$

Thus,

$$\overline{\lim_{n\to\infty}} \frac{1}{2}\|\xi_n\|_X^2 \le 2ce^{-\alpha M} + \frac{1}{2}\|\xi\|_X^2.$$

Passing to the limit as $M \to \infty$, we get

$$\overline{\lim_{n\to\infty}} \|\xi_n\|_X \le \|\xi\|_X.$$

Combining this inequality with weak convergence (5.8), we obtain that the sequence $\{\xi_n\}$ is precompact in X. Theorem is proved. □

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