JOURNAL OF OPTIMIZATION, DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS (JODEA)
Volume **31**, Issue **1**, June **2023**, pp. 74–94, DOI 10.15421/142304

ISSN (print) 2617–0108
ISSN (on-line) 2663–6824

MODEL OF NON-ISOTHERMAL CONSOLIDATION IN THE PRESENCE OF GEOBARRIERS AND THE TOTAL APPROXIMATION PROPERTIES OF ITS FINITE ELEMENT SOLUTIONS

Petro M. Martyniuk*, Oksana V. Ulianchuk-Martyniuk †

Abstract. The boundary value problem for the system of quasi-linear parabolic equations in the presence of integral conjugation conditions is considered. The boundary value problem is a mathematical model of the process of non-isothermal filtration consolidation of the soil mass which contains a thin geobarrier. Geobarriers exposed to non-isothermal conditions are a component of waste storage facilities. The change of hydromechanical and thermal properties the geobarriers, as well as the phenomenon of thermal osmosis, require modification of both the equations in the mathematical model and the conjugation conditions. The finite element method is used to find approximate solutions of the corresponding system of quasi-linear parabolic equations. The existence and uniqueness of the approximate generalized solution is proved. The accuracy of finite element solutions in the sense of total approximation are also estimated. The differences in the values of pressure and temperature distributions for the classical case and the case considered in the article were analyzed on the test model example.

Key words: system of quasi-linear parabolic equations, finite element method, generalized solution, accuracy of finite element solutions, thermo-osmosis, geobarrier, conjugation condition.

2010 Mathematics Subject Classification: 35K05, 35K59, 35Q35, 65M60, 86-08.

Communicated by Prof. V. Ye. Belozyorov

1. Introduction

Waste dumps have become integral elements of the daily life of people on planet Earth [22]. This applies to both the industrial scale and the household life. [22] state: "Disposal of municipal solid waste (MSW) in engineered landfills is one of the most widely used waste management practices in the USA and worldwide." It is clear that the consequence of this most common practice is the problems of the impact of waste landfills on the environment. One of the engineering elements of waste storage facilities designed to reduce the level of such negative impact is geobarriers, both of natural geomaterials (mainly clay and artificial geotextiles. The use of geosynthetic clay liners (GCLs) is noted in [5] as a common practice in geologic engineering for waste storage facilities to protect groundwaters from contamination.

*Department of Computer Science and Applied Mathematics, National University of Water and Environmental Engineering, 11, Soborna st., Rivne, 33028, Ukraine, p.m.martyniuk@nuwm.edu.ua
†Department of Computer Science and Applied Mathematics, National University of Water and Environmental Engineering, 11, Soborna st., Rivne, 33028, Ukraine, o.v.ulianchukmartyniuk@nuwm.edu.ua

Complex processes that take place in the waste storage facilities can affect the physical, mechanical and chemical properties of geobarrier materials. The internal temperature in waste dumps can reach $55-60^oC$ due to the biochemical reactions present there [27]. Geobarriers are also mandatory components of radioactive waste and spent fuel storage [24] and are exposed to temperatures over 60^oC. Geobarriers are also used in bioreactors during the operation of which a significant amount of thermal energy is released [9]. Therefore, studying and modeling the behavior of geobarriers in non-isothermal conditions is a problem of practical importance.

One of the main issues of the impact of waste storage facilities on the environment is the spread of harmful substances due to their filtration through the soil base of the facilities into underground water. Therefore, the factors that affect the parameters of filtration through geobarriers should be taken into account both in field experiments and in predictive mathematical models. The importance of researching heat transfer processes in geobarriers in relation to the impact on the environment was shown in field experiments [4]. The interactions of thermal, hydrological and mechanical properties of porous media are manifested in the change in the filtration coefficient of the porous material of the geobarrier; presence of thermo-osmosis [10, 36]; dependence of the thermal conductivity coefficient of the porous medium on the porosity. For example, the problem of thermal conductivity of porous media from the point of view of the important task of burying radioactive waste which still emits thermal energy for many years is considered in [37]. Specifically, eight mathematical formula models for determining the thermal conductivity coefficient were analyzed in detail, depending on the characteristics of the porous medium (porosity, saturation, geometry of the structure). We will consider each particular factor separately on the examples of the analysis of scientific literature.

The results of experimental studies of the effect of temperature on the filtration coefficient of illite clay are presented in [27]. The research is summarized in the form of analytical dependence

$$k\left(T\right)=\frac{\rho_w\, g\, \exp\left(-30.894-0.0109\, T\right)}{0.2601+1.517\, \exp\left(-0.034688\, T\right)}.$$

Here $k\left(T\right)$ is the filtration coefficient (cm/s); ρ_w is the density of water; g is the gravity acceleration; T is temperature (^oC). For instance, when the temperature increases from $25°C$ to 60^oC, the filtration coefficient increases monotonously and non-linearly from $3.2\cdot 10^{-9} cm/s$ to $4.5\cdot 10^{-9} cm/s$.

A detailed review of experimental studies of thermo-osmotic properties of membrane systems is provided in [2]. It is the clay soils used in geobarriers that are known to have the properties of semipermeable membranes [7, 28]. A new theoretical explanation of thermo-osmotic filtration of solutions in clays is proposed in [11]. However, the authors do not deviate from the law for the rate of filtration known from the scientific literature which takes into account thermal osmosis:

$$q = -\frac{k}{\eta}\left(\nabla p + \rho g \nabla z\right) - \frac{k\nabla H}{\eta T}\nabla T,$$

where q is the pore fluid-specific discharge (m/s), k is the Darcy permeability (m^2), η is the dynamic viscosity $(Pa \cdot s)$, p is the pressure (Pa), ρ is the fluid density (kg/m^3), g is the gravity acceleration (m/s^2), ∇z is $(0, 0, 1)$ if the z axis is vertical upward, T is the temperature (K), and ∇H is the macroscopic volume-averaged excess specific enthalpy due to fluid-solid interactions (J/m^3). According to the equation, the thermo-osmotic permeability is $k_T = k\nabla H/T$ $(Pa \cdot m^2/K)$ [11]. Most often, $\nabla H > 0$ in clays, and fluid flow occurs in the direction of decreasing temperature, but negative values have also been reported.

Attention not just on the effect of thermal osmosis in the presence of a temperature gradient, but especially on the dependence of the thermo-osmotic coefficient on the porosity of the medium is focused in [12]. Qualitatively, the thermo-osmotic coefficient decreases monotonously with increasing porosity. However, in the vicinity of porosity values of 0.4, the authors observed an anomalous slight increase when porosity increases to 0.5. This effect is reasonably explained by the authors. It is important that quantitative indicators are also given (e.g. see Figure 8 of the work). For instance, the thermo-osmotic coefficient is approximately halved with an increase in porosity from 0.35 to 0.55.

An extensive review of thermal conductivity models of sands is presented in [16], and the model performance is evaluated for different types of sand, from dry to saturated. A total of 14 models were evaluated to predict the thermal conductivity of sands using a large data set collection consisting of 1025 measurements on 62 samples from 20 studies. According to the results of research, the authors selected two models of thermal conductivity of sands which show the best agreement with the data of field experiments. This article is important from the point of view that such studies are relevant not only for clays, but also for sands. After all, sands are used in nuclear power plant waste repositories, and for them, too, the issues of thermal conductivity and its non-linear dependence on the parameters of the porous medium (porosity and moisture saturation) are important.

The coefficient of thermal conductivity of the soil, as shown in [26], depends non-linearly on many factors. Among these factors, soil moisture, density, soil organic matter (SOM), as well as clay content, were singled out. The following trends were noted and quantified: the coefficient of thermal conductivity of the soil monotonously increases with humidity; the coefficient of thermal conductivity decreases with increasing clay content; the coefficient of thermal conductivity increases with an increase in SOM content. In particular, an analytical dependence is proposed

$$\lambda = -2.35 + 3.58\,S - 2.04\,S^2 + 1.82\,BD + 2.88\,SOM - 1.48\,clay\,S,$$

where λ is soil thermal conductivity, $[\lambda] = \frac{W}{m \cdot K}$; SOM is soil organic matter,

$[SOM] = 1$; BD is bulk density, $[BD] = \frac{g}{cm^3}$; $clay$ is relative content of clay particles, $[clay] = \frac{kg}{kg}$; S is degree of pore water saturation, $[S] = 1$. Similarly, experimental studies of thermal conductivity of sands, sandy loams and clays depending on humidity, the presence of salts ($NaCl$ and $CaCl_2$) and SOM were performed in [1]. The data of field experiments showed that these influencing factors cannot be neglected. For instance, for clays, the coefficient of thermal conductivity varies from 0.36 to $0.65\frac{W}{m \cdot K}$ when the water content varies from 1.4 to 21.2 %.

Based on the analysis of scientific sources and data [35], eight models were selected for determining the thermal conductivity of a fully saturated porous medium, depending on the coefficients of thermal conductivity of solid particles, pore fluid and porosity. Qualitative conclusions from research are: 1) the thermal conductivity coefficient of the medium increases with water saturation; 2) thermal conductivity decreases with increasing porosity and vice versa. In the context of our problem, it is important that in the process of consolidation of a fully saturated porous medium, its porosity changes. Therefore, the coefficient of thermal conductivity will also change.

Mathematical modeling of various processes in porous media is also being developed [13, 14], taking into account the influence of non-isothermal conditions and thermal osmosis. The effect of thermal osmosis on the occurrence of excess pressures in the pores of a porous medium and the displacements of the skeleton of the porous medium resulting from the change in pressures and temperature were investigated in [3]. This was done by modifying Darcy's law, constructing the appropriate mathematical model and performing numerical experiments. The authors confirm that taking thermal osmosis into account can lead to negative pressures in the pore fluid. The effect of thermo-osmotic effects on pressure in the pore fluid of saturated clays was also modeled in [38]. Unfortunately, the authors disregarded that similar effects had been studied earlier [33, 34]. It is shown in all these works that under certain conditions thermal effects can change the field of pressures and concentrations of chemical substances. However, these reports do not take into account the presence of geobarriers. Additionally, the work [3] did not take into account the dependence of the filtration coefficient on temperature, and the study was conducted for homogeneous soils without the presence of geobarriers.

The importance of the phenomenon of thermal osmosis in the study of deformations of clay soils was proved in [21] on the basis of numerical simulations. However, the presence of thin geobarriers in a porous medium was not considered.

Analyzing the results of the above field experiments, the non-linear dependences of parameters and influencing factors should be noted (e.g. porosity, temperature, filtration coefficient, thermal conductivity coefficient). In the presence of thin inclusions, the conjugation conditions of non-ideal contact for inclusions should take into account the change in the physical and mechanical parameters of the inclusion under the effect of the studied factors. It was shown in [6] that the conjugation conditions of non-ideal contact can apply to the contact problems of heterogeneous media even without the presence of fine inclusions. Quite detailed studies of the effect of modified conjugation conditions in the presence of geobarriers on forecast distributions of moisture and excess pressures under the action of chemical and biological (bioclogging) factors were performed in [7, 18–20, 28–32]. However, problems with the effect of thermal factors have not yet been considered.

2. Formulation of the problem in the physical domain

Consider a soil massif with a total thickness of l which consists of two subregions Ω_1 and Ω_2. Moreover $\Omega_1 \cap \Omega_2 = \emptyset$. We consider the area $\Omega = \Omega_1 \cup \Omega_2$ to be inhomogeneous. By inhomogeneity, we mean the presence of a contact boundary $\omega = \overline{\Omega}_1 \cap \overline{\Omega}_2$ which, from a physical viewpoint, is a thin inclusion of the third material of the thickness d (Fig. 1). From a mathematical viewpoint, the thickness of the inclusion itself is neglected and the value d appears only in the so-called conjugation conditions of non-ideal contact for unknown functions.

Let us investigate the process of consolidation of this fully saturated porous medium in the region $\Omega = \Omega_1 \cup \Omega_2$ under the effect of temperature. The heterogeneous soil layer is considered the base of the waste repository, and the fine inclusion is the geobarrier. Soil consolidation is a consequence of applying an external load in the form of solid waste in storage. Chemical reactions in the storage result in the release of thermal energy. Therefore, the pressure $h(x,t)$ and temperature $T(x,t)$ functions are unknown. Although the processes in the geobarrier itself are not investigated, both the thickness of the inclusion and its characteristics appear in the conjugation conditions. The classical conjugation condition of non-ideal contact for pressures serves as an example [8, 25]

$$u^{\pm}\big|_{x=\xi} = -\frac{k_\omega}{d}\left(h^+ - h^-\right),$$

where $k_\omega = const$ is the filtration coefficient of the porous inclusion material, u is the filtration rate, h^+ and h^- are the values of the pressure at the inclusion at $x = \xi + 0$ and $x = \xi - 0$, respectively. However, the filtration coefficient depends on the parameters of the state of the environment (in the context of the considered problem, porosity and temperature). Then such dependences should be taken into account in conjugation conditions and their modification. The method of modification of the conjugation conditions and the conjugation conditions themselves for the cases of considering the effect of chemical and biological factors are given in [7, 29, 31, 32].

3. Mathematical model of the problem in the domain with a thin inclusion

The main elements of the mathematical model are known from the classical theory of filtration consolidation and heat transfer in porous media. However,

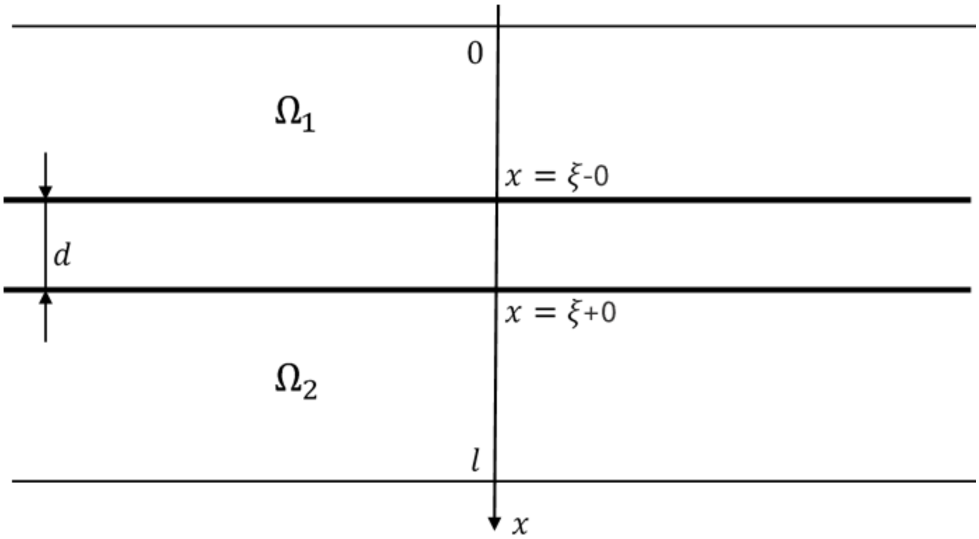

Fig. 2.1. A layer of soil of thickness l with a thin inclusion ω of thickness d ($d \ll l$).

some dependences will require further explanation and clarification. The interrelated process of changes in pressure and temperature of a completely saturated inhomogeneous soil mass in the one-dimensional case is described by the following boundary value problem:

$$\frac{\partial h}{\partial t} = \frac{1+e}{\gamma a}\frac{\partial}{\partial x}\left(k\left(h,T\right)\frac{\partial h}{\partial x} + \mu\left(h\right)\frac{\partial T}{\partial x}\right), \quad x \in \Omega_1 \cup \Omega_2, t > 0; \tag{3.1}$$

$$h(x,t)|_{x=0} = \overline{h}_0(t), t \geq 0; \tag{3.2}$$

$$u(x,t)|_{x=l} = \left(-k\left(h,T\right)\frac{\partial h}{\partial x} - \mu\left(h\right)\frac{\partial T}{\partial x}\right)\Big|_{x=l} = 0, t \geq 0; \tag{3.3}$$

$$h(x,0) = h_0(x), x \in \overline{\Omega}_1 \cup \overline{\Omega}_2; \tag{3.4}$$

$$c_s\left(h\right)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda(h)\frac{\partial T}{\partial x}\right) - \rho_w c_w u\left(h,T\right)\frac{\partial T}{\partial x}, \quad x \in \Omega_1 \cup \Omega_2, t > 0; \tag{3.5}$$

$$T(x,t)|_{x=0} = \overline{T}_0(t), t \geq 0; \tag{3.6}$$

$$q_T(x,t)|_{x=l} = -\lambda(h)\frac{\partial T}{\partial x}\Big|_{x=l} = 0, t \geq 0; \tag{3.7}$$

$$T(x,0) = T_0(x), \in \overline{\Omega}_1 \cup \overline{\Omega}_2; \tag{3.8}$$

$$u^{\pm}\big|_{x=\xi} = -\frac{[h]}{\int_0^d \frac{dx}{k_\omega(h,T)}} - \frac{[T]}{\int_0^d \frac{dx}{\mu_\omega(h)}}; \tag{3.9}$$

$$q_T^{\pm}\big|_{x=\xi} = -\frac{[T]}{\int_0^d \frac{dx}{\lambda_\omega(h)}}. \tag{3.10}$$

Here $\Omega_1 = (0;\xi)$, $\Omega_2 = (\xi;l)$, $0 < \xi < l$; $\overline{h}_0(t)$, $h_0(x)$, $\overline{T}_0(t)$, $T_0(x)$ are known functions; a is the soil compressibility coefficient; h is pressure; k, k_ω are the filtration coefficients of the main soil and inclusion soil, respectively; λ, λ_ω are the thermal conductivity coefficients of the main soil and inclusion soil, respectively; μ, μ_ω are the thermo-osmotic coefficients of of the main soil and inclusion soil, respectively; u is the filtration rate; e is the soil void ratio, with $e = \frac{n}{1-n}$, where n is the soil porosity; q_T is the thermal energy flow; $u^{\pm}$, $q_T^{\pm}$ are the values of filtration rates and flows at $x = \xi - 0$ and $x = \xi + 0$, respectively; $[h] = h^+ - h^-$, $[T] = T^+ - T^-$ are the pressure and temperature jumps on the thin inclusion; $c_s = \rho_w c_w n + \rho_{solid} c_{solid}(1-n)$ is the volume heat capacity coefficient of the soil; ρ_w, ρ_{solid} are the densities of pore fluid and solid soil particles; c_w, c_{solid} are specific heat capacities of pore fluid and solid soil particles.

Eq. (3.1) is the filtration consolidation equation for variable temperature and the presence of thermo-osmotic effects [33,34]. Conjugation conditions (3.9), (3.10) differ from classical ones [8,25] and take into account the dependence of the geobarrier filtration coefficient on porosity and temperature, the presence of thermal osmosis, as well as the dependence of the thermal conductivity coefficient on porosity. Conditions (3.9), (3.10) are derived similarly to those in [7,29,31,32].

We note that according to known field experiments the filtration, thermo-osmotic, and thermal conductivity coefficients depend on porosity n. However, in the consolidation problem, as explained in [20], $n = n(h)$. This is taken into account in the problem (3.1)–(3.10). Eqs. (3.1), (3.5) neglect internal sources (sinks) of pore fluid and thermal energy. Note that in (3.1)–(3.10) each of the functions $\alpha(x,t)$ is defined as

$$\alpha(x,t) = \{\alpha_i(x,t),\ x \in \Omega_i\ ,\quad i = 1,2.$$

Purely for convenience in this article $i = 2$. Generally, for $i > 2$ all calculations will be similar.

4. A system of quasi-linear equations of the parabolic type with homogeneous boundary conditions of the first kind and its generalized solution

To simplify the theoretical statements, which in principle do not reduce the generality of the problem, in problem (3.1)–(3.10) we will consider the boundary conditions of the first kind (3.2), (3.6) to be homogeneous for the time being. That is, let the conditions be fulfilled:

$$\overline{h}_0(t) \equiv 0, \overline{T}_0(t) \equiv 0, \quad t \geq 0. \tag{4.1}$$

How to take into account the inhomogeneity of boundary conditions of the first kind will be discussed in a separate section of the article.

Similarly to [8], we introduce the following notations: $Q_{\mathcal{T}} = \Omega \times (0; \mathcal{T}]$, $Q^1_{\mathcal{T}} = \Omega_1 \times (0; \mathcal{T}]$, $Q^2_{\mathcal{T}} = \Omega_2 \times (0; \mathcal{T}]$.

Suppose that the functions $h_0(x)$, $T_0(x)$ are continuous on each of the closures $\overline{\Omega}_1$, $\overline{\Omega}_2$. Also regarding the coefficients k, k_ω, μ, μ_ω, λ, λ_ω, c_s suppose that
1)

$$0 < k_{min} \le k(s_1, s_2) \le k_{max} < \infty,$$
$$0 < k_{\omega,min} \le k_\omega(s_1, s_2) \le k_{\omega,max} < \infty,$$
$$0 < p_{min} \le p(s) \le p_{max} < \infty,$$

for all $s,\ s_1,\ s_2 \in \mathbb{R}$; k_{min}, k_{max},$k_{\omega,min}$, $k_{\omega,max}$, p_{min}, p_{max} are positive constants; $p \in \{\mu,\ \lambda,\ \mu_\omega,\ \lambda_\omega, c_s\ \}$;
2)

$$|p\,(s_1) - p\,(s_2)| \le p_L\,|s_1 - s_2|\,,\ 0 < p_L < \infty;$$
$$\left|k\,(s_1, s_2) - k\,(s'_1, s'_2)\right| \le k_L\left(\left|s_1 - s'_1\right| + \left|s_2 - s'_2\right|\right),\quad 0 < k_L < \infty;$$
$$\left|k_\omega\,(s_1, s_2) - k_\omega\,(s'_1, s'_2)\right| \le k_{\omega,L}\left(\left|s_1 - s'_1\right| + \left|s_2 - s'_2\right|\right),\quad 0 < k_{\omega,L} < \infty;$$

for all $s_1,\ s_2,\ s'_1, s'_2 \in \mathbb{R}$; $p \in \{\mu,\ \lambda,\ \mu_\omega,\ \lambda_\omega\}$.
3)

$$|u\,(s_1, s_2)| \le u_1,\ \forall s_1, s_2 \in \mathbb{R};$$
$$\left|u\,(s_1, s_2) - u\,(s'_1, s'_2)\right| \le u_L\left(\left|s_1 - s'_1\right| + \left|s_2 - s'_2\right|\right),\ 0 < u_1,\ u_L < \infty,$$

for all $s_1,\ s_2,\ s'_1, s'_2 \in \mathbb{R}$.

Definition 4.1. The classical solution of the initial-boundary value problem (3.1)–(3.10), which admits a discontinuity of the first kind at the point $x = \xi$, is called a pair of functions $h(x,t) \in \Psi_h$, $T(x,t) \in \Psi_T$, which satisfy $\forall (x,t) \in \overline{Q}_{\mathcal{T}}$ equations (3.1), (3.5) and initial conditions (3.4), (3.8) respectively.

In the above definition Ψ_h, Ψ_T are the sets of functions $\psi_h(x,t)$, $\psi_T(x,t)$, which together with $\frac{\partial(\cdot)}{\partial x}$, are continuous on each of the closures $\overline{Q}^1_{\mathcal{T}}$, $\overline{Q}^2_{\mathcal{T}}$, have bounded continuous partial derivatives $\frac{\partial(\cdot)}{\partial t}$, $\frac{\partial^2(\cdot)}{\partial x^2}$ on $Q^1_{\mathcal{T}}$, $Q^2_{\mathcal{T}}$, and satisfy conditions (3.2), (3.3), (3.9) and (3.6), (3.7), (3.10) respectively.

For further explanations, similarly to the work [20], we note the following. Given condition 1), we get

$$k_{\omega,min}\frac{[h]}{d} \le \frac{[h]}{\int_0^d \frac{dx}{k_\omega(h,T)}} \le k_{\omega,max}\frac{[h]}{d},$$

$$\lambda_{\omega,min}\frac{[T]}{d} \le \frac{[T]}{\int_0^d \frac{dx}{\lambda_\omega(h)}} \le \lambda_{\omega,max}\frac{[T]}{d},$$

$$\mu_{\omega,min}\frac{[T]}{d}\leq\frac{[T]}{\int_0^d\frac{dx}{\mu_\omega(h)}}\leq\mu_{\omega,max}\frac{[T]}{d}.$$

The above estimates make it possible to apply well-known theoretical calculations [8, 25] for the classical conjugation condition (see [25], page 291, formula (7.4))

$$\left(\varkappa\left(x,u\right)\frac{\partial u}{\partial x}\right)\bigg|_{x=\xi}=r\left[u\right],$$

in which r is some known constant; u is an unknown function. The classical conjugation condition and theoretical explanations for it require that

$$0<r_0\leq r<\infty.$$

Similarly to [8], let H_0 be the space of functions $s\left(x\right)$ that on each of the domains Ω_i belong to the Sobolev space $W_2^1\left(\Omega_i\right)$, $i=1,2$, and satisfy the condition

$$s(x)|_{x=0}=0.$$

Let $h(x,t)\in\Psi_h$, $T(x,t)\in\Psi_T$ be the classical solution of the initial-boundary value problem (3.1)–(3.10). Take $s\left(x\right)\in H_0$. We multiply equation (3.1) and initial condition (3.4) by $s\left(x\right)$, and similarly, equation (3.5) and initial condition (3.8). Integrating them over the segment $[0;l]$ and taking into account the conjugation conditions (3.9), (3.10), we obtain

$$\int_0^l\frac{\gamma a}{1+e}\frac{\partial h}{\partial t}s\left(x\right)dx+\int_0^l k(h,T)\frac{\partial h}{\partial x}\frac{ds}{dx}\,dx+\int_0^l\mu(h)\frac{\partial T}{\partial x}\frac{ds}{dx}\,dx$$
$$+\frac{[h]\,[s]}{\int_0^d\frac{dx}{k_\omega(h,T)}}+\frac{[T]\,[s]}{\int_0^d\frac{dx}{\mu_\omega(h)}}=0, \tag{4.2}$$

$$\int_0^l h\left(x,0\right)s\left(x\right)\,dx=\int_0^l h_0\left(x\right)s\left(x\right)\,dx, \tag{4.3}$$

$$\int_0^l c_s\frac{\partial T}{\partial t}s\left(x\right)dx+\int_0^l\lambda(h)\frac{\partial T}{\partial x}\frac{ds}{dx}\,dx+\int_0^l\rho_w c_w u\frac{\partial T}{\partial x}s\left(x\right)\,dx+\frac{[T]\,[s]}{\int_0^d\frac{dx}{\lambda_\omega(h)}}=0, \tag{4.4}$$

$$\int_0^l h\left(x,0\right)s\left(x\right)\,dx=\int_0^l h_0\left(x\right)s\left(x\right)\,dx. \tag{4.5}$$

Therefore, if $h(x,t)\in\Psi_h$, $T(x,t)\in\Psi_T$ is a classical solution of the initial-boundary value problem (3.1)–(3.10), then $h(x,t)$, $T(x,t)$ is the solution of the problem (4.2)–(4.5) in the weak formulation.

Let H be the space of functions $v\left(x,t\right)$ that are square integrable together with their first derivatives $\frac{\partial v}{\partial t}$, $\frac{\partial v}{\partial x}$ on each of the intervals $(0;\xi)$, $(\xi;l)$, $\forall t\in(0\,;\mathcal{T}]$, $\mathcal{T}>0$, and they satisfy homogeneous boundary conditions of the first kind

$$v(x,t)|_{x=0}=0,\quad t\geq0.$$

Definition 4.2. The functions $h(x,t) \in H$, $T(x,t) \in H$, which for any $s\,(x) \in H_0$ satisfy the integral relation (4.2)–(4.5) are called the generalized solution of the initial-boundary problem (3.1)–(3.10) (if condition (4.1) is fulfilled).

5. Approximate generalized solution: its existence and uniqueness

An approximate generalized solution of the initial-boundary value problem (3.1)–(3.10) will be sought in the form

$$\widehat{h}\,(x,t) = \sum_{i=1}^{N} h_i\,(t)\,\varphi_i\,(x), \quad \widehat{T}\,(x,t) = \sum_{i=1}^{N} T_i\,(t)\,\varphi_i\,(x), \tag{5.1}$$

where $\{\varphi_i(x)\}_{i=1}^{N}$ is the basis of the finite-dimensional subspace $M_0 \subset H_0$; $h_i\,(t)$, $T_i\,(t)$, $i = \overline{1,N}$ are unknown coefficients that depend only on time.

A set of functions that can be represented in the form (5.1) generate a finite-dimensional subspace $M \subset H$.

Definition 5.1. An approximate generalized solution of the initial-boundary value problem (3.1)–(3.10) is a pair of functions $\widehat{h}(x,t) \in M$, $\widehat{T}(x,t) \in M$ which for an arbitrary function $S(x) \in M_0$ satisfy the integral relations

$$\int_0^l \frac{\gamma a}{1+e}\frac{\partial \widehat{h}}{\partial t} S\,(x)\;dx + \int_0^l k(\widehat{h},\widehat{T})\frac{\partial \widehat{h}}{\partial x}\frac{dS}{dx}\,dx + \int_0^l \mu(\widehat{h})\frac{\partial \widehat{T}}{\partial x}\frac{dS}{dx}\,dx + \frac{\left[\widehat{h}\right][S]}{\int_0^d \frac{dx}{k_\omega(\widehat{h},\widehat{T})}} + \frac{\left[\widehat{T}\right][S]}{\int_0^d \frac{dx}{\mu_\omega(\widehat{h})}} = 0, \tag{5.2}$$

$$\int_0^l \widehat{h}\,(x,0)\,S\,(x)\;dx = \int_0^l h_0\,(x)\,S\,(x)\;dx, \tag{5.3}$$

$$\int_0^l c_s\frac{\partial \widehat{T}}{\partial t} S\,(x)\;dx + \int_0^l \lambda(\widehat{h})\frac{\partial \widehat{T}}{\partial x}\frac{dS}{dx}\,dx + \int_0^l \rho_w c_w u\frac{\partial \widehat{T}}{\partial x} S\,(x)\;dx + \frac{\left[\widehat{T}\right][S]}{\int_0^d \frac{dx}{\lambda_\omega(\widehat{h})}} = 0, \tag{5.4}$$

$$\int_0^l \widehat{T}\,(x,0)\,S\,(x)\;dx = \int_0^l T_0\,(x)\,S\,(x)\;dx. \tag{5.5}$$

Next, from the weak formulation (5.2)–(5.5) of the problem (3.1)–(3.10), taking into account (5.1) (setting the function $S(x)$ equal to each basis function $\varphi_i(x)$, $i = \overline{1,N}$), we obtain the Cauchy problem for the system of non-linear differential equations

$$\mathbf{M_1}\left(\mathbf{H}\right)\frac{d\mathbf{H}}{dt}+\mathbf{L_1}\left(\mathbf{H},\mathbf{T}\right)\mathbf{H}\left(t\right)+\mathbf{L_{12}}\left(\mathbf{H}\right)\mathbf{T}\left(t\right)=\mathbf{0}, \tag{5.6}$$

$$\widetilde{\mathbf{M}}_\mathbf{1}\mathbf{H^{(0)}}=\widetilde{\mathbf{F}}_\mathbf{1}, \tag{5.7}$$

$$\mathbf{M_2}\left(\mathbf{H}\right)\frac{d\mathbf{T}}{dt}+\mathbf{L_2}\left(\mathbf{H},\mathbf{T}\right)\mathbf{T}\left(t\right)=\mathbf{0}, \tag{5.8}$$

$$\widetilde{\mathbf{M}}_\mathbf{2}\mathbf{T^{(0)}}=\widetilde{\mathbf{F}}_\mathbf{2}, \tag{5.9}$$

where

$$\widetilde{\mathbf{F}}_k=\left(\widetilde{f}_i^{(k)}\right)_{i=1}^N,\widetilde{\mathbf{M}}_k=\left(\widetilde{m}_{ij}^{(k)}\right)_{i,j=1}^N,\mathbf{M}_k=\left(m_{ij}^{(k)}\right)_{i,j=1}^N,\mathbf{L}_k=\left(l_{ij}^{(k)}\right)_{i,j=1}^N,k=1,2;$$

$$\mathbf{L}_{12}=\left(l_{ij}^{(12)}\right)_{i,j=1}^N\widetilde{m}_{ij}^{(k)}=\int_0^l\varphi_i\varphi_j\,dx,\ \ \widetilde{f}_i^{(1)}=\int_0^0 h_0\varphi_i\,dx,\widetilde{f}_i^{(2)}=\int_0^0 T_0\varphi_i\,dx,$$

$$\mathbf{H}=\left(h_i\left(t\right)\right)_{i=1}^N,\mathbf{T}=\left(T_i\left(t\right)\right)_{i=1}^N,\mathbf{H^{(0)}}=\left(h_i\left(0\right)\right)_{i=1}^N,\mathbf{T^{(0)}}=\left(T_i\left(0\right)\right)_{i=1}^N,$$

$$m_{ij}^{(1)}=\int_0^l\frac{\gamma a}{1+e}\varphi_i\varphi_j\,dx,l_{ij}^{(1)}=\int_0^l k\left(\widehat{h},\widehat{T}\right)\frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}\,dx+\frac{[\varphi_i]\,[\varphi_j]}{\int_0^d\frac{dx}{k_\omega(\widehat{h},\widehat{T})}},$$

$$l_{ij}^{(12)}=\int_0^l\mu\left(\widehat{h}\right)\frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}\,dx+\frac{[\varphi_i]\,[\varphi_j]}{\int_0^d\frac{dx}{\mu_\omega(\widehat{h})}},m_{ij}^{(2)}=\int_0^l c_s\varphi_i\varphi_j\,dx,$$

$$l_{ij}^{(2)}=\int_0^l\lambda(\widehat{h})\frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}\,dx+\int_0^l\rho_w c_w u\frac{d\varphi_j}{dx}\varphi_i\,dx+\frac{[\varphi_i]\,[\varphi_j]}{\int_0^d\frac{dx}{\lambda_\omega(\widehat{h})}}.$$

The system of equations (5.6), (5.8) can be written in the form

$$\mathbf{M}\frac{d\mathbf{V}}{dt}+\mathbf{L}\left(\mathbf{V}\right)\mathbf{V}\left(t\right)=\mathbf{0}, \tag{5.10}$$

where

$$\mathbf{M}=\begin{pmatrix}\mathbf{M_1} & \mathbf{0}\\ \mathbf{0} & \mathbf{M_2}\end{pmatrix},\quad \mathbf{V}=\begin{pmatrix}\mathbf{H}\\ \mathbf{T}\end{pmatrix},\quad \mathbf{M}=\begin{pmatrix}\mathbf{L_1} & \mathbf{L_{12}}\\ \mathbf{0} & \mathbf{L_2}\end{pmatrix}.$$

If the above conditions 1)–3) are met, the square matrix $\mathbf{M}$ is symmetric and positive definite, $\forall(x,t)\in\overline{Q}_\mathcal{T}$. Therefore, there exists a unique inverse matrix $\mathbf{M^{-1}}$. Then we write (5.10) in the form

$$\frac{d\mathbf{V}}{dt}=\mathbf{\Phi}\left(\mathbf{V}\right), \tag{5.11}$$

where $\mathbf{\Phi}\left(\mathbf{V}\right)=-\mathbf{M^{-1}}\mathbf{L}\left(\mathbf{V}\right)\mathbf{V}\left(t\right)$. Despite the fact that the matrix $\mathbf{L}\left(\mathbf{V}\right)$ will not be symmetric and positive definite, the functions $\mathbf{\Phi}\left(\mathbf{H}\right)$, $\frac{\partial\mathbf{\Phi}}{\partial\mathbf{V}}$ will be continuous and bounded. Then, similarly to [8] (Chapter 8, point 6), the solution $\mathbf{V}$ of the

Cauchy problem for the system of equations (5.11) exists and is unique. That is, there exists a single approximate generalized solution of the problem (3.1)–(3.10) with homogeneous boundary conditions of the first kind.

Let us introduce the following norms [8, page 380]:

$$\|u\|_{L_2}^2 = \int_0^l u^2(x,t)\,dx, \|u\|_{H_0^1}^2 = \left\|\frac{\partial u}{\partial x}\right\|_{L_2}^2, \|u\|_{L_2\times L_2}^2 = \|u\|_{L_2(Q_T)}^2 = \int_0^{\mathcal{T}}\int_0^l u^2\,dxdt,$$

$$\|u\|_{H_0^1\times L_2}^2 = \int_0^{\mathcal{T}} \|u\|_{H_0^1}^2\,dt = \int_0^{\mathcal{T}}\int_0^l \left(\frac{\partial u}{\partial x}\right)^2 dxdt, \|u\|_{L_2\times L_\infty} = \sup_{t\in(0,\mathcal{T}]} \|u(\cdot,t)\|_{L_2},$$

$$\|\nabla_x u\|_{L_\infty\times L_\infty} = \sup_{(x,t)\in Q_{\mathcal{T}}} \left|\frac{\partial u(x,t)}{\partial x}\right|, \|u\|_{W_2^1\times L_2}^2 = \int_0^{\mathcal{T}}\int_0^l \left(u^2 + \left(\frac{\partial u}{\partial x}\right)^2\right) dxdt,$$

$$\|[u]\|_{L_2}^2 = \int_0^{\mathcal{T}} [u]^2\,dt = \int_0^{\mathcal{T}} (u(\xi+0,t) - u(\xi-0,t))^2\,dt.$$

Theorem 5.1. *Let $h(x,t)$, $T(x,t)$ be the generalized solution of the initial-boundary value problem (3.1)–(3.10), and $\widehat{h}\,(x,t)$, $\widehat{T}(x,t)$ be the approximate generalized solution of this problem. Then, if conditions 1)–3) are fulfilled, there exist such positive constant values c, δ_1, δ_2, that for arbitrary functions $\tilde{h}\,(x,t) \in M$, $\tilde{T}\,(x,t) \in M$ the inequality holds*

$$\begin{gathered}\left\|h-\widehat{h}\right\|_{L_2\times L_\infty} + \left\|T-\widehat{T}\right\|_{L_2\times L_\infty} + \delta_1\left(\left\|h-\widehat{h}\right\|_{H_0^1\times L_2} + \left\|T-\widehat{T}\right\|_{H_0^1\times L_2}\right) \\ +\delta_2\left(\left\|\left[h-\widehat{h}\right]\right\|_{L_2} + \left\|\left[T-\widehat{T}\right]\right\|_{L_2}\right) \le c\left(\left\|h-\tilde{h}\right\|_{L_2\times L_\infty}\right. \\ +\left\|h-\tilde{h}\right\|_{H_0^1\times L_2} + \left\|\left[h-\tilde{h}\right]\right\|_{L_2} + \left\|\frac{\partial\left(h-\tilde{h}\right)}{\partial t}\right\|_{L_2\times L_2} + \left\|T-\tilde{T}\right\|_{L_2\times L_\infty} \\ \left. +\left\|T-\tilde{T}\right\|_{H_0^1\times L_2} + \left\|\left[T-\tilde{T}\right]\right\|_{L_2} + \left\|\frac{\partial\left(T-\tilde{T}\right)}{\partial t}\right\|_{L_2\times L_2}\right) \quad (5.12)\end{gathered}$$

Proof. The theorem is proved similarly to [8, p. 380, Theorem 1; p. 438, Theorem 19]. However, there is one difference. At the initial stage of the proof, the equality obtained from (4.2), (5.2) for the functions $\left(h-\widehat{h}\right)$, $\left(\tilde{h}-\widehat{h}\right)$ and the equality obtained from (4.4), (5.4) for the functions $\left(T-\widehat{T}\right)$, $\left(\tilde{T}-\widehat{T}\right)$ must be added. As a result, estimate (5.12) is cumulative with respect to both functions $h(x,t)$, $T(x,t)$. The estimate (5.12) can be generalized to an arbitrary finite number of functions as a generalized solution of the initial-boundary value problem, the structure of which coincides with (3.1)–(3.10). □

Inequality (5.12) is used in estimating the accuracy of the finite element method.

6. Finite element method

Cover the region $\Omega = \overline{\Omega}_1 \cup \overline{\Omega}_2$ with a finite element mesh with the total number of nodes N. Moreover, the point $x = \xi$ should have double numbering, of the node on the left $x = \xi - 0$ and the node on the right $x = \xi + 0$. Let in (5.1) $\varphi_i(x)$ be the basis functions of the finite element method which admit a discontinuity of the first kind at a point $x = \xi$ and are polynomials of degree m. Then the space of functions of the form (5.1) with the indicated basis functions is denoted by H_m^N.

Theorem 6.1. *Let the classical solution $h(x,t)$, $T(x,t)$ of the boundary value problem* (3.1)–(3.10) *have partial derivatives $\frac{\partial^{m+1}(\cdot)}{\partial x^{m+1}}$, $\frac{\partial^{m+2}(\cdot)}{\partial x^{m+1}\partial t}$, that are bounded on $Q_{\mathcal{T}}^i$, $i = 1, 2$. Then for the approximate generalized solution $\widehat{h}(x,t) \in H_m^N$, $\widehat{T}(x,t) \in H_m^N$, the following estimate is valid:*

$$\left\| h - \widehat{h} \right\|_{W_2^1 \times L_2} + \left\| T - \widehat{T} \right\|_{W_2^1 \times L_2} \le c\, h_{max}^m,$$

where m is the degree of finite element polynomials, $c = const > 0$, $h_{max} = \max_{i=\overline{0,N-1}} (x_{i+1} - x_i)$, $[x_{i+1}; x_i]$ are finite elements.

Proof. The validity of the theorem follows from the estimate (5.12) of the previous theorem, taking into account the interpolation estimates [8, p. 387, Theorem 2]. Note that if the basis functions of different degrees m_1 and m_2 are used for the sought $\widehat{h}(x,t)$, $\widehat{T}(x,t)$, then $m = min\,(m_1; m_2)$. □

Problem (5.6)–(5.9) is a Cauchy problem for a system of non-linear differential equations of the first order. Finding its solution also requires the use of appropriate discretization schemes. [8, 25] substantiate the use of the Crank-Nicolson method

$$\mathbf{M_1}\left(\mathbf{H}^{(j+\frac{1}{2})}\right)\frac{\mathbf{H}^{(j+1)} - \mathbf{H}^{(j)}}{\tau} + \mathbf{L_1}\left(\mathbf{H}^{(j+\frac{1}{2})}, \mathbf{T}^{(j+\frac{1}{2})}\right)\mathbf{H}^{(j+\frac{1}{2})} + \mathbf{L_{12}}\left(\mathbf{H}^{(j+\frac{1}{2})}\right)\mathbf{T}^{(j+\frac{1}{2})} = \mathbf{0},$$

$$\mathbf{M_2}\left(\mathbf{H}^{(j+\frac{1}{2})}\right)\frac{\mathbf{T}^{(j+1)} - \mathbf{T}^{(j)}}{\tau} + \mathbf{L_2}\left(\mathbf{H}^{(j+\frac{1}{2})}, \mathbf{T}^{(j+\frac{1}{2})}\right)\mathbf{T}^{(j+\frac{1}{2})} = \mathbf{0},$$

$j = 0, 1, 2, ..., m_\tau - 1$. Here time segment $[0, \mathcal{T}]$ is split into m_τ equal parts with step $\tau = \frac{\mathcal{T}}{m_\tau}$; $\mathbf{H}^{(j)}$, $\mathbf{T}^{(j)}$ is the approximate solution of the Cauchy problem for $t = j\tau$, $\mathbf{H}^{(j+\frac{1}{2})} = \frac{1}{2}\left(\mathbf{H}^{(j+1)} + \mathbf{H}^{(j)}\right)$, $\mathbf{T}^{(j+\frac{1}{2})} = \frac{1}{2}\left(\mathbf{T}^{(j+1)} + \mathbf{T}^{(j)}\right)$. We also introduce the following notations: $h^{(j)}$, $T^{(j)}$ is the classical solution of the initial-boundary value problem (3.1)–(3.10) for $t = j\tau$; $\widehat{h}^{(j)}$, $\widehat{T}^{(j)}$ is an approximate generalized solution of the initial-boundary value problem (3.1)–(3.10) for $t = j\tau$; $\phi^{(j+\frac{1}{2})} = \frac{1}{2}(\phi^{(j+1)} + \phi^{(j)})$; $z_h^{(j)} = h^{(j)} - \widehat{h}^{(j)}$, $z_T^{(j)} = T^{(j)} - \widehat{T}^{(j)}$.

Similarly to Theorem 5 [8, Chap. 8],

Theorem 6.2. *Let $h(x,t)$, $T(x,t)$ be the classical solution of the initial boundary value problem* (3.1)–(3.10)*. Let the first derivatives $\frac{\partial(\cdot)}{\partial t}$, $\frac{\partial(\cdot)}{\partial x}$ of the classical solution be twice continuously differentiable with respect to time on $\overline{Q}^i_{\mathcal{T}}$, $i = 1, 2$. Also assume that the derivatives $\frac{\partial^3(\cdot)}{\partial t^3}$, $\frac{\partial^3(\cdot)}{\partial t^2 \partial x}$ are uniformly bounded in modulus by a constant c_1 $\forall (x,t) \in \overline{Q}_{\mathcal{T}}$. If conditions 1)–3) are fulfilled, then there exist positive constants c, δ_1, r_0, τ_0, which depend on the constants from conditions 1)–3), as well as $\mathcal{T}$, l, such that $\forall \tau \le \tau_0$ for the classical solution $h(x,t)$, $T(x,t)$ and for the approximate generalized solution $\widehat{h}(x,t) \in M$, $\widehat{T}(x,t) \in M$ obtained using the Crank-Nicolson method, of the problems* (3.1)–(3.10) *and* (5.6)–(5.9)*, respectively, the following inequality is valid:*

$$
\begin{aligned}
&\left\|z_h^{(m_\tau)}\right\|^2_{L_2} + \left\|z_T^{(m_\tau)}\right\|^2_{L_2} + \delta_1 \tau \left(\sum_{j=0}^{m_\tau - 1} \left\| z_h^{(j+\frac{1}{2})} \right\|^2_{H_0^1} + \sum_{j=0}^{m_\tau - 1} \left\| z_T^{(j+\frac{1}{2})} \right\|^2_{H_0^1} \right) \\
&\quad + r_0 \tau \left(\sum_{j=0}^{m_\tau - 1} \left[z_h^{(j+\frac{1}{2})} \right]^2 + \sum_{j=0}^{m_\tau - 1} \left[z_T^{(j+\frac{1}{2})} \right]^2 \right) \\
&\le c \Bigg(\tau \sum_{j=0}^{m_\tau - 1} \left\| \left(h - \tilde{h}\right)^{(j+\frac{1}{2})} \right\|^2_{H_0^1} + \tau \sum_{j=0}^{m_\tau - 1} \left\| \left(T - \tilde{T}\right)^{(j+\frac{1}{2})} \right\|^2_{H_0^1} \\
&\quad + \tau \sum_{j=1}^{m_\tau - 1} \left\| \frac{\left(h - \tilde{h}\right)^{(j+\frac{1}{2})} - \left(h - \tilde{h}\right)^{(j-\frac{1}{2})}}{\tau} \right\|^2_{L_2} \\
&\quad + \tau \sum_{j=1}^{m_\tau - 1} \left\| \frac{\left(T - \tilde{T}\right)^{(j+\frac{1}{2})} - \left(T - \tilde{T}\right)^{(j-\frac{1}{2})}}{\tau} \right\|^2_{L_2} \\
&\quad + \tau \sum_{j=0}^{m_\tau - 1} \left[\left(h - \tilde{h}\right)^{(j+\frac{1}{2})} \right]^2 + \left\| \left(h - \tilde{h}\right)^{(0)} \right\|^2_{L_2} + \left\| \left(h - \tilde{h}\right)^{(m_\tau - \frac{1}{2})} \right\|^2_{L_2} \\
&\quad + \left\| \left(h - \tilde{h}\right)^{(\frac{1}{2})} \right\|^2_{L_2} \\
&\quad + \tau \sum_{j=0}^{m_\tau - 1} \left[\left(T - \tilde{T}\right)^{(j+\frac{1}{2})} \right]^2 + \left\| \left(T - \tilde{T}\right)^{(0)} \right\|^2_{L_2} + \left\| \left(T - \tilde{T}\right)^{(m_\tau - \frac{1}{2})} \right\|^2_{L_2}
\end{aligned}
$$

$$+\left\|\left(T-\tilde{T}\right)^{\left(\frac{1}{2}\right)}\right\|_{L_2}^2+O\left(\tau^4\right)\Bigg), \quad \forall\, \tilde{h}\in M, \forall\, \tilde{T}\in M. \tag{6.1}$$

Similarly to [8, Theorem 6, Chap. 8], taking into account estimate (6.1), it holds

Theorem 6.3. *Let the classical solution $h(x,t)$, $T(x,t)$ of the problem* (3.1)–(3.10) *satisfy the conditions of Theorem 6.2. Then for the errors z of the approximate generalized solution $\widehat{h}(x,t)\in H_m^N$, $\widehat{T}(x,t)\in H_m^N$ of the problem* (5.6)–(5.9) *obtained using the Crank-Nicolson method, the following estimate is valid:*

$$\left\|z_h^{(m_\tau)}\right\|_{L_2}^2+\left\|z_T^{(m_\tau)}\right\|_{L_2}^2+\delta_1\tau\left(\sum_{j=0}^{m_\tau-1}\left\|z_h^{\left(j+\frac{1}{2}\right)}\right\|_{H_0^1}^2+\sum_{j=0}^{m_\tau-1}\left\|z_T^{\left(j+\frac{1}{2}\right)}\right\|_{H_0^1}^2\right)$$

$$\leq c\cdot\left(h_{max}^{2m}+\tau^4\right).$$

7. Using finite element method in the case of inhomogeneous boundary conditions of the first kind

We now abandon the assumptions that the boundary conditions of the first kind (3.2), (3.6) are homogeneous. Then the approximate generalized solution of the initial-boundary value problem (3.1)–(3.10) is sought in a slightly modified form compared to (5.1)

$$\begin{aligned}\widehat{h}\left(x,t\right)&=\sum_{i=1}^{N}h_i\left(t\right)\varphi_i\left(x\right)+W_h\left(x,t\right),\\ \widehat{T}\left(x,t\right)&=\sum_{i=1}^{N}T_i\left(t\right)\varphi_i\left(x\right)+W_T\left(x,t\right),\end{aligned} \tag{7.1}$$

where $W_h\left(x,t\right)$, $W_T\left(x,t\right)$ are some known functions such that

$$W_h\left(x,t\right)|_{x=0}=\overline{h}_0(t), \quad W_T\left(x,t\right)|_{x=0}=\overline{T}_0(t), t\geq 0. \tag{7.2}$$

As a result, all theoretical statements, including the formulation and proof of Theorems 5.1-6.3, will not change in substance. However, they will become more bulky. Given conditions (7.2), the presence of the functions $W_h\left(x,t\right)$, $W_T\left(x,t\right)$ will not affect the accuracy estimates in Theorems 6.1 and 6.3.

In the practical application of the finite element method functions $W_h\left(x,t\right)$, $W_T\left(x,t\right)$ are approximated by expressions

$$W_h\left(x,t\right)\approx h_0\left(t\right)\varphi_0\left(x\right), \quad W_h\left(x,t\right)\approx T_0\left(t\right)\varphi_0\left(x\right), \tag{7.3}$$

where $\varphi_0(x)$ is the piecewise polynomial basis function of the finite element method defined at the node $x = 0$. Next, substituting (7.3) into (7.1), we get

$$\widehat{h}(x,t) = \sum_{i=0}^{N} h_i(t)\,\varphi_i(x), \quad \widehat{T}(x,t) = \sum_{i=0}^{N} T_i(t)\,\varphi_i(x).$$

Given that according to the properties of the basis functions of the finite element method

$$\varphi_0(x)|_{x=0} = 1$$

from (7.2) and (7.3) we obtain

$$h_0(t) = \overline{h}_0(t), \quad T_0(t) = \overline{T}_0(t), t \geq 0.$$

8. Results of numerical experiments

Soil parameters for the numerical experiments were taken from the Hydrus-1D freeware [15]. Specifically, sandy-clay loam was considered as the main soil, with $k_0 = 0.108\ \frac{m}{day}$, $n_0 = 0.45$. Clay with the following parameters was used as the fine inclusion soil: $k_{\omega,0} = 0.0048\ \frac{m}{day}$, $n_{\omega,0} = 0.36$.

Chung and Horton model was used the dependence of thermal conductivity coefficient of saturated soil on porosity according to [23]. In this model $\lambda = p_1 + p_2 n + p_3\sqrt{n}$, where $p_1 = b_1,\ p_2 = b_2,\ p_3 = b_2\sqrt{0.75n + 2b_1/b_2}$. According to Hydrus-1D models [15], for clays $b_1 = 17020.86\frac{J}{day\cdot m\cdot{}^\circ C}$, $b_2 = 83676.27\frac{J}{day\cdot m\cdot{}^\circ C}$, and for loams $b_1 = 20995.15\frac{J}{day\cdot m\cdot{}^\circ C}$, $b_2 = 33955.17\frac{J}{day\cdot m\cdot{}^\circ C}$. Also, we used for both clays and loams, similar to Hydrus-1D,

$$\rho_w c_w = 419995.50\frac{J}{m^3\cdot{}^\circ C},\ \rho_{solid}c_{solid} = 1919996.90\frac{J}{m^3\cdot{}^\circ C}.$$

Regarding the dependence of the filtration coefficient on temperature, the experimental dependences of [27] were used. The value of the filtration coefficient at $T = 20°C$ was taken as a standard. Then, using the Kozeny-Carman formula [20],

$$k(n,T) = k_0\frac{\overline{k}(T)}{\overline{k}(20°C)}\frac{1+e_0}{1+e}\left(\frac{e}{e_0}\right)^3.$$

Here k_0, e_0 are the initial values of the filtration coefficient and the void ratio; k, e are their variable values over time, with $e = \frac{n}{1-n}$, and according to [27]

$$\overline{k}(T) = \frac{\rho_w\,g\,\exp(-30.894 - 0.0109\,T)}{0.2601 + 1.517\,\exp(-0.034688\,T)}.$$

The conclusions of [12] were used for the dependence of thermo-osmotic coefficient on porosity. Particularly, in the model example

$$\mu_\omega(n_\omega) = \begin{cases} 2\mu_{\omega,0} & ,\ n_\omega < 0.75,\ n_{\omega,0}; \\ \mu_{\omega,0} & ,\ 0.75\,n_{\omega,0} \leq n_\omega \leq 1.25\,n_{\omega,0}; \\ 0.5\mu_{\omega,0} & ,\ n_\omega > 1.25\,n_{\omega,0}. \end{cases}$$

The initial value of the thermo-osmotic coefficient was taken as 10% of the value of the filtration coefficient.

The model problem considered a soil layer of $l = 10$ m thickness. The depth of the inclusion $\xi = 2$ m, and its thickness $d = 0.2$ m. The variable x step was 0.02 m. Time step $\tau = 3$ day. Initial pressure distribution $h_0(x) = 20$ m. Initial temperature distribution $T_0(x) = 14°C$. Functions in boundary conditions of the first kind on the soil surface $\overline{h}_0(t) = 0$ m, $\overline{T}_0(t) = 55$ $°C$. Boundary conditions of the second kind were set at the lower boundary. The results of numerical experiments are shown in Table 1 and Table 2. Case I corresponds to numerical experiments for the problem of filtration consolidation under conditions of variable porosity, but without taking into account the effect of temperature. Case II additionally considers the phenomena of thermal osmosis for the geobarrier and the effect of temperature on the hydraulic conductivity parameters of the entire porous media. Note that temperature values and jumps for both cases are practically the same. Such temperature values will differ if the classical conjugation condition is used, without accounting for the effect of variable porosity on the thermal conductivity coefficient.

Taking into account the effects of temperature changes the values of pressure below and above the inclusion, as well as their jumps. Such changes vary within 10% of the values for the isothermal case. By the time of 240 *days*, the pressure jump for Case II is greater than when neglecting thermal effect. However, later, as the soil warms up (because the temperature at the upper limit reaches 55 $°C$), such jumps become smaller. That is, the effect of temperature on the $[h]$ values will not be uniform (definite and predictable increase or decrease of pressure jumps compared to classical and isothermal cases). The following points should also be noted: 1. The values of pressure from the bottom and top of the inclusion are always smaller for the case of non-isothermal conditions. 2. If we take into consideration the classical conjugation condition with constant inclusion parameters, the pressures and their jumps will differ both for Case I and for Case II. 3. Temperature variation at the upper limit will lead to pressure fluctuations, the amplitude of which will depend on the ratio between the filtration and thermo-osmotic coefficients.

The results of the experiments in the assumption that the excess pressures in the area of the soil mass has already dissipated, i.e. the initial pressure distribution $h_0(x) = 0$ m, are also informative. The presence of non-isothermal conditions and taking into account thermo-osmotic properties of the geobarrier material leads to the appearance of a stable pressure jump in the vicinity of a thin inclusion. The pressures above the inclusion become negative, and under the inclusion, positive. If the value of the thermo-osmotic coefficient is 10% of the value of the filtration coefficient, then the maximum pressure jump is 11 cm, increasing to 17 cm if it is 20%. If the values of the filtration and thermo-osmotic coefficients are equal, the maximum pressure jump reaches fully 1.14 m. Such a situation is dangerous in case of the complex geometry of the base of a waste storage facility and the presence of slopes. After all, the stability of slopes decreases in the presence of high humidity and stable pressure fields [17].

Generally, the results of the model examples show that the distribution of pressures in soil structures and natural masses of porous media with fine inclusions depends on the temperature factor. The quantitative indicators of such effects may vary, depending on the ratio of the values of thermo-osmotic and filtration coefficients.

Table 1.Results of numerical experiments - Case I

Time moment	h^-	h^+	$[h]$
$t = 30$ *days*	6,40	12,99	6,59
$t = 60$ *days*	6,20	11,17	4,97
$t = 120$ *days*	6,06	9,86	3,80
$t = 180$ *days*	5,81	9,16	3,35
$t = 240$ *days*	5,50	8,60	3,10
$t = 360$ *days*	4,93	7,71	2,78
$t = 540$ *days*	4,24	6,65	2,41
$t = 720$ *days*	3,70	5,80	2,10
$t = 900$ *days*	3,27	5,08	1,81
$t = 1080$ *days*	2,91	4,48	1,57

Table 2.Results of numerical experiments - Case II

Time moment	h^-	h^+	$[h]$
$t = 30$ *days*	5,90	13,02	7,12
$t = 60$ *days*	5,73	10,98	5,25
$t = 120$ *days*	5,31	9,44	4,13
$t = 180$ *days*	4,91	8,50	3,59
$t = 240$ *days*	4,65	7,86	3,21
$t = 360$ *days*	4,20	6,97	2,77
$t = 540$ *days*	3,60	5,94	2,34
$t = 720$ *days*	3,12	5,10	1,98
$t = 900$ *days*	2,72	4,39	1,67
$t = 1080$ *days*	2,40	3,80	1,40

References

1. N.H. Abu-Hamdeh, R.C. Reeder, *Soil Thermal Conductivity Effects of Density, Moisture, Salt Concentration, and Organic Matter*, Soil Science Society of America Journal, **64**(4) (2000), 1285–1290.
2. V.Maria Barragan, Signe Kjelstrup, *Thermo-osmosis in Membrane Systems: A Review*, Journal of Non-Equilibrium Thermodynamics, **42**(3) (2017), 217–236.
3. Bin Zhu, Zhigang Ye, Lujun Wang, Wenjie Xu, Deqiong Kong, Thomas Nagel, Olaf Kolditz, Yunmin Chen, *Theoretical Investigation into Thermo-Osmosis and Thermofiltration Effects on Hydromechanical Behavior of Saturated Soils*, Journal of Engineering Mechanics, **147**(4) (2021), 04021005.

4. A. Bouazza, M. Asgar Ali, R. Kerry Rowe, Will P. Gates, A. El-Zein, *Heat mitigation in geosynthetic composite liners exposed to elevated temperatures*, Geotextiles and Geomembranes, **45**(5) (2017), 406–417.
5. Bowei Yu, Abbas El-Zein, *Challenges and solutions in using geosynthetic clay liners exposed to thermal desiccation risks*, Japanese Geotechnical Society Special Publication, **9**(2) (2021), 61–66.
6. O.Y. Chernukha, Y.I. Bilushchak, *Mathematical Modeling of the Mean Concentration Field in Random Stratified Structures with Regard for the Jumps of Sought Function on the Interfaces*, J Math Sci,**240** (2019), 70–85.
7. Yu. Chui, P. Martynyuk, M. Kuzlo, O. Ulianchuk-Martyniuk, *The conditions of conjugation in the tasks of moisture transfer on a thin clay inclusion taking into account salt salutions and temperature*, J. of Theoretical and Applied Mechanics(Bulgaria), **49**(1) (2019), 28–38.
8. V.S. Deineka, I.V. Sergienko, V.V. Skopetsky, *Models and methods for solving problems with conjugation conditions*, Naukova dumka, Kiev, 1998.
9. G. Dolle, O. Duran, N. Feyeux, E. Frenod, M. Giacomini, C. Prud'homme, *Mathematical modeling and numerical simulation of a bioreactor landfill using Feel++*, ESAIM: Proceedings and Surveys, **55**(1) (2017), 83–110.
10. J. Goncalves, G. de Marsily, J. Tremosa, *Importance of thermo-osmosis for fluid flow and transport in clay formations hosting a nuclear waste repository*, Earth and Planetary Science Letters, **339-340** (2012), 1–10.
11. J. Goncalves, C.J. Yu, J.-M. Matray, J. Tremosa, *Analytical expressions for thermo-osmotic permeability of clays*, Geophysical Research Letters,**45**(2) (2018), 691–698.
12. B. Hafskjold, D. Bedeaux, S. Kjelstrup, O. Wilhelmsen, *Soret separation and thermo-osmosis in porous media*, Eur. Phys. J. E, **45** (2022), Article number: 41.
13. V.A. Herus, N.V. Ivanchuk, P.M. Martyniuk, *A System Approach to Mathematical and Computer Modeling of Geomigration Processes Using Freefem++ and Parallelization of Computations*, Cybernetics and Systems Analysis, **54**(2) (2019), 284–294.
14. N. Ivanchuk, P. Martynyuk, T. Tsvetkova, O. Michuta, *Mathematical modeling and computer simulation of the filtration processes in earth dams*, Eastern-European Journal of Enterprise Technologies, **2** (6) (2017), 63–69.
15. D. Jacques,J. Simunek, D. Mallants, M.Th. van Genuchten, *The HPx software for multicomponent reactive transport during variably-saturated flow: Recent developments and applications*, J. Hydrol. Hydromech., **66**(2) (2018), 211–226.
16. Jiaming Wang, Hailong He, Miles Dyck, Jialong Lv, *A Review and Evaluation of Predictive Models for Thermal Conductivity of Sands at Full Water Content Range*, Energies, **13** (2020), 1083.
17. T.V. Kutya, P.N. Martynyuk, *Mathematical Simulation of Humidification of Earth on a Slope and Calculation of Its Safety Factor*, J. of Eng. Phys. and Thermophysics, **91** (2018), 1189–1198.
18. P.M. Martyniuk, O.R. Michuta, O.V. Ulianchuk-Martyniuk, M.T. Kuzlo, *Numerical investigation of pressure head jump values on a thin inclusion in one-dimensional non-linear soil moisture transport problem*, Int. J. of Appl. Mathem., **31**(4) (2018), 649–660.
19. O. Michuta, N. Ivanchuk, P. Martyniuk, O. Ostapchuk, *A finite element study of elastic filtration in soils with thin inclusions*, Eastern-European Journal of Enterprise Technologies, **5**(5-107) (2020), 41–48.

20. O. R. Michuta, P. M. Martyniuk, *Nonlinear Evolutionary Problem of Filtration Consolidation With the Non-Classical Conjugation Condition*, J. of Optimization, Differential Equations and their Applications, **30**(1) (2022), 71–87.
21. Mohammadreza Mir Tamizdoust, Omid Ghasemi-Fare, *Assessment of thermal, hydraulic, and mechanical constitutive relations on the temperature-induced stress and pore fluid pressure in saturated clays*, Computers and Geotechnics, **145** (2022), 104686.
22. K.R. Reddy, G. Kumar, R.K. Giri, *Modeling Coupled Processes in Municipal Solid Waste Landfills: An Overview with Key Engineering Challenges*, Int. J. of Geosynth. and Ground Eng., **3** (2017), Article number: 6.
23. M. Sadeghi, B. Ghanbarian, R. Horton, *Derivation of an explicit form of the percolation-based effective medium approximation for thermal conductivity of partially saturated soils*, Water Resources Research, **54**(2) (2018), 1389–1399.
24. M. Sanchez, B. Pomaro, A. Gens, *Coupled THM analysis of a full-scale test for high-level nuclear waste and spent fuel disposal under actual repository conditions during 18 years of operation*, Geotechnique, (2022), 1–21. https://doi.org/10.1680/jgeot.21.00106
25. I.V. Sergienko, V.V. Skopetsky, V.S. Deineka, *Mathematical modeling and research of processes in inhomogeneous environments*, Naukova dumka, Kiev, 1991.
26. Per Schjonning, *Thermal conductivity of undisturbed soil - Measurements and predictions*, Geoderma, **402** (2021), 115188.
27. Shenwei Wang, Wei Zhu, Xuede Qian, Haoqing Xu, Xihui Fan, *Temperature effects on non-Darcy flow of compacted clay*, Applied Clay Science, **135** (2017), 521–525.
28. O.V. Ulianchuk-Martyniuk, *Numerical simulation of the effect of semipermeable properties of clay on the value of concentration jumps of contaminants in a thin geochemical barrier*, Eurasian Journal of Mathematical and Computer Applications, **8**(1) (2020), 91–104.
29. O. Ulianchuk-Martyniuk, O. Michuta, *Conjugation conditions in the problem of filtering chemical solutions in the case of structural changes to the material and chemical suffusion in the geobarrier*, JP Journal of Heat and Mass Transfer, **19** (1) (2020), 141–154.
30. O.V. Ulianchuk-Martyniuk, O.R. Michuta, *Numerical Simulation of the Effect of Chemical Osmosis on the Value of the Jumps of Pollution in the Geochemical Barrier*,Int. J. of Appl. Mathematics, **33**(6) (2020), 1067–1082.
31. O. Ulianchuk-Martyniuk, O. Michuta, N. Ivanchuk, *Biocolmatation and the finite element modeling of its influence on changes in the head drop in a geobarrier*, Eastern-European Journal of Enterprise Technologies, **4** (10-106) (2020), 18–26.
32. O.V. Ulianchuk-Martyniuk, O.R. Michuta, N.V. Ivanchuk, *Finite element analysis of the diffusion model of the bioclogging of the geobarrier*, Eurasian Journal of Mathematical and Computer Applications, **9**(4) (2021), 100–114.
33. A.P. Vlasyuk, P.M. Martynyuk, *Numerical solution of three-dimensional problems of filtration consolidation with regard for the influence of technogenic factors by the method of radial basis functions*, J. of Math. Sciences, **171**(5) (2010), 632–648.
34. A.P. Vlasyuk, P.M. Martynyuk, O.R. Fursovych, *Numerical solution of a one-dimensional problem of filtration consolidation of saline soils in a nonisothermal regime*, J. of Math. Sciences, **160**(4) (2009), 525–535.
35. Wei Zhu, XiaoSi Su, Qiang Liu, *Analysis of the relationships between the thermophysical properties of rocks in the Dandong Area of China*, European Journal of Remote Sensing, **54**(2) (2021), 122–131.

36. Weiqiang Chen, Majid Sedighi, Andrey P. Jivkov, *Thermo-osmosis in silica nanochannels*, Japanese Geotechnical Society Special Publication, **9**(5) (2021), 210–214.
37. Yanjun Shen, Xu Wang, Yongzhi Wang, Keping Zhou, Jinyuan Zhang, Huan Zhang, Jielin Li, *Thermal conductivity models of sandstone: applicability evaluation and a newly proposed model*, Heat Mass Transfer, **57** (2021), 985–998.
38. R. Zagorscak, M. Sedighi, H. R. Thomas, *Effects of Thermo-Osmosis on Hydraulic Behavior of Saturated Clays*, Int. J. of Geomechanics, **17**(3) (2017), Article num.: 04016068.

Received 21.04.2023