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On the Superabundance of Singular Varieties in Positive Characteristic

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ON THE SUPERABUNDANCE OF SINGULAR VARIETIES IN POSITIVE CHARACTERISTIC

by

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ON THE SUPERABUNDANCE OF SINGULAR VARIETIES IN POSITIVE CHARACTERISTIC

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The geproci property is a recent development in the world of geometry. We call a set of points $Z \subseteq \mathbb{P}^3_k$ an (a, b)-geproci set (for GEneral PROjection is a Complete Intersection) if its projection from a general point P to a plane is a complete intersection of curves of degrees a and b. Examples known as grids have been known since 2011.

Previously, the study of the geproci property has taken place within the characteristic 0 setting; prior to the work in this thesis, a procedure has been known for creating an (a, b)-geproci half-grid for $4 \leq a \leq b$, but it was not known what other examples there can be. Furthermore, before the work in this thesis, only a few examples of geproci nontrivial non-grid non-half-grids were known and there was no known way to generate more. Here, we use geometry in the positive characteristic setting to give new methods of producing geproci half-grids and non-half-grids.

We also pick up work that had been done in 2017 by Solomon Akesseh, who had proven that there are no unexpected cubics in characteristic 3 with distinct points and gave examples involving infinitely near points based on quasi-elliptic fibrations in characteristic 2. Each quasi-elliptic fibration has a Dynkin diagram. Here, in contrast, for each possible Dynkin diagram for a quasi-elliptic fibration in characteristic 3, we give an example of the fibration but show it does not give rise to an unexpected cubic.

DEDICATION

To my parents, and to my wife.

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First, I would like to thank my advisor Brian Harbourne. This work would not be possible without him. Thank you for always being patient and understanding, and for helping fuel my mathematical curiosity.

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Chapter 1

Introduction to the Geproci Property

While complete intersections have been a topic of much study for many years in algebraic geometry, the study of the geproci property has emerged relatively recently. Much of the groundwork in this study has been laid in the works [3], [4], and [14], which will be cited often in this dissertation. We will begin with the definition of geproci (from: **ge**neral **pro**jection **c**omplete **i**ntersection).

Definition 1. Let K be an algebraically closed field. A finite set Z in $\mathbb{P}^n_K$ is *geproci* $\left(\widehat{\text{dʒ}}\text{əˈprot}\widehat{\text{ʃ}}\text{i}\right)$ if the projection $\overline{Z}$ of Z from a general point $P \in \mathbb{P}^n_K$ to a hyperplane H is a complete intersection in $H \cong \mathbb{P}^{n-1}_K$.

In this thesis, we are specifically interested in geproci configurations in $\mathbb{P}^3_K$. (No nontrivial examples are known in $\mathbb{P}^n$, $n > 3$.) In the three-dimensional setting, we will specify that a configuration $Z \subseteq \mathbb{P}^3_K$ is (a, b)-geproci (where $a \leq b$) if the image of Z under a general projection into $\mathbb{P}^2_K$ is the complete intersection of a degree a curve and a degree b curve. We will use the notation $\{a, b\}$-geproci in instances when we do not want to require $a \leq b$.

There are two easy-to-understand types of geproci sets. One type is any complete intersection in a plane: it will project from a general point isomorphically to another complete intersection in any other plane, and so is geproci. The other type is a grid, which we will now define.

Definition 2. Given a finite set of a pairwise-disjoint lines A and a finite set of b pairwise-disjoint lines B, such that every line in A intersects every line in B transversely, the ab points of intersection form an (a, b)-**grid**.

The set of points Z of an (a, b)-grid is (a, b)-geproci. The image $\overline{Z}$ of Z under a general projection is equal to the intersection of the images $\overline{A}$ and $\overline{B}$ of A and B, which are unions of a lines in the plane and b lines in the plane respectively, and thus $\overline{A}$ and $\overline{B}$ are curves of degrees a and b, respectively, meeting at ab points. Thus $\overline{Z}$ is a complete intersection.

These two types (sets of coplanar points and grids) are well understood, so are called *trivial*. What is not yet well understood is how non-trivial geproci sets can arise. The existing work on the geproci property has been done over fields of characteristic 0. What is new with this thesis are the results in characteristic $p > 0$, starting in Chapter 2. For the rest of this chapter we will only discuss work which has been done in characteristic 0.

The first non-trivial examples of geproci sets came from the root systems D_4 (pictured below, clickable) and F_4 [4]. These are configurations in $\mathbb{P}^3$ containing 12 points and 24 points, respectively [9]. It was also shown that D_4 is the *smallest* non-trivial geproci set [4], and the only non-trivial $(3, b)$-geproci set [3].

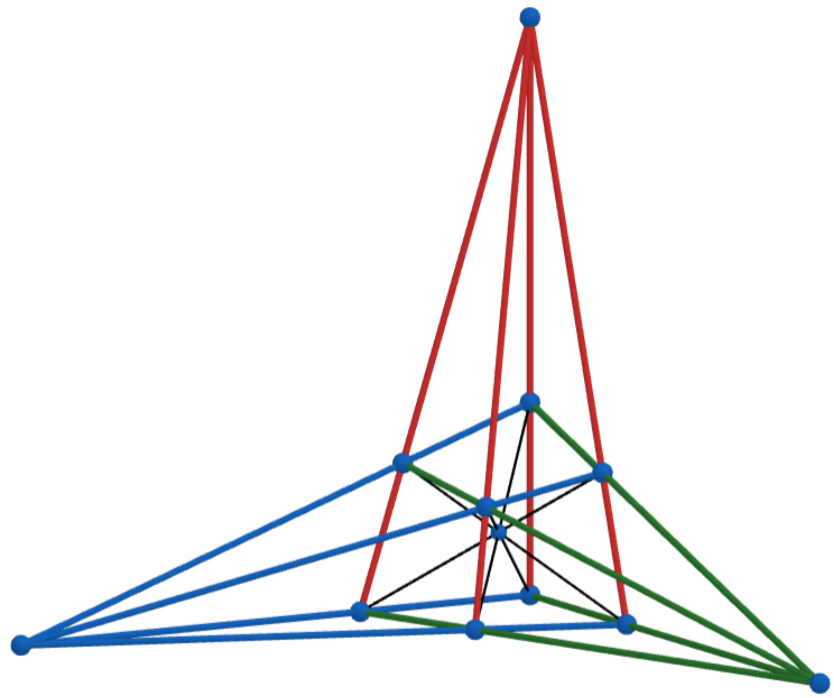


Figure 1.1: D_4

The configurations D_4 and F_4 are examples of *half-grids*.

Definition 3. A non-grid $\{\mu, \lambda\}$-geproci set $Z \subseteq \mathbb{P}^3$ is a **half-grid** if Z is contained in a set of μ mutually-skew lines, with each line containing λ points of Z.

For example, the D4 configuration pictured above is (3, 4)-geproci and can be covered by four mutually-skew lines, with each line containing three points, as Figure 1.2 shows. The general projection of an $\{a, b\}$-geproci half-grid is a complete intersection of a union of a lines and a degree b curve that is not a union of lines. It is known that there is a half-grid (a, b)-geproci set for each $4 \leq a \leq b$ [3]. No other infinite families of non-trivial geproci sets were known before the results in this dissertation, and only finitely many (indeed, three [3]) non-half-grid nontrivial geproci sets were known before the results in the next chapter.

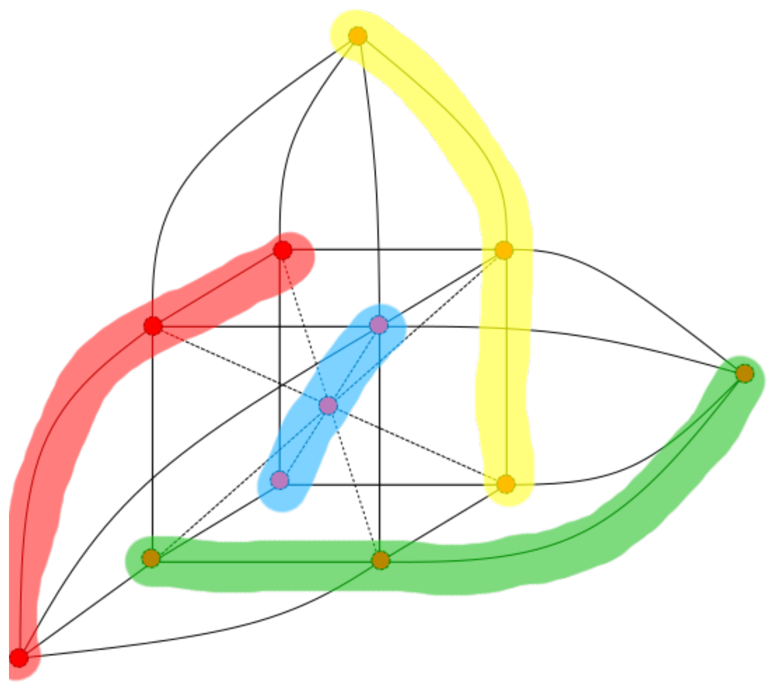


Figure 1.2: D_4 covered by four skew lines

There are strong links between geproci sets Z and sets Z admitting an **unexpected cone** [3, 4].

Definition 4. For $Z \subseteq \mathbb{P}^n_k$ for $n \geq 3$, Z admits an **unexpected** cone of degree d when

$$\dim[I(Z) \cap I(P)^d]_d > \max\left(0, [I(Z)]_d - \binom{d+n-1}{n}\right)$$

for a general point $P \in \mathbb{P}^n_k$, where $I(Z)$ is the homogeneous ideal of Z in $k[\mathbb{P}^n]$ and $[I(Z)]_d$ is its homogeneous component of degree d [9, 10].

This is said to be unexpected because one expects by a naive dimension count that the vector subspace of homogeneous polynomials in $[I(Z)]_d$ that are singular with multiplicity d at a general point P would have codimension $\binom{n+d-1}{n}$ (since being singular at P to order d imposes $\binom{n+d-1}{n}$ conditions on $[I(Z)]_d)$. Therefore it is called unexpected when more such hypersurfaces exist than a naive dimension count would lead one to expect. Chiantini and Migliore showed that every (a, b)-grid with $3 \leq a \leq b$ admits unexpected cones of degrees a and b [4].

Chapter 2

The Geproci Property over Finite Fields

While examples of nontrivial geproci configurations (especially nontrivial non-half-grids) have proven rather elusive in the characteristic 0 setting, we will see in this thesis that they arise quite naturally over finite fields. In the finite field setting, we make generous use of the study of spreads over projective space, which we will define now.

Definition 5. Let $\mathbb{P}_k^{2t-1}$ be a projective space of odd dimension over a field k. Let S be a set of $(t-1)$-dimensional linear subspaces of $\mathbb{P}_k^{2t-1}$. We call S a **spread** if each point of $\mathbb{P}_k^{2t-1}$ is contained in one and only one member of S.

Over a finite field, spreads always exist for each t [2]. In our three-dimensional case, we have $t = 2$. Therefore a spread in $\mathbb{P}_k^3$ will be a set of mutually-skew lines that cover $\mathbb{P}_k^3$.

Example 1. Here we show an example of a spread based on [2]. Given a field extension $k \subseteq L$ with (as vector spaces) $\dim_k L = t$, we get a map

$$\mathbb{P}_k^{2t-1} = \mathbb{P}_k(k^{2t}) = \mathbb{P}_k(L^2) \longrightarrow \mathbb{P}_L(L^2) = \mathbb{P}_L^1$$

with linear fibers $\mathbb{P}_k(L) = \mathbb{P}(k^t) = \mathbb{P}_k^{t-1}$, giving a spread. When we take $k = \mathbb{R}$, $t = 2$, and $L = \mathbb{C}$, we get

$$\mathbb{P}^3_{\mathbb{R}} \longrightarrow \mathbb{P}^1_{\mathbb{C}} = S^2.$$

Composing with the antipodal map $S^3 \to \mathbb{P}^3_{\mathbb{R}}$ gives the well-known Hopf fibration $S^3 \to S^2$ with fibers S^1.

Example 2. Here we give another construction of spreads for $\mathbb{P}^3$ over fields of positive characteristic based on [2] and [11]. Let $\mathbb{F}_q$ be a finite field of size q and characteristic p, where p is an odd prime. Let $r \in \mathbb{F}_q$ be such that the polynomial $x^2 - r \in \mathbb{F}_q[x]$ is irreducible; that is, r has no square root in $\mathbb{F}_q$. Denote by $L_r(a, b)$ the line in $\mathbb{P}^3_{\mathbb{F}_q}$ connecting the points $(1, 0, a, b)$ and $(0, 1, rb, a)$. Denote by $L(\infty)$ the line connecting the points $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$. Then the set of lines

$$S_r = \{L_r(a, b), L(\infty) : a, b \in \mathbb{F}_q\}$$

is a spread in $\mathbb{P}^3_{\mathbb{F}_q}$ (since the lines are skew and there are $q^2 + 1$ lines).

In the case $\operatorname{char} \mathbb{F}_q = 2$, we want to choose $r \in \mathbb{F}_q$ to be such that the polynomial $x^2 + x + r$ is irreducible in $\mathbb{F}_q[x]$. Then define $L_r(a, b)$ to be the line in $\mathbb{P}^3_{\mathbb{F}_q}$ connecting the points $(1, 0, a, b)$ and $(0, 1, br, a + b)$. Then $S_r = \{L_r(a, b), L(\infty) : a, b \in \mathbb{F}_q\}$ is a spread.

Theorem 1. *Let $\mathbb{F}_q$ be the field of size q, where q is some power of a prime. Then $Z = \mathbb{P}^3_{\mathbb{F}_q} \subseteq \mathbb{P}^3_{\overline{\mathbb{F}}_q}$ is a $(q + 1, q^2 + 1)$-geproci half-grid.*

Proof. First we will show that there is a degree $(q + 1)$ cone containing Z having a singularity of multiplicity $q + 1$ at a general point $P \in \mathbb{P}^3_{\overline{\mathbb{F}}_q}$. Let $P = (a, b, c, d) \in \mathbb{P}^3_{\overline{\mathbb{F}}_q}$.

Let

$$M = \begin{pmatrix} a & b & c & d \\ a^q & b^q & c^q & d^q \\ x & y & z & w \\ x^q & y^q & z^q & w^q \end{pmatrix}.$$

Then we claim $F = \det M$ is such a cone.

First note that F contains every point of Z, because $x^q = x$ for each $x \in \mathbb{F}_q$. Furthermore, the terms of F can be combined into groups of 4 so that F is the sum of terms of the form

$$\begin{aligned}(x^q y c^q d - x^q w c^q b) - (z^q y a^q d - z^q w a^q b) &= x^q c^q (yd - wb) - z^q a^q (yd - wb) \\ &= (x^q c^q - z^q a^q)(yd - wb) = (xc - za)^q (yd - wb) \in I^{q+1}((a, b, c, d))\end{aligned}$$

Thus F is a cone C_1 of degree $q+1$ with vertex (a, b, c, d) of multiplicity $q+1$.

Now we will show there is a degree q^2+1 cone C_2 containing Z having a general point P of multiplicity q^2+1. By Example 1, the space $\mathbb{P}^3_{\mathbb{F}_q}$ admits a spread of q^2+1 mutually-skew lines that covers all of $\mathbb{P}^3_{\mathbb{F}_q}$. Each line together with a fixed general point P determines a plane. The union of the planes gives C_2.

Projecting the q^2+1 lines from a general point $P \in \mathbb{P}^3_{\overline{\mathbb{F}}_q}$ onto a general plane $\Pi = \mathbb{P}^2_{\overline{\mathbb{F}}_q}$ yields a set of q^2+1 lines in $\mathbb{P}^2_k$ containing the $(q+1)(q^2+1)$ points of the image of Z.

Now we will show that C_1 and C_2 do not have components in common; to this end, we will show that C_1 contains no line in $\mathbb{P}^3_{\overline{\mathbb{F}}_q}$ defined over $\mathbb{P}^3_{\mathbb{F}_q}$. Note that C_1

vanishes on such a line if and only if $F = 0$, where $F = \det M$ and

$$M = \begin{pmatrix} a & b & c & d \\ a^q & b^q & c^q & d^q \\ X & Y & Z & W \\ X^q & Y^q & Z^q & W^q \end{pmatrix}$$

for $X = \eta_0 u + \mu_0 v$, $Y = \eta_1 u + \mu_1 v$, $Z = \eta_2 u + \mu_2 v$, and $W = \eta_3 u + \mu_3 v$ for all $(u, v) \in \mathbb{P}^1_{\mathbb{F}_q}$ where $(\eta_0, \eta_1, \eta_2, \eta_3)$ and $(\mu_0, \mu_1, \mu_2, \mu_3)$ are points on the line. If r_1, r_2, r_3, and r_4 are the rows of a 4×4 matrix, we will denote the determinant of that matrix by $|r_1, r_2, r_3, r_4|$. In particular, taking the r_i be the the rows of M, we have $F = |r_1, r_2, r_3, r_4| = |r_1, r_2, \eta u + \mu v, \eta u^q + \mu v^q| = 0$ for all (u, v).

Since determinants are multilinear, we have

$$\begin{aligned} & |r_1, r_2, \eta u + \mu v, \eta u^q + \mu v^q| \\ =& |r_1, r_2, \eta u, \eta u^q| + |r_1, r_2, \eta u, \mu v^q| + |r_1, r_2, \mu v, \eta u^q| + |r_1, r_2, \mu v, \mu v^q| \\ =& |r_1, r_2, \eta u, \eta u| u^{q-1} + |r_1, r_2, \eta u, \mu v^q| + |r_1, r_2, \mu v, \eta u^q| + |r_1, r_2, \mu v, \mu v| v^{q-1} \\ =& |r_1, r_2, \eta u, \mu v^q| + |r_1, r_2, \mu v, \eta u^q| = |r_1, r_2, \eta, \mu| u v^q + |r_1, r_2, \mu, \eta| u^q v \\ =& |r_1, r_2, \eta, \mu| u v^q - |r_1, r_2, \eta, \mu| u^q v = |r_1, r_2, \eta, \mu| (v^{q-1} - u^{q-1}) u v. \end{aligned}$$

But $v^{q-1} - u^{q-1} \neq 0$ unless $u = v \in \mathbb{F}_q$. Therefore F can be 0 for all (u, v) only if $|r_1, r_2, \eta, \mu| = 0$. By an appropriate choice of coordinates we get $\eta = (1, 0, 0, 0)$, $\mu = (0, 1, 0, 0)$, $r_1 = (a', b', c', d')$, and $r_2 = (a'^q, b'^q, c'^q, d'^q)$ for some point (a', b', c', d') which is general since (a, b, c, d) is general. Then F is nonzero for $a' = b' = 0$, $c' = 1$, $d' \in \overline{\mathbb{F}}_q \setminus \mathbb{F}_q$, we see $F \neq 0$ for general (a', b', c', d'). We conclude that C_1 does not contain a line of $\mathbb{P}^3_{\overline{\mathbb{F}}_q}$ defined over $\mathbb{P}^3_{\mathbb{F}_q}$, and so C_1 has no components in common with

C_2. (In fact, since C_1 contains the $q+1$ points of each line of $\mathbb{P}^3_{\overline{\mathbb{F}}_q}$ defined over $\mathbb{P}^3_{\mathbb{F}_q}$ but does not contain the line, C_1 meets each line of $\mathbb{P}^3_{\overline{\mathbb{F}}_q}$ defined over $\mathbb{P}^3_{\mathbb{F}_q}$ transversely.) Thus $C_1 \cap C_2$ is a curve of degree $(q+1)(q^2+1)$ and contains the $(q+1)(q^2+1)$ lines through P and points of Z, hence $C_1 \cap C_2$ is exactly this set of lines.

So $\overline{Z}$ is a set of $(q+1)(q^2+1)$ points, which is the intersection of the curves $C_1 \cap \Pi$ (of degree $q+1$) and $C_2 \cap \Pi$ (of degree q^2+1), so $\overline{Z}$ is a $(q+1, q^2+1)$-complete intersection. Thus Z is $(q+1, q^2+1)$-geproci.

□

Of particular interest to the hunt for geproci configurations is the existence of *maximal partial spreads*.

Definition 6. A **partial spread** of $\mathbb{P}^3_{\mathbb{F}_q}$ with **deficiency** d is a set of q^2+1-d mutually-skew lines of $\mathbb{P}^3_{\mathbb{F}_q}$. A **maximal partial spread** is a partial spread of positive deficiency that is not contained in any larger partial spread. We will denote the set of points of $\mathbb{P}^3_{\mathbb{F}_q}$ contained in the lines in a spread S by $\mathcal{P}(S)$.

Maximal partial spreads allow us to construct examples of many geproci sets as subsets of $\mathbb{P}^3_{\mathbb{F}_q}$, using the following corollary.

Corollary 1. *Let S be a partial spread of s lines in $\mathbb{P}^3_{\mathbb{F}_q}$. Then the set of points $\mathcal{P}(S) \subseteq \mathbb{P}^3_{\mathbb{F}_q}$ is $\{s, q+1\}$-geproci.*

Proof. The same degree $q+1$ cone C_1 from the proof of Theorem 1 works in this case. The degree s cone is the join of the s lines with the general point P. It follows from the proof of Theorem 1 that C_1 meets every line of $\mathbb{P}^3_{\mathbb{F}_q}$ transversely and thus that $\mathcal{P}(S)$ is geproci. □

Lemma 1. *Let Z be an $\{a, b\}$-geproci set and let $Z' \subseteq Z$ be a $\{c, b\}$-geproci subset, whose general projection shares with the general projection of Z a minimal generator of degree b. Then the residual set $Z'' = Z \setminus Z'$ is $\{a - c, b\}$-geproci.*

Proof. This is Lemma 4.5 of [3], and the proof still works in positive characteristic. □

Theorem 2. *The complement Z of a maximal partial spread of deficiency d is a non-trivial $\{q + 1, d\}$-geproci set. Furthermore, when $d > q + 1$, Z is also not a half-grid.*

Proof. The first sentence of the Theorem comes directly from the Corollary 1, except for being nontrivial. To demonstrate that Z is nontrivial, suppose Z is contained in a plane H. Let Z' be the complement of Z. Then Z' consists of $q+1$ points on q^2+1-d lines. At most one of those lines can be in H, but each of the lines meet H. Thus Z' has at least q^2+1-d points in H, so Z consists of at most $q^2+q+1-(q^2+1-d) = q+d$ points. This is impossible since $|Z| = (q + 1)d > q + d$.

Now suppose that Z it a grid. Thus is consists of $q + 1$ points on each of d lines. But Z' comes from a maximal partial spread, so Z contains no set of $q + 1$ collinear points. Thus Z cannot be a grid, so Z is nontrivial.

So we will prove that Z is a non-trivial non-half-grid when $d > q + 1$. Recall that every line in $\mathbb{P}^3_{\mathbb{F}_q}$ consists of $q + 1$ points. Suppose Z were a half-grid. Then Z would be contained in $q + 1$ mutually-skew lines, with each line containing d points of Z (we know that it cannot be the other way around because then Z couldn't be the complement of a maximal partial spread). This cannot happen when $d > q + 1$, because each line contains only $q + 1$ points. □

Example 3. By [11], if $q \geq 7$ and q is odd, then $\mathbb{P}^3_{\mathbb{F}_q}$ has a maximal partial spread of size n for each integer n in the interval $\dfrac{q^2+1}{2} + 6 \leq n \leq q^2 - q + 2$. In terms of deficiency $d = q^2 + 1 - n$, we get the inequalities $q - 1 \leq d \leq \dfrac{q^2+1}{2} - 6$. Thus for every odd $q \geq 7$ there is a maximal partial spread in $\mathbb{P}^3_{\mathbb{F}_q}$ of deficiency $d > q + 1$ and thus a nontrivial non-half-grid $(q + 1, d)$-geproci set.

Remark 1. In addition to Heden's bounds [11] showing the existence of maximal partial spreads, Mesner has provided a lower bound for the size of the deficiency d at $\sqrt{q} + 1 \leq d$ [13]. Glynn has provided an upper bound for d at $d \leq (q - 1)^2$ [7].

Example 4. By Lemma 1, for any line $L \subseteq \mathbb{P}^3_{\mathbb{F}_2}$, the set $Z = \mathbb{P}^3_{\mathbb{F}_2} \setminus L$ is a $(3, 4)$-geproci half-grid. In fact, Z has the same combinatorics as D_4, shown in Figure 2.1 (that is, Z is a configuration of 12 points, each of which is on 4 lines, with each line containing 3 points).

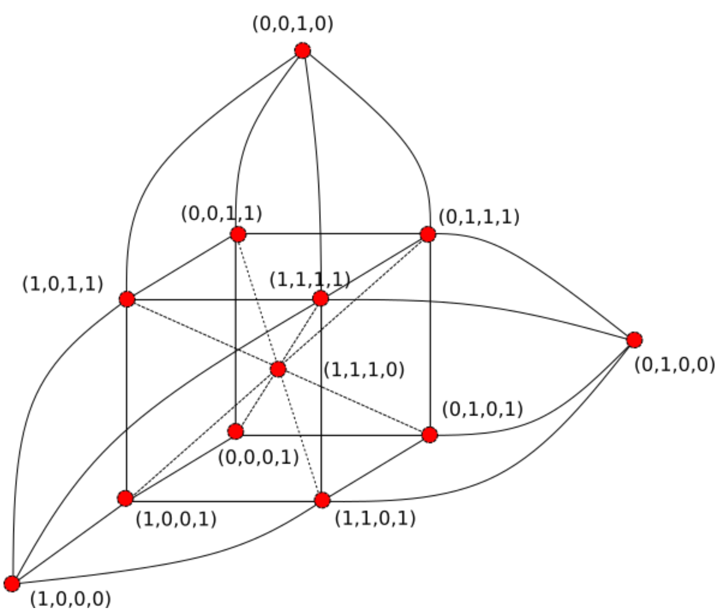


Figure 2.1: $\mathbb{P}^2_{\mathbb{Z}/2\mathbb{Z}} \setminus V(x + y + z, w)$ is a D_4

Example 5. There is (up to isomorphism) a unique maximal partial spread in $\mathbb{P}^3_{\mathbb{Z}/3\mathbb{Z}}$. This spread contains seven lines (as opposed to a complete spread, which contains ten). The complement Z of the points of the maximal partial spread is a set of 12 points in $\mathbb{P}^3_{\mathbb{Z}/3\mathbb{Z}}$ that is $(3,4)$-geproci and non-trivial. Furthermore, Z has the same combinatorics as the D_4 configuration (that is, Z is a configuration of 12 points, each of which is on 4 lines, with each line containing 3 points). Note that Z is then a half-grid, as shown in Figure 2.2. Coordinates can be chosen (as in Figure 2.2) so that the 12 points are $(3,4)$-geproci over any field of characteristic $\neq 2$.

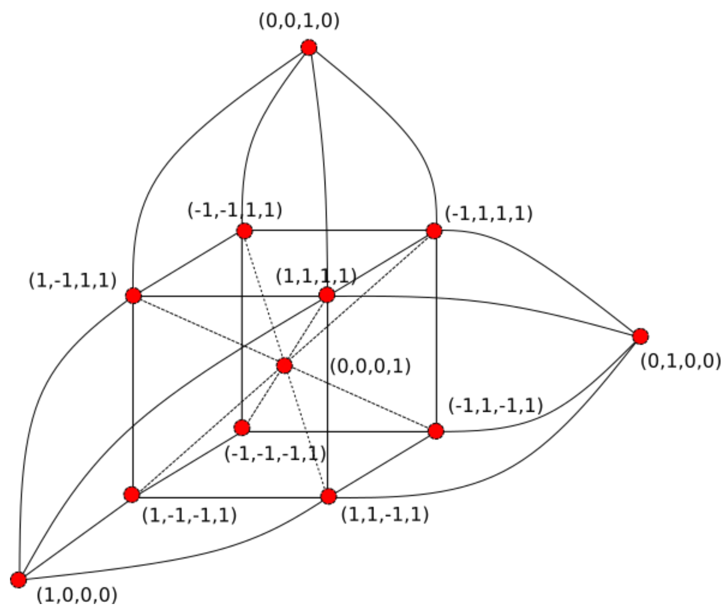


Figure 2.2: A D_4 in $\mathbb{P}^3_k$, $\operatorname{char} k \neq 2$

Example 6. There are (up to isomorphism) fifteen maximal partial spreads in $\mathbb{P}^3_{\mathbb{Z}/7\mathbb{Z}}$ that consist of 45 lines (as opposed to a complete spread, which contains 50). Let Z be the complement of the set of points of any of these maximal partial spreads. Then Z is a set of 40 points that is a non-trivial $(5,8)$-geproci non-half-grid. Furthermore, Z has the same combinatorics as the Penrose configuration of 40 points.

Note that if we look at two non-isomorphic maximal partial spreads M and M', and consider their complements Z and Z', then Z and Z' are non-isomorphic non-trivial non-half grid $(5, 8)$-geproci configurations. In fact, some such configurations have stabilizers of different sizes! Of the fifteen up to isomorphism, there are nine with stabilizers of size 10, there is one with a stabilizer of size 20, there is one with a stabilizer of size 60, and there are four with stabilizers of size 120.

An example of such a geproci set is

$$\begin{aligned}
\{&(0,0,1,3),(0,1,3,3),(0,1,3,5),(0,1,4,6),\\
&(0,1,6,5),(1,0,1,3),(1,0,2,6),(1,0,4,5),\\
&(1,0,4,6),(1,1,0,1),(1,1,0,4),(1,1,1,4),\\
&(1,1,5,2),(1,2,1,6),(1,2,3,3),(1,2,5,2),\\
&(1,2,6,5),(1,3,2,1),(1,3,4,4),(1,3,5,2),\\
&(1,3,6,0),(1,4,0,5),(1,4,2,4),(1,4,4,1),\\
&(1,4,6,2),(1,5,0,4),(1,5,1,0),(1,5,2,0),\\
&(1,5,3,0),(1,5,3,1),(1,5,3,3),(1,5,3,6),\\
&(1,5,4,5),(1,5,5,0),(1,5,5,2),(1,5,6,3),\\
&(1,6,0,3),(1,6,1,5),(1,6,2,1),(1,6,6,6)\}.
\end{aligned}$$

This example is the complement of a maximal partial spread of size 45 with a stabilizer of size 60.

We also used Macaulay2 to check that at least one configuration of each size stabilizer is Gorenstein. This contrasts with the case in characteristic 0, where only one non-trivial Gorenstein configuration is known, up to isomorphism: the Penrose configuration. [3]

One can determine this using the following commands in Macaulay2 with the example set of points from above.

```
i1:K=toField(ZZ/7[a,b,c,d]);

i2:R=K[x,y,z,w];

i3:V={{0,0,1,3},{0,1,3,3},{0,1,3,5},{0,1,4,6},
{0,1,6,5},{1,0,1,3},{1,0,2,6},{1,0,4,5},
{1,0,4,6},{1,1,0,1},{1,1,0,4},{1,1,1,4},
{1,1,5,2},{1,2,1,6},{1,2,3,3},{1,2,5,2},
{1,2,6,5},{1,3,2,1},{1,3,4,4},{1,3,5,2},
{1,3,6,0},{1,4,0,5},{1,4,2,4},{1,4,4,1},
{1,4,6,2},{1,5,0,4},{1,5,1,0},{1,5,2,0},
{1,5,3,0},{1,5,3,1},{1,5,3,3},{1,5,3,6},
{1,5,4,5},{1,5,5,0},{1,5,5,2},{1,5,6,3},
{1,6,0,3},{1,6,1,5},{1,6,2,1},{1,6,6,6}};

i4:IV={};

i5:for i from 0 to V-1 do {A=trim ideal(V_i_0*y-V_i_1*x,
V_i_0*z-V_i_2*x,V_i_0*w-V_i_3*x,V_i_1*z-V_i_2*y,V_i_1*w-V_i_3*y,
V_i_2*w-V_i_3*z);IV=IV|{A}};

i6:I=intersect(IV);

i7:betti res I
```

o7:

	0	1	2	3
total:	1	5	5	1
0:	1	·	·	·
1:	·	·	·	·
2:	·	·	·	·
3:	·	5	·	·
4:	·	·	5	·
5:	·	·	·	·
6:	·	·	·	·
7:	·	·	·	1

We can see from the Betti table that this configuration of points is Gorenstein. A similar calculation works to show the other geproci configurations are Gorenstein.

This pattern leads us to the following question:

Question 1. *Given the complement of a maximal partial spread $Z \subseteq \mathbb{P}^3_{\mathbb{F}_q}$, when does Z correspond to a non-trivial geproci set that exists in $\mathbb{P}^3_{\mathbb{C}}$?*

Chapter 3

The Geproci Property with Infinitely-Near Points

We can also consider configurations of points that include *infinitely-near points*.

Definition 7. Let P be a point on an algebraic set X. Let $\mathrm{Bl}_P(X)$ denote the **blowup** of X at P. Then the point $Q \in \mathrm{Bl}_P(X)$ is **infinitely-near** P if Q maps to P under the standard blowup map $\pi_P : \mathrm{Bl}_P(X) \to X$.

On the other hand, if $Q \in X$ and $Q \neq P$, then Q and P are **distinct**.

Intuitively, Q corresponds to the direction of a line through P. In the plane, we can consider how a point P and a point Q that is infinitely-near P can uniquely determine a line, the same way a line can be uniquely determined by two completely different points. This is akin to determining a line from a point and a slope. In $\mathbb{P}^3$, we will consider how infinitely-near points impose conditions on varieties the same way distinct points can.

We can extend the definition of geproci to include configurations with infinitely-near points by realizing Z as a non-reduced 0-dimensional subscheme of $\mathbb{P}^3$. We have Z is geproci if the projection $\overline{Z}$ of Z from a general point P to a plane is a complete intersection as a subscheme of $\mathbb{P}^2$.

In the following configurations of points in $\mathbb{P}^3_{\mathbb{F}_2}$, we will denote a point P together with a point infinitely-near P as $P \times 2$. We will then specify what line the infinitely-near point corresponds to.

Example 7. We will consider the set of nine (not distinct) points in $\mathbb{P}^3_{\mathbb{Z}/2\mathbb{Z}}$:

$$Z = \{(1,0,0,0) \times 2, (0,1,0,0) \times 2, (0,0,1,0) \times 2, (0,0,0,1) \times 2, (1,1,1,1)\}$$

by choosing as our infinitely-near points for $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, and $(0,0,0,1)$ as the point that corresponds to the (respective) direction of the line through the given point and the point $(1,1,1,1)$.

The projection $\overline{Z}$ of these 9 points to the plane $w = 0$ from a general point takes $(0,0,1)$, $(0,1,0)$, $(1,0,0)$ to themselves and $(1,1,1,1)$ and $(0,0,0,1)$ to general points. After a change of coordinates we can map the image of $(1,1,1,1)$ to $(1,1,1)$ and the image of $(0,0,0,1)$ to (a,b,c). We will denote

$$Z' = \{(0,0,1) \times 2, (0,1,0) \times 2, (1,0,0) \times 2, (a,b,c) \times 2, (1,1,1)\},$$

where the tangent directions of each point of multiplicity 2 correspond to the line connecting the point with $(1,1,1)$.

We can see that the conic $C_1 = V(xy + xz + yz)$ contains the points $(0,0,1)$, $(0,1,0)$, and $(1,0,0)$, and the tangent lines of the three points all meet $(1,1,1)$. Additionally, the line L_1 connecting (a,b,c) and $(1,1,1)$ has the appropriate slope to contain the remaining infinitely-near point. Therefore the cubic given by $C_1 \cup L_1$ contains Z'.

Similarly, we can also construct a conic $C_2 = V(cxy + bxz + ayz + (a+b+c)y^2)$ that contains the points $(0,0,1)$, $(0,1,0)$, (a,b,c), and their respective infinitely-near points. Letting L_2 denote the line connecting $(1,0,0)$ and $(1,1,1)$, we get another cubic $C_2 \cup L_2$ containing Z'. The two cubics share no components in common, and so Z' is a complete intersection of two cubics.

Since Z' is projectively equivalent to $\overline{Z}$, we get $\overline{Z}$ is a complete intersection. Therefore Z is $(3,3)$-geproci. Note that Z is a nontrivial non-half-grid.

Example 8. Let $\operatorname{char} k = 2$. Now consider the 6 points

$$Y = \{(1,0,0,0) \times 2, (0,1,0,0) \times 2, (0,0,1,0) \times 2\},$$

where the infinitely near point for each is in the direction of $(0,0,0,1)$. We will show that this is $(2,3)$-geproci.

Proof. First we will look at a configuration of points in $\mathbb{P}^2$:

$$Y' = \{(1,0,0) \times 2, (0,1,0) \times 2, (0,0,1) \times 2\}$$

where the infinitely-near point for each is in the direction of $(1,1,1)$. We will show that this set of 6 points is a complete intersection of a conic and a cubic, and then show that a general projection of Y onto any plane is isomorphic to Y'. Note that Y' is contained in the conic $A = V(xy+xz+yz)$ and the cubic $B = V((x+y)(x+z)(y+z))$. Also note that A and B, have no components in common, since A is an irreducible conic and B is the union of three lines. Therefore Y' is a complete intersection of a conic and a cubic.

Now let us return to $Y \subseteq \mathbb{P}^3$. Let us project Y from a general point $P \in \mathbb{P}^3$ onto a general plane $\Pi \subseteq \mathbb{P}^3$. Since the lines corresponding to each infinitely-near point meet at $(0,0,0,1)$, and since projection from a point preserves lines (and therefore the intersection of lines), the images of the three infinitely-near points under the projection $\pi_{P,\Pi}$ will also correspond to three concurrent lines. In other words, Y will map to the set

$$\pi_{P,\Pi}(Y) = \{\pi_{P,\Pi}(1,0,0,0) \times 2, \pi_{P,\Pi}(0,1,0,0) \times 2, \pi_{P,\Pi}(0,0,1,0) \times 2\}$$

where each infinitely-near point is in the direction of $\pi_{P,\Pi}(0,0,0,1)$. For a general point P, the images of the three ordinary points in Y and the point $\pi_{P,\Pi}(0,0,0,1)$ will not be collinear. Therefore we can map Π to $\mathbb{P}^2$ and use an automorphism of the plane to map $\pi_{P,\Pi}(1,0,0,0)$ to $(1,0,0)$, $\pi_{P,\Pi}(0,1,0,0)$ to $(0,1,0)$, $\pi_{P,\Pi}(0,0,1,0)$ to $(0,0,1)$, and $\pi_{P,\Pi}(0,0,0,1)$ to $(1,1,1)$. Then we are in the same situation as Y', which is a complete intersection of a conic and a cubic.

Note that Y is a half-grid, since the cubic containing Y is a union of three lines, but the conic is irreducible.

The unique quadric cone containing $Y + 2 \times (a,b,c,d)$ is given by $cdxy + bdxz + adyz + abw^2$. $\square$

Example 9. Now consider the 9 points

$$X = \{(1,0,0,0) \times 2, (1,1,0,0) \times 2, (0,1,0,0) \times 2, (0,0,1,0) \times 2, (0,0,0,1)\} \subseteq \mathbb{P}^3_{\mathbb{Z}/2\mathbb{Z}},$$

by choosing as our infinitely-near points for $(1,0,0,0)$, $(1,1,0,0)$, $(0,1,0,0)$, and $(0,0,1,0)$ the point that corresponds to the respective direction of the point $(0,0,0,1)$. First we will look at the following configuration of points in $\mathbb{P}^2_{\mathbb{F}_2}$:

$$X' = \{(1,0,0) \times 2, (a,0,1) \times 2, (0,0,1) \times 2, (1,1,1) \times 2, (0,1,0)\}$$

where $a \in \overline{\mathbb{F}}_2 \setminus \mathbb{F}_2$ and the infinitely-near point is in the direction of $(0,1,0)$. These nine points are a complete intersection of $(y^2+xz)(x+az)$ and $y^2(x+z)$. Since every set of four points, no three of which are collinear, maps can be mapped to every other such set of four points by a linear automorphism, every projection of X onto any plane Π will be isomorphic to the configuration X' for some $a \in k \setminus \{1,0\}$, and so X is a non-trivial $(3,3)$-geproci set.

The preceding example is particularly interesting because the general projection of X is not only $(3,3)$-geproci, but it is also the set of base points of a quasi-elliptic fibration (specifically one with Dynkin diagram $\widetilde{A}_1^{\oplus 4} \oplus \widetilde{D}_4$), which we will investigate more in the following chapters. This raises an interesting question: does every quasi-elliptic fibration correspond to some geproci set of points? The answer is no, with a counter example provided by the following case:

Example 10. Let k be a field of characteristic 2. Let

$$Z' = \{(0,0,1,0),(0,1,0,0),(0,1,1,0),(1,0,0,0),(1,0,1,0),(1,1,0,0),(1,1,1,0)\} \subseteq \mathbb{P}^3_k.$$

In other words, $Z' = V(w) \cap \mathbb{P}^3_{\mathbb{Z}/2\mathbb{Z}} \subseteq \mathbb{P}^3_k$. Then for any point $P_8' \in \mathbb{P}^3_k$ and any P_9' infinitely-near P_8', the set $Z = Z' \cup \{P_8', P_9'\}$ is not geproci.

Proof. First consider the 7 points of the Fano plane $\mathbb{P}^2_{\mathbb{Z}/2\mathbb{Z}} \subseteq \mathbb{P}^2_k$. Let P_8' and Q be general After a change of coordinates we may attain $P_8' = (a,b,c,d)$ and $Q = (0,0,0,1)$. The image P_8 of P_8' under the projection π_Q from Q to the plane $w=0$ is $P_8 = (a,b,c,0)$. The points $P_1,\dots,P_7,P_8$ yield a pencil of cubics in the plane $w=0$

whose ninth base point P_9 is infinitely-near P_8 in the direction of

$$bc(b+c)x + ac(a+c)y + ab(a+b)z = 0.$$

In order to be a complete intersection, $\pi_Q(P_9')$ would have to be on the line $bc(b+c)x+ac(a+c)y+ab(a+b)z = 0$ on the plane $w = 0$. The preimage of this line is the plane $bc(b+c)x + ac(a+c)y + ab(a+b)z = 0$ in $\mathbb{P}^3_k$. Therefore P_9' must correspond to some line on this plane through P_8'.

Now project from a general point R. The image P_8 of P_8' will be some general point $(a', b', c', 0)$ and P_9 will have to be the tangent at P_8 to

$$b'c'(b'+c')x + a'c'(a'+c')y + a'b'(a'+b')z = 0,$$

so P_9' has to be a tangent at P_8' in the plane this defines in $\mathbb{P}^3$.

Now project from a general point S. The image P_8 of P_8' will be some general point $(a'', b'', c'', 0)$ and P_9 will have to be the tangent at P_8 to

$$b''c''(b''+c'')x + a''c''(a''+c'')y + a''b''(a''+b'')z = 0,$$

so P_9' has to be a tangent at P_8 in the plane this defines in $\mathbb{P}_3$. But for general points (a, b, c), (a', b', c'), (a'', b'', c''), these planes intersect only at the point $P_8' = (a, b, c, d)$. Thus there is no choice for the infinitely-near point P_9' that will make Z a (3,3)-geproci configuration.

$\square$

Chapter 4

Introduction to Quasi-Elliptic Fibrations

The rest of this thesis concerns the notion of a quasi-elliptic fibration and is a departure from the topic of the geproci property from chapters 1 to 3 (although quasi-elliptic fibrations are related to the geproci property due to their mutual connection to unexpected varieties). The main result of these chapters is Theorem 3 in chapter 7.

In this chapter, we prove some general results about quasi-elliptic fibrations. In chapter 5, we describe the Dynkin diagram associated to a quasi-elliptic fibration in characteristic 2. In chapter 6, we will connect the quasi-elliptic fibrations from chapter 5 to unexpected cubics. In chapter 7, we describe the Dynkin diagram associated to a quasi-elliptic fibration in characteristic 3, and go on to show that these quasi-elliptic fibrations are not connected to any unexpected cubics.

Definition 8. Let k be an algebraically closed field, and let X be a smooth projective surface. A **quasi-elliptic fibration** $f : X \to \mathbb{P}^1_k$ is a morphism whose general fiber is isomorphic to a singular cubic curve.

(Note that if X is a surface admitting such a morphism f, we will often simply refer to X as a quasi-elliptic fibration.) It is known that quasi-elliptic fibrations can occur only in characteristics 2 and 3 [12]. Also, any quasi-elliptic surface is also the consecutive blowing-up of the projective plane $\mathbb{P}^2$ at nine (possibly infinitely-near) points [8].

We will show that every quasi-elliptic fibration X is extremal; that is, every quasi-elliptic fibration has only finitely many sections.

Each section of X corresponds to a linear combination of the base points under the group law of each fibre, with a fixed base point chosen as the group identity. We will show such a set of points is well-defined.

Proposition 1. *Let $A, B, C, D \in F \subset \mathbb{P}^2$, a general cubic curve. Let $\langle P_1, \ldots, P_n \rangle_Q$ denote the subgroup of F generated by the points $P_1, \ldots, P_n \in F$ with $Q \in F$ chosen as the zero point. Let $P_1 +_Q P_2$, $-_Q P$, and $n \cdot_Q P$ denote addition, negation, and $\mathbb{Z}$-module multiplication with Q as the zero point, respectively. Then $A +_B C \in \langle A, B, C \rangle_D$, $A +_B C \in \langle A, B \rangle_C$, $-_A B \in \langle A \rangle_B$, and $-_A B \in \langle A, B \rangle_C$.*

Proof. Note that

$$A +_B C = A +_D C -_D B, \tag{4.1}$$

$$A +_B C = A -_C B, \tag{4.2}$$

$$-_A B = 2 \cdot_B A, \tag{4.3}$$

$$-_A B = A +_B A = A +_C A -_C B \tag{4.4}$$

all follow from manipulations of line bundles in the Picard group, as we will show. We shall rewrite the above expressions as equations in the degree 0 Picard group $\mathrm{Pic}^0 F$.

(4.1): For the left hand of the first equation, we have $A +_B C$, which is equivalent to the expression $A + C - 2B \in \mathrm{Pic}^0 F$. Let $P_1 \in F$ be such that $P_1 - B \sim A + C - 2B$, where $\sim$ denotes linear equivalence. The right side of the equation is $A +_D C -_D B$, which is equivalent to $A + C - D - B \in \mathrm{Pic}^0 F$. Let $Q_1 \in F$ be such that $Q_1 - D \sim$

$A+C-D-B$. We want to show that $P_1 = Q_1$.

The equivalence $P_1 - B \sim A+C-2B$ gives us $P_1 + B \sim A + C$, which means that the line connecting P_1 and B intersects the line connecting A and C at a point $R_1 \in F$. Similarly, the equivalence $Q_1 - D \sim A+C-D-B$ gives us $Q_1+B \sim A+C$, which means that the line connecting Q_1 and B intersects the line connecting A and C at a point on F. This point must be R_1, since $\deg F = 3$. Thus we must have $Q_1 = P_1$, as desired, and so we get the equivalence $A +_B C = A +_D C -_D B = P_1$.

(4.2): Now let us investigate the second equation $A +_B C = A -_C B$. Let $P_2 \in F$ be such that $A + C - 2B \sim P_2 - B$, and let $Q_2 \in F$ be such that $A - B \sim Q_2 - C$. The first equivalence gives us $A + C \sim P_2 + B$, and so the line connecting A and C intersects the line connecting P_2 and B at a point $R_2 \in F$. The second equivalence gives us $A + C = Q_2 + B$, and so the line connecting A and C intersects the line connecting Q_2 and B at a point on F. This point must be R_2, since $\deg F = 3$. Thus we have $P_2 = Q_2$, and we get the equivalence $A +_B C = A -_C B = P_2$.

(4.3): Now let $P_3 \in F$ be such that $A - B \sim P_3 - A$, giving us the equivalence $2A \sim P_3 + B$. Thus the line tangent to F at A intersects the line connecting P_3 and B at a point $R_3 \in F$. Let $Q_3 \in F$ be such that $2A - 2B \sim Q_3 - B$, giving us the equivalence $2A \sim Q_3 + B$. Thus the line tangent to F at A intersects the line connecting Q_3 and B at a point on F. This point must be R_3, which forces $P_3 = Q_3$. Thus we have the equation $-_A B = 2 \cdot_B A = P_3$.

(4.4): The final equation $-_A B = A +_B A = A +_C A -_C B$ comes from combining equations (4.3) and (4.1). $\square$

Therefore for any nine points $P_1, \ldots, P_9 \in F$, we have $\langle P_1, \ldots, P_9 \rangle_{P_i} = \langle P_1 \ldots, P_9 \rangle_{P_j}$ for all $1 \leq i, j \leq 9$.

Corollary 2. *Let X be a blow-up of $\mathbb{P}^2$ that gives a quasi-elliptic fibration. Then X*

is extremal.

Proof. By Proposition 1, it is well-defined to say each section of X corresponds to a linear combination of the base points under the group law of each fibre, with some fixed base point chosen as the zero.

Let F_η be the fiber of $\eta \in \mathbb{P}^1_k$, and define G_η as $\langle P_1, \ldots, P_9 \rangle_{P_i}$ as a subgroup of F_η. Since each fibre F_η is of additive type [8], and $\operatorname{char} k \in \{2, 3\}$, the subgroup G_η is finite. Since each (-1)-curve is a section, they correspond to a curve given by a point in G_η as η runs through $\mathbb{P}^1_k$, there must be finitely many (-1)-curves. Therefore X is extremal. $\square$

Let $Z \subseteq \mathbb{P}^2_k$ be a set of points. Let $I_{Z+dP} \subseteq k[\mathbb{P}^2]$ be the ideal $\bigcap_{q \in Z} I(q) \cap I(P)^d$. We say Z admits an unexpected curve of degree $d+1$ if

$$h^0\left(\mathbb{P}^2_k, I_{Z+dP}(d+1)\right) > \max\left(0, I_Z(d+1) - \binom{d+1}{2}\right),$$

for a general point $P \in \mathbb{P}^2_k$.

For a finite set $Z \subseteq \mathbb{P}^n_k$ for $n \geq 3$, we consider whether Z admits an unexpected cone of degree d, which is when

$$h^0\left(\mathbb{P}^n_k, I_{Z+dP}(d)\right) > \max\left(0, I_Z(d) - \binom{d+n-1}{n}\right)$$

for a general point $P \in \mathbb{P}^n_k$. We will borrow notation from Chiantini and Migliore [4], who denote Z admitting an unexpected cone of degree d as Z having the $C(d)$ property, in $\mathbb{P}^3_k$ (or $C_Z(d)$ if it is not clear which Z is intended).

In 2017, Solomon Akesseh [1] proved that a finite set $Z \subseteq \mathbb{P}^2$ of distinct points has an unexpected cubic curve only when Z consists of the seven points of the Fano plane in characteristic 2. Farnik, Galuppi, Sodomaco, and Trok [6] recovered the result that there is no unexpected cubic in characteristic 0, and in addition showed in characteristic 0 that unexpected quartics arise for a unique (up to choice of coordinates) configuration of 9 points in $\mathbb{P}^2$.

To begin discussing unexpected cubic curves, it is necessary to detail their connection to quasi-elliptic fibrations with the following lemma.

Lemma 2. *Let Z be a set of seven (not necessarily distinct) points in $\mathbb{P}^2_k$ admitting an unexpected cubic. Then the points impose independent conditions on cubics, every cubic curve through Z is singular, the general cubic through Z is reduced and irreducible, and for a general point P there is a unique cubic C_P singular at P; it is reduced and irreducible.*

Proof. The proof of Lemma 3.2.9 in Akesseh [1] carries over to the case that Z contains infinitely near points. □

Proposition 2. *Let $Z = \{P_1, \dots, P_7\}$ be a set of seven (not necessarily distinct) points in $\mathbb{P}^2_k$ admitting an unexpected cubic. Let $P \in \mathbb{P}^2$ be general. Then the set $\mathcal{P}_P$ of cubic curves containing $Z \cup \{P\}$ is a quasi-elliptic fibration. Furthermore, it is isomorphic to the blowup of $\mathbb{P}^2_k$ at nine points: the seven points of Z, the point P and a point infinitely-near P.*

Proof. Since Z imposes independent conditions on cubics by Lemma 2 and since P is general, $\mathcal{P}_P$ is a linear pencil of cubics. By Lemma 2, the cubics are singular but fixed component free, so blowing up the vase points gives a quasi-elliptic fibration. Since the general member of the pencil is a cubic, there are 9 base points. By Lemma 2, the cubics containing $Z \cup \{P\}$ contain C_P and the general one, C, is reduced and irreducible. Since $C.C_P = P_1 + \cdots + P_7 + 2P$, the 9 base points are $P_1, \dots, P_7, P, P_9$, where P_9 is infinitely near P. □

Chapter 5

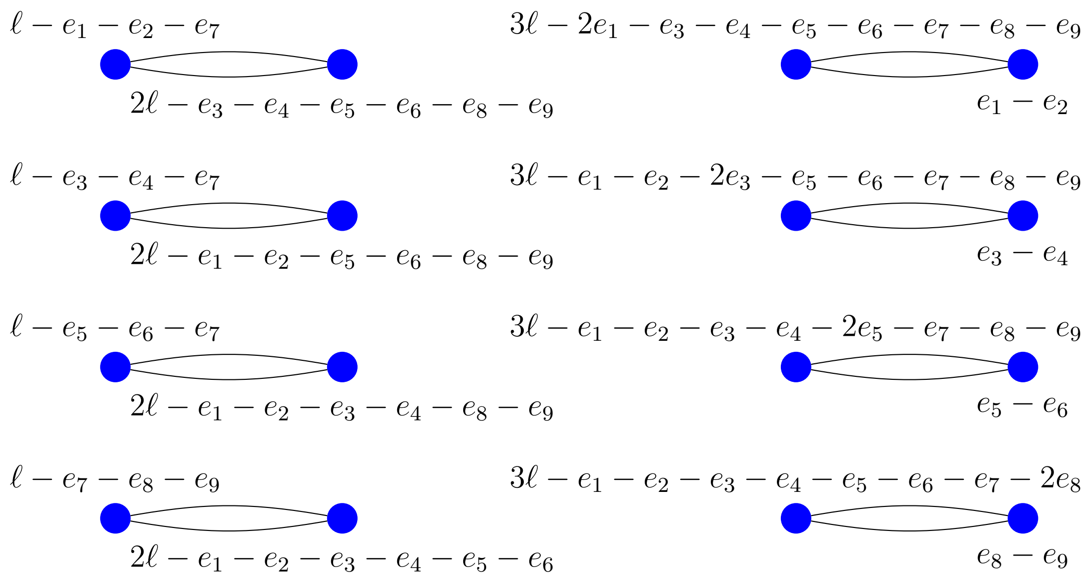
Quasi-Elliptic Fibrations in Characteristic 2

Given a quasi-elliptic fibration X, one may construct a multigraph D the following way: the vertex set is the set of curves on X with a self-intersection of -2, and the number of edges connecting two vertices is equal to the multiplicity of the intersection of their respective curves. The multigraph D constructed this way is called the **Dynkin diagram** of X.

The possible Dynkin diagrams for quasi-elliptic fibrations in characteristic 2 are $\widetilde{A}_1^{\oplus 8}$, $\widetilde{A}_1^{\oplus 4} \oplus \widetilde{D}_4$, $\widetilde{A}_1^{\oplus 2} \oplus \widetilde{D}_6$, $\widetilde{D}_4^{\oplus 2}$, $\widetilde{A}_1 \oplus \widetilde{E}_7$, $\widetilde{D}_8$, and $\widetilde{E}_8$. We will show that there are 2 ways to blow down $\widetilde{A}_1^{\oplus 8}$ to $\mathbb{P}^2$, 4 ways to blow down $\widetilde{A}_1^{\oplus 4} \oplus \widetilde{D}_4$, 4 ways to blow down $\widetilde{A}_1^{\oplus 2} \oplus \widetilde{D}_6$, 2 ways to blow down $\widetilde{D}_4^{\oplus 2}$, 2 ways to blow down $\widetilde{A}_1 \oplus \widetilde{E}_7$, 2 ways to blow down $\widetilde{D}_8$, and 1 way to blow down $\widetilde{E}_8$.

Case 1: $\widetilde{A}_1^{\oplus 8}$

Let us start with $\widetilde{A}_1^{\oplus 8}$. We can get this Dynkin diagram by blowing up $P_1, \ldots, P_9$ where P_1, P_3, P_5, P_7, and P_8 are ordinary points in $\mathbb{P}^2$, and P_2 is infinitely-near P_1 in the direction of P_7, P_4 is infinitely-near P_3 in the direction of P_7, P_6 is infinitely-near P_5 in the direction of P_7, and P_9 is infinitely-near P_8 in the direction of P_7. Let us call this surface X. Note $\mathrm{Pic}X = \langle \ell, e_1, \ldots, e_9 \rangle$. We will draw the Dynkin Diagram below:



The 16 (-1)-curves are e_2, e_4, e_6, e_7, e_9, $\ell - e_1 - e_3$, $\ell - e_1 - e_5$, $\ell - e_1 - e_8$, $\ell - e_3 - e_5$, $\ell - e_5 - e_8$, $2\ell - e_1 - e_2 - e_3 - e_5 - e_8$, $2\ell - e_1 - e_3 - e_4 - e_5 - e_8$, $2\ell - e_1 - e_3 - e_5 - e_6 - e_8$, $2\ell - e_1 - e_3 - e_5 - e_6 - e_7 - e_8$, and $2\ell - e_1 - e_3 - e_5 - e_6 - e_8 - e_9$. Now we will draw the complete adjacency diagram of the (-2)-curves and (-1)-curves of X. Note that the (-2)-curves will remain blue and the (-1)-curves will be colored red. Also note that a black edge between two vertices means that those two curves intersect with multiplicity 1, and a red edge means that those curves intersect with multiplicity 2.

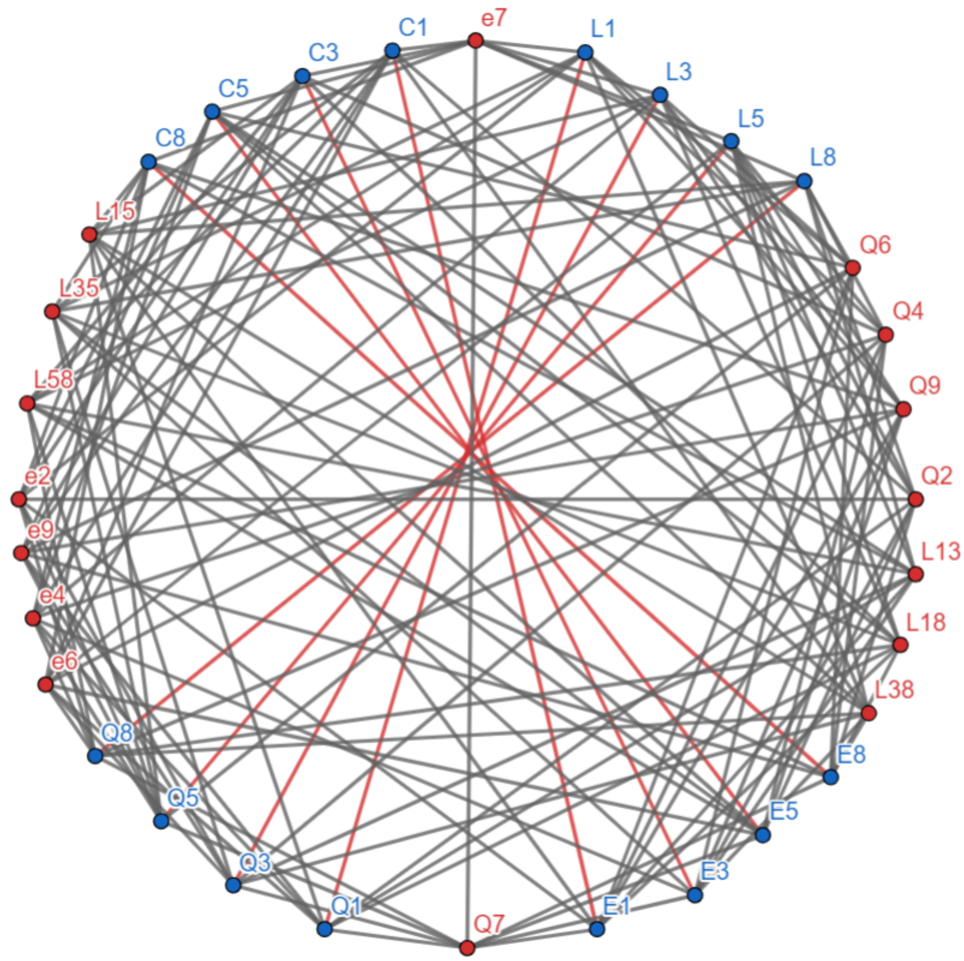


Figure 5.1: The graph of (-2)-curves and (-1)-curves is 9-regular

This is a 9-regular graph on 32 vertices (counting the red edges as black). We will show that there are only 2 ways of blowing this diagram down to $\mathbb{P}^2$.

By Castelnuovo's contraction theorem, one can blow down a curve C on a smooth projective surface X to get a smooth projective surface if and only if the self-intersection number of C is -1.

First choose one red point R_1 to blow down. Note that since we are blowing down to $\mathbb{P}^2$, we will need to blow down exactly nine curves. Once we blow down a red vertex, the single adjacent red vertex cannot be blown down since its self-intersection number would increase. There are eight pairs of adjacent red vertices. Therefore, if we blow down eight disjoint red vertices, we would still need to blow down one of the blue vertices as the ninth. Thus there must be at least one blue vertex present in the blow-down. Choose B_1 adjacent to R_1 to blow down.

Now let us take an inventory of which vertices cannot be blown down. The single blue vertex adjacent to B_1 cannot be blown down, because it intersects B_1 with multiplicity 2, so its self-intersection number has increased to 0. In addition, the other 7 red vertices adjacent to B_1 cannot be blown down, because their self-intersection have also increased to 0. Finally, the other 7 blue vertices adjacent to R_1 cannot be blown down, because their self intersection increases to (-1) after R_1 is blown down, but since B_1 also intersects R_1, their self intersection would then increase to 0 after B_1 is blown down.

What is left is a three-regular bipartite graph on 14 vertices containing 7 disjoint red vertices and 7 disjoint blue vertices.

In this subgraph, the two possibilities for blow-downs are to blow down the 7 disjoint red vertices, or to blow down one red vertex and three red-blue pairs.

It is impossible to blow down three disjoint red vertices and two red-blue pairs because there do not exist three red vertices that no two blue vertices are connected to.

It is impossible to blow down five disjoint red vertices and one red-blue pair because there every blue vertex is connected to three red vertices: if one has already been blown down, then there are two of the remaining six connected to blue, leaving four out of six not connected to blue. This is less than the five required, so this blow down is impossible.

Thus there are only two types of blow-downs.

For those interested, the graph can be read from the following adjacency matrix:

$$\begin{pmatrix} B & M \\ M^{\mathsf{T}} & R \end{pmatrix},$$

where

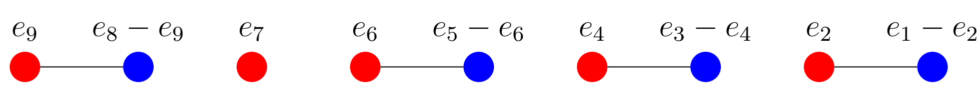
$$B = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

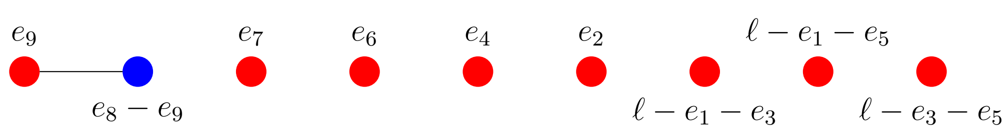
$$R = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

and

$$M = \begin{pmatrix}
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0
\end{pmatrix}.$$

The two blow-downs are given by the following diagrams.

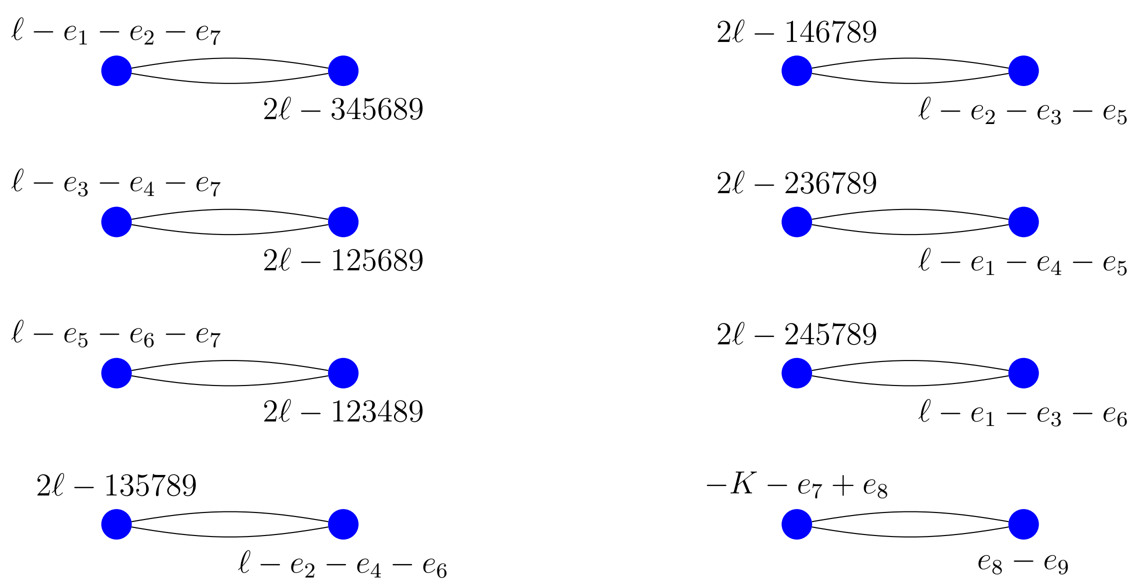




We can define a map $A : \mathrm{Pic}X \longrightarrow \mathrm{Pic}X$ given by the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

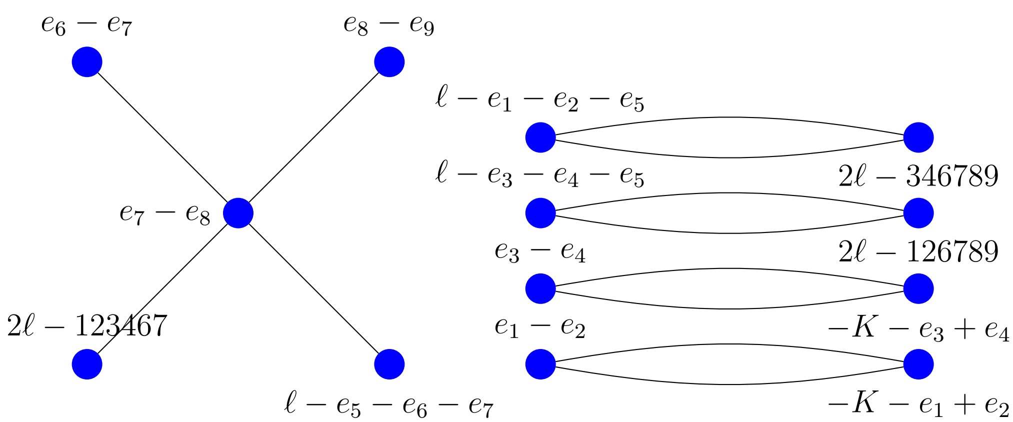
Applying A to the Dynkin Diagram gives us a new Dynkin Diagram shown below. To save space, we will denote $3\ell - e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 - 2e_8$ as $-K - e_7 + e_8$, where $K = -3\ell + e_1 + \cdots + e_9$ is the canonical divisor and rename $2\ell - e_{i_1} - e_{i_2} - e_{i_3} - e_{i_4} - e_{i_5} - e_{i_6}$ as $2\ell - i_1 i_2 i_3 i_4 i_5 i_6$.



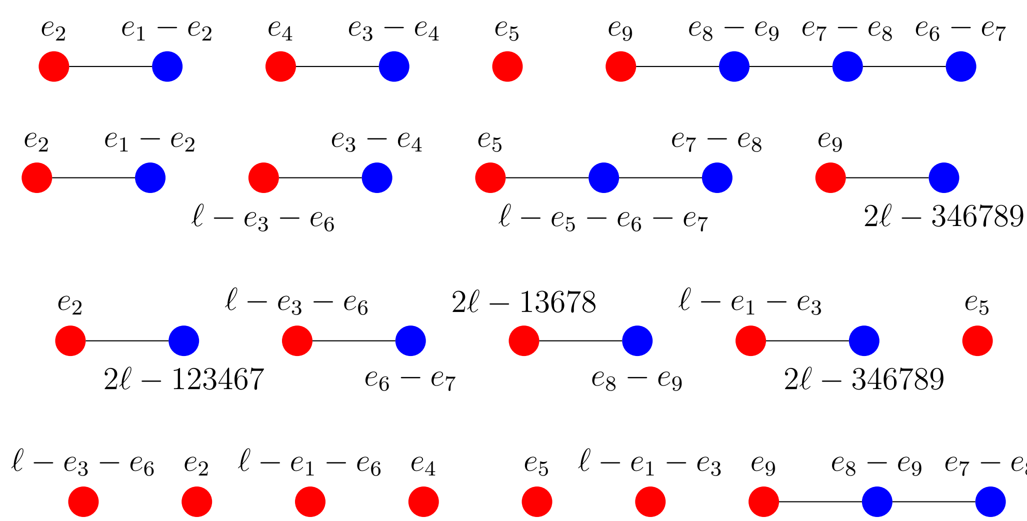
This diagram corresponds to seven points blown up in a Fano plane configuration, and a general eighth point blown up twice.

Case 2: $\widetilde{A}_1^{\oplus 4} \oplus \widetilde{D}_4$

Let's now look at the Dynkin diagram $\widetilde{A}_1^{\oplus 4} \oplus \widetilde{D}_4$, as depicted below.



The eight (-1)-curves are $e_2,\ e_4,\ e_5,\ e_9,\ \ell - e_1 - e_3,\ \ell - e_1 - e_6,\ \ell - e_3 - e_6$, and $2\ell - e_1 - e_3 - e_6 - e_7 - e_8$. The four ways to blow down this diagram (up to isomorphism) are given by the following diagrams.



The three bottom blow-downs give us the matrices

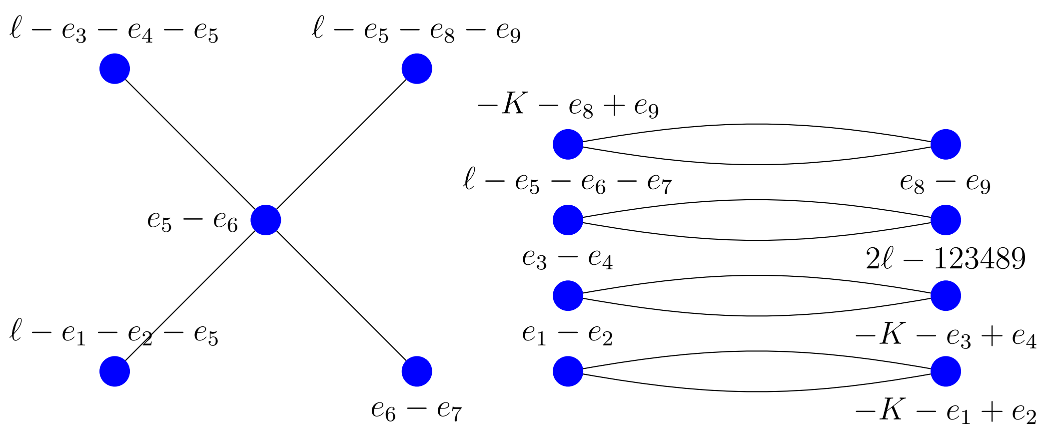
$$\begin{pmatrix} 3 & 0 & 0 & 1 & 1 & 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & -1 & -1 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} : \mathrm{Pic}X \to \mathrm{Pic}X,$$

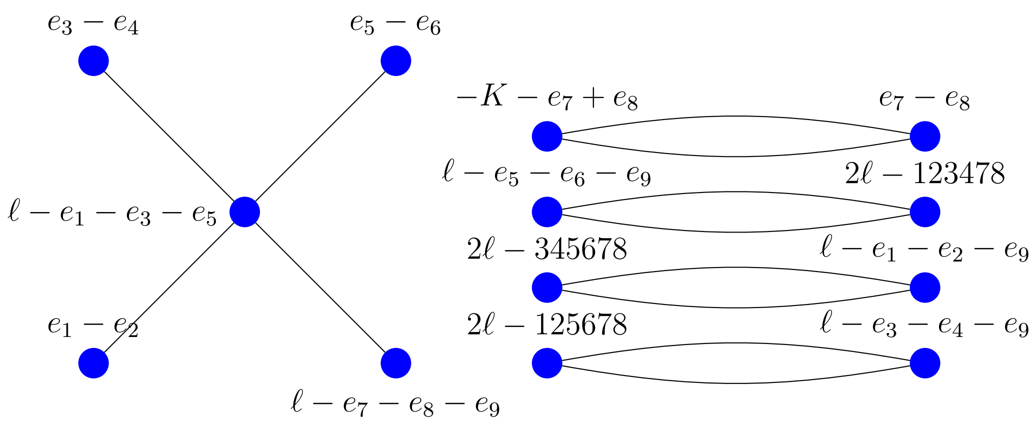
$$\begin{pmatrix} 5 & 2 & 0 & 3 & 1 & 0 & 2 & 2 & 1 & 1 \\ -2 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ -2 & -1 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & -1 \\ -2 & -1 & 0 & -1 & 0 & 0 & -1 & -1 & -1 & 0 \\ -3 & -1 & 0 & -2 & -1 & 0 & -1 & -1 & -1 & -1 \\ -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} : \mathrm{Pic}X \to \mathrm{Pic}X,$$

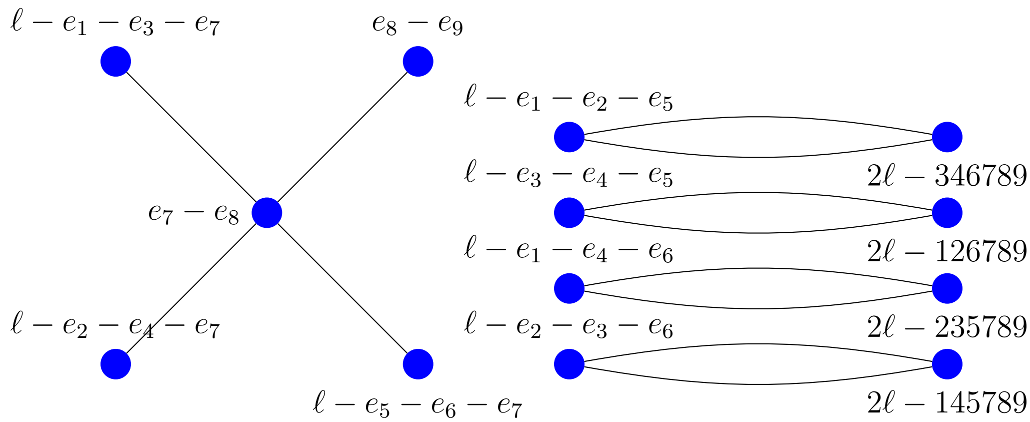
and

$$\begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} : \mathrm{Pic}X \to \mathrm{Pic}X$$

respectively. Applying these change-of-basis matrices to the original Dynkin diagram gives us the following diagrams.







The original diagram is a pencil defined by the union of a conic C_1 and a tangent line L_1, and the union of a conic C_2 that intersects C_1 at two points with multiplicity 2 and a line L_2 that intersects C_1 at one point with muliplicity 2. An example of such a pencil is $x^2z + xy^2$ and $x^2z + \varphi xz^2 + \varphi^2 y^2 z$ where φ satisfies $\varphi^2 + \varphi + 1 = 0$.

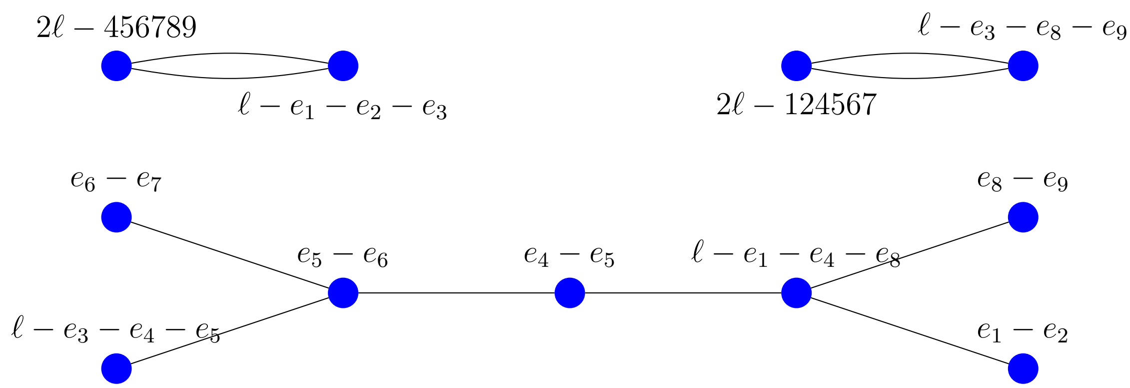
The second diagram is a pencil of the union of a conic and a line with the union of three concurrent lines, each tangent to the conic. An example of such a pencil is that spanned by $(y^2 + xz)(x + \varphi z)$ and $xz(x + z)$.

The third diagram is a pencil spanned by the union of a conic and a line, and the union of a line that is tangent to the conic and a double line. An example of such a pencil is that spanned by $(y^2 + xz)(x + \varphi z)$ and $y^2(x + z)$.

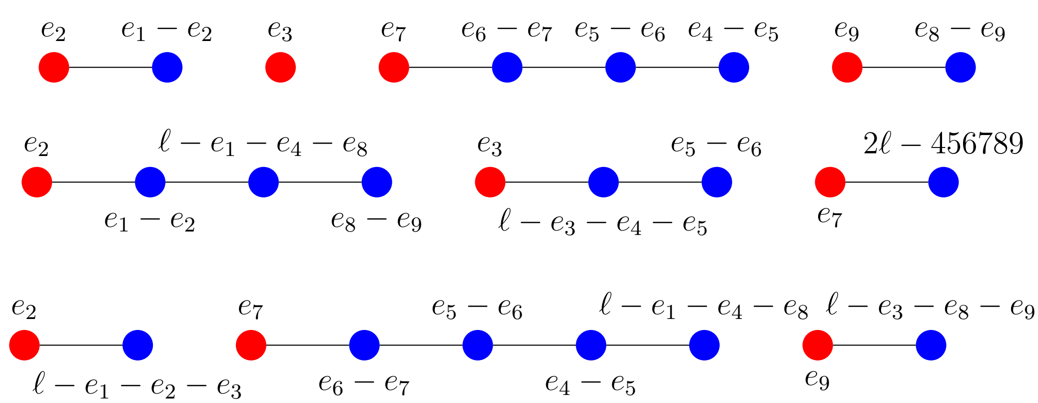
The fourth diagram corresponds to a pencil spanned by the union of three concurrent lines, and the union of a line and a conic that contains that contains the point where the three lines intersect. The seven ordinary points in the intersection of these two cubics form a Fano plane. An example of such a pencil is that spanned by $(x + y)(x + z)(y + z)$ and $(x + y + z)(Ax(y + z) + By(x + z))$ where $(A, B) \in \mathbb{P}^1$.

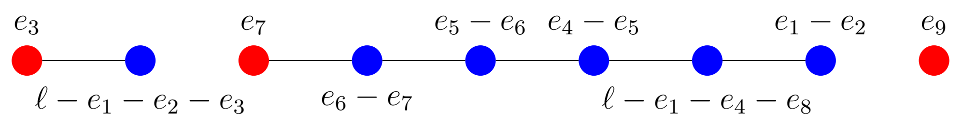
Case 3: $\widetilde{A}_1^{\oplus 2} \oplus \widetilde{D}_6$

Let's now look at the Dynkin diagram $\widetilde{A}_1^{\oplus 2} \oplus \widetilde{D}_6$, as depicted below.



The four (-1)-curves are e_2, e_3, e_7, and e_9. The four ways to blow down this diagram (up to isomorphism) are given by the following diagrams.





The bottom three blow-downs give us the change-of-basis matrices

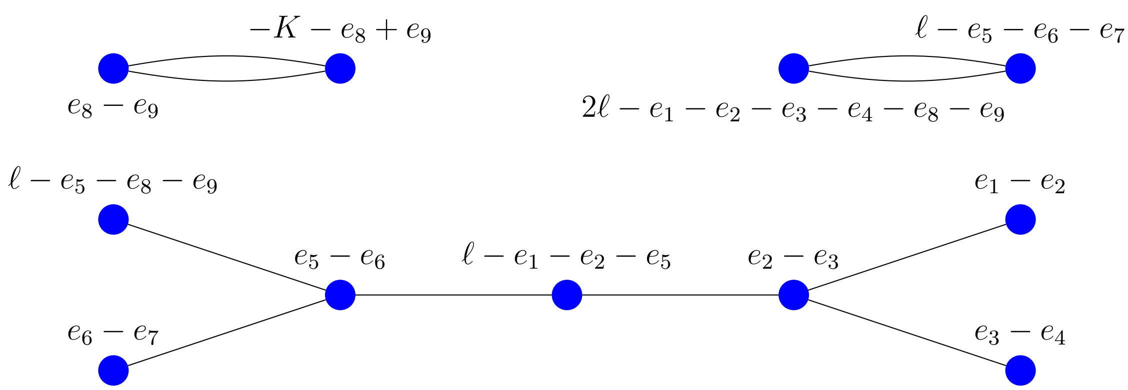
$$\begin{pmatrix} 3 & 0 & 0 & 0 & 2 & 1 & 1 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} : \mathrm{Pic} X \to \mathrm{Pic} X,$$

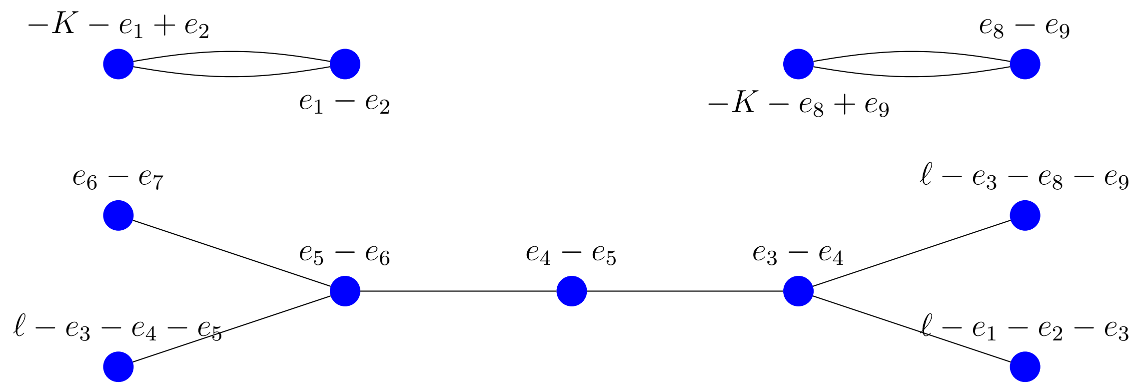
$$\begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} : \mathrm{Pic} X \to \mathrm{Pic} X,$$

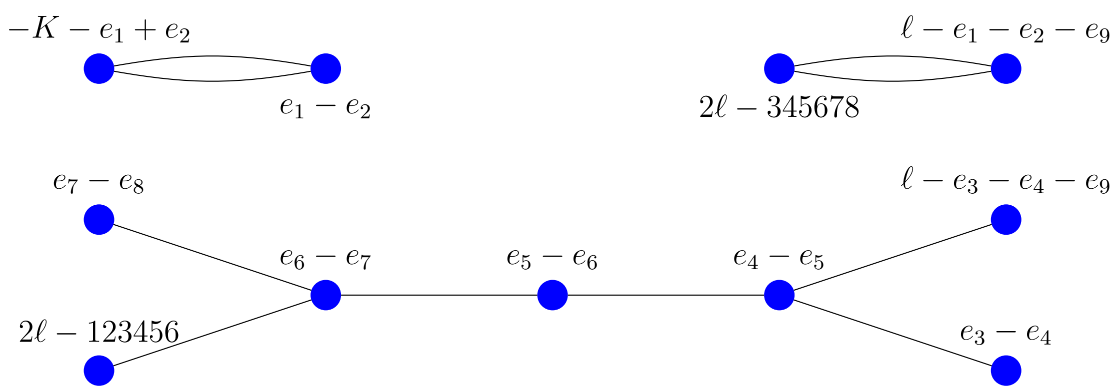
and

$$\begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} : \mathrm{Pic} X \to \mathrm{Pic} X$$

respectively. Applying these change-of-basis matrices to the original Dynkin diagram gives us the following diagrams.




$-K - e_1 + e_2$
$e_1 - e_2$
$e_8 - e_9$
$-K - e_8 + e_9$
$e_6 - e_7$
$\ell - e_3 - e_8 - e_9$
$e_5 - e_6$
$e_4 - e_5$
$e_3 - e_4$
$\ell - e_3 - e_4 - e_5$
$\ell - e_1 - e_2 - e_3$


$-K - e_1 + e_2$
$\ell - e_1 - e_2 - e_9$
$e_1 - e_2$
$2\ell - 345678$
$e_7 - e_8$
$\ell - e_3 - e_4 - e_9$
$e_6 - e_7$
$e_5 - e_6$
$e_4 - e_5$
$2\ell - 123456$
$e_3 - e_4$

The original diagram corresponds to a pencil spanned by the union of a conic and a line, and the union of a line tangent to the conic and a double line. An example of such a pencil is that spanned by $(y^2 + xz)(x + z)$ and xy^2.

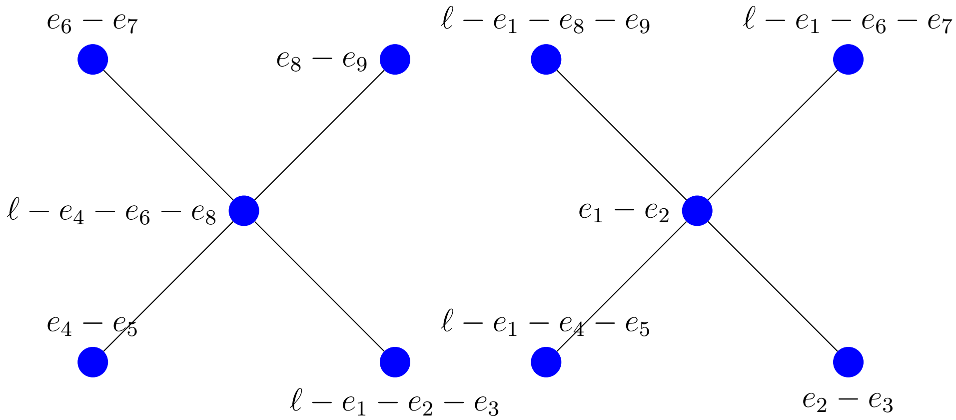
The second diagram corresponds to a pencil spanned by the union of a conic and a line, and the union of two lines tangent to the conic and intersect on the first line, one of which is double. An example of such a pencil is $(y^2 + xz)(x + z)$ and x^2z.

The third diagram corresponds to a pencil spanned by the union of three concurrent lines, and a cuspidal cubic that has a cusp on one of the lines, is tangent to another line at the concurrent intersection point, and is tangent to the third line elsewhere. An example of such a pencil is that spanned by $x^3 + y^2z$ and $xz(x + z)$.

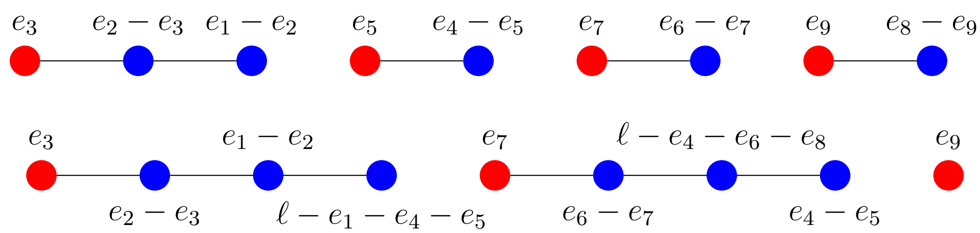
The fourth diagram corresponds to a pencil spanned by the union of a conic and a tangent line, and the union of a line that is tangent to the first conic and a conic that intersects the first conic with multiplicity 4 at the original tangent point, creating a multiplicity-6 intersection of the two cubics at that point. An example of such a pencil is that spanned by $(y^2 + xz)x$ and $z(y^2 + x^2 + xz)$. The modulus is 1 because there is a pencil of conics that meet each other at a point of multiplicity 4.

Case 4: $\widetilde{D}_4^{\oplus 2}$

Let's now look at the Dynkin diagram $\widetilde{D}_4^{\oplus 2}$, as depicted below.



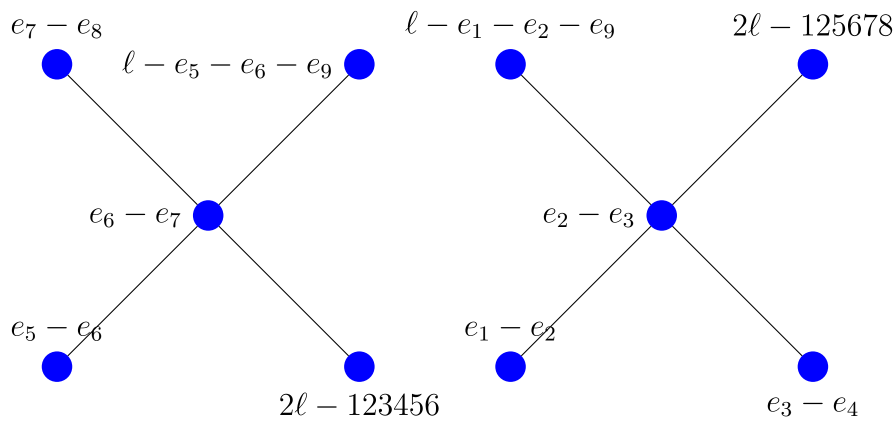
The four (-1)-curves of this diagram are e_3, e_5, e_7, and e_9. The two blow-downs of the diagram (up to isomorphism) are represented by the diagrams below.



The second blow-down gives us the following change-of-basis matrix.

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} : \mathrm{Pic} X \to \mathrm{Pic} X.$$

Applying this matrix to the Dynkin diagram gives us the following relabeling.

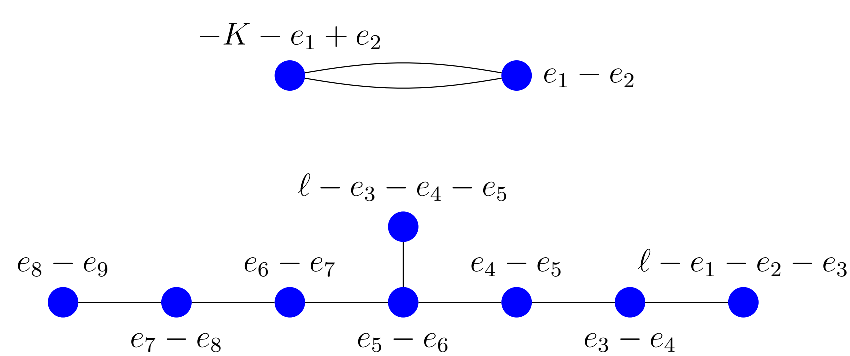


The first diagram corresponds to a pencil spanned by the union of a line and a double line, and the union of three concurrent lines which all intersect on the single line. An example of such a pencil is that spanned by $x(x+z)z$ and $y^2(x+\varphi z)$.

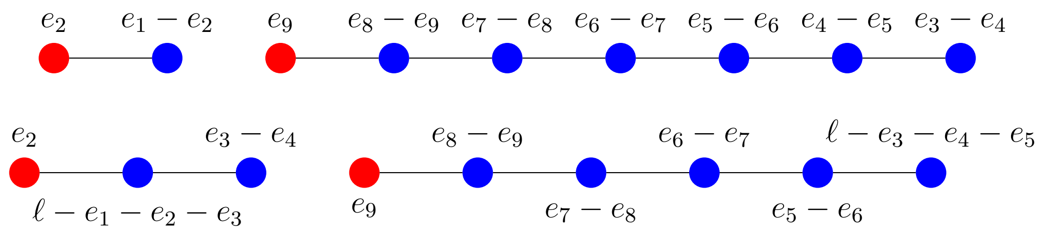
The second diagram corresponds to a pencil spanned by the union of a conic C_1 and a tangent line L_1, and the union of a conic C_2 which intersects C_1 at two points with multiplicity 2, one of which is the intersection point of C_1 with L_1, and a line L_2 which is tangent to C_1 and C_2 at the other point where they meet with multiplicity 2. An example of such a pencil is that spanned by $x(y^2+xz)$ and $z(\varphi y^2 + xz)$.

Case 5: $\widetilde{A}_1 \oplus \widetilde{E}_7$

Let's now look at the Dynkin diagram $\widetilde{A}_1 \oplus \widetilde{E}_7$, as depicted below.



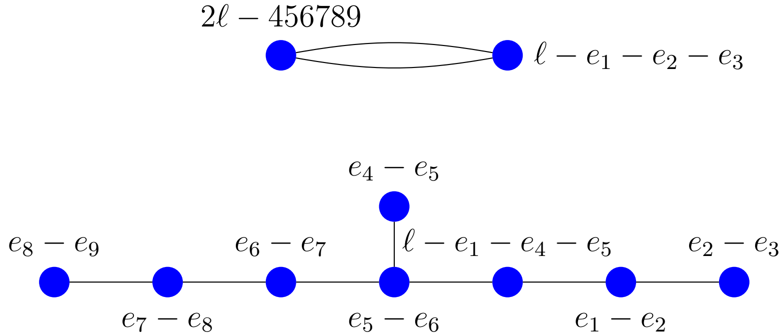
The two (-1)-curves of this diagram are e_2 and e_9. The two blow-downs of the diagram (up to isomorphism) are represented by the diagrams below.



The second blow-down gives us the following change-of-basis matrix.

$$\begin{pmatrix} 2 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} : \mathrm{Pic} X \to \mathrm{Pic} X.$$

Applying this matrix to the Dynkin diagram gives us the following relabeling.

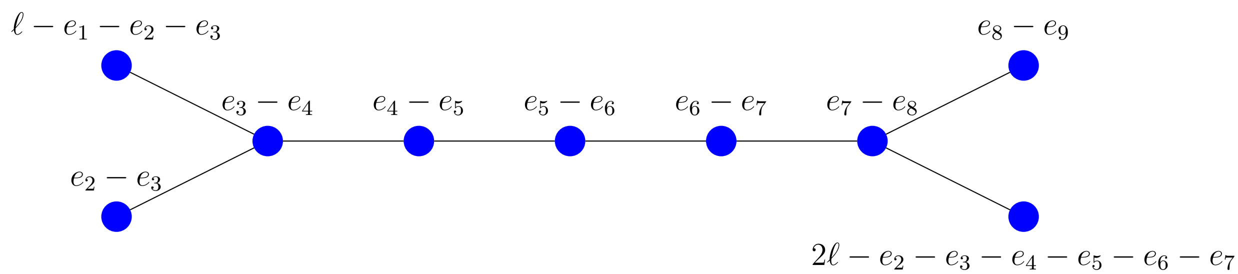


The first diagram gives us a pencil spanned by a cuspidal cubic, and the union of a line and a double line, where the double line is tangent to the flex point of the cuspidal cubic and the line goes through the cusp and the flex point. An example of such a pencil is that spanned by $x^2 + y^2z$ and xz^2.

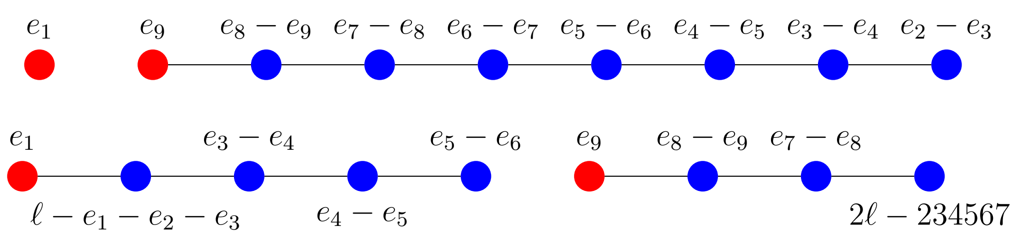
The second diagram corresponds to a pencil spanned by the union of a conic and a line, and a triple line that is tangent to the conic. An example of such a pencil is that spanned by $(y^2 + xz)z$ and x^3.

Case 6: $\widetilde{D}_8$

Let's now look at the Dynkin diagram $\widetilde{D}_8$, as depicted below.



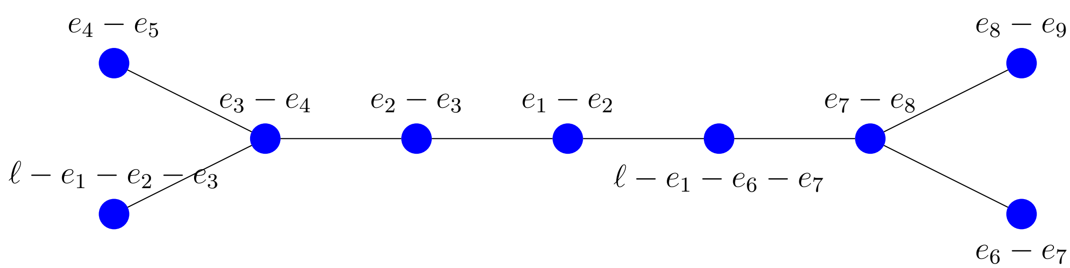
The two (-1)-curves of this diagram are e_1 and e_9. The two blow-downs of the diagram (up to isomorphism) are represented by the diagrams below.



The second blow-down yields the following change-of-basis matrix.

$$\begin{pmatrix} 3 & 0 & 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} : \mathrm{Pic} X \to \mathrm{Pic} X.$$

Applying the matrix to the Dynkin diagram gives us the following relabeling.

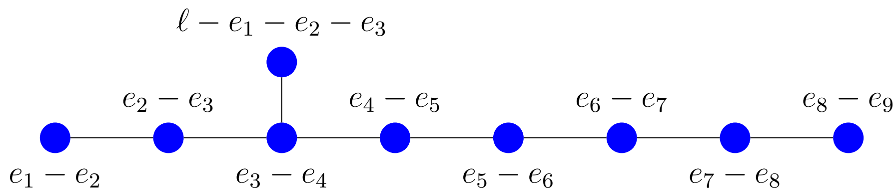


The first diagram corresponds to a pencil spanned by a cuspidal cubic and the union of a line tangent to the cubic at somewhere other than the cusp or flex point, and a conic that meets the cubic at the same point with multiplicity 6. (Note: in characteristic 2, all points other than the flex and cusp on a cuspidal cubic are sextactic.) An example of such a conic is that spanned by $(y^2 + x^2 + xz + z^2)(x + z)$ and $x^3 + y^2 z$.

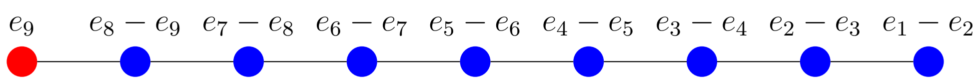
The second diagram corresponds to a pencil spanned by a cuspidal cubic and the union of a line and double line, where the line is tangent to the cubic at the flex point and the double line is tangent to a sextactic point. An example of such a pencil is that spanned by $x^3 + y^2z$ and $(x + z)^2z$.

Case 7: $\widetilde{E}_8$

Let's now look at the Dynkin diagram $\widetilde{E}_8$, as depicted below.



The only (-1)-curve of this diagram is e_9. The only way to blow down this diagram is depicted in the following diagram.



Since there is only one way to blow down this surface, there are no change of basis matrices.

This diagram corresponds to a pencil spanned by a cuspidal cubic and a triple line that meets the cubic at its cusp with multiplicity 3. An example of such a pencil is that spanned by $x^3 + y^2z$ and y^3.

Chapter 6

Unexpected Cubics in Characteristic 2

Let k be an algebraically closed field of characteristic 2, and let $\varphi \in k$ satisfy $\varphi^2 + \varphi + 1 = 0$.

We found in total eight types of configurations of points in $\mathbb{P}^2_k$ that yield unexpected cubics. Two come from the two blow-downs of $\widetilde{A}_1^{\oplus 8}$, three come from the four blow-downs of $\widetilde{A}_1^{\oplus 4} \oplus \widetilde{D}_4$, and three come from the four blow-downs of $\widetilde{A}_1^{\oplus 2} \oplus \widetilde{D}_6$.

In the first $\widetilde{A}_1^{\oplus 8}$ case, the sheet of cubics spanned by $x^2(y+z)$, $y^2(x+z)$, and $z^2(x+y)$ produces cubic curves that contain the seven points $(1,0,0) \times 2$, $(0,1,0) \times 2$, $(0,0,1) \times 2$, and $(1,1,1)$, and have cusps at generic points.

In the second $\widetilde{A}_1^{\oplus 8}$ case (outlined in Akesseh's thesis), the sheet of cubics spanned by $xy(x+y)$, $xz(x+z)$, and $yz(y+z)$ produce cubic curves that contain the seven points of the Fano plane and have cusps at generic points.

In the first $\widetilde{A}_1^{\oplus 4} \oplus \widetilde{D}_4$ case, the sheet of cubics spanned by $x^2z + xy^2$, $x^2z + \varphi xz^2 + \varphi^2 y^2 z$, and $x^2z + xz^2 + \varphi y^2 z$ produce cubic curves containing the seven points $(0,0,1) \times 4$, $(1,0,0) \times 2$, and $(0,1,0)$, and have cusps at generic points.

In the second $\widetilde{A}_1^{\oplus 4} \oplus \widetilde{D}_4$ case, the sheet of cubics spanned by $(y^2 + xz)(x + \varphi z)$, $xz(x+z)$, and $xz(x + \varphi^2 z)$ produces cubic curves that contain the seven points $(1,0,0) \times 2$, $(0,0,1) \times 2$, and $(0,1,0) \times 3$, and have cusps at generic points.

In the third $\widetilde{A}_1^{\oplus 4} \oplus \widetilde{D}_4$ case, the sheet of cubics spanned by $(y^2+xz)(x+\varphi z)$, $y^2(x+$

z), and $y^2(x+\varphi^2 z)$ produces cubic curves that contain the seven points $(0,0,1)\times 2$, $(\varphi,0,1)\times 2$, $(1,0,0)\times 2$, and $(0,1,0)$, and have cusps at generic points.

In the first $\widetilde{A}_1^{\oplus 2}\oplus\widetilde{D}_6$ case, the sheet of cubics spanned by $(y^2+xz)(x+z)$, x^2z, and $x^2(x+z)$ produces cubic curves that contain the seven points $(0,0,1)\times 4$ and $(0,1,0)\times 2$, and have cusps at generic points.

In the third $\widetilde{A}_1^{\oplus 2}\oplus\widetilde{D}_6$ case, the sheet of cubics spanned by x^3+y^2z, $xz(x+z)$, and $xz(x+\varphi z)$ produces a cubic curve that contains the seven points $(0,0,1)\times 2$, and $(0,1,0)\times 5$, and has a cusp at a generic point.

In the fourth $\widetilde{A}_1^{\oplus 2}\oplus\widetilde{D}_6$ case, the sheet of cubics spanned by $x(y^2+xz)$, $z(y^2+x^2+xz)$, and $(x+z)(y^2+\varphi x^2+xz)$ produces a cubic curve that contains the seven points $(0,0,1)\times 6$ and $(0,1,0)$, and has a cusp at a generic point.

The cuspidal curves produced by each sheet above is an unexpected cubic for a set Z of seven points. The examples where some of the points of Z are infinitely near are new.

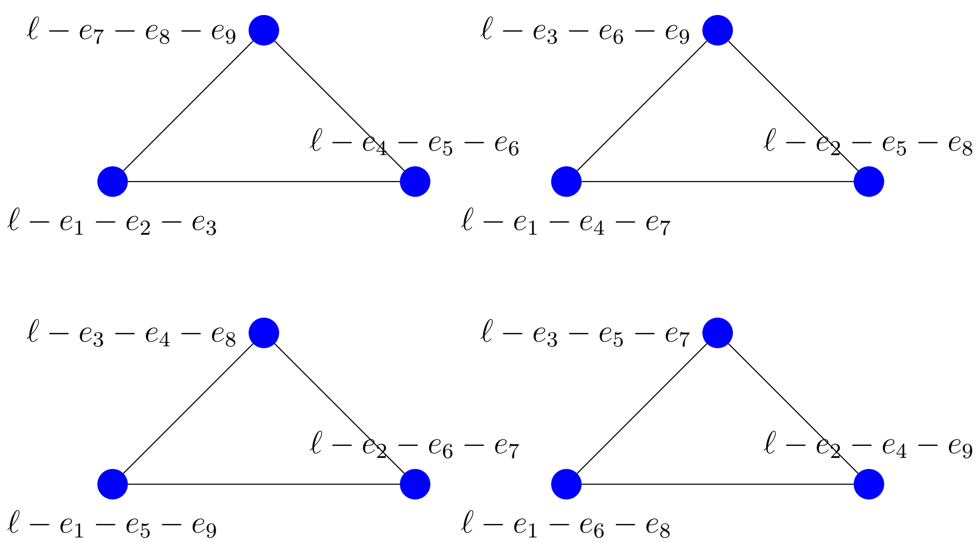
Chapter 7

Quasi-Elliptic Fibrations in Characteristic 3

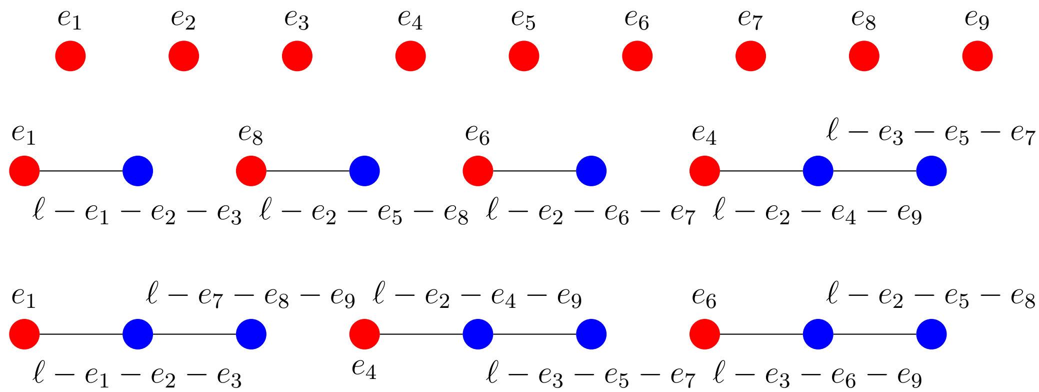
Now let us consider quasi-elliptic fibrations in characteristic 3. According to Cossec and Dolgachev [5], a quasi-elliptic fibration in characteristic 3 will have a Dynkin diagram of $\widetilde{A}_2^{\oplus 4}$, $\widetilde{A}_2 \oplus \widetilde{E}_6$, or $\widetilde{E}_8$.

Case 1: $\widetilde{A}_2^{\oplus 4}$

Let us first consider the diagram $\widetilde{A}_2^{\oplus 4}$. We can label this diagram as follows.



The nine (-1)-curves according to this labeling are $e_1, \ldots, e_9$. The three blow-downs of the diagram (up to isomorphism) are represented by the diagrams below.



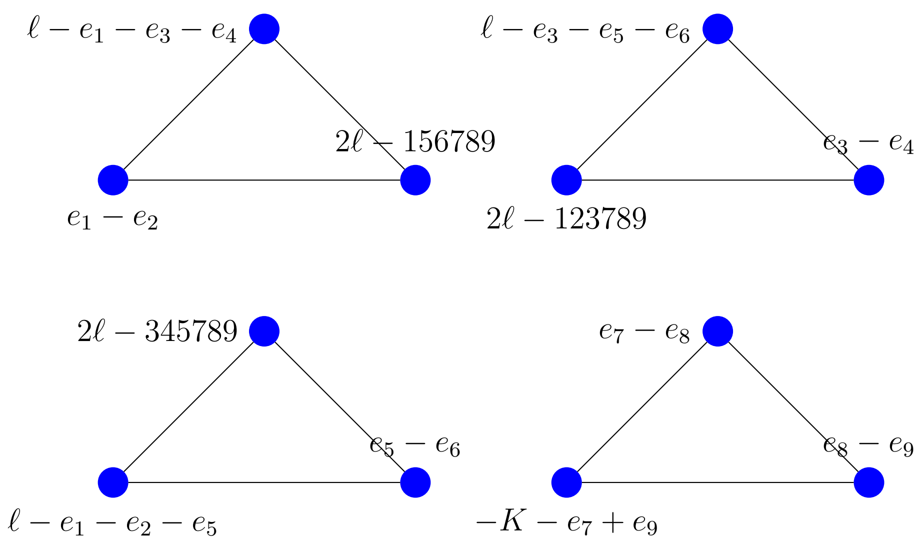
The second two blow-downs give us the change-of-basis matrices

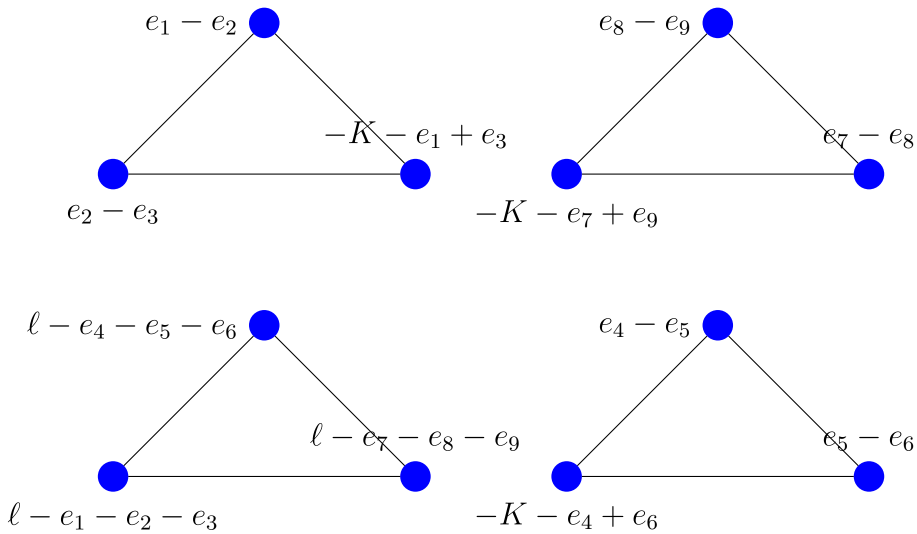
$$\begin{pmatrix} 3 & 0 & 2 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} : \mathrm{Pic} X \to \mathrm{Pic} X$$

and

$$\begin{pmatrix} 4 & 0 & 2 & 2 & 0 & 1 & 0 & 1 & 1 & 2 \\ -2 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} : \mathrm{Pic} X \to \mathrm{Pic} X$$

respectively. Applying these matrices to the Dynkin diagram above gives us the following relabelings.





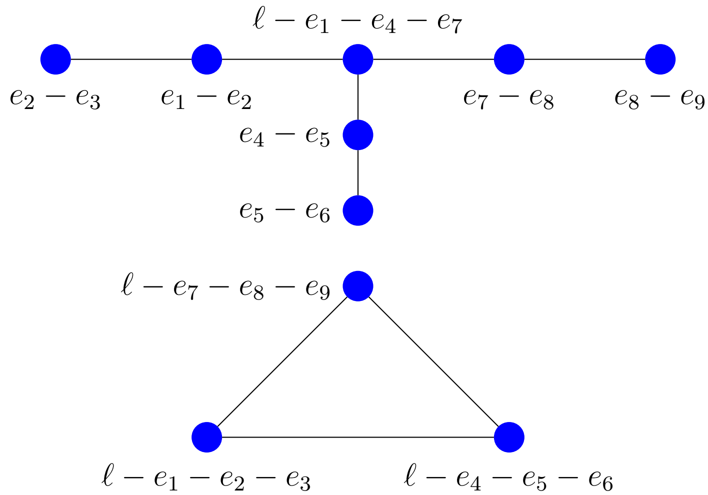
The first Dynkin diagram corresponds to a pencil spanned by the union of three concurrent lines and the union of three different concurrent lines. An example of such a pencil is that spanned by $x(x+z)(x-z)$ and $y(y+z)(y-z)$.

The second diagram corresponds to a pencil spanned by the union of a conic and a line, and the union of a line tangent to the first conic and a conic that contains the intersection of the two lines and the intersection of the first conic and the first line. An example of such a pencil is that spanned by $2x^2z + yz^2$ and $xy^2 + 2yz^2$.

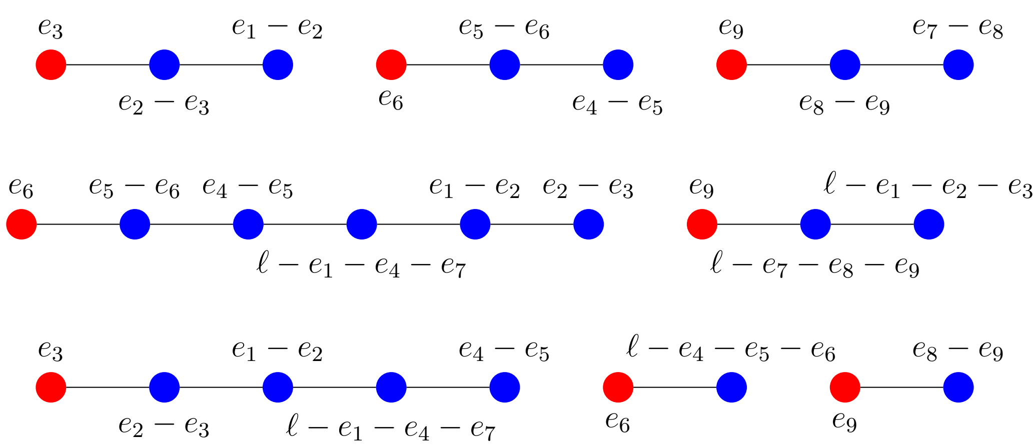
The third diagram corresponds to a pencil spanned by a flexible cuspidal cubic and the union of three lines, two of which are tangent to flex points on the cubic and one which intersects the cusp with multiplicity 3. An example of such a pencil is that spanned by $x^3 - y^2z$ and $yz(y-z)$.

Case 2: $\widetilde{A}_2 \oplus \widetilde{E}_6$

Let us now consider the diagram $\widetilde{A}_2 \oplus \widetilde{E}_6$. A labeling is pictured below.



This surface has the three (-1)-curves e_3, e_6, and e_9. There are three ways (up to isomorphism) of blowing this surface down to $\mathbb{P}^2$. They are depicted below.



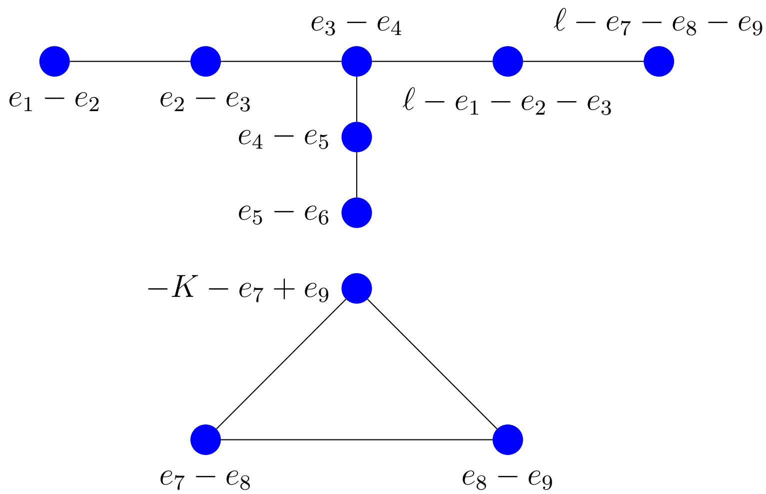
The second two blow-downs give us the change-of-basis matrices

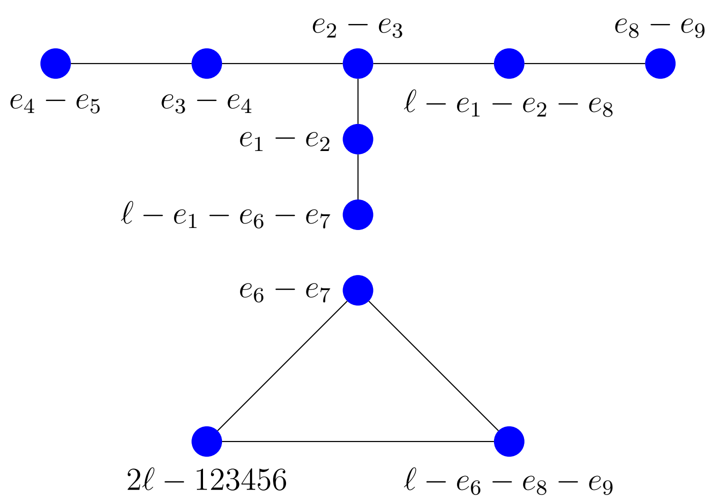
$$\begin{pmatrix} 3 & 1 & 1 & 1 & 0 & 0 & 0 & 2 & 1 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -2 & -1 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} : \mathrm{Pic}X \to \mathrm{Pic}X$$

and

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} : \mathrm{Pic}X \to \mathrm{Pic}X$$

respectively. Applying these matrices to the Dynkin diagram gives us the relabelings depicted below.





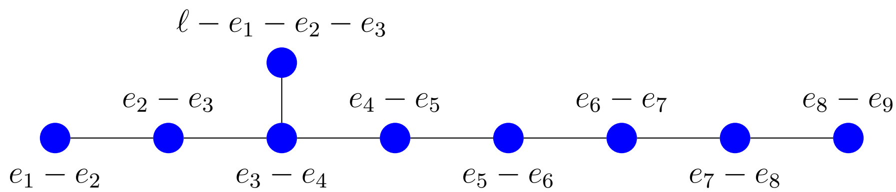
The first diagram corresponds to a pencil spanned by the union of three lines and a triple line. An example of such a pencil is that spanned by xyz and $(x+y+z)^3$.

The second diagram corresponds to a pencil spanned by a flexible cuspidal cubic and the union of a line and a double line, with the line meeting the cubic at a flex point with multiplicity three, and the double line intersecting the cusp of the cubic with multiplicity three.

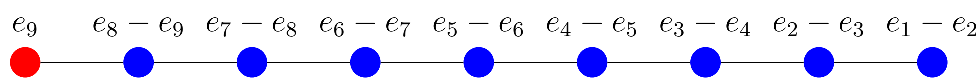
The third diagram corresponds to a pencil spanned by the union of a conic and a line and the union of a line and a double line, with the line containing one of the intersection points of the conic and the first line, and the double line is tangent to the conic at the other point it meets the second line. An example of such a pencil is that spanned by $x^2(x-y)$ and $(2xy+2xz+y^2)z$.

Case 3: $\widetilde{E}_8$

Let us now consider the Dynkin diagram $\widetilde{E}_8$, as depicted below.



The only (-1)-curve of this diagram is e_9. The only way to blow down this diagram is depicted in the following diagram.

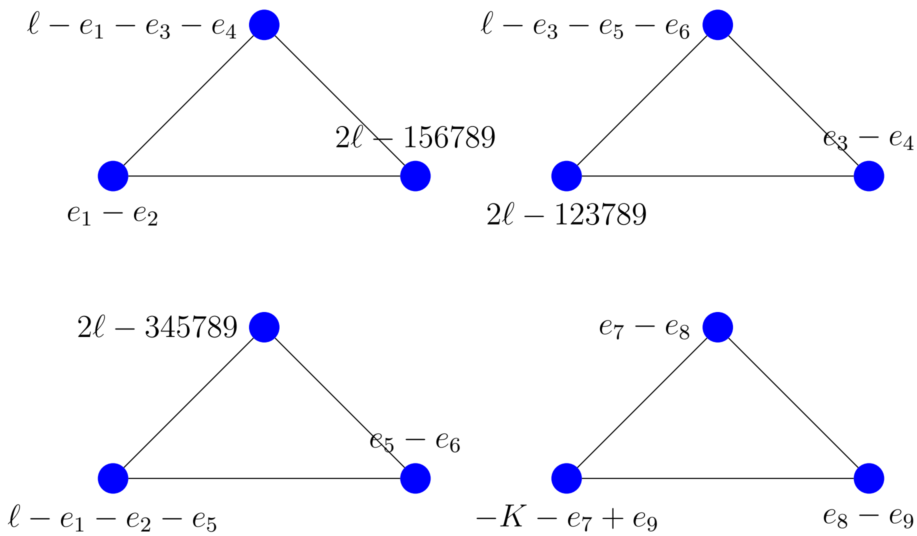


Since there is only one way to blow down this surface, there are no change of basis matrices.

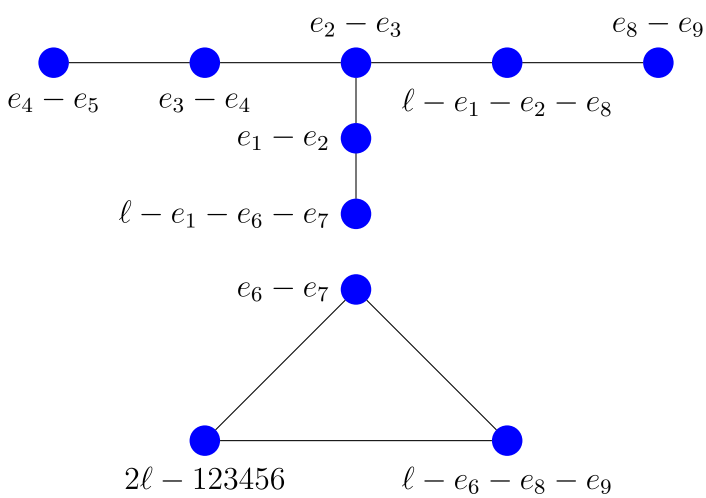
This diagram corresponds to a pencil spanned by a cuspidal cubic and a triple line that meets the cubic at its cusp with multiplicity 3. An example of such a pencil is that spanned by x^3+y^2z and y^3.

Theorem 3. *Let k be a field of characteristic 3. There are no unexpected cubic curves in $\mathbb{P}^2_k$ coming from the quasi-elliptic fibrations listed above.*

Proof. Recall that an unexpected cubic curve comes from choosing seven points $P_1, P_2, \ldots, P_7 \in \mathbb{P}^2$ such that there is a cubic with a double point at the general point P. By Proposition 2, an unexpected cubic yields a quasi-elliptic fibration that is isomorphic to the blowup of $\mathbb{P}^2_k$ at the following nine points: the seven points $P_1, \ldots, P_7$, a general point P and a point infinitely-near P. Since P is general, it will not be equal to any of the P_i. Therefore we need only consider the quasi-elliptic fibrations that are attained by blowing up $\mathbb{P}^2_k$ at some point exactly twice, with the other seven blowup points elsewhere. Out of all of the quasi-elliptic fibrations we've seen in this chapter, there are only two that come from blowing up $\mathbb{P}^2_k$ at some point exactly twice. We need only check that quasi-elliptic fibrations with the Dynkin diagrams



and



will not yield an unexpected cubic.

In the $\widetilde{A}_2^{\oplus 4}$ diagram, we have an example of a pencil spanned by $2x^2z + yz^2$ and $xy^2 + 2yz^2$. This pencil has a base points of multiplicity 2 at $(0, 0, 1)$, $(0, 1, 0)$, and $(1, 0, 0)$ and has a base point of multiplicity 3 at $(1, 1, 1)$. We can create a two-dimensional linear system of cubics by introducing the third polynomial $(xz+2y^2)(x+2y)$. This linear system has two base points of multiplicity 2: one at $(0, 0, 1)$ and one at $(1, 0, 0)$. It also has a base point of multiplicity 3 at $(1, 1, 1)$. These seven points are all the base points of the linear system. Since unexpected cubics have a general double point, we must now determine whether this linear system has a general double point.

Let us examine the linear combination

$$F(xz + 2y^2)(x + 2y) + G(2x^2z + yz^2) + H(xy^2 + 2yz^2)$$

and the generic point $P = (a, b, c)$. It is sufficient to reduce to the affine case $z = 1$, and we want to determine whether there are values for F, G, and H such that

$$F(x + 2y^2)(x + 2y) + G(2x^2 + y) + H(xy^2 + 2y)$$

has a double point at $(A, B) = (a/c, b/c)$. By shifting $x \mapsto x + A$ and $y \mapsto y + B$, we get that (F, G, H) annihilates the constant and linear components of the resulting polynomial if and only if

$$(F, G, H) \in \ker \begin{pmatrix} -A^2B + B^3 & -A^2 + B & AB^2 - B \\ -B^2 - A - B & A & B^2 \\ AB - A & 1 & -AB - 1 \end{pmatrix}.$$

But this matrix has a determinant

$$A^4B^2 - A^3B^3 - A^2B^4 + A^4B + A^2B^3 - AB^4 - B^5 + A^3B - AB^3 + A^3 \neq 0$$

and so this two-dimensional linear system of cubics does not has a generic double point. Therefore the seven base points do not admit an unexpected cubic.

Next we will look at the $\widetilde{A}_2 \oplus \widetilde{E}_6$ configuration. This is a pencil of cubics with base points α of multiplicity 2, β of multiplicity 2, and γ of multiplicity 5. We will denote the nine points as α^1, α^2, β^1, β^2, γ^1, γ^2, γ^3, γ^4, and γ^5, where α^1, α^2, and γ^1 are collinear, γ^1, γ^2, and β^1 are collinear, and β^1, β^2, and α^1 are collinear.

In order to get a two-dimensional linear system of cubics, we can choose to either take the seven base points $\alpha^1, \alpha^2, \gamma^1, \dots, \gamma^5$ or $\beta^1, \beta^2, \gamma^1, \dots, \gamma^5$.

In the first scenario, we want a cubic with a generic double point P given seven base points $\alpha^1, \alpha^2, \gamma^1, \dots, \gamma^5$. In order for the resulting pencil to still be quasi elliptic, P needs to be on the line determined by γ^1 and γ^2, and so it turns out P cannot be generic after all. If P is not on this line, then we will not have the $\widetilde{A}_2 \oplus \widetilde{E}_6$ diagram, and there is no quasi-elliptic Dynkin diagram for the configuration we would get.

Similarly, in the second scenario, we want a cubic with a generic double point P given seven base points $\beta^1, \beta^2, \gamma^1, \dots, \gamma^5$. In order for the resulting pencil to still be quasi elliptic, P needs to be on the line determined by β^1 and β^2, and so it turns out P cannot be generic after all. Therefore the quasi-elliptic fibration $\widetilde{A}_2 \oplus \widetilde{E}_6$ does not yield an unexpected cubic.

Therefore these $\widetilde{A}_2^{\oplus 4}$ and $\widetilde{A}_2 \oplus \widetilde{E}_6$ diagrams do not yield unexpected cubics. No other diagram of a blow-down of a quasi-elliptic fibration could yield an unexpected cubic because no other diagram involves a point being blown up exactly twice, which is necessary for the generic double point. Therefore there are no unexpected cubics in characteristic 3 coming from the quasi-elliptic fibrations enumerated above.

□

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