Spring 5-4-2023

Gordian Distance and Complete Alexander Neighbors

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GORDIAN DISTANCE AND COMPLETE ALEXANDER NEIGHBORS

by

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A DISSERTATION

Presented to the Faculty of

The Graduate College at the University of Nebraska

In Partial Fulfilment of Requirements

For the Degree of Doctor of Philosophy

Major: Mathematics

Under the Supervision of Professor Mark Brittenham and Professor Alex Zupan

Lincoln, Nebraska

May, 2023

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We call a knot K a *complete Alexander neighbor* if every possible Alexander polynomial is realized by a knot one crossing change away from K. It is unknown whether there exists a complete Alexander neighbor with nontrivial Alexander polynomial. We eliminate infinite families of knots with nontrivial Alexander polynomial from having this property and discuss possible strategies for unresolved cases.

Additionally, we use a condition on determinants of knots one crossing change away from unknotting number one knots to improve KnotInfo's [10] unknotting number data on 11 and 12 crossing knots. Lickorish introduced an obstruction to unknotting number one in [9], which proves the same result. However, we show that Lickorish's obstruction does not subsume the obstruction coming from the condition on determinants.

ACKNOWLEDGMENTS

I am so grateful to my advisors, Mark and Alex, for all their support and advice over the years. Thank you Mark for always being willing to answer my questions and chat about math or anything else. I always feel like you are in my corner and I really appreciate that. Thank you Alex for all the opportunities you have given me. You have included me in workshops, your travel to the Max Planck Institute of Mathematics, and two Polymath Jr. projects. Thank you for your mentorship and for your support.

I am grateful to the other members of my supervisory committee, Susan Hermiller, Jamie Radcliffe, and Vinod Variyam for the time you have all spent helping me through this program from my comprehensive exams to now.

I am also grateful to Chuck Livingston and Peter Feller for productive mathematical discussions. Thank you for taking time to talk to me about knot theory in general and about my work.

Thank you to Josh Brummer, Kristin Pfabe, Brian Harborne, and Eloísa Grifo for supporting my endeavors and for giving me advice and opportunities.

I would not be where I am today without my friends, especially my fellow math graduate students throughout my time at the University of Nebraska-Lincoln. I would like to thank everyone who has been there for me. Graduate school has been a really joyful time in my life because of all of you.

I would like to thank my undergraduate math professors at Willamette University whose inspiring teaching made me want to study math in the first place. They made me believe I could do math and see the value it could have in my life.

I would also like to thank my family for all their support and positivity. Thank you for all your advice and enthusiasm.

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Chapter 1

Introduction

In this dissertation, we will investigate the relationship between the Alexander polynomial and crossing changes. Alexander defined the Alexander polynomial, the first polynomial knot invariant, in 1923 [1]. The Alexander polynomial is a strong invariant which has been of interest since its development.

There is a long history of work on the connection between the Alexander polynomial and unknotting number or Gordian distance (See Definitions 2.6 and 2.7). In 1978, Kondo showed that given any Alexander polynomial $p(t)$, there exists a knot with unknotting number one realizing $p(t)$ as its Alexander polynomial [7]. This means that given any Alexander polynomial $p(t)$, the unknot has a Gordian neighbor realizing $p(t)$ as its Alexander polynomial. In 2012 Kawauchi proved that for any Alexander polynomial of slice type $p(t)$ and any Alexander polynomial $q(t)$, there exists a pair of knots K_p and K_q realizing $p(t)$ and $q(t)$ respectively as their Alexander polynomials such that K_p and K_q have Gordian distance one [6]. On the other hand, Kawauchi also proved that for an infinite family of pairs of Alexander polynomials, any pair of knots realizing the polynomials have Gordian distance at least two. So Kawauchi was able to produce pairs of Alexander polynomials whose associated knots must have Gordian distance one and pairs of Alexander polynomials whose associated knots must have Gordian distance at least two, which is the farthest we can hope for in such a result due to Kondo's theorem. In 2012, Nakanishi and Okada investigated the collection of Alexander polynomials realized by the Gordian neighbors of a knot. They gave characterizations of this collection, including an algorithm for knots with monic Alexander polynomial [14], [15].

We define a **complete Alexander neighbor** as a knot K where given any Alexander polynomial $p(t)$, there exists a Gordian neighbor of K realizing $p(t)$ as its Alexander polynomial. Kondo showed that the unknot is a complete Alexander neighbor and since Alexander polynomials are multiplicative under connected sum, any knot with trivial Alexander polynomial is a complete Alexander neighbor. It is unknown whether there exists a complete Alexander neighbor with nontrivial Alexander polynomial. Kawauchi produced *polynomials* whose associated knots have particular Gordian distances, but we are concerned with fixed *knots* whose Gordian neighbors can reach every possible Alexander polynomial. In this dissertation we build on the work of Nakanishi and Okada [15] to prove that infinitely many knots with nontrivial Alexander polynomial are not complete Alexander neighbors in the following theorem.

Theorem 1.1. *Let K be a knot with unknotting number 1, where $\det(K) > 1$ and where $\det(K)$ is composite or $\det(K) \equiv 1 \mod 4$. Then K is not a complete Alexander neighbor.*

In Proposition 3.6 (Corollary 4.2 from [6]), Kawauchi defines sets $S_{p,n,\ell}$ of breadth 2 Alexander polynomials and proves that given any two knots K_a and K_b realizing a pair of distinct Alexander polynomials $a(t)$ and $b(t)$ such that $a(t), b(t) \in S_{p,n,\ell}$ for some p, n, and ℓ, then K_a and K_b have Gordian distance at least two. We characterize the Alexander polynomials in Kawauchi's result in the following theorem.

Theorem 1.2. *An Alexander polynomial of breadth 2, $q(t) = n(t + t^{-1}) + 1 - 2n$ is contained in $S_{p,n,\ell}$ for some $p, n,$ and ℓ as defined in Proposition 3.6 (Corollary 4.2 from [6]) if and only if $1 - 4n$ is not a square.*

Together, Theorem 1.1 and Theorem 1.2 yield the following corollary.

Corollary 1.3. *Let K be a knot with a breadth 2 Alexander polynomial $\triangle_K(t) = n(t + t^{-1}) + 1 - 2n$. If K has unknotting number one or $1 - 4n$ is not a square, then K is not a complete Alexander neighbor.*

Together, Theorem 1.1 and Corollary 1.3 eliminate 1,940 of the 2,977 prime knots with crossing number 12 or less from being a complete Alexander neighbor with non-trivial Alexander polynomial. We give possible strategies to improve these techniques.

We also use conditions from Nakanishi and Okada [15] as well as Lickorish [9] to improve the unknotting number information in KnotInfo [10] for the knots $11n_{163}$, $12n_{805}$, $12n_{814}$, $12n_{844}$, and $12n_{856}$.

Theorem 1.4. *The knots $11n_{162}, 12n_{805}, 12n_{814}, 12n_{844},$ and $12n_{856}$ have unknotting number greater than one.*

We can think of unknotting number as measuring how easily a knot can be untied. This is an intuitive invariant which is simple to describe and visualize, but is not well understood. There are many methods to bound unknotting number, but of the 2,977 prime knots with crossing number 12 or less, for 668 of them the unknotting number is unknown in KnotInfo. In 2016, McCoy proved that any alternating diagram of a knot with unknotting number one has an unknotting crossing [11]. This means that it is algorithmic to detect unknotting number one in alternating knots, but not for nonalternating knots. In fact, in 2001 Stoimenow produced a minimal crossing diagram of $14n_{17214}$ and $14n_{17224}$, which have unknotting number one, with no unknotting crossing [20]. In Chapter 4 we describe two distinct obstructions to unknotting number one, one described by Lickorish in [9] and one coming from a result by Nakanishi and Okada in [15], which improve the data in KnotInfo for five nonalternating knots and compare the obstructions.

Chapter 2

Background

Knot theory is the study of how circles can be tangled in three dimensional space. Intuitively we can imagine a knot as a closed loop of string which we can move freely in space without cutting it or passing a strand through itself. We will study properties of knots by defining knot invariants and thinking about how these properties interact with each other and with alterations such as crossing changes.

Throughout this thesis we will assume familiarity with topics from a basic graduate algebraic topology course.

2.1 Knots and Knot Diagrams

We begin by defining knots more precisely.

Definition 2.1. *A* ***knot*** *is a smooth embedding of the circle S^1 in the 3-sphere S^3 considered up to ambient isotopy.*

Knots exist in 3-dimensional space, but we often represent them with 2-dimensional diagrams.

Definition 2.2. *A* ***knot diagram*** *is a projection of the knot to a plane where any intersections in the plane are of two strands crossing transversely and the crossing* *information is recorded with a break in the undercrossing.*

For example, in Figure 2.1 we see two knot diagrams of the same knot, since we can move the knot on the right to look like the knot on the left by untwisting the loop at the top of the diagram. This untwisting is an ambient isotopy.

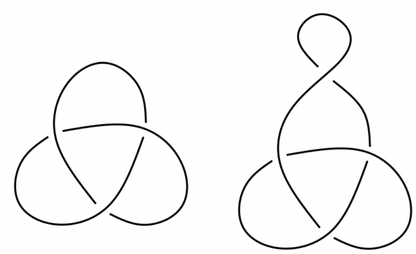

Figure 2.1: Two diagrams of a knot called the left-handed trefoil

The simplest knot, which we call the **unknot**, is the knot which can be represented by a circle with no crossings. For example, Figure 2.2 includes two diagrams of the unknot with the standard diagram on the left.

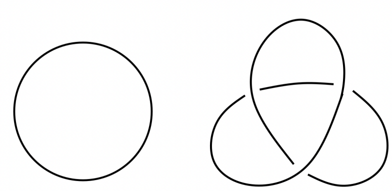

Figure 2.2: Two diagrams of the unknot

One of the most important results in knot theory gives us a small collection of moves to describe any ambient isotopy. This theorem is powerful because it gives us a simple handful of moves to build an isotopy between any two diagrams for the same knot. This gives us a managable way to think about all the possible ambient isotopies of a particular smooth embedding of S^1 in S^3.

Theorem 2.3 (Reidemeister's Theorem). *([16]) Any two diagrams for the same knot are related by a sequence of Reidemeister moves as pictured in Figure 2.3.*

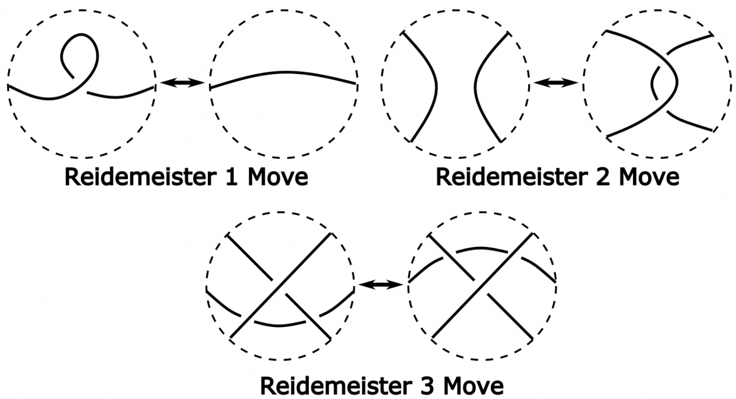

Figure 2.3: The three local Reidemeister moves for knot diagrams

This theorem implies that any manipulation of a knot through 3-dimensional space can be broken down into a sequence of these three moves on diagrams of the knot. For example, Figure 2.4 shows a sequence of moves between the two diagrams of the unknot in Figure 2.2.

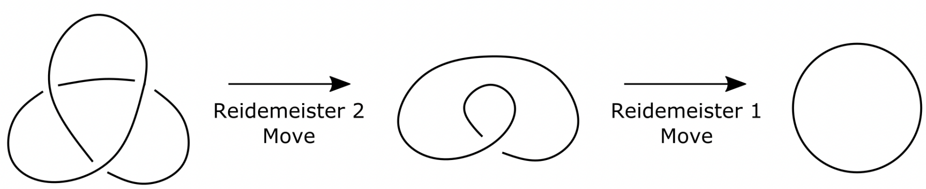

Figure 2.4: A sequence of two Reidemeister moves demonstrating that the diagrams in Figure 2.2 are equivalent

2.2 Knot Invariants

Given two knot diagrams, a fundamental question in knot theory is whether they are equivalent, meaning that they represent the same knot. If we can find an ambient isotopy between the diagrams, or equivalently, find a sequence of Reidemeister moves between them, then we have our answer, but it is more difficult to show that no such sequence of moves exists. There are infinitely many possible sequences of Reidemeister moves, so we cannot use an argument by exhaustion. Instead, we must define knot invariants to distinguish knots.

Definition 2.4. *A **knot invariant** is a well-defined function f from the set of all knots to some set S.*

Knot invariants give us a way to "measure," in a very general sense, properties of the knot. If the function sends one knot K or knot diagram to a different element of the codomain than another knot K' or knot diagram, meaning that $f(K) \neq f(K')$, then we can conclude they are distinct knots. For example, the Alexander polynomial is a knot invariant that maps each knot K to a Laurent polynomial $\triangle_K(t) \in \mathbb{Z}[t, t^{-1}]$. This polynomial is invariant over Reidemeister moves, so we can conclude that any pair of knots with distinct Alexander polynomials must be distinct knots. We will define the Alexander polynomial and compute examples in Section 2.3.

Knot invariants can either be computed from knot diagrams and be invariant under Reidemeister moves, like polynomial invariants including the Alexander polynomial, or be a measurement which is minimized over all knot diagram representations of a knot, like crossing number.

Definition 2.5. *The **crossing number** of a knot K is the minimal number of crossings in a any diagram of K.*

In general this type of invariant is more difficult to compute, but is useful for describing the complexity of knots in some way. The unknot is represented by a diagram with 0 crossings, so the unknot has crossing number 0 and is the only knot to have crossing number 0. We can use a different invariant such as the Alexander polynomial (see Section 2.3) to show that the trefoil is distinct from the unknot. The trefoil has a diagram with 3 crossings, and we can see through exhaustion that any diagram with one or two crossings is a diagram of the unknot. Therefore, we can conclude that the crossing number of the trefoil is 3. A more standard name for the trefoil is 3_1 because it is the first knot with crossing number 3 in the knot table (see KnotInfo [10] for an online database of the knot table through 12 crossings).

Another classical way to measure the complexity of a knot is how easily we can unknot it by a series of crossing changes. A crossing change is a swap in the crossing information of a crossing in a knot diagram.

Definition 2.6. *The* ***unknotting number*** $u(K)$ *of a knot* K *is the minimal number of crossing changes required to transform* K *into the unknot.*

For example, the unknotting number of the trefoil is one since we know the trefoil is distinct from the unknot and there exists a diagram of the trefoil where we can swap the crossing information of one crossing to obtain a diagram of the unknot. Figure 2.5 shows this crossing change.

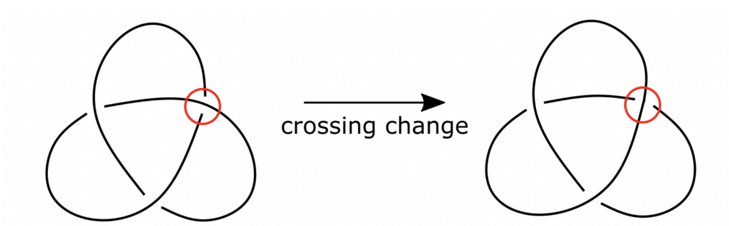

Figure 2.5: A single crossing change in a diagram of the trefoil to obtain a diagram of the unknot

Not every knot diagram will contain a minimal collection of unknotting crossing changes. For example, Figure 2.6 shows two diagrams of the knot $14n_{17214}$, one with no crossing which would unknot the knot when reversed and one with a crossing circled which would transform the diagram to a diagram of the unknot when reversed. These diagrams were given by Stoimenow in [20].

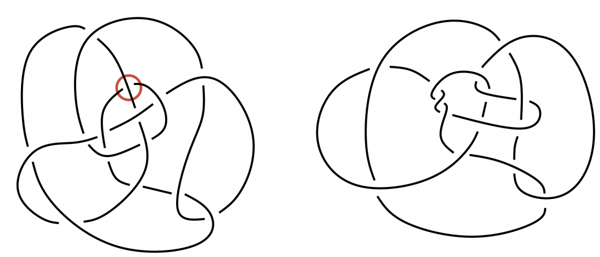

Figure 2.6: Two diagrams of $14n_{17214}$. On the left the circled crossing transforms $14n_{17214}$ into the unknot when it is changed. On the right is a diagram with no unknotting crossing. This example is given in [20].

A more general way to think about crossing changes is Gordian distance between a pair of knots rather than the unknotting number of a single knot.

Definition 2.7. *The **Gordian distance** between two knots K and K' is the minimal number of crossing changes necessary to go from a diagram of K to a diagram of K'.*

The unknotting number of a knot K is the Gordian distance between K and the unknot.

2.3 The Alexander Polynomial

There are several polynomial invariants which assign each knot a polynomial. Alexander defined the first polynomial invariant in 1923, called the Alexander polynomial. To define the Alexander polynomial, we need to introduce a definition first.

Definition 2.8. *A **Seifert surface** of a knot K is an orientable surface whose boundary is K.*

Given any knot K, we can use Seifert's algorithm to construct a Seifert surface of K [18] so every knot has a Seifert surface. There are many equivalent ways to define the Alexander polynomial. In this dissertation we will use the infinite cyclic cover X_∞ of the complement of a knot K in S^3 to compute the Alexander polynomial $\triangle_K(t)$ of K. We can construct X_∞ by first cutting the complement of K in S^3 along a Seifert surface Γ of K. Since Seifert surfaces are orientable, we can assign a positive and a negative orientation to the two copies of Γ after cutting the complement of K. For each $i \in \mathbb{Z}$, let X_i be the complement of K in S^3 cut along Γ so the boundary of X_i contains a copy Γ_i^+ of Γ assigned positive orientation and a copy Γ_i^- of Γ assigned negative orientation. Then we glue countably infinitely many of these cut complements in a row along the Seifert surfaces, by identifying Γ_i^- to Γ_{i+1}^+ for all $i \in \mathbb{Z}$. This construction is described and illustrated on pages 128-130 of [17].

Definition 2.9. *The* ***Alexander polynomial*** $\triangle_K(t)$ *of a knot* K *is the determinant of a presentation matrix (known as an* ***Alexander matrix****) for* $H_1(X_\infty)$ *as a module (known as the* ***Alexander module****) over* $\mathbb{Z}[t, t^{-1}]$ *where* t *is a covering transformation along the infinite cyclic cover* X_∞ *from one lift* X_i *of the complement of* Γ *in* S^3 *to the next adjacent lift* X_{i+1}*.*

This polynomial is unique up to multiplication by a unit in $\mathbb{Z}[t, t^{-1}]$. Note that the units of $\mathbb{Z}[t, t^{-1}]$ are $\pm t^n$ for any integer n.

We will compute the Alexander polynomial of the unknot, trefoil, and 5_1. The complement of the unknot in S^3 is $S^1 \times D^2$. A disk D^2 is a Seifert surface of the unknot, so we can cut the complement $S^1 \times D^2$ along D^2 and glue infinitely many of these cut complements in a row to obtain $X_\infty = \mathbb{R} \times D^2$. This is illustrated in Figure 2.7. Then for the unknot, $H_1(X_\infty)$ is trivial so the unknot has Alexander matrix (1) and Alexander polynomial $\det(1) = 1$ which we call the **trivial Alexander polynomial**.

It is difficult to visualize cutting the complement of a nontrivial knot K along a

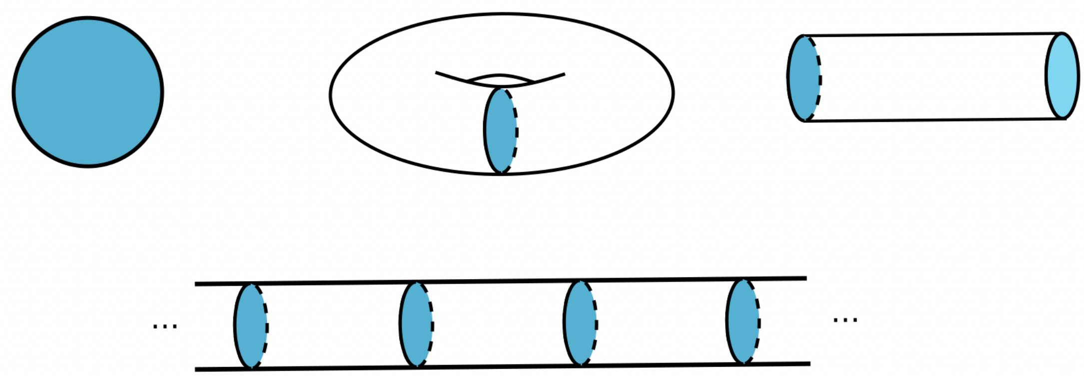

Figure 2.7: In the upper left is the unknot with a Seifert surface disk colored blue, followed by the complement of the unknot $S^1 \times D^2$ with a Seifert surface disk colored blue, then the complement of the unknot cut along the Seifert surface disk. In the bottom center is $X_\infty \cong \mathbb{R} \times D^2$.

Seifert surface since the Seifert surface of K is not a disk, so it is helpful to define a construction called Dehn surgery, introduced by Dehn in [5], to make K look like the unknot.

Definition 2.10. *A 3-manifold M' is obtained by **Dehn surgery** on a 3-manifold M along a solid torus T in M with a surgery framing curve γ on the boundary of T by removing the interior T from M and gluing in a solid torus such that γ bounds a disk.*

We can reverse crossings to transform K into the unknot and compensate for these crossing changes with Dehn surgeries. Then we can more easily construct an Alexander matrix of K. Levine [8] and Rolfsen [17] introduced this surgery view of the Alexander matrix, which Nakanishi and Okada also describe in Section 2 of [15]. We know K can be transformed into the unknot with a series of n crossing changes where n is the unknotting number of K. Every crossing change can be described with a ± 1 surgery along a solid torus around the crossing with linking number 0 with

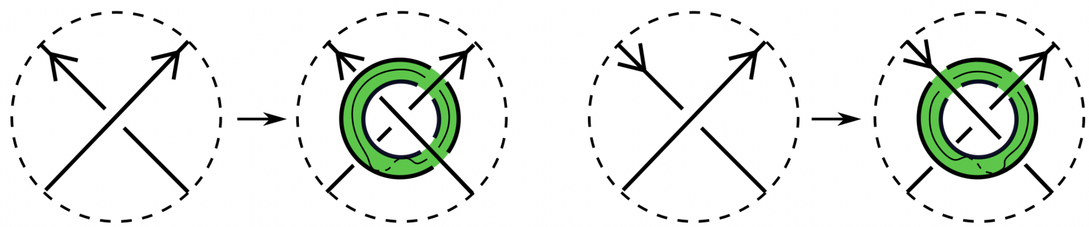

Figure 2.8: On the left is a positive crossing followed by this crossing reversed by a Dehn surgery along a solid torus illustrated in green. On the right is the same process illustrated for a negative crossing.

the knot. Illustrated in Figure 2.8 is the appropriate surgery to reverse a positive or negative crossing. It is necessary for the solid torus to have linking number 0 with K in order for the surgery solid tori to lift to solid tori in X_∞. So we know there exists a collection of these surgeries describing crossing changes resulting in the unknot. Therefore we can represent K as an unknot with n Dehn surgeries along solid tori T_i in the knot complement. Let γ_i be the framing curve representing the surgery on the boundary of T_i. Then we can cut along a disk whose boundary is this unknot and glue infinitely many copies of this cut space to form X_∞. Now we have infinitely many lifts of the n surgery tori T_i. In one lift of each T_i, we will call the meridian of T_i, μ_i. The lifts of these meridians μ_i for $1 \leq i \leq n$ in X_∞ generate $H_1(X_\infty)$ as a module over $\mathbb{Z}[t, t^{-1}]$ where t is a covering transformation along X_∞ from a lift in the complement of K in S^3 to the next adjacent lift. We can write the surgery framing curve γ_i of T_i as an element of $H_1(X_\infty)$ in the form

$$\gamma_i = \sum_{j=1}^{n} p_{i,j}(t) \cdot \mu_i.$$

The matrix $A_k(t) = (p_{i,j}(t))_{1 \leq i,j \leq n}$ is an Alexander matrix of K.

To illustrate this, we will compute the Alexander polynomial $\triangle_{3_1}(t)$ of the trefoil 3_1. We will use the surgery view described above to compute an Alexander matrix and from the matrix, the Alexander polynomial. We begin by deleting a solid torus around a crossing of 3_1, taking care that this solid torus has linking number 0 with the trefoil (this solid torus is illustrated in green in Figure 2.9). Then we glue a 2-disk D^2 along a curve on the boundary of the solid torus (illustrated in Figure 2.9 as a curve on the green surgery solid torus). Then there is a unique way to glue a 3-disk D^3 to fill in the solid torus again. The result is a Dehn surgery that twists our complement space to compensate for changing the crossing in 3_1. Now we can construct X_∞ similar to the way we did for the unknot, while keeping track of the surgery. This process is illustrated in Figure 2.9. Now we observe that $H_1(X_\infty)$ is generated by the lifts of the meridian μ of the surgery torus, which we will call $t^n\mu$ for $n \in \mathbb{Z}$, as illustrated in Figure 2.9. We can construct our Alexander matrix by describing a lift of our surgery curve γ, pictured in red in Figure 2.9, using our generators. In the case of 3_1, we can describe γ illustrated in Figure 2.9 as $\gamma = t\mu - \mu + t^{-1}\mu = (t - 1 + t^{-1})\mu$ in $H_1(X_\infty)$, so $\triangle_{3_1}(t) = \det(t - 1 + t^{-1}) = t - 1 + t^{-1}$ defined up to multiplication by $\pm t^n$ for $n \in \mathbb{Z}$.

We will repeat this process with 5_1, a knot with unknotting number 2, to see how we can use this surgery view with 2 Dehn surgeries to build a 2×2 Alexander matrix. We see this process illustrated in Figure 2.10. In Figure 2.10, X_∞ contains a blue Dehn surgery with meridian μ_1 and surgery framing curve γ_1 and a green Dehn surgery with meridian μ_2 and surgery framing curve γ_2. We have that $\gamma_1 = (t - 1 + t^{-1})\mu_1 + (-t^{-1} + 1)\mu_2$ and $\gamma_2 = (-t + 1)\mu_1 + (t - 1 + t^{-1})\mu_2$, so

$$\triangle_{5_1}(t) = \det \begin{pmatrix} t - 1 + t^{-1} & -t^{-1} + 1 \\ -t + 1 & t - 1 + t^{-1} \end{pmatrix} = t^2 - t + 1 - t^{-1} + t^{-2}.$$

This process is described in general and illustrated for other examples on pages 160-

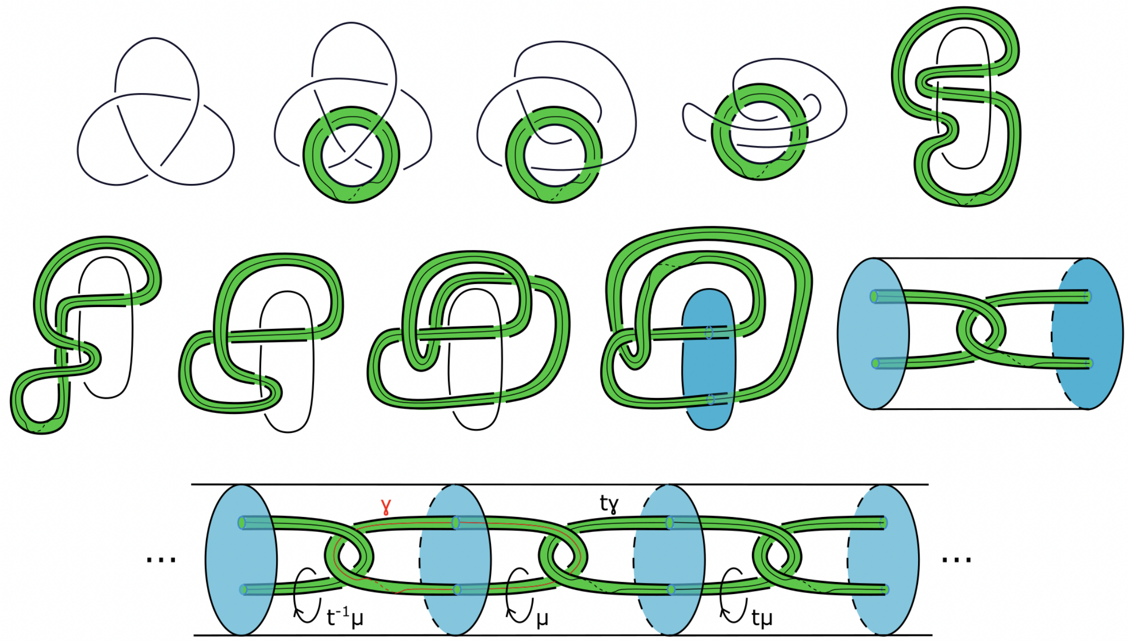

Figure 2.9: This illustrates a way to construct the infinite cyclic cover X_∞ of the complement of 3_1 in S^3. We begin with a diagram of 3_1, introduce a Dehn surgery, illustrated in green, then we use a series of isotopies to move our knot so that we can more clearly see a Seifert surface, illustrated in blue. Then we cut along this blue surface and glue countable infinite copies of this cut complement to form X_∞.

170 of [17].

It will be useful to define an integer invariant from the Alexander polynomial.

Definition 2.11. *The **determinant*** $\det(K)$ *of a knot* K *is* $|\triangle_K(-1)|$.

Because the Alexander polynomial is defined up to multiplication by a unit in $\mathbb{Z}[t, t^{-1}]$, this is a well-defined knot invariant.

Now we can use the Alexander matrix and Alexander polynomial to introduce two more knot invariants that minimize over all possible knot diagrams. The first is algebraic unknotting number introduced by Murakami in [12].

Definition 2.12. *The **algebraic unknotting number*** $u_a(K)$ *of a knot* K *is the minimal number of crossing changes necessary to change* K *to a knot with trivial Alexander polynomial.*

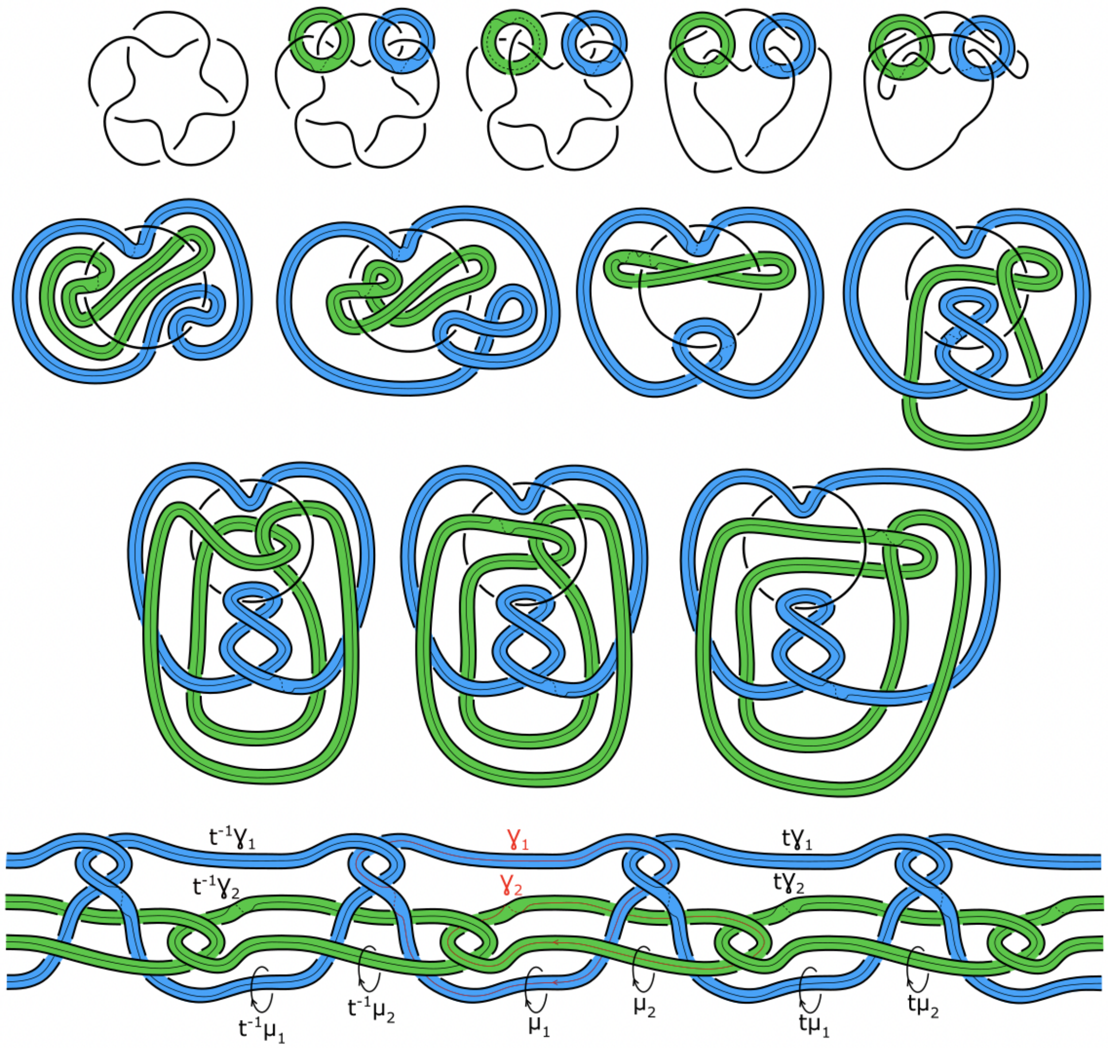

Figure 2.10: This illustrates a way to construct the infinite cyclic cover X_∞ of the complement of 5_1 in S^3. We begin with a diagram of 5_1, introduce two Dehn surgeries, illustrated in green and blue, then we use a series of isotopies to move our knot so that we can more clearly see a Seifert surface. Then we cut along this surface and glue countable infinite copies of this cut complement to form X_∞.

Definition 2.13. *The* ***Nakanishi index*** $n(K)$ *of a knot* K *is the minimal* n *such that the Alexander module of* K *is presented by an* $n \times n$ *matrix [13].*

We can use the surgery view of the Alexander matrix discussed above with a collection of unknotting surgeries, so it is always possible to build an $n \times n$ Alexander matrix where n is the unknotting number. However, in some cases there exists a smaller Alexander matrix. Also notice that algebraic unknotting number is a lower bound for unknotting number since the unknot has trivial Alexander polynomial. Furthermore, Nakanishi index is a lower bound for algebraic unknotting number, though the proof is more subtle (see Section 4.1 of [2]), so

$$n(K) \leq u_a(K) \leq u(K)$$

for any knot K.

Chapter 3

Gordian Distance and the Alexander Polynomial

Unknotting number and Alexander polynomials are classical knot invariants, so it is natural to consider the interaction between crossing changes of a knot K and the Alexander polynomial $\triangle_K(t)$. The unknotting number is hard to compute in general and not well understood, so it is useful to learn about its relationship to invariants which are algorithmic to compute, like the Alexander polynomial.

First we need to characterize the set of possible Alexander polynomials. Every Alexander polynomial can be written (after multiplication by a unit in $\mathbb{Z}[t, t^{-1}]$) as $p(t) \in \mathbb{Z}[t, t^{-1}]$ such that

(a) $p(1) = \pm 1$ and

(b) $p(t^{-1}) = p(t)$.

Conversely, every such polynomial is the Alexander polynomial of some knot (see Theorem 6.10 in [9] and page 171 in [17]). Notice that this characterization of the Alexander polynomial implies that the set of knot determinants is exactly the positive odd integers. An Alexander polynomial $p(t)$ can be written in the form

$$p(t) = a_0 + \sum_{i=1}^{n} a_i(t + t^{-1})$$

for some nonnegative integer n and some $a_i \in \mathbb{Z}$ by condition (b). By condition (a), a_0 is odd, so

$$|p(-1)| = \left|a_0 + \sum_{i=1}^{n} -2a_i\right| = \left|a_0 - 2\sum_{i=1}^{n} a_i\right|,$$

is a positive odd integer. Conversely, let d be a positive odd integer. Then consider the case where $d \equiv 1 \mod 4$, so $d = 1 + 4n$ for some nonnegative integer n. Notice that $p(t) = n(t+t^{-1}) - 2n - 1$ is a polynomial satisfying conditions (a) and (b) where $|p_n(-1)| = |-1-4n| = 1+4n = d$. Otherwise we have $d \equiv 3 \equiv -1 \mod 4$, so $d = -1 + 4n$ for some positive integer n. Notice that $p(t) = n(t+t^{-1}) - 2n + 1$ is a polynomial satisfying conditions (a) and (b) where $|p_n(-1)| = |1-4n| = -1+4n = d$.

In 1978, Kondo gave a constructive proof that there exists a knot with unknotting number one realizing any given Alexander polynomial [7]. Therefore, any pair of Alexander polynomials $p(t)$ and $q(t)$ is realized by a pair of knots K_p and K_q with unknotting number one, so their Gordian distance is at most two.

A natural next question to ask is whether there exists a nontrivial Alexander polynomial such that, given any second Alexander polynomial, there exist a pair of knots with Gordian distance one realizing the two polynomials. In 2012, Kawauchi proved that this is the case for Alexander polynomials of slice type, meaning a polynomial $p(t) = c(t)c(t^{-1})$ for some Laurent polynomial $c(t)$ (Corollary 5.2 in [6]). For example, $-2t + 5 - 2t^{-1} = (2t-1)(2t^{-1}-1)$ is a slice type Alexander polynomial, so given any Alexander polynomial $q(t)$, there exists a pair of knots one crossing change apart realizing $q(t)$ and $-2t + 5 - 2t^{-1}$.

Jong's problem (Pg. 954 of [6]) asks whether there exists a pair of Alexander polynomials such that any two knots realizing the polynomials have Gordian distance at least two. In 2012, Kawauchi found a family of pairs of polynomials for which this is the case [6]. One example is $t - 1 + t^{-1}$ and $t - 3 + t^{-1}$ which are the Alexander polynomials of the trefoil and figure eight knot respectively. Any pair of knots realizing this pair of polynomials have Gordian distance at least two, including the trefoil and figure eight knot.

This brings us to a knot property that we will study:

Definition 3.1. *A knot K is a* ***complete Alexander neighbor*** *if for any Alexander polynomial $p(t)$, there exists a knot K' such that K and K' are one crossing change apart and $\triangle_{K'}(t) = p(t)$.*

Since the Alexander polynomial is multiplicative under connected sum of knots, Kondo's result stating that there exists a knot with unknotting number one realizing any given Alexander polynomial implies that any knot with trivial Alexander polynomial is a complete Alexander neighbor [7]. Let K be a knot with trivial Alexander polynomial and let $p(t)$ be an Alexander polynomial. By Kondo's result, there exists a knot K' with unknotting number one such that $\triangle_{K'}(t) = p(t)$. Then the connected sum $K\#K'$ is a knot one crossing change away from K such that $\triangle_{K\#K'}(t) = p(t)$. However, it is unknown whether any knot with nontrivial Alexander polynomial has this property.

Question 3.2. *(Pg. 1017 of [15]) Does there exist a complete Alexander neighbor with nontrivial Alexander polynomial?*

While Kawauchi found *polynomials* which are realized by knots with particular Gordian distances, this question asks for a *knot* whose Gordian neighbors realize all Alexander polynomials.

3.1 Obstructing Knots from Being Complete Alexander Neighbors

We can obstruct knots from this property in a variety of ways. One is by considering the algebraic unknotting number (see Definition 2.12). A complete Alexander neighbor K must have algebraic unknotting number at most one because otherwise K would not have a Gordian neighbor realizing the trivial Alexander polynomial. The database Knotorious eliminates 1,526 knots of the 2,977 prime knots with 12 crossings or fewer from being complete Alexander neighbors using algebraic unknotting number [3].

Now we will use other techniques to eliminate families of knots from being complete Alexander neighbors. First, we need to introduce a definition and prove a lemma.

Definition 3.3. *An integer q is a* ***quadratic residue*** *mod n if there exists an integer x such that $q \equiv x^2 \mod n$. Otherwise, q is a* ***quadratic nonresidue*** *mod n.*

Lemma 3.4. *Let $n > 1$ be an odd integer. Then n is composite or $n \equiv 1 \mod 4$ if and only if there exists some integer d such that both d and $-d$ are quadratic nonresidues mod n.*

Proof. Let $n > 1$ be an odd integer and let $f : \mathbb{Z}_n \to \mathbb{Z}_n$ be given by $f(x) = x^2$ for all $x \in \mathbb{Z}_n$, so the image of f is the set of quadratic residues mod n.

First notice that for every nonzero y such that there exists $x \in \mathbb{Z}_n$ where $f(x) = y$ (equivalently, every nonzero quadratic residue y mod n), we have

$$f(n-x) = (n-x)^2 = n^2 - 2nx + x^2 \equiv x^2 = f(x) = y \mod n.$$

Since n is odd, $n - x \not\equiv x \mod n$, so at least two distinct elements of $\mathbb{Z}_n$ map to each quadratic residue. Therefore, at most half the nonzero elements of $\mathbb{Z}_n$ are quadratic residues.

Consider the case where n is prime. Then by the law of quadratic reciprocity, if $n \equiv 3 \mod 4$, then the negative of a residue modulo n is a nonresidue and the negative of a nonresidue is a residue, as desired. Also by the law of quadratic reciprocity, if $n \equiv 1 \mod 4$, then the negative of a residue modulo n is a residue and the negative of a nonresidue is a nonresidue. Since at most half the nonzero elements of $\mathbb{Z}_n$ are quadratic residues and $n \geq 3$, there exists a nonzero quadratic nonresidue d, so d and $-d$ are quadratic nonresidues mod n as desired when $n \equiv 1 \mod 4$.

Otherwise, n is composite, so $n = ab$ for some positive odd integers a and b both greater than one. Assume without loss of generality that $a \leq b$. First we will show that one of the following must be true.

(a) $1 \leq b - a < b + a \leq \frac{ab}{2}$.

(b) $a = b$.

(c) $a = 3$ and $b = 5$.

We will assume (a) is false and show that (b) or (c) must be true. Let $1 > b - a$ or $b + a > \frac{ab}{2}$. In the case where $b - a < 1$ we have $b - 1 < a \leq b$, so (b) holds.

In the case where $b + a > \frac{ab}{2}$, we have that $a < \frac{b}{\frac{b}{2} - 1}$. Then for $b \geq 6$ we have

$$1 < a < \frac{b}{\frac{b}{2} - 1} \leq 3$$

which is impossible since a is an odd integer, so $b < 6$. Since b is odd and greater than one, we have that $b = 3$ or $b = 5$. Since $1 < a \leq b$ and a is odd, in the case that

$b = 3$ (b) holds, and in the case that $b = 5$ either $a = 3$ so (c) holds or $a = 5$ so (b) holds.

Consider the case (a) where $1 \leq b - a \leq \frac{ab}{2}$ and $1 < a + b \leq \frac{ab}{2}$. Notice that $f(b-a)$, $f(a+b)$, $f(n-(b-a))$, and $f(n-(a+b))$ are all congruent to $a^2 + b^2$ mod $n = ab$. Since $1 \leq b - a < b + a \leq \frac{ab}{2}$, we have that

$$1 \leq b - a < b + a \leq \frac{ab}{2} = n - \frac{ab}{2} \leq n - (a + b) < n - (b - a) \leq n - 1,$$

so at least three distinct elements of $\mathbb{Z}_n$ map to the same quadratic residue $a^2 + b^2$ in $\mathbb{Z}_n$. Therefore, strictly less than half of the nonzero elements of $\mathbb{Z}_n$ are quadratic residues mod n. Thus, there exists some quadratic nonresidue d, so d and $-d$ are quadratic nonresidues mod n as desired.

Consider the case (b) where $a = b$. Then we have $a \not\equiv 0 \mod n$ and $f(a) = a^2 = n \equiv 0 \mod n$, so strictly less than half of the nonzero elements of $\mathbb{Z}_n$ are quadratic residues mod n. Therefore, there exists some quadratic nonresidue d, so d and $-d$ are quadratic nonresidues mod n as desired.

Consider the case (c) where $a = 3$ and $b = 5$. Then notice that 2 and -2 are quadratic nonresidues mod 15 as desired. □

We will use Lemma 3.4 to improve the following result by Nakanishi and Okada.

Lemma 3.5. *(Case $n = 1$ of propositions 5 and 6 in [15]) Let K be a knot with unknotting number one. Then a Laurent polynomial $p(t)$ is the Alexander polynomial of some knot K' one crossing change away from K if and only if there exist Laurent polynomials $r(t)$, and $m(t)$ such that*

(a) $m(t) = m(t^{-1})$, $m(1) = \pm 1$, $r(1) = 0$, *and*

(b) $p(t) = m(t)\triangle_K(t) - r(t)r(t^{-1})$

Note that Lemma 3.5 is proved using the surgery view of the Alexander polynomial described in Section 2.3. This result states that the only Alexander polynomials realized by a Gordian neighbor of a knot K with unknotting number one are the determinant of a 2×2 Alexander matrix obtained by the surgery view described in Section 2.3 with two Dehn surgeries. For example, note that in our computation of $\triangle_{5_1}(t)$ in Section 2.3, we found Alexander matrix $\begin{pmatrix} t-1+t^{-1} & -t^{-1}+1 \\ -t+1 & t-1+t^{-1} \end{pmatrix}$ using a surgery view with two surgeries, one of which transforms 5_1 into 3_1, both of which transform 5_1 into the unknot. So

$$\triangle_{5_1}(t) = \det \begin{pmatrix} t-1+t^{-1} & -t^{-1}+1 \\ -t+1 & t-1+t^{-1} \end{pmatrix} = m(t)\triangle_{3_1}(t) - r(t)r(t^{-1})$$

where $m(t) = t-1+t^{-1}$ and $r(t) = -t+1$.

We now have all the tools to prove Theorem 1.1, which states that any knot K with unknotting number one, where $\det(K)$ is composite or $\det(K) \equiv 1 \mod 4$ is not a complete Alexander neighbor.

Proof of Theorem 1.1. Let K be a knot with unknotting number one, where $\det(K) \geq 3$ and $\det(K)$ is composite or $\det(K) \equiv 1 \mod 4$. Since knot determinants are odd, by Lemma 3.4, there exists some quadratic nonresidue $d \mod \det(K)$ such that $-d$ is also a quadratic nonresidue $\mod \det(K)$.

Let K' be a knot one crossing change away from K. Then, by Lemma 3.5, we have $\triangle_{K'}(t) = \triangle_K(t) \cdot m(t) - r(t) \cdot r(t^{-1})$ for some $m(t), r(t) \in \mathbb{Z}[t, t^{-1}]$ such that

$m(t^{-1}) = m(t)$, $|m(1)| = 1$, and $r(1) = 0$. Then

$$\det(K') = |\triangle_{K'}(-1)| = |\triangle_K(-1) \cdot m(-1) - (r(-1))^2|$$
$$\det(K') = |\pm \det(K) \cdot m(-1) - (r(-1))^2|$$
$$\text{so } \det(K') = |a \det(K) - b^2|$$

for some $a, b \in \mathbb{Z}$.

Consider the case where $a \det(K) \geq b^2$. Then

$$\det(K') = a \det(K) - b^2$$
$$b^2 = -\det(K') + a \det(K)$$

so $-\det(K')$ is a quadratic residue $\mod \det(K)$. Therefore, $-\det(K') \not\equiv d \mod \det(K)$ and $-\det(K') \not\equiv -d \mod \det(K)$, so $\det(K') \not\equiv |d| \mod \det(K)$.

Otherwise, $a \det(K) < b^2$. Then

$$\det(K') = b^2 - a \det(K)$$
$$b^2 = \det(K') + a \det(K)$$

so $\det(K')$ is a quadratic residue $\mod \det(K)$. Therefore, $\det(K') \not\equiv d \mod \det(K)$ and $\det(K') \not\equiv -d \mod \det(K)$, so $\det(K') \not\equiv |d| \mod \det(K)$.

Since the knot determinants are exactly the odd natural numbers, there exists an Alexander polynomial $p(t)$ such that $|p(-1)| \equiv |d| \mod \det(K)$. As argued above, this Alexander polynomial is not realized by any knot one crossing change away from K. □

Kawauchi also eliminated families of knots from being complete Alexander neighbors in the following result.

Proposition 3.6. *(Corollary 4.2 from [6]) Let p be any prime number, and n, ℓ integers coprime to p. If p is an odd prime, then assume that p is coprime to $1-4n$ and that $1-4n$ is a quadratic nonresidue mod p. Consider a set of Alexander polynomials*

$$S_{p,n,\ell} = \{n(t+t^{-1})+1-2n\} \cup \{(n+\ell p^{2s+1})(t+t^{-1})+1-2(n+\ell p^{2s+1}) | s \in \mathbb{N}_0\}$$

and let $a, b \in S_{p,n,\ell}$ such that $a \neq b$. Then for any knots K_a, K_b such that $\triangle_{K_a} = a$ and $\triangle_{K_b} = b$, we have that K_a and K_b must have Gordian distance at least two.

We can characterize the knots Kawauchi has shown not to be complete Alexander neighbors here. First notice that any Alexander polynomial of breadth 2 of a knot K can be written in the form $\triangle_K(t) = n(t+t^{-1})+1-2n$ for some nonzero integer n. Now we can prove Theorem 1.2, which states that a breadth 2 Alexander polynomial $q(t) = n(t+t^{-1})+1-2n$ is contained in $S_{p,n,\ell}$ for some p, n, ℓ as defined in Proposition 3.6 (Corollary 4.2 in [6]) if and only if $1-4n$ is not a square.

Proof of Theorem 1.2. Let $q(t) = n(t+t^{-1})+1-2n$ be an Alexander polynomial of breadth 2 for some $n \in \mathbb{Z}$.

Assume $1-4n$ is not a square. First notice that for all non-square x, there exist infinitely many primes p such that x is a quadratic nonresidue mod p. Since $1-4n$ is not a square, there exist infinitely many primes p_i such that $1-4n$ is a quadratic nonresidue mod p_i, so there exists such a prime p_k such that $|1-4n| < p_k$ and $n < p_k$, so $1-4n$ and n are coprime to p_k. Therefore, $q(t) \in S_{p_k,n,\ell}$ for some ℓ as defined in Proposition 3.6.

Conversely, assume $q(t) \in S_{p,n,\ell}$ for some p, n, and ℓ as defined in Proposition 3.6. Then notice that either $1-4n$ is a quadratic nonresidue mod p where p is prime or

$p = 2$ and n is coprime to p, meaning that n is odd.

In the case where $1 - 4n$ is a quadratic nonresidue mod p, then $1 - 4n$ must not be a square.

In the case where n is odd, $1 - 4n \equiv 5 \mod 8$. Notice that odd squares are congruent to 1 mod 8 since $(2x + 1)^2 = 4x^2 + 4x + 1 = 4x(x + 1) + 1$ and $x(x + 1)$ must be even for any positive integer x. Therefore, $1 - 4n$ is not a square. □

Theorems 1.1 and 1.2 yield Corollary 1.3, which states that if K is a knot with breadth 2 Alexander polynomial $\triangle_K(t) = n(t + t^{-1}) + 1 - 2n$ such that K has unknotting number one or $1-4n$ is not a square, then K is not a complete Alexander neighbor.

Proof of Corollary 1.3. Let $\triangle_K(t) = n(t+t^{-1})+1-2n$ be an Alexander polynomial of breadth 2 of a knot K for some nonzero $n \in \mathbb{Z}$.

In the case where $1-4n$ is not a square, K is not a complete Alexander neighbor by Theorem 1.2 together with Proposition 3.6.

Consider the case where K has unknotting number one. Notice that

$$\det K = \begin{cases} 1 - 4n & n < 0 \\ -1 + 4n & n > 0 \end{cases},$$

so in the subcase where n is negative, $\det K \equiv 1 \mod 4$ and $5 \leq 1 - 4n$, meaning that K is not a complete Alexander neighbor by Theorem 1.1. In the subcase where n is positive, we have $1 - 4n < 0$, so $1 - 4n$ is not a square. Therefore, K is not a complete Alexander neighbor by Theorem 1.2 and Proposition 3.6. □

Theorem 3 from [7] by Kondo states that every Alexander polynomial is realized by a knot with unknotting number one and thus algebraic unknotting number one.

Thus, Theorem 1.1 proves that infinitely many knots are not complete Alexander neighbors. As an example,

$$1 + \sum_{i=1}^{n} \left((t^{2i} + t^{-2i}) - (t^{2i-1} + t^{-2i+1}) \right)$$

for $n \in \mathbb{N}$ is an infinite class of Alexander polynomials with breadth $4n$ and determinant $1 + 4n$, so there exist infinitely many knots with unknotting number one, and thus algebraic unknotting number one, realizing this class of Alexander polynomials which are eliminated from being a complete Alexander neighbor by Theorem 1.1 and not by Corollary 1.3 or their algebraic unknotting number.

Similarly, since every Alexander polynomial is realized by a knot with unknotting number one, Corollary 1.3 also proves that infinitely many knots are not complete Alexander neighbors. For example,

$$n(t + t^{-1}) + 1 - 2n$$

for all nonzero integers n is a collection of Alexander polynomials with breadth 2 and determinant $|1 - 4n|$ including infinitely many prime determinants congruent to $3 \mod 4$, which are each realized by a knot with unknotting number one, and thus algebraic unknotting number one. Therefore, infinitely many knots are eliminated by Corollary 1.3 and not by Theorem 1.1 or their algebraic unknotting number.

Together, these three methods of proving that a knot is not a complete Alexander neighbor apply to any knot K which meets at least one of the following criteria:

(a) K has algebraic unknotting number greater than one (which applies to 1,546 of the 2,977 prime knots with crossing number 12 or less),

(b) K has unknotting number one and determinant which is composite or congruent to 1 mod 4 (which applies to 384 of the 2,977 prime knots with crossing number 12 or less), or

(c) K has Alexander polynomial of breadth 2, $\triangle_K(t) = n(t + t^{-1}) + 1 - 2n$, where K has unknotting number one or $1 - 4n$ is not a square (which applies to 25 of the 2,977 prime knots with crossing number 12 or less).

All together, this eliminates 1,940 of the 2,977 prime knots with 12 crossings or fewer and there are 4 nontrivial prime knots with trivial Alexander polynomial and 12 crossings or fewer. There are many very small candidates for a complete Alexander neighbor with nontrivial Alexander polynomial which are not yet eliminated. Through eight crossings these are 6_2, 7_6, 8_3, 8_4, 8_6, 8_7, 8_{10}, 8_{12}, and 8_{14}.

3.2 Future Directions

It may be possible to improve Lemma 3.5. Lemma 3.5 is the $n = 1$ case of the following result.

Proposition 3.7. *(Propositions 5 and 6 in [15]) Let K be a knot and let $A_K(t) = (a_{ij}(t))_{1 \leq i,j \leq n}$ be an $n \times n$ Alexander matrix of K obtained through a collection of n unknotting surgeries as described in Section 2.3. Then*

(a) $a_{ij}(t) = a_{ij}(t^{-1})$ *for all* $1 \leq i, j \leq n$ *and*

(b) $a_{ij}(1) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Moreover, a Laurent polynomial $p(t)$ is the Alexander polynomial of some knot K' one crossing change away from K if and only if there exist Laurent polynomials $r_1(t), ..., r_n(t)$, and $m(t)$ such that

(1) $m(t) = m(t^{-1})$, $m(1) = \pm 1$, *and* $r_i(1) = 0$ *for all* $1 \leq i \leq n$, *and*

(2) $p(t) = \pm \det \begin{pmatrix} & & & r_1(t^{-1}) \\ & A_K(t) & & \vdots \\ & & & r_n(t^{-1}) \\ r_1(t) & \dots & r_n(t) & m(t) \end{pmatrix}$

Let K be a knot with Nakanishi index one (see Definition 2.13). Then $(\triangle_K(t))$ is an Alexander matrix of K which satisfies (a) and (b) of Proposition 3.7. But the characterization of Alexander polynomials of the Gordian neighbors of K described in Proposition 3.7 doesn't necessarily hold. For example, consider the knots 5_1 and 10_{132}. We have that 10_{132} has unknotting number one so Lemma 3.5 gives a characterization of the Alexander polynomials of the Gordian neighbors of 10_{132}. The knot 5_1 has Nakanishi index one and $\triangle_{5_1}(t) = \triangle_{10_{132}}(t)$, so we might expect 5_1 and 10_{132} to have the same set of Alexander polynomials realized by their Gordian neighbors by Proposition 3.7, but 5_1 has algebraic unknotting number 2, so the trivial Alexander polynomial is not realized by a Gordian neighbor of 5_1, while the trivial Alexander polynomial is realized by a Gordian neighbor of 10_{132}. This difference in the Alexander polynomials realized by Gordian neighbors is because Proposition 3.7 requires the Alexander matrix $A_K(t)$ of K to be obtained through a collection of unknotting surgeries as described in Section 2.3. We cannot take $A_K(t)$ to be any Alexander matrix of K satisfiying (a) and (b) from Proposition 3.7.

However, we believe the result in Proposition 3.7 will hold (and that the surgery view of the Alexander polynomial described in Section 2.3 will hold), for any collection of surgeries that transform a knot into a knot with trivial Alexander polynomial rather than a collection of surgeries that transform a knot into the unknot. This leads us to the following conjecture.

Conjecture 3.8. *Let K be a knot with algebraic unknotting number one. Then a Laurent polynomial $p(t)$ is the Alexander polynomial of some knot K' one crossing change away from K if and only if there exist Laurent polynomials $r(t)$ and $m(t)$ such that*

(a) $m(t) = m(t^{-1})$, $m(1) = \pm 1$, $r(1) = 0$, *and*

(b) $p(t) = m(t)\triangle_K(t) - r(t)r(t^{-1})$

First note that if the suppositions for the surgery view of the Alexander polynomial can be relaxed as conjectured, then we have an intuitive proof that $n(K) \leq u_a(K)$ for any knot K. Also, if this conjecture is true, then we can improve Theorem 1.1 and Corollary 1.3 to the conjectured statements below.

Conjecture 3.9. *Let K be a knot with algebraic unknotting number one, where* $\det(K) \geq 3$ *and where* $\det(K)$ *is composite or* $\det(K) \equiv 1 \mod 4$. *Then K is not a complete Alexander neighbor.*

Conjecture 3.10. *Let K be a knot whose Alexander polynomial $\triangle_K(t)$ has breadth 2. Then K is not a complete Alexander neighbor.*

Conjecture 3.8 implies Conjecture 3.9 by a similar argument to the proof of Theorem 1.1 in Section 3.1. Notice that since knots with algebraic unknotting number greater than one are not complete Alexander neighbors by definition, we can restate Conjecture 3.9 without the supposition that K must have algebraic unknotting number one.

Lemma 3.11. *Conjecture 3.9 implies Conjecture 3.10.*

Proof. Let K be a knot with breadth 2. Then we can write $\triangle_K(t) = n(t+t^{-1}+1-2n$ for some nonzero $n \in \mathbb{Z}$.

In the case where K has algebraic unknotting number greater than one, K is not a complete Alexander neighbor.

Otherwise K has algebraic unknotting number one. Notice that

$$\det(K) = \begin{cases} 1 - 4n & n < 0 \\ -1 + 4n & n > 0 \end{cases},$$

so in the subcase where n is negative, $\det(K) \equiv 1 \mod 4$ and $5 \leq 1 - 4n$, meaning that K is not a complete Alexander neighbor by Conjecture 3.9. In the subcase where n is positive, we have $1 - 4n < 0$, so $1 - 4n$ is not a square. Therefore, K is not a complete Alexander neighbor by Theorem 1.2 and Proposition 3.6. □

In conversation with Peter Feller, he made similar conjectures based on work in progress, which further bolsters our confidence in Conjecture 3.8. The conjecture below is due to Peter Feller and states that any two knots with algebraic unknotting number one and the same Alexander polynomial realize the same set of Alexander polynomials with their Gordian neighbors.

Conjecture 3.12 (Feller)**.** *Let K and J be knots with algebraic unknotting number one and the same Alexander polynomial. If $p(t)$ is an Alexander polynomial realized by a knot K' one crossing change away from K, then there exists a knot J' one crossing change away from J with Alexander polynomial $p(t)$.*

Then, Conjecture 3.12 together with the characterization of the set of Alexander polynomials realized by the Gordian neighbors of a knot with unknotting number one in Lemma 3.5, gives us Conjecture 3.8.

Lemma 3.13. *Conjecture 3.12 implies Conjecture 3.8.*

Proof. Let K be a knot with algebraic unknotting number one. By Theorem 3 from [7], there exists a knot J with the same Alexander polynomial as K with unknotting number one and note that this implies J also has algebraic unknotting number one.

Let K' be a knot one crossing change away from K. By Conjecture 3.12, there exists a knot J' one crossing change away from J with the same Alexander polynomial as K'. By Lemma 3.5, $\triangle_{J'}(t) = \triangle_J(t) \cdot m(t) - r(t) \cdot r(t^{-1})$ for some $m(t), r(t) \in \mathbb{Z}[t, t^{-1}]$ such that $m(t^{-1}) = m(t)$, $|m(1)| = 1$, and $r(1) = 0$. Therefore we have $\triangle_{K'}(t) = \triangle_K(t) \cdot m(t) - r(t) \cdot r(t^{-1})$.

Let $r(t)$ and $m(t)$ be Laurent polynomials such that $m(t^{-1}) = m(t)$, $|m(1)| = 1$, and $r(1) = 0$. Let $p(t) = \triangle_K(t) \cdot m(t) - r(t) \cdot r(t^{-1}) = \triangle_J(t) \cdot m(t) - r(t) \cdot r(t^{-1})$. By Lemma 3.5, there exists some J' one crossing change away from J with Alexander polynomial $p(t)$. Then by Conjecture 3.12, there exists a knot K' one crossing change away from K with the same Alexander polynomial as J'. □

For another possible method of investigating knots for which it is unresolved whether they are a complete Alexander neighbor, consider those with monic Alexander polynomial. For example, the knots 6_2, 7_6, and 8_7 have monic Alexander polynomial, which means we can use Nakanishi and Okada's algorithm used on the knots 3_1 and 4_1 in [15] and the knots 5_1 and 10_{132} in [14] to characterize the set of Alexander polynomials realized by their Gordian neighbors. This characterization might be useful to determine whether this set includes all Alexander polynomials.

Chapter 4

Obstructions to Unknotting Number One

We can leverage the effect of a single crossing change in a knot K on the determinant or on the double cover M_K of S^3 branched over K to obtain two obstructions to unknotting number one. One was described by Lickorish in 1985 [9]. Another follows from work by Nakanishi and Okada in 2012 [15]. Using these obstructions, we can show that many knots have unknotting number greater than one through simple computations. The two obstructions are similar, however Lickorish's obstruction does not subsume the other obstruction.

These obstructions improve the KnotInfo [10] data for five knots, pictured in Figure 4.1, in Theorem 1.4.

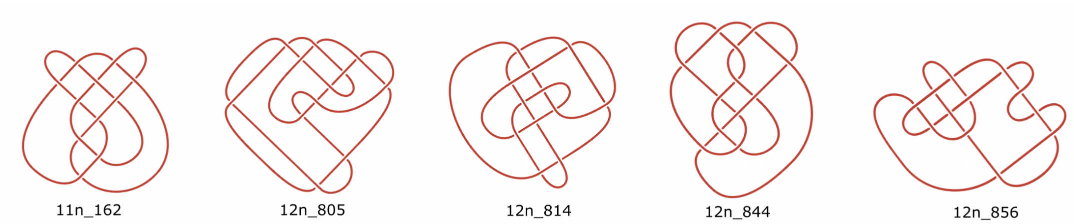

Figure 4.1: Knots $11n_{162}, 12n_{805}, 12n_{814}, 12n_{844}$, and $12n_{856}$

4.1 Condition on Determinants

In their work on the relationship between crossing changes and Alexander polynomials, Nakanishi and Okada proved in [15] a condition on the determinants $|\triangle_K(-1)|$ and $|\triangle_{K'}(-1)|$ of knots K and K' one crossing change apart where K has unknotting number one.

Lemma 4.1. *(Proposition 13 in [15]) Let K be a knot and K' be a knot one crossing change away from K. If K has unknotting number 1, then $\pm \det K' \equiv -n^2 \mod \det K$ for some integer n.*

Therefore, by the contrapositive of Lemma 4.1, given any knot K where there exists a knot K' one crossing change away such that $\det K'$ and $-\det K'$ are quadratic nonresidues mod $\det K$, we have that K has unknotting number greater than one.

Note that it is necessary for both $\det K'$ and $-\det K'$ to be a quadratic non-residue mod $\det K$ to conclude that K has unknotting number greater than one. For example, 3_1 and 5_2 are unknotting number one knots one crossing change apart with determinants 3 and 7 respectively. We have that 3 is a quadratic nonresidue mod 7, but $-3 \equiv 4 \mod 7$ is a quadratic residue mod 7. We also have that $-7 \equiv 2 \mod 3$ is a quadratic nonresidue mod 3 and $7 \equiv 1 \mod 3$ is a quadratic residue mod 3.

In the KnotInfo database [10], we can use this observation to show that $11n_{162}$, $12n_{805}$, $12n_{814}$, $12n_{844}$, and $12n_{856}$ have unknotting number greater than one, where this was previously unknown in the database. This shows that $11n_{162}$ has unknotting number 2 and constrains the others to have unknotting number 2 or 3.

Proof of Theorem 1.4. The knot $11n_{162}$ has determinant 55 and DT code $[6, -10, 12, 22, 16, -18, 8, 20, -4, 2, 14]$ in KnotInfo [10]. We can change the sign of the first entry in the DT code to obtain 9_{45}, a knot one crossing change away from $11n_{162}$.

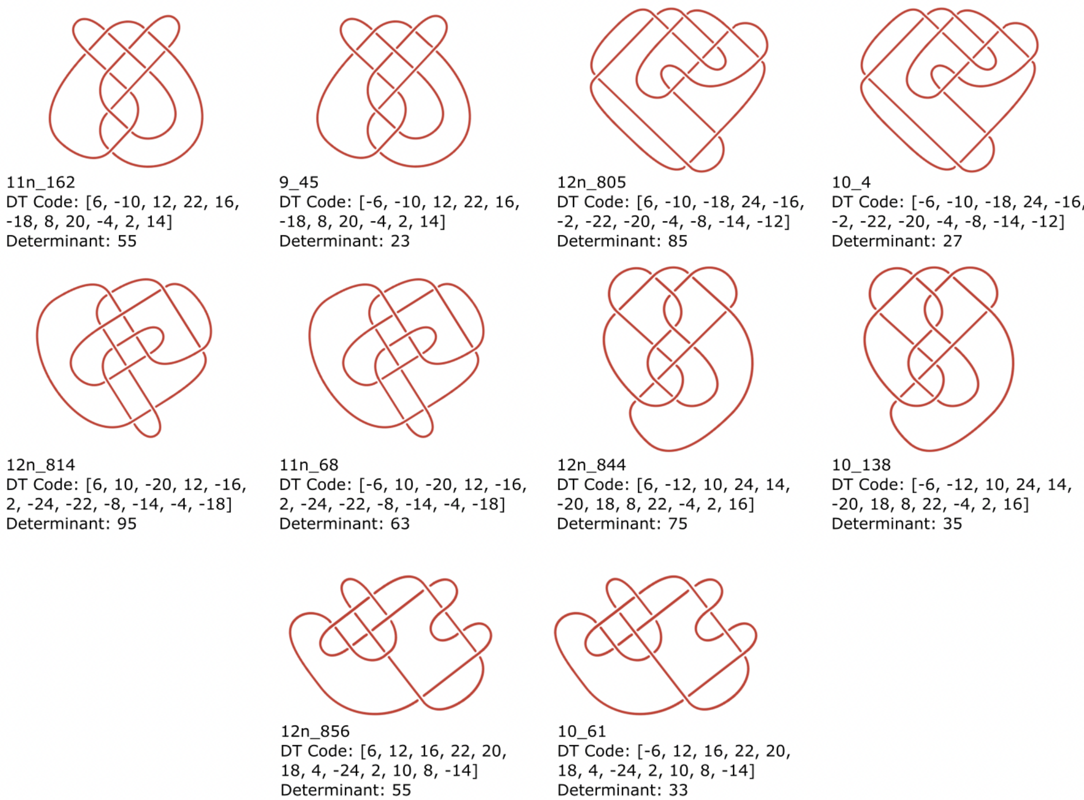

Figure 4.2: The knots $11n_{162}, 12n_{805}, 12n_{814}, 12n_{844}$, and $12n_{856}$ along with a knot one crossing change away from each of these. Under each knot is their name, a DT Code for the pictured diagram, and the knot's determinant.

The determinant of 9_{45} is 23. Since 23 and -23 are both quadratic nonresidues mod 55, by Lemma 4.1, $11n_{162}$ has unknotting number greater than one.

In Figure 4.2, we see a knot one crossing change away from $11n_{162}, 12n_{805}, 12n_{814}, 12n_{844}$, and $12n_{856}$ whose determinant satisfies the contrapositive of Lemma 4.1. Therefore, using a similar argument to the one above for $11n_{162}$, we conclude the proof. □

We identify these knots by performing a search with code using SnapPy [4] in Sage to compute the determinant of each knot for which it is unknown in KnotInfo [10] whether the unknotting number is one, and compute the determinant of each knot obtained by changing the sign of one number in the DT code recorded in KnotInfo

[10]. Then we check whether the determinants satisfy the condition in Lemma 4.1 (Proposition 13 in [15]).

We may expand this search by checking more crossing changes than just swapping the crossings in the diagram described by the DT code recorded in KnotInfo.

4.2 Condition on Linking Form

We can also use Lickorish's obstruction to show that these knots do not have unknotting number one [9]. To describe Lickorish's obstruction, we need to introduce some definitions.

Definition 4.2. *Let M be an oriented 3-manifold where $H_1(M)$ is finite. Then the* ***linking form*** *of M is $\lambda : H_1(M) \times H_1(M) \to \mathbb{Q}/\mathbb{Z}$ as defined below. Let $[\alpha], [\beta] \in H_1(M)$ be represented by 1-cycles α and β in M respectively. Then $n\alpha$ bounds an orientable surface Σ for some integer n. Define $\lambda([\alpha], [\beta]) = \frac{1}{n} i(\Sigma, \beta)$ where $i(\Sigma, \beta)$ is the intersection number of Σ and β.*

Definition 4.3. *Let D be a connected, checkerboard colored diagram of a knot K. Let $R_0, R_1, ..., R_n$ be the white regions of D. Assign each crossing of D a sign ± 1 as in Figure 4.3. Let g_{ij} be the sum of the signs of the crossings abutted by the white regions R_i and R_j for $0 \leq i, j \leq n$ where $i \neq j$ and let $g_{ii} = -\sum_{i \neq j} g_{ij}$. A* ***Goeritz matrix*** *G_K of K is the $n \times n$ matrix $(g_{ij})_{1 \leq i,j \leq n}$. Note that this eliminates all g_{ij} where $i = 0$ or $j = 0$.*

Lickorish proved the following condition on the linking form of the cyclic double cover M_K of the complement of K in S^3.

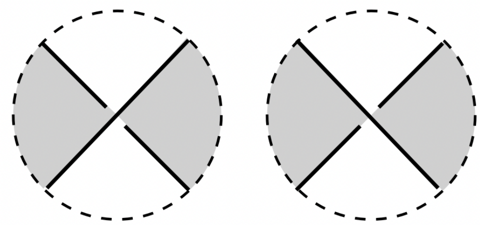

Figure 4.3: These are the sign conventions used in the definition of a Goeritz matrix.

Lemma 4.4. *(Lemmas 1 and 2 in [9]) If K is a knot with unknotting number one, then M_K is obtained by $\pm\frac{\det K}{2}$-surgery on a knot in S^3 and $H_1(M_K)$ is cyclic with a generator g such that $\lambda(g,g) = \frac{2}{\det K} \in \mathbb{Q}/\mathbb{Z}$.*

The following lemma allows us to use a Goeritz matrix to check for the condition on the linking form in Lemma 4.4.

Lemma 4.5. *(page 253 in [21], page 761 of [19]) Given a knot K, the linking form λ of M_K is given by $\pm(G_K)^{-1}$, meaning that there exists a generating set $\{g_1, g_2, ..., g_n\}$ of $H_1(M_K)$ such that $\lambda(g_i, g_j) = \pm(G_K^{-1})_{i,j}$.*

Now we can use Lickorish's obstruction to give an alternate proof of Theorem 1.4 with Goeritz matrices.

Alternate proof of Theorem 1.4. First notice that in SnapPy [4], we can see that there exist Goeritz matrices

$$G_{11n_{162}}^{-1} = \begin{pmatrix} \frac{16}{55} & \frac{8}{55} & \frac{1}{5} & \frac{3}{55} & \frac{6}{55} \\ \frac{8}{55} & \frac{4}{55} & \frac{3}{5} & \frac{29}{55} & \frac{3}{55} \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \\ \frac{3}{55} & \frac{29}{55} & \frac{3}{5} & \frac{4}{55} & \frac{8}{55} \\ \frac{6}{55} & \frac{3}{55} & \frac{1}{5} & \frac{8}{55} & \frac{16}{55} \end{pmatrix}, G_{12n_{805}}^{-1} = \begin{pmatrix} \frac{4}{17} & \frac{3}{17} & \frac{2}{17} & \frac{4}{17} \\ \frac{3}{17} & \frac{41}{85} & \frac{16}{85} & \frac{32}{85} \\ \frac{2}{17} & \frac{16}{85} & -\frac{29}{85} & \frac{27}{85} \\ \frac{4}{17} & \frac{32}{85} & \frac{27}{85} & \frac{54}{85} \end{pmatrix},$$

$$G_{12n_{814}}^{-1} = \begin{pmatrix} \frac{36}{95} & \frac{3}{19} & -\frac{2}{95} & \frac{22}{95} & \frac{12}{95} \\ \frac{3}{19} & \frac{6}{19} & \frac{3}{19} & \frac{5}{19} & \frac{1}{19} \\ -\frac{2}{95} & \frac{3}{19} & -\frac{21}{95} & \frac{41}{95} & \frac{31}{95} \\ \frac{22}{95} & \frac{5}{19} & \frac{41}{95} & \frac{24}{95} & \frac{39}{95} \\ \frac{12}{95} & \frac{1}{19} & \frac{31}{95} & \frac{39}{95} & \frac{4}{95} \end{pmatrix}, G_{12n_{844}}^{-1} = \begin{pmatrix} \frac{4}{15} & \frac{1}{15} & \frac{1}{5} & \frac{2}{15} & \frac{2}{15} \\ \frac{1}{15} & \frac{1}{15} & \frac{3}{5} & \frac{8}{15} & \frac{2}{15} \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \\ \frac{2}{15} & \frac{8}{15} & \frac{3}{5} & \frac{1}{15} & \frac{1}{15} \\ \frac{2}{15} & \frac{2}{15} & \frac{1}{5} & \frac{1}{15} & \frac{4}{15} \end{pmatrix}, \text{ and}$$

$$G_{12n_{856}}^{-1} = \begin{pmatrix} -\frac{14}{55} & \frac{27}{55} & \frac{2}{5} & \frac{19}{55} \\ \frac{27}{55} & \frac{54}{55} & \frac{4}{5} & \frac{38}{55} \\ \frac{2}{5} & \frac{4}{5} & \frac{4}{5} & \frac{3}{5} \\ \frac{19}{55} & \frac{38}{55} & \frac{3}{5} & \frac{41}{55} \end{pmatrix}.$$

Assume for contradiction that $11n_{162}$ has unknotting number one. Then, by Lemma 4.4, $H_1(M_{11n_{162}})$ is cyclic with a generator g such that $\lambda(g,g) = \frac{2}{55} \in \mathbb{Q}/\mathbb{Z}$. Since $(G_{11n_{162}}^{-1})_{1,1} = \frac{16}{55}$, we have by Lemma 4.5 that there exists $g_1 \in H_1(M_{11n_{162}})$ such that $\lambda(g_1, g_1) = \pm\frac{16}{55}$. Since $H_1(M_{11n_{162}})$ is cyclic with a generator g, we have that $g_1 = tg$ for some integer t, so

$$\pm\frac{16}{55} = \lambda(g_1, g_1) = \lambda(tg, tg) = t^2\lambda(g, g) = t^2\frac{2}{55}$$

in $\mathbb{Q}/\mathbb{Z}$. Therefore $t^2 \equiv \pm 8 \mod 55$, but 8 and -8 are not quadratic residues mod 55, which is a contradiction.

Using a similar argument to the one above for $11n_{162}$, and the Goeritz matrices above, we conclude the proof. □

4.3 Comparing Obstructions

These two obstructions are similar in the sense that to prove that a knot K has unknotting number greater than one, they both depend on the existence of an integer d such that both d and $-d$ are quadratic nonresidues mod $\det(K)$. However, they are not the same; Lickorish's obstruction does not subsume the obstruction coming from Lemma 4.1 (Proposition 13 in [15]).

It is difficult to show that the first obstruction to unknotting number one from Lemma 4.1 does not apply to a particular knot since there are infinitely many crossing changes to check for the condition on determinants in Lemma 4.1; however, when we only check each crossing change done by a single sign change in the DT code for each knot recorded in KnotInfo [10] up to 12 crossings, this obstruction shows that 1,273 knots have unknotting number greater than one. Note that there are 2,977 prime knots with crossing number 12 or less, and of those, 505 are known to have unknotting number one, so there are 2,472 knots which are not known to have unknotting number one.

To show that Lickorish's obstruction does not apply to a particular knot K we must check whether $\lambda(g,g)/2$ or $-\lambda(g,g)/2$ is a quadratic residue mod $\det(K)$ for each $g \in H_1(M_K)$. In the case where $H_1(M_K)$ is not cyclic, we know by Lemma 4.4 that K must have unknotting number greater than one and in the case where $H_1(M_K)$ is cyclic, checking the diagonal entries of G_K is sufficient to determine whether Lickorish's obstruction is applicable to K. However, just checking that for each entry $(G_K^{-1})_{i,i}$ along the main diagonal of the inverse of a Goeritz matrix of K for each prime K with up to 12 crossings, shows that 1,269 knots have unknotting number greater than one out of 2,472 knots which are not known to have unknotting number one. We also have that 11 of the remaining knots which are not known to have unknotting number one have non-cyclic $H_1(M_K)$, so must have unknotting number greater than one.

In the prime knots up to 13 crossings, there are 17 examples ($11a_{47}$, $11n_{170}$, $12a_{166}$, $12a_{615}$, $12a_{886}$, $13a_{947}$, $13a_{1237}$, $13a_{1602}$, $13a_{1853}$, $13a_{1995}$, $13a_{2005}$, $13a_{2006}$, $13a_{2649}$, $13a_{4258}$, $13n_{1663}$, $13n_{2937}$, and $13n_{2955}$) where changing some crossing in the DT code recorded in KnotInfo [10] yields a knot one crossing change away which satisfies the condition on determinants from Lemma 4.1 to show that the unknotting number must be greater than one, but Lickorish's obstruction does not apply using any of the diagonal entries of the inverse of a Goeritz matrix. However, all of these examples except $11n_{170}$ and $13a_{2649}$ have non-cyclic first homology of the double cover of S^3 branched over the knot, which also demonstrates that these knots have unknotting number greater than one. In the prime knots up to 13 crossings, there are 4 examples ($12n_{553}$, $13a_{1448}$, $13a_{2142}$, and $13n_{3264}$) where Lickorish's argument applies to one of the diagonal entries of the inverse of a Goeritz matrix, but no crossing change in KnotInfo's saved DT code [10] gives a knot satisfying the condition on determinants from Lemma 4.1.

We can use other methods to prove that many of these knots have unknotting number greater than one, but we see here that when we use the diagonal entries of a Goeritz matrix and the crossing changes from a sign change in the DT code, on small knots these obstructions are very similar, but not the same, and apply to many knots. Also, using the condition on determinants from Lemma 4.1 has the advantage that it is possible to expand the search to different crossing changes than those from sign changes in a particular DT code.

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