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Advances in Mathematics of Communications
Vol. 17, No. 2, April 2023, pp. 465–483
doi:10.3934/amc.2021003

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(Communicated by Vitaly Skachek)

Abstract. Absorbing sets are combinatorial structures in the Tanner graphs of low-density parity-check (LDPC) codes that have been shown to inhibit the high signal-to-noise ratio performance of iterative decoders over many communication channels. Absorbing sets of minimum size are the most likely to cause errors, and thus have been the focus of much research. In this paper, we determine the sizes of absorbing sets that can occur in general and left-regular LDPC code graphs, with emphasis on the range of b for a given a for which an (a, b)-absorbing set may exist. We identify certain cases of extremal absorbing sets that are elementary, a particularly harmful class of absorbing sets, and also introduce the notion of minimal absorbing sets which will help in designing absorbing set removal algorithms.

1. **Introduction.** Codes based on low-density parity-check (LDPC) matrices have been at the forefront of research in coding theory due to their low-complexity iterative message-passing decoders and near-capacity performance at long block lengths [3, 13, 21]. An LDPC code may be represented graphically using a bipartite Tanner graph [22]; iterative decoders operate on this graph using message-passing algorithms that transmit information along its edges. Unfortunately, at finite block lengths, these decoders are suboptimal compared to a maximum likelihood (ML) decoder, and can fail to converge to a codeword.

Failure of belief propagation (BP) and other iterative decoding algorithms operating on an LDPC code graph can be characterized by combinatorial substructures in the graph representation. Among these substructures are stopping sets (in the case of the binary erasure channel), trapping sets, absorbing sets, and pseudocodewords [5, 20, 6, 7, 19, 10, 16, 17]. The performance of an LDPC code, as illustrated by its bit error rate (BER) curve, is typically divided into two regions: the waterfall region and the error floor region. The degree distribution of the vertices in the

2020 *Mathematics Subject Classification.* Primary: 94A05; Secondary: 68P30, 68R10.

Key words and phrases. Absorbing sets, decoding, iterative decoding, low-density parity-check code, Tanner graph.

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Tanner graph influences the decoding threshold (i.e. the channel quality at which the iterative decoder can start to converge) and waterfall performance, which is measured as a probability of decoding failure as a function of the channel quality (i.e. signal-to-noise ratio). The slope of the BER curve eventually flattens, even as channel quality continues to increase, a phenomenon known as the error floor in iterative decoding. For many practical channels, the error floor behavior of the LDPC code is dominated by the harmful (with respect to the decoder) trapping sets in the graph [19]. Absorbing sets have been identified as combinatorial structures that inhibit decoder performance at high signal-to-noise ratios regardless of the particular choice of channel and iterative decoder [1, 6, 7, 24, 14]. Thus, absorbing sets are universal in that they are harmful graphical structures for a wide range of channel conditions and iterative decoders.

In this paper, we study the structure of absorbing sets and determine the sizes of absorbing sets that can occur in Tanner graphs. Smaller absorbing sets are considered more harmful when decoders such as the sum-product iterative decoder are used, as they are more likely to have all variable nodes in error. Determining the possible parameters for small absorbing sets in particular families of codes is of importance, since it is often the case that only a few targeted types of absorbing sets can be removed in the code design process. We first focus on general bipartite graphs with no restrictions on their vertex degrees, and later consider the class of left-regular LDPC codes used in many practical applications. LDPC codes are used widely, including in (distributed) data storage and wireless communications, and have been incorporated in 5G standards [2]. In particular, left-regular LDPC codes are prominent in data storage due to the importance of a lower error floor. We begin by showing that, in the class of Tanner graphs with girth at least g, (a, b)-absorbing sets where b is maximum (in the class) for the given a belong to a special sub-class of absorbing sets called elementary absorbing sets. These are known for being particularly harmful in iterative decoding [1]. We use this result to simplify the characterization of extremal absorbing sets in the Tanner graphs. We also introduce the concept of minimal absorbing sets, subsets of variable nodes that do not properly contain a smaller absorbing set. This characterization is useful for absorbing set removal algorithms, since minimal absorbing sets form the building blocks of larger absorbing sets.

The paper is organized as follows. Section 2 presents definitions and necessary notation. Section 3 considers general LDPC Tanner graphs with lower-bounded girth but no restrictions on the degrees of the nodes in the graph, and in Section 4, we consider classes of left-regular Tanner graphs. We also show how results from the theory of cages may be used to bound some extremal absorbing sets. In Section 5, we introduce the notion of minimal absorbing sets and provide a characterization of one class of minimal absorbing sets for general graphs. Section 6 concludes the paper.

2. Preliminaries.

Let $\mathcal{C}$ be an $[n, k]$ binary linear code defined by an $m \times n$ sparse parity check matrix H. The *Tanner graph* of $\mathcal{C}$ corresponding to H is the bipartite graph $G = (V, W; E)$ with n vertices in V, called *variable nodes*, corresponding to the columns of H, and m vertices in W, called *check nodes*, corresponding to the rows of H, such that there is an edge between $v_i \in V$ and $w_j \in W$ if and only if $H_{j,i} = 1$. In other words, H is the (simplified) adjacency matrix of the graph G. Tanner graphs of LDPC codes have historically been drawn with vertex set V

on the left and vertex set W on the right (i.e. in left-right representation), and so it has become standard to refer to the variable nodes as the "left nodes" and the check nodes as the "right nodes". Correspondingly, an LDPC code defined by a particular parity-check matrix is (j, ℓ)*-regular*, or j*-left regular* and ℓ*-right regular*, if its Tanner graph has variable nodes of degree j and check nodes of degree ℓ.

For a subset S of V, let $\mathcal{N}(S) \subseteq W$ denote the set of check node neighbors of S. Let $G_S = (S, \mathcal{N}(S); E_S)$ denote the subgraph induced by $S \cup \mathcal{N}(S)$ in G, where E_S is the set of edges between S and $\mathcal{N}(S)$. We can now define absorbing sets.

Definition 2.1. An (a, b)*-absorbing set* is a subset $D \subseteq V$ of variable nodes in a code's Tanner graph $G = (V, W; E)$ such that $|D| = a$, $|O(D)| = b$, and each variable node in D has strictly fewer neighbors in $O(D)$ than in $W \setminus O(D)$, where $O(D)$ is the subset of check nodes of odd degree in the subgraph G_D. For completeness, we define $E(D)$ to be the subset of check nodes of even degree in G_D.

Definition 2.2. An (a, b)*-absorbing set* is *fully absorbing* if it has the additional property that each variable node not in D has strictly fewer neighbors in $O(D)$ than in $W \setminus O(D)$, and is *elementary* if all check nodes in $\mathcal{N}(D)$ have degree 1 or 2 in G_D.

Consider the smallest value of a, and the smallest value of b given that a, for which an (a, b)-absorbing set exists in the Tanner graph of a code. An absorbing set with these parameters is called a *smallest* (a, b)-absorbing set of the code. High error floors in BER curves of LDPC codes over the Binary Symmetric Channel (BSC) and Additive White Gaussian Noise (AWGN) channels have been attributed to the presence of small absorbing sets in the corresponding Tanner graph [14, 24]. Thus, small absorbing sets are regarded as the most harmful due to their increased likelihood to cause decoding errors, and much work has focused on determining and eliminating the smallest absorbing sets in a code's graph.

The set of possible parameters of absorbing sets in a code's graph depends on characteristics of the Tanner graph, including its girth. Recall that the girth g of a graph G is the length of a shortest cycle in G. Since iterative decoding is optimal on cycle-free Tanner graphs, large girth is desirable [13, 23]. For code rates and block lengths used in practical applications (namely, rates $\leq .85$ and block lengths $\geq 10,000$ bits), the girth of a Tanner graph is often at least 8, and many known algebraic constructions have girth 12.

A Tanner graph with a fixed variable node degree (i.e., left degree) j and a girth g has the smallest absorbing set bounded as below. This result uses a tree-based argument that is also used to bound parameters such as minimum distance, minimum stopping set size, etc.

Theorem 2.3 ([6])**.** *Let G be a Tanner graph with girth $g \geq 6$ and variable node degree j. Define $\ell = \left\lfloor \frac{g}{4} \right\rfloor - 1$ and $t = \left\lceil \frac{j+1}{2} \right\rceil$. Then, for a smallest (a, b)-absorbing set in G,*

$$a \geq \begin{cases} 1 + \sum_{i=0}^{\ell} t(t-1)^i & \text{if } g \equiv 2 \pmod 4, \\ 1 + \sum_{i=0}^{\ell-1} t(t-1)^i + (t-1)^\ell & \text{if } g \equiv 0 \pmod 4. \end{cases}$$

Furthermore, $b \geq a \left\lfloor \frac{j-1}{2} \right\rfloor$.

Notice that for an (a, b)-absorbing set in a j-left regular graph, it is always the case that $b \leq a \left\lfloor \frac{j-1}{2} \right\rfloor$; thus, Theorem 2.3 asserts equality for smallest absorbing

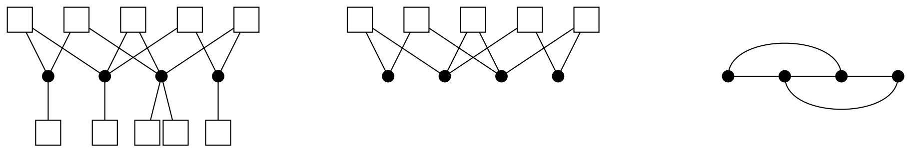


FIGURE 1. (a) A $(4,5)$-absorbing set graph. (b) The even part of the absorbing set graph in (a). (c) The normal graph of the graph in (a).

sets in that class. When the Tanner graph is irregular, the above bound may be modified by replacing j with the smallest left degree in the graph.

In the remainder of the paper, we will investigate the largest b for which an (a,b)-absorbing set may occur in Tanner graphs with certain parameters, and will make use of the following definitions.

Definition 2.4. The *absorbing set graph* of an absorbing set D in a Tanner graph G is the subgraph G_D[1]. We define the *even part* of the absorbing set graph to be the subgraph of G_D induced by $D \cup E(D)$, where $E(D)$ is the set of neighbors of D that have even degree in G_D.

Definition 2.5. Let D be an elementary absorbing set. We define the *normal graph* of the absorbing set graph G_D to be the graph obtained from G_D by (1) removing all degree-one check nodes and their edges, and (2) replacing each degree-two check node (and its incident edges) with a single edge.

The following example illustrates these definitions.

Example 2.6. In Figure 1(a), an absorbing set graph G_D is shown for a $(4,5)$-absorbing set D. The four circular vertices represent the variable nodes in D, and the square vertices represent check nodes ($|O(D)| = 5$ of which are odd-degree); notice that each variable node has strictly more even-degree than odd-degree check neighbors, so that D does in fact form an absorbing set. Observe that the girth of this absorbing set is 6, and it is also elementary, since each check node neighbor of D has degree one or two in G_D. The even part of G_D is shown in (b), and (c) shows the normal graph corresponding to G_D.

3. **Extremal absorbing sets in general graphs.** In this section, we determine the maximal size of b for a given value a for which an (a,b)-absorbing set may exist in a specified class of Tanner graphs. This will give insight to the structure of the absorbing sets that may occur. Let b_a^* denote the smallest possible value of b such that there exists an (a,b)-absorbing set in some graph in a given Tanner graph class, and let B_a^* indicate the largest such b. Notice that not every graph within a particular class will exhibit an (a,b)-absorbing set with $b = b_a^*$ or B_a^*. Theorem 2.3 gives bounds on b_a^* for smallest absorbing sets (i.e., minimal values of a) in the class of left-regular codes with girth $g \geq 6$. Since B_a^* depends on the class of graphs under consideration, many of our results will apply to the broad class of Tanner graphs with girth at least g. For classes with more restrictions (such as degree

[1] In an abuse of terminology, the graph G_D is sometimes itself called an absorbing set.

constraints), the results may differ; some such cases are considered in Section 4. In this section, we give results on the value of B_a^* for classes based purely on girth. To do so, we first show that this extremal case yields elementary absorbing sets.

Lemma 3.1. *Consider the collection of Tanner graphs with girth at least g. Every (a, b)-absorbing set such that $b = B_a^*$ is an elementary absorbing set.*

Proof. Let D be a set of a variable nodes giving an (a, b)-absorbing set in G with $b = B_a^*$.

First, suppose there is an odd-degree check node, C, in G_D of degree $d_C > 1$. We will exhibit an $(a, b+d_C-1)$-absorbing set whose girth is at least g (and thus is present in the considered class of Tanner graphs). Change the graph G_D as follows: break C into d_C distinct check nodes, each of degree 1, so that the union of the neighbors of those degree-1 check nodes is exactly equal to the neighborhood of C. This alteration forms an $(a, b + d_C - 1)$-absorbing set of girth at least g. This contradicts the fact that $b = B_a^*$. We conclude that every odd-degree check node in G_D has degree 1.

Now, suppose we have an even-degree check node, C, in G_D of degree $d_C > 2$. We will exhibit an $(a, b+d_C-2)$-absorbing set whose girth is at least g. The node C is adjacent to d_C variable nodes, $V_1, \dots, V_{d_C}$. Break C into $d_C - 1$ check nodes, each of degree two so that V_i is connected to V_{i+1} via a degree two check node for all $i \in [d_C - 1]$. This does not decrease the girth of the subgraph, as all variable nodes originally adjacent to C are still at least distance 2 from each other. We now add a single degree one check node adjacent to each V_i for $2 \le i \le d_C - 1$. In this way, we have created an $(a, b + d_C - 2)$-absorbing set with girth at least g, a contradiction. We conclude that every even-degree check node in G_D must have had degree 2 to begin with. $\square$

Remark 1. If a Tanner graph has girth 4, then for any value of $a \ge 2$, B_a^* is unbounded, as an absorbing set graph can be created by adding an arbitrary number of degree 2 check nodes adjacent to variable nodes. We therefore do not consider the girth 4 case in this section.

If $a = 1$, no absorbing sets of any size exist. If $a = 2$, any adjacent even degree check node is adjacent to both vertices in the absorbing set. Hence any absorbing set either has one even-degree check node and no odd-degree check nodes, or has a 4-cycle. The former case would necessitate degree-one variable nodes. We therefore assume $a \ge 3$ in Theorem 3.2 and Corollary 2.

Remark 2. Suppose D is an (a, B_a^*)-absorbing set in a graph G of girth at least g. By Lemma 3.1, this absorbing set is elementary. Hence, the even graph of G_D is right 2-regular, and so it has a (unique) normal graph on a nodes, and this normal graph has girth $g/2$. Further, each odd-degree check node in G_D has degree 1, so we may conclude that $B_a^* = 2m - a$ where m is the number of edges in the normal graph of G_D.

We next use Lemma 3.1 to establish the value of B_a^* in Tanner graphs of girth 6.

Theorem 3.2. *Consider the class of Tanner graphs with girth $g = 6$, and let $a \ge 3$. For any (a, b)-absorbing set appearing in a graph in this class, $b \le B_a^* = a(a-2)$.*

Proof. Let D be an (a, B_a^*)-absorbing set. By Lemma 3.1, D is elementary. As noted in Remark 2, if D is an (a, B_a^*)-absorbing set of girth $g = 6$, then its normal graph has girth $g/2 = 3$ and $B_a^* = 2m - a$, where m is the number of edges in its normal graph. The graph on a vertices with girth 3 and maximum number of edges is K_a, which has $a(a-1)/2$ edges. Hence, $B_a^* = a(a-1) - a = a(a-2)$. □

In [15], Hoory relates the average left and right degrees of a bipartite graph to the graph's girth, the size of each part, and the number of edges. Of particular relevance to our work is the following result:

Theorem 3.3 ([15]). *Consider a bipartite graph of girth $g = 2r$ with n_L vertices on the left side, n_R vertices on the right side, and m edges. Let $d_L = \frac{m}{n_L}$ be the average degree of the left vertices, and $d_R = \frac{m}{n_R}$ the average degree of the right vertices. Then,*

$$n_L \geq \sum_{i=0}^{r-1} (d_R - 1)^{\lceil i/2 \rceil} (d_L - 1)^{\lfloor i/2 \rfloor}.$$

Theorem 3.3 allows us to relate a and B_a^* for a given girth, as shown in the following corollary.

Corollary 1. *Consider the class of graphs of girth at least g. For an (a, B_a^*)-absorbing set in this class,*

$$a \geq \sum_{i=0}^{\frac{g}{2}-1} \left(\frac{B_a^*}{a}\right)^{\lfloor i/2 \rfloor}.$$

Proof. Let D be an (a, B_a^*)-absorbing set in a graph G of girth at least g. By Lemma 3.1, this absorbing set is elementary. Thus, the even part of G_D is right 2-regular, and contains a variable nodes. Furthermore, each odd-degree check node in G_D has degree 1, and so we may conclude that $B_a^* = m - a$, where m is the number of edges in the even part of G_D. Thus, by Theorem 3.3 applied to the even part of G_D,

$$\begin{aligned} a &\geq \sum_{i=0}^{\frac{g}{2}-1} (2-1)^{\lceil i/2 \rceil} \left(\frac{m}{a} - 1\right)^{\lfloor i/2 \rfloor} \\ &= \sum_{i=0}^{\frac{g}{2}-1} \left(\frac{m}{a} - 1\right)^{\lfloor i/2 \rfloor} \\ &= \sum_{i=0}^{\frac{g}{2}-1} \left(\frac{B_a^*}{a}\right)^{\lfloor i/2 \rfloor}. \end{aligned}$$

□

Remark 3. Corollary 1, along with the fact that graphs with girth exactly g are a subclass of graphs with girth at least g, gives the following bounds on B_a^*. Equality is shown for the first two cases by Theorem 3.2 and Corollary 2, respectively. See Table 1 for a concise statement of these results.

- For $g = 6$, $B_a^* \leq a(a-2)$.
- For $g = 8$, $B_a^* \leq \left\lfloor \frac{a(a-2)}{2} \right\rfloor$.
- For $g = 10$, $B_a^* \leq \left\lfloor a(\sqrt{a-1} - 1) \right\rfloor$.
- For $g = 12$, $B_a^* \leq \left\lfloor \frac{a}{2}\left(\sqrt{2a-3} - 1\right) \right\rfloor$.

As stated in the remark, equality with the upper bound is achieved when the girth is 8. To show this, we begin by recalling a fundamental theorem on triangle-free graphs introduced by Mantel in 1907 [18], and then use the result in Corollary 2 to show equality holds for (a, B_a^*)-absorbing sets of girth 8.

Theorem 3.4 ([18]). *The maximum number of edges in an n-vertex triangle-free graph is* $\left\lfloor \frac{n^2}{4} \right\rfloor$.

Corollary 2. *Consider the class of Tanner graphs with girth $g = 8$ and let $a \geq 3$. For any (a, b)-absorbing set appearing in a graph in this class, $b \leq B_a^* = \left\lfloor \frac{a(a-2)}{2} \right\rfloor$.*

Proof. Let D be an (a, B_a^*)-absorbing set in G. Let m be the number of edges in its normal graph. Then by Theorem 3.4 and Remark 2, $m = \frac{B_a^*+a}{2} = \left\lfloor \frac{a^2}{4} \right\rfloor$. Rearranging, we see that

$$B_a^* = \left\lfloor \frac{a^2}{2} \right\rfloor - a = \left\lfloor \frac{a(a-2)}{2} \right\rfloor .$$

□

B_a^*	Girth \ a	2	3	4	5	6	7	8	9	10	11	12
$= a(a-2)$	6	0	3	8	15	24	35	48	63	80	99	120
$= \left\lfloor \frac{a(a-2)}{2} \right\rfloor$	8	0	1	4	7	12	17	24	31	40	49	60
$\leq \lfloor a(\sqrt{a-1}-1) \rfloor$	10	0	1	2	5	6	9	12	15	20	21	24
$\leq \left\lfloor \frac{a}{2}(\sqrt{2a-3}-1) \right\rfloor$	12	0	1	2	3	6	7	10	11	14	15	18

TABLE 1. Exact values for B_a^* for relevant values of a and practical girths are given. Our formulas and bounds for B_a^* are provided in the first column for comparison. Shaded entries indicate values that are smaller than the bound given in the first column.

B_a^*	Girth \ a	2	3	4	5	6	7	8	9	10	11	12
$\leq \lfloor a(\sqrt{a-1}-1) \rfloor$	10	0	1	2	5	7	10	13	16	20	23	27
error		-	-	-	-	1	1	1	1	-	2	3
$\leq \left\lfloor \frac{a}{2}(\sqrt{2a-3}-1) \right\rfloor$	12	0	1	2	4	6	8	10	12	15	18	21
error		-	-	-	1	-	1	-	1	1	3	3

TABLE 2. Values of our bounds for B_a^* are provided for the girth 10 and 12 cases and relevant values of a. The first row for each girth indicates the predicted value from the bound, and the second row indicates the deviation of the bound from the actual value.

Remark 4. Theorem 4.5 in the survey paper [12] may be applied to obtain exact values of B_a^* for certain large values of a and g. The smallest case is $g = 12$ and $a = 14$ and yields a value equal to our upper bound on B_a^*, showing that our bound is tight in this case. Since larger values given by the theorem are not relevant to our work, we refer the interested reader to the survey paper.

In the following example, we illustrate our strategy for finding the values of B_a^* that do not meet the bounds given in Table 1 by studying the specific example where the girth of the graph is 10 and our absorbing set has size $a = 7$.

Example 3.5. A graph of girth 5 on 7 variable nodes is shown in Figure 2(a). By replacing each edge in this graph by a degree-2 check node and adding as many degree-1 check nodes as possible to each variable node (while maintaining an absorbing set), we obtain the $(7, 9)$-absorbing set graph shown in Figure 2(b). The bound given in Table 1 indicates that $B_7^* \leq 10$. We will show that no absorbing set on 7 variable nodes with girth 10 can have 10 degree-1 check nodes, and hence that $B_7^* = 9$.

· First, suppose the normal graph corresponding to our absorbing set has a vertex of degree 6. This normal graph then has 7 vertices, 6 of degree 1 and 1 of degree 6, and corresponds to a $(7, 5)$-absorbing set.

· Suppose the normal graph corresponding to our absorbing set has a vertex of degree 5. This accounts for six vertices. We need one more vertex, which must be added to one of the degree 1 vertices attached to the degree 5 vertex. Notice this seventh vertex cannot be adjacent to any other vertices while maintaining a girth of at least 5. Thus, the result is a normal graph that corresponds to a $(7, 5)$-absorbing set.

· Suppose the normal graph corresponding to our absorbing set has a vertex of degree 4. Then five vertices are already represented: one of degree 4, and its four neighbors. If we add the remaining two variable nodes so that the normal graph is a tree, it again corresponds to a $(7, 5)$-absorbing set. If we add the variable nodes so that there exists a cycle, we must add both between two of the nodes adjacent to the degree 4 node, otherwise, we would create a 4-cycle, contradicting the girth being at least 5. Such a normal graph would correspond to a $(7, 7)$-absorbing set.

· Suppose the normal graph corresponding to our absorbing set has vertices of degree at most 3. Let x be the number of degree 1 nodes, y be the number of degree 2 nodes, and z be the number of degree 3 nodes. Assume by way of contradiction that there exists a $(7, 10)$-absorbing set of girth 10 (whose normal graph then has girth 5). Then (x, y, z) must be a nonnegative integer solution to the system

$$\begin{cases} x + y + z & = 7 \\ y + 2z & = 10 \end{cases}$$

where the equation $x + y + z = 7$ is the constraint on the number of variable nodes and $y + 2z = 10$ is the constraint on the number of degree 1 check nodes we can add to variable nodes of degree 2 or 3 in the normal graph. The nonnegative integer solutions of this system are $(0, 4, 3)$, $(1, 2, 4)$, and $(2, 0, 5)$. Each of these solutions has an odd number of odd-degree vertices, so no graphs with these degree distributions can exist. We conclude that no normal graphs corresponding to a $(7, 10)$-absorbing set exist.

Hence there do not exist any normal graphs that correspond to a $(7, 10)$-absorbing set with girth 10. So in this case, $B_7^* = 9$.

We end this section with bounds on B_a^* in terms of corresponding values for smaller absorbing set sizes. Although these bounds are generally loose, we will use them in Section 5 when we consider minimal absorbing sets.

Lemma 3.6. *Consider the class of Tanner graphs of girth at least g. Then for $a, c \geq 2$,*

1. $B_a^* + 1 \leq B_{a+1}^*$, *and*

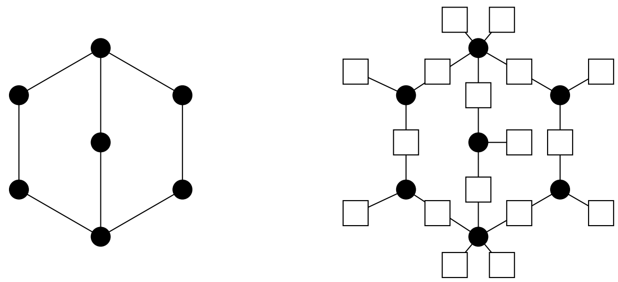


FIGURE 2. (a) A graph of girth 5 on 7 vertices. (b) A $(7, 9)$-absorbing set of girth 10 whose normal graph is the graph in (a).

2. $B_a^* + B_c^* + 2 \leq B_{a+c}^*$.

Proof. Let $a \geq 2$ and D be an (a, B_a^*)-absorbing set of girth at least g. Further, let $c \geq 2$ and S be a (c, B_c^*)-absorbing set of girth at least g. Recall that by Lemma 3.1, both absorbing sets are elementary.

1. Consider the absorbing set graph G_D. Replace a degree 2 check node in G_D with two degree 2 check nodes with a single adjacent variable node between them. Append a single degree one check node to this new variable node. This is an $(a + 1, B_a^* + 1)$-absorbing set graph with girth at least the girth of G_D. Hence $B_a^* + 1 \leq B_{a+1}^*$.
2. Consider absorbing set graphs G_D and G_S. We may assume G_D and G_S are disjoint, so that $G_D \cup G_S = G_{D\cup S}$ is an $(a+c, B_a^* + B_c^*)$-absorbing set graph. To this graph, add a single degree 2 check node with variable node neighbors $v \in G_D$ and $v' \in G_S$. To each of v and v', add a single degree one check node neighbor. This resulting graph has the same girth as (the lowest of the girths of) G_D and G_S, but has two additional odd degree check nodes, and is an absorbing set graph. Hence we have constructed an $(a + c, B_a^* + B_c^* + 2)$-absorbing set. So $B_a^* + B_c^* + 2 \leq B_{a+c}^*$.

□

We can tighten these bounds in classes where we have an exact expression for B_a^*, as shown for Tanner graphs of girths 6 and 8 in the following lemma.

Lemma 3.7. *Let $a, c \geq 3$. For the class of Tanner graphs of girth 6, $B_{a+1}^* = B_a^* + (2a - 1)$, and $B_{a+c}^* = B_a^* + B_c^* + 2ac$. For girth 8,*

$$B_{a+1}^* = \begin{cases} B_a^* + a & \text{if } a \text{ is odd,} \\ B_a^* + a - 1 & \text{if } a \text{ is even,} \end{cases}$$

$$B_{a+c}^* = \begin{cases} B_a^* + B_c^* + ac + 1 & \text{if both } a \text{ and } c \text{ are odd,} \\ B_a^* + B_c^* + ac & \text{else.} \end{cases}$$

Proof. Begin by recalling that Tanner graphs with girth at least 6 and variable node degree at least two cannot contain absorbing sets of size 1 or 2; hence, we restrict our focus to $a, c \geq 3$.

By Theorem 3.2, $B_a^* = a(a-2)$ for the class of graphs of girth 6. Then,

$$B_{a+1}^* = (a+1)(a-1) = a(a-2) + (2a-1) = B_a^* + (2a-1), \text{ and}$$
$$B_{a+c}^* = (a+c)(a+c-2) = a(a-2) + c(c-2) + 2ac = B_a^* + B_c^* + 2ac.$$

For the class of graphs of girth 8, Corollary 2 gives us $B_a^* = \left\lfloor \frac{a(a-2)}{2} \right\rfloor$, and so

$$\begin{aligned} B_{a+1}^* &= \left\lfloor \frac{(a+1)(a-1)}{2} \right\rfloor \\ &= \left\lfloor \frac{a(a-2)}{2} - \frac{1}{2} \right\rfloor + a \\ &= \begin{cases} B_a^* + a & \text{if } a \text{ is odd,} \\ B_a^* + a - 1 & \text{if } a \text{ is even,} \end{cases} \end{aligned}$$

and,

$$\begin{aligned} B_{a+c}^* &= \left\lfloor \frac{(a+c)(a+c-2)}{2} \right\rfloor \\ &= \left\lfloor \frac{a(a-2)}{2} + \frac{c(c-2)}{2} \right\rfloor + ac \\ &= \begin{cases} B_a^* + B_c^* + ac + 1 & \text{if both } a \text{ and } c \text{ are odd,} \\ B_a^* + B_c^* + ac & \text{else.} \end{cases} \end{aligned}$$

□

4. **Extremal absorbing sets in left-regular graphs.** Recall that [6] gives an exact value of b for a given a in *smallest* (a, b)-absorbing sets of left-regular Tanner graphs with fixed girth. That is, for smallest absorbing sets in such a class, $b_a^* = B_a^*$. However, for non-minimal a values, it is not necessarily the case that $b_a^* = B_a^*$. We extend to larger values of a by presenting results on B_a^* for classes of left-regular graphs. We also note a connection between smallest (a, b)-absorbing sets and graph structures known as *cages*.

We begin by presenting a general upper bound on B_a^*.

Theorem 4.1. *Consider the class of j-left regular Tanner graphs for $j \geq 2$. For any (a, b)-absorbing set in a graph in this class, $b \leq B_a^* \leq a \left\lfloor \frac{j-1}{2} \right\rfloor$. Moreover, if both a and $\left\lceil \frac{j+1}{2} \right\rceil$ are odd, $B_a^* \leq a \left\lfloor \frac{j-1}{2} \right\rfloor - 1$.*

Proof. Let D be a set of a variable nodes giving an (a, b)-absorbing set in G. Because G is j-left regular and D gives an absorbing set, each variable node has strictly more neighbors in $E(D)$ than $O(D)$. The largest possible number of odd degree check nodes neighboring a single variable node is $\left\lfloor \frac{j-1}{2} \right\rfloor$, which occurs when the variable node has $\left\lceil \frac{j+1}{2} \right\rceil$ even degree check node neighbors. Hence, $B_a^* \leq a \left\lfloor \frac{j-1}{2} \right\rfloor$.

If both $\left\lceil \frac{j+1}{2} \right\rceil$ and a are odd, we can improve the upper bound. Note that if a is odd and all variable nodes have $\left\lceil \frac{j+1}{2} \right\rceil$ even degree check node neighbors, then the even part of G_D has an odd number of vertices of odd degree, which is not possible. So at least one of the a variable nodes must have at least $\left\lceil \frac{j+1}{2} \right\rceil + 1$ even degree check node neighbors. Such a vertex would have at most $\left\lfloor \frac{j-1}{2} \right\rfloor - 1$ odd degree check node neighbors. Hence, in this case, $B_a^* \leq a \left\lfloor \frac{j-1}{2} \right\rfloor - 1$. □

There are values of j for which these bounds are tight, as shown in the following two corollaries to Theorem 4.1.

Corollary 3. *Consider the class of* 2*-left regular Tanner graphs with girth* g*. An absorbing set of size* a *exists in some graph in this class if and only if* $a \geq \frac{g}{2}$*. For* $a \geq \frac{g}{2}$, $B_a^* = 0$*.*

Proof. Let $a \geq \frac{g}{2}$, and observe that there exists a graph G in the considered class that contains a cycle of length $2a \geq g$. The variable nodes in such a cycle form an absorbing set, showing that an (a, b)-absorbing set can occur. Furthermore, each of the variable nodes in such a graph have two even degree neighbors in G_D, resulting in $b = 0$. By Theorem 4.1, $B_a^* = 0$.

Now assume $a < \frac{g}{2}$ and let D be a set of a variable nodes giving an (a, b)-absorbing set. By the definition of an absorbing set, every vertex in the even part of G_D has degree at least 2. So, the even part of G_D contains a cycle, a contradiction to the fact that $a < \frac{g}{2}$. □

Corollary 4. *Consider the class of* 3*-left regular Tanner graphs with girth* g*. An absorbing set of size* a *exists in some graph in this class if and only if* $a \geq \frac{g}{2}$*. For* $a \geq \frac{g}{2}$, $B_a^* = a$*.*

Proof. Let $a \geq \frac{g}{2}$. Consider the cycle graph on $2a$ vertices, where every other node is regarded as a variable or check node. Append a single degree one check node to each variable node in the cycle. We have formed a 3-left regular (a, a)-absorbing set graph of girth $2a \geq g$. By Theorem 4.1, $B_a^* = a$.

Assume now that $a < \frac{g}{2}$ and that D is an absorbing set of size a. Every vertex in the even part of G_D has degree at least 2, so G_D contains a cycle. Because $a < \frac{g}{2}$, no (a, b)-absorbing set can exist, because any cycle would have length less than g, a contradiction. □

If we restrict our attention to j-left regular graphs of girth 4, we see that the general upper bounds given in Theorem 4.1 are again met with equality.

Theorem 4.2. *Consider the class of* j*-left regular Tanner graphs of girth* $g = 4$*, where* $j \geq 2$*. Then for* $a \geq 2$*, the Tanner graphs in this class can contain absorbing sets of size* a*. Furthermore, for any* $a \geq 2$*,*

1. *if at least one of* $\left\lceil \frac{j+1}{2} \right\rceil$ *and* a *is even, then* $B_a^* = a \left\lfloor \frac{j-1}{2} \right\rfloor$*, and*
2. *if both* $\left\lceil \frac{j+1}{2} \right\rceil$ *and* a *are odd, then* $B_a^* = a \left\lfloor \frac{j-1}{2} \right\rfloor - 1$*.*

Proof. Let $j \geq 2$. First, we note that no absorbing set of size $a = 1$ can exist. Let D be a set of $a \geq 2$ variable nodes giving an absorbing set in a j-left regular graph of girth 4.

If $a = 2$, it is easy to see that $B_a^* = \left\lfloor \frac{j-1}{2} \right\rfloor$. We can construct an absorbing set graph by connecting two variable nodes with $\left\lceil \frac{j+1}{2} \right\rceil$ degree two check nodes and adding $\left\lfloor \frac{j-1}{2} \right\rfloor$ degree one check nodes to each of the variable nodes. Such a structure has girth 4 and is j-left regular. For the remainder of the proof, we assume that $a \geq 3$.

Suppose, first, that at least one of $\left\lceil \frac{j+1}{2} \right\rceil$ and a is even.

- If $\left\lceil \frac{j+1}{2} \right\rceil$ is even, construct a multigraph by connecting a vertices in a cycle, where each pair of adjacent vertices in the cycle is connected by $\left\lceil \frac{j+1}{2} \right\rceil/2$ edges.
- If a is even and $\left\lceil \frac{j+1}{2} \right\rceil$ is odd, construct a multigraph by connecting a vertices in a cycle, where adjacent pairs of vertices in the cycle are connected by $\left\lceil \frac{j+3}{2} \right\rceil/2$ or $\left\lceil \frac{j-1}{2} \right\rceil/2$ in an alternating pattern.

In both of the preceding multigraphs, each vertex has degree $\left\lceil \frac{j+1}{2} \right\rceil$. Create a bipartite graph from each multigraph by considering each of the a vertices as variable nodes and adding a degree 2 check node along each edge. These are $\left\lceil \frac{j+1}{2} \right\rceil$-left regular, 2-right regular bipartite graphs of girth 4. By adding $\left\lfloor \frac{j-1}{2} \right\rfloor$ degree 1 check nodes to each of the a left nodes in each case, we obtain $\left(a, a\left\lfloor \frac{j-1}{2} \right\rfloor\right)$-absorbing sets. By Theorem 4.1, we conclude that $B_a^* = a\left\lfloor \frac{j-1}{2} \right\rfloor$.

Now assume that both $\left\lceil \frac{j+1}{2} \right\rceil$ and a are odd. Construct a multigraph by connecting a vertices in a cycle, where adjacent pairs of vertices in the cycle are connected by $\left\lceil \frac{j+3}{2} \right\rceil/2$ or $\left\lceil \frac{j-1}{2} \right\rceil/2$ edges in an alternating pattern, starting with $\left\lceil \frac{j+3}{2} \right\rceil/2$, so that the first vertex in the cycle is incident to $\left\lceil \frac{j+3}{2} \right\rceil = \left\lceil \frac{j+1}{2} \right\rceil + 1$ edges. This multigraph has $a-1$ vertices of degree $\left\lceil \frac{j+1}{2} \right\rceil$ and one vertex of degree $\left\lceil \frac{j+1}{2} \right\rceil + 1$. Create a bipartite graph from this multigraph by considering each of the a vertices as variable nodes and adding a degree 2 check node along each edge. This bipartite graph has girth 4, is 2-right regular, and has the previously stated left degree distribution. We add enough degree 1 check nodes to each of the a variable nodes so that each of the variable nodes has degree j. In other words, we add $\left\lfloor \frac{j-1}{2} \right\rfloor$ degree 1 check nodes to $a-1$ variable nodes, and $\left\lfloor \frac{j-1}{2} \right\rfloor - 1$ degree 1 check nodes to one variable node. In this way, we obtain an $\left(a, a\left\lfloor \frac{j-1}{2} \right\rfloor - 1\right)$-absorbing set. By Theorem 4.1, $B_a^* = a\left\lfloor \frac{j-1}{2} \right\rfloor - 1$. ☐

4.1. **Relation to cages.** Next, we examine the relationship between absorbing sets and a class of regular graphs called *cages* [9, 11]. The connection between these structures leads to results on the existence of absorbing sets of certain parameters in j-left regular Tanner graphs of fixed girth. Indeed, recall that Theorem 2.3 presents a lower bound on absorbing set size in such a graph; the construction of absorbing sets from cages we present in the proof of Proposition 1 below explicitly shows that the lower bound is, in fact, tight for certain girths and values of j. We begin by defining cages.

Definition 4.3. A $(k, \tilde{g})$-*cage* is a k-regular graph G of girth $\tilde{g}$ of minimum order (i.e. minimum number of vertices). This minimum order is denoted $n(k, \tilde{g})$.

Table 3 gives known sizes of cages. Note that a cage need not be bipartite; however, we may view a $(k, \tilde{g})$-cage as the normal graph of an $(a = n(k, \tilde{g}), 0)$ elementary absorbing set in a Tanner graph of girth $g = 2\tilde{g}$. Adding between 0 and $k-1$ odd degree check nodes to each variable node maintains the absorbing set property. Note that adding only degree one check nodes ensures that the absorbing set retains girth $2\tilde{g}$ and remains elementary. If we fix the number of odd degree check nodes added to each variable node to be t, then the absorbing set belongs to a $(k+t)$-left regular Tanner graph. That is, there is a connection between a $(k, \tilde{g})$-cage of size $a = n(k, \tilde{g})$ ($k \geq 2$, $\tilde{g} \geq 3$) and smallest (a, b)-absorbing sets of girth $g = 2\tilde{g}$, where $0 \leq b \leq a(k-1)$. As we will show in Proposition 1, by adding $t = k - 1$ degree one check nodes to each variable node in the absorbing set, one obtains an extremal absorbing set in a $(2k - 1)$ left-regular graph, with $B^*_a = a(k - 1)$. A similar connection between cages and the more general trapping sets was explored in [4].

$\tilde{g}$	$n(2,\tilde{g})$	$n(3,\tilde{g})$	$n(4,\tilde{g})$	$n(5,\tilde{g})$	$n(6,\tilde{g})$	$n(7,\tilde{g})$
3	3	4	5	6	7	8
4	4	6	8	10	12	14
5	5	10	19	30	40	50
6	6	14	26	42	62	90
7	7	24	67	108*	189*	304*

Table 3. Known results on the sizes of cages of relatively small girth and degree [9]. Entries marked with * denote cases for which $n(k,\tilde{g})$ is not known; these values correspond instead to lower bounds from [8].

Proposition 1. *Consider the class of j-left regular Tanner graphs of girth $g \geq 6$. If $a = n\left(\lceil \frac{j+1}{2} \rceil, \frac{g}{2}\right)$, an absorbing set of size a exists in some graph in this class, and $B^*_a = \lfloor \frac{j-1}{2} \rfloor a$.*

Proof. First, let $a = n\left(\left\lceil \frac{j+1}{2} \right\rceil, \frac{g}{2}\right)$. Let G_a be a $\left(\left\lceil \frac{j+1}{2} \right\rceil, \frac{g}{2}\right)$-cage. Replace each edge in G_a with a check node of degree 2 and consider each vertex in G_a as a variable node. Then we have a $\left(\left\lceil \frac{j+1}{2} \right\rceil, 2\right)$-regular bipartite graph of girth g. We can add $\left\lfloor \frac{j-1}{2} \right\rfloor$ degree 1 check nodes to each variable node to create an $\left(a, \left\lfloor \frac{j-1}{2} \right\rfloor a\right)$-absorbing set that is j-left regular. By Theorem 4.1, we conclude that $B^*_a = \left\lfloor \frac{j-1}{2} \right\rfloor a$. □

For certain cases, the absorbing sets constructed as in the proposition proof meet the bounds given by Theorem 2.3 [6]. These cases are unshaded in Table 4.

Remark 5. If $g \geq 6$ and $j \geq 2$, then it is possible that absorbing sets of size $a > n\left(\left\lceil \frac{j+1}{2} \right\rceil, \frac{g}{2}\right)$ exist in the class described in Proposition 1. If such an absorbing set exists, then by Theorem 4.1, $B^*_a \leq \left\lfloor \frac{j-1}{2} \right\rfloor a$.

$g \setminus \left\lceil \frac{j+1}{2} \right\rceil$	2	3	4	5	6	7
6	3	4	5	6	7	8
8	4	6	8	10	12	14
10	5	10	17	26	37	50
12	6	14	26	42	62	86
14	7	22	53	106	189	302

Table 4. The lower bounds on a from Theorem 2.3 given girth g and variable node degree j. Unshaded values are those for which absorbing sets constructed in Proposition 1 achieve the lower bounds in Theorem 2.3, thereby demonstrating tightness.

4.2. **Extensions to regular graphs and fully absorbing sets.** We conclude this section by showing that the results that hold in the j-left regular case coincide with those for the (j, k)-regular case and the fully absorbing set case.

Proposition 2. *Consider the class of (j, k)-regular Tanner graphs of girth at least g, where $j \geq 2$ and $k \geq 2$. Then for a given a, B_a^* in this class is the same as in a general j-left regular graph. In other words, all the previous bounds on B_a^* given for j-left regular Tanner graphs still hold.*

Proof. Let G be a (j, k)-regular Tanner graph of girth g with $j \geq 2$ and $k \geq 2$, and fix a value a. Let β be the maximum value of b for this a in the class of j-left regular Tanner graphs of girth at least g, and let B_a^* denote the maximum value of b for this a in the class of (j, k)-regular Tanner graphs of girth at least g. Note that because the class of (j, k)-regular Tanner graphs of girth at least g is contained in the class of j-left regular Tanner graphs of girth at least g, we have that $B_a^* \leq \beta$. We will show that $B_a^* = \beta$ by exhibiting an (a, β)-absorbing set absorbing set in a (j, k)-regular Tanner graph.

Let D be an (a, β)-absorbing set in a j-left regular Tanner graph. Because β is maximal, D is an elementary absorbing set. Hence all nodes in $E(D)$ have degree 2 and all nodes in $O(D)$ have degree 1. Consider only the absorbing set graph G_D, which is necessarily itself j-left regular with girth at least g.

We will show that G_D embeds in a (j, k)-regular Tanner graph G'. To do so, we will start with G_D and enumerate a tree-like structure as follows. The elements of D are in Layer 0, and the elements of $E(D) \cup O(D)$ are in Layer 1. Since Layers 0 and 1 replicate G_D, the girth among nodes in those layers is at least g. Viewing each check node in Layer 1 as a root of a tree, we can enumerate each tree for $g/2$ more layers by creating new nodes of appropriate degrees as needed. Specifically, for each check node c in $E(D)$ (resp., $O(D)$), add $k - 2$ (resp., $k - 1$) variable node children in Layer 2. For each variable node in Layer 2, add $j - 1$ new check node children in Layer 3. Each of these now receive $k - 1$ variable node children in Layer 4, etc. Continue in this way so that the nodes in Layers 0 to $g/2$ are distinct. In Layer $g/2 + 1$, when the girth of g may occur, some nodes from branches stemming from different neighbors of D may coincide. Regardless, this structure may be completed into a (j, k)-regular Tanner graph containing G_D. □

We now consider the case of fully absorbing sets.

Proposition 3. *Consider the class of j-left regular Tanner graphs of girth at least g, where $j \geq 3$. For a given a, let B_a^* be the maximum b value for a graph in this class. Then there exists an (a, B_a^*) fully absorbing set. In other words, the previous bounds on B_a^* for j-left regular Tanner graphs also hold for fully absorbing sets.*

Proof. Let G be a j-left regular Tanner graph of girth g with $j \geq 3$. Let B_a^* be the maximum value of b for a in the class of j-left regular Tanner graphs of girth at least g such that an (a, b)-absorbing set exists in some graph in the class. We will exhibit an (a, B_a^*) fully absorbing set in the class of j-left regular Tanner graphs of girth at least g with $j \geq 3$.

Let D be an (a, B_a^*)-absorbing set in a j-left regular Tanner graph. Because B_a^* is maximal, D is an elementary absorbing set. Consider only the absorbing set graph G_D. If we view G_D as its own Tanner graph, then vacuously, D is a fully absorbing set in G_D. Because G_D is a j-left regular graph of girth at least g, we have shown that there exists an (a, B_a^*) fully absorbing set in this class. □

5. **Minimal absorbing sets.** In this section, we introduce the notion of minimal absorbing sets, which are absorbing sets that do not properly contain smaller absorbing sets. Thus, any absorbing set will contain at least one minimal absorbing set. By understanding minimal absorbing sets, search algorithms and absorbing set removal algorithms may be designed to target these fundamental structures. Namely, if minimal absorbing sets are removed, then any absorbing set containing that minimal absorbing set is also removed.

Definition 5.1. An (a, b)-absorbing set D is *minimal* if no proper subset of D forms an absorbing set.

Our first result in this section identifies a class of absorbing sets that is always minimal.

Theorem 5.2. *Every (a, B_a^*)-absorbing set in the class of graphs of girth at least g is minimal.*

Proof. Let D be an (a, B_a^*)-absorbing set of girth at least g. Let D' be a strict subset of D. Notice that D is an elementary absorbing set by Lemma 3.1.

Suppose D' also forms an absorbing set. We claim it must then be the case that $G_D = G_{D'} \oplus G_{D \setminus D'}$, where $\oplus$ denotes a disjoint union of graphs. In other words, $G_{D'}$ and $G_{D \setminus D'}$ have no variable nor check nodes in common. To prove this claim, assume not: then, there is a degree 2 check node in G_D with one neighbor in D' (denote this variable node by v) and the other in $D \setminus D'$. The number of even-degree neighbors of v cannot increase from G_D to $G_{D'}$ by virtue of D being an elementary absorbing set; thus, the number of odd-degree neighbors of v increases by at least one from G_D to $G_{D'}$. Since D' is an absorbing set, it must be the case that v can tolerate an additional degree 1 check node neighbor in G_D while maintaining D as an absorbing set. This contradicts the fact that D is an (a, B_a^*)-absorbing set, and we conclude that $G_D = G_{D'} \oplus G_{D \setminus D'}$.

Since D and D' are absorbing sets, $D \setminus D'$ must also form an absorbing set. However, by Lemma 3.6, $B_{c+d}^* > B_c^* + B_d^*$. Thus, D could not have formed an (a, B_a^*)-absorbing set to begin with, since the number of odd degree check nodes in G_D is at most the sum of the numbers of odd degree check nodes in G'_D and $G_{D \setminus D'}$, which is bounded above by $B_{|D'|}^* + B_{|D|-|D'|}^*$. $\square$

Thus, for the class of graphs of girth at least g, absorbing sets that are extremal in the number of odd degree check nodes in the corresponding absorbing set graph provide one classification type of minimal absorbing sets.

In general, minimal absorbing sets are not always elementary, and elementary absorbing sets are not always minimal. However, in the following proposition, we note a specific case when all minimal absorbing sets are elementary.

Proposition 4. *If a Tanner graph G has variable nodes of degree at most 3, then all minimal absorbing sets in G are elementary.*

Proof. Let G be a Tanner graph with all variable nodes of degree at most 3. Let D be a minimal absorbing set in G and G_D be its induced subgraph. Assume by way of contradiction that D is not elementary. Then there exists some check node $c \in G_D$ such that $d_{G_D}(c) \geq 3$.

(i) Case 1: there exist variable nodes $v_1, v_2 \in \mathcal{N}_{G_D}(c)$ such that v_1 and v_2 are connected in $G_D \setminus \{c\}$. Consider the set $\mathcal{N}_{G_D}(c) = \{v_i\}_{i=1}^{d_{G_D}(c)}$. Let P be the shortest path in $G_D \setminus \{c\}$ between any two variable nodes v_i and v_j in $\mathcal{N}_{G_D}(c)$. We will assume without loss of generality that these are v_1 and v_2. Let Q be the set of variable nodes in P. Note that G_D contains all check node neighbors of variable nodes in Q, and so may not be equal to the path P.

We claim that $G_Q \setminus \{c\}$ has no odd degree check nodes of degree more than 1. Assume by way of contradiction there exists a check node $t \in G_Q$ with $d_{G_Q}(t) \geq 3$. Let $w_1, w_2, w_3 \in \mathcal{N}_{G_Q}(t)$. Then w_1, w_2, and w_3 all occur along the shortest path between v_1 and v_2, and we can assume they occur in this order, i.e., that $P = v_1 \cdots w_1 \cdots w_2 \cdots w_3 \cdots v_2$. We can remove the portion of the path between w_1 and w_3 by instead visiting t to obtain the path $v_1 \cdots w_1\, t\, w_3 \cdots v_2$, which is a path from v_1 to v_2 that is shorter than P, a contradiction. Note that it is possible that $v_1 = w_1$ and/or $v_2 = w_3$, but this does not affect the contradiction. Hence all odd degree check nodes in $G_Q \setminus \{c\}$ have degree 1.

We will show that $d_{G_Q}(c) = 2$. Assume by way of contradiction that $d_{G_Q}(c) \geq 3$. Then there is some variable node $s \in \mathcal{N}_{G_Q}(c)$ such that $P = v_1\, R_1 s\, R_2\, v_2$, where R_1 and R_2 are paths. But then, both $v_1\, R_1 s$ and $s\, R_2\, v_2$ are shorter paths than P between two variable nodes in $\mathcal{N}_{G_Q}(c)$, contradicting our choice of P. Hence $d_{G_Q}(c) = 2$.

Because all odd degree check nodes in $G_Q \setminus \{c\}$ have degree 1, all check nodes along the path P must be even degree. Hence each variable node $v \in G_Q \setminus \{c\}$ where $v \notin \mathcal{N}_{G_Q}(c)$ has two even degree neighbors because it is part of a path. If $v \notin \mathcal{N}_{G_Q}(c)$ and $d_G(v) = 2$, then both of its neighbors are even, and if $d_G(v) = 3$, at least two of its neighbors are even. It remains only to consider when $v \in \mathcal{N}_{G_Q}(c)$. In this case, v has one even degree neighbor in G_D from the path P, and c is a second even degree neighbor in G_Q. The degree of its third check node neighbor does not matter, as its neighborhood maintains a strict majority of even degree check node neighbors regardless. Hence all variable nodes in G_Q have at strictly more even degree check node neighbors than odd degree check node neighbors. So Q is an absorbing set. Because $d_{G_Q}(c) = 2 < d_{G_D}(c)$, Q necessarily contains fewer variable nodes than D. Hence Q is an absorbing set strictly contained in D, so D is not minimal.

(ii) Case 2: no path exists in $G_D \setminus \{c\}$ between any two elements of $\mathcal{N}_{G_D}(c)$. Let S be the set of variable nodes in the connected component containing some $v_1 \in \mathcal{N}_{G_D}(c)$ in $G_D \setminus \{c\}$ and T be the set of variable nodes in the connected component containing some $v_2 \in \mathcal{N}_{G_D}(c)$ in $G_D \setminus \{c\}$. Consider the set $S \cup T$. Because G_S and G_T are disconnected in $G_D \setminus \{c\}$, the neighbors of all variable nodes in both G_S and G_T are the same as in G_D, except potentially v_1 and v_2. The same holds for $G_{S\cup T}$. Further, $d_{G_{S\cup T}}(c) = 2$, as it is connecting only v_1 and v_2. Hence both v_1 and v_2 have at least as many even degree check node neighbors as they had in G_D, and so still have a strict majority of even degree check node neighbors. Hence $G_{S\cup T}$ is an absorbing set graph. Because $d_{G_D}(c) \geq 3$, there exists some $v_3 \in \mathcal{N}_{G_D}(c) \subseteq D$ which is not in $S \cup T$. Hence $S \cup T$ is an absorbing set strictly contained in D, so D is not minimal.

In both cases, we reach a contradiction. Thus, D must be elementary. □

The variable node degree requirements given in Proposition 4 cannot be loosened without further restrictions. We illustrate this next.

Example 5.3.

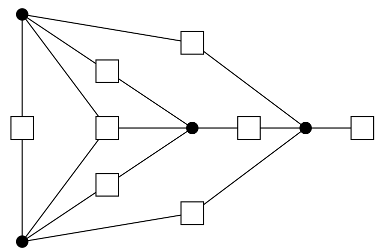
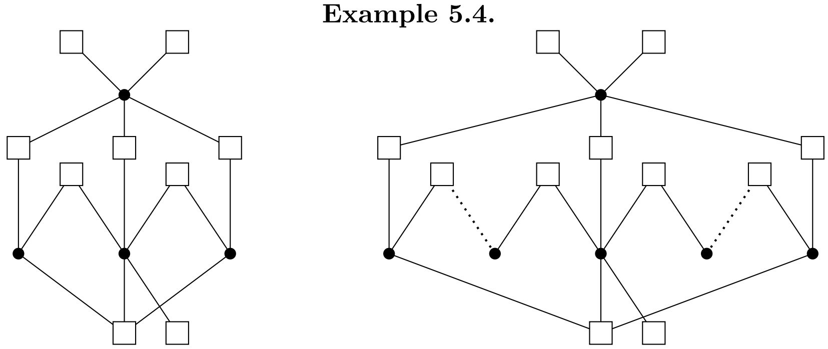


FIGURE 3. A minimal $(4, 2)$-absorbing set with degree 4 variable nodes that is not elementary.

Example 5.4.



(a) (b)

FIGURE 4. (a) A minimal $(4, 4)$-absorbing set of girth 4 that is not elementary. (b) As shown, a minimal $(6, 4)$-absorbing set of girth 6 that is not elementary. The dotted edges represent where paths could be extended to generalize to an absorbing set of arbitrary girth.

In Figure 3, a non-elementary $(4, 2)$-absorbing set with degree 4 variable nodes is shown. We can see that this absorbing set is minimal by noticing that any proper subset of its variable nodes induces a graph that is not an absorbing set graph.

Moreover, we note that minimal absorbing sets in graphs of arbitrary girth need not be elementary via the construction in the following example.

In Figure 4, two minimal absorbing sets that are not elementary are shown. Part (a) shows a minimal absorbing set of girth 4, and (b) a minimal absorbing set of girth 6. In part (b), two edges are dotted. These edges could be augmented by alternating variable and check nodes to create an absorbing set of arbitrarily large girth that would still not be minimal.

6. **Conclusions.** We derived ranges for the parameters of absorbing sets that may exist in general classes of LDPC codes. In particular, we gave exact and upper bounds on the largest b for a given a for which an (a, b)-absorbing set may occur. We show that these extremal absorbing sets are elementary. Moreover, we show that this extremal class is also minimal, a notion we introduce as a means to break down and remove fundamental absorbing sets.

Acknowledgments. We thank the anonymous reviewers for their helpful comments that improved the quality of the paper, including a nicer proof of Theorem 3.2.

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Received June 2020; revised November 2020.

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