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Abstract

Recently Chase determined the maximum possible number of cliques of size t in a graph on n vertices with given maximum degree. Soon afterward, Chakraborti and Chen answered the version of this question in which we ask that the graph have m edges and fixed maximum degree (without imposing any constraint on the number of vertices). In this paper we address these problems on hypergraphs. For s-graphs with $s \geq 3$ a number of issues arise that do not appear in the graph case. For instance, for general s-graphs we can assign degrees to any i-subset of the vertex set with $1 \leq i \leq s-1$.

We establish bounds on the number of t-cliques in an s-graph $\mathcal{H}$ with i-degree bounded by Δ in three contexts: $\mathcal{H}$ has n vertices; $\mathcal{H}$ has m (hyper)edges; and (generalizing the previous case) $\mathcal{H}$ has a fixed number p of u-cliques for some u with $s \leq u \leq t$. When Δ is of a special form we characterize the extremal s-graphs and prove that the bounds are tight. These extremal examples are the shadows of either Steiner systems or packings. On the way to proving our uniqueness results, we extend results of Füredi and Griggs on uniqueness in Kruskal-Katona from the shadow case to the clique case.

1 Introduction

There has been recent interest in generalized Turán problems: determining the maximum (or minimum) number of copies of a fixed graph T that a graph G can contain, subject to a variety of constraints. The roots of this problem go back to Turán's Theorem [22] and its extension by Zykov [24] which determine, respectively, the maximum number of copies of K_2 and K_t in a graph on n vertices containing no K_{r+1}. The paper of Alon and Shikhelman [1] proved many foundational results and introduced the general problem to a wider audience.

1.1 Many cliques in bounded-degree graphs

We will focus on hypergraph versions of two generalized Turán problems: determining the maximum number of cliques in graphs of bounded degree, using either vertices or edges as a "resource." We discuss the graph problems below; for a more complete history see [2, 4, 5, 6, 11, 16, 17]. The first phase of progress in these problems consisted of "signpost" results: estimates that are best possible infinitely often, but not for all values of the parameters.

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We write $k(G)$ for the total number of cliques in G and $k^t(G)$ for the number of cliques of size t (and always insist that $t \geq 1$). Similarly $k^{\geq t}(G)$ is the number of cliques of size at least t in G. The next two theorems are due to Wood.

Theorem 1 (Wood [23])**.** *If G is a graph on n vertices with $\Delta(G) \leq r-1$ then*

$$k^t(G) \leq \frac{n}{r}\binom{r}{t} \quad \text{and} \quad k^{\geq 1}(G) \leq \frac{n}{r}\left(2^r - 1\right)$$

with equality when $G = aK_r$.

Theorem 2 (Wood [23])**.** *If G is a graph on n vertices having m edges with $\Delta(G) \leq r-1$ then*

$$k^t(G) \leq \frac{m}{\binom{r}{2}}\binom{r}{t} \quad \text{and} \quad k^{\geq 2}(G) \leq \frac{m}{\binom{r}{2}}\left(2^r - r - 1\right),$$

with equality when $G = aK_r$.

Quite recently results in this direction were proved that are best possible for all values of the parameters. The vertex problem was solved by Chase [4]. Building on Chase's theorem, Chakraborti and Chen [2] solved the edge problem.

Theorem 3 (Chase [4])**.** *Let G be a graph with $\Delta(G) \leq r-1$ on n vertices. Let a and b satisfy $n = ar + b$ with $0 \leq b < r$. Then*

$$k^t(G) \leq a\binom{r}{t} + \binom{b}{t},$$

with equality for the graph $G = aK_r \cup K_b$, the disjoint union of a copies of K_r and one copy of K_b.

In 2022 Chao and Dong [3] announced a new proof of Theorem 3 that proves the result for all t simultaneously, unlike Chase's proof.

Theorem 4 (Chakraborti and Chen [2])**.** *Let G be a graph with $\Delta(G) \leq r-1$ having m edges. Let a and b satisfy $m = a\binom{r}{2} + b$ with $0 \leq b < \binom{r}{2}$. Then*

$$k^t(G) \leq a\binom{r}{t} + k^t(\mathcal{C}_2(b)),$$

with equality for the graph $G = aK_r \cup \mathcal{C}_2(b)$. Here, $\mathcal{C}_2(b)$ is the colex graph having b edges: the graph on vertex set $\mathbb{N}$ whose edges are the first b pairs in colexicographic order.

In this paper we are concerned with hypergraph versions of these problems. To state the questions we need to introduce our notation for hypergraphs and discuss the issue of degrees in hypergraphs. This we do next.

In Section 2 we discuss various versions of the Kruskal-Katona Theorem, which is central in this area. In Section 3 we introduce constructions which, in some cases, give optimal examples, and in Section 4 we prove general results for arbitrary degree bounds and prove the existence of some optimal and near-optimal examples. Finally, in Section 5 we mention some open problems.

1.2 Hypergraph definitions and questions

Our notation is mostly standard.

Definition 5. An *s-graph* $\mathcal{H}$ is a pair $(V, \mathcal{E})$ consisting of a set of vertices V together with a subset $\mathcal{E} \subseteq \binom{V}{s}$. Frequently we'll suppress mention of the vertex set and simply use $\mathcal{H}$ to refer to the edge set. If $I \subseteq V$ has size i then we define the *neighborhood* $\mathcal{H}(I)$ of I to be the $(s-i)$-graph with edge set

$$\mathcal{E}\big(\mathcal{H}(I)\big) = \{E \setminus I \,:\, I \subseteq E \in \mathcal{E}(\mathcal{H})\}.$$

The *degree of I in $\mathcal{H}$* is the number of these edges, i.e.,

$$d_{\mathcal{H}}(I) = \big|\{E \in \mathcal{E}(\mathcal{H}) \,:\, I \subseteq E\}\big|.$$

We let the vertex set of $\mathcal{H}(I)$ be the union of all the edges in $\mathcal{E}(\mathcal{H}(I))$, i.e., we omit all vertices not contained in an edge of $\mathcal{H}(I)$. The *maximum i-degree* of $\mathcal{H}$ is simply

$$\Delta_i(\mathcal{H}) = \max\Big\{d_{\mathcal{H}}(I) \,:\, I \in \binom{V}{i}\Big\}.$$

We now define shadows and cliques in hypergraphs.

Definition 6. Suppose that $\mathcal{A}$ is an s-graph. The *shadow of $\mathcal{A}$ on level q* (where $q < s$) is given by

$$\partial_q(\mathcal{A}) = \Big\{B \,:\, |B| = q \text{ and } \exists\, A \in \mathcal{A} \text{ s.t. } B \subseteq A\Big\} = \bigcup_{A \in \mathcal{A}} \binom{A}{q}.$$

The *set of cliques on level t* (where $t > s$) is

$$K^t(\mathcal{A}) = \Big\{C \,:\, |C| = t \text{ and } \binom{C}{s} \subseteq \mathcal{A}\Big\}.$$

Since we're interested in the number of cliques, we let $k^t(\mathcal{A}) = \big|K^t(\mathcal{A})\big|$.

We can now state the questions we address in this paper.

Question 1. Suppose that an s-graph $\mathcal{H}$ has n vertices, and that for some $1 \le i \le s-1$ and $D > 0$ we have $\Delta_i(\mathcal{H}) \le D$. Given $t \ge s$, what is the maximum possible value of $k^t(\mathcal{H})$? In other words we aim to determine

$$\max\{k^t(\mathcal{H}) \,:\, \mathcal{H} \text{ an } s\text{-graph with } n \text{ vertices and } \Delta_i(\mathcal{H}) \le D\}.$$

Question 2. Suppose that an s-graph $\mathcal{H}$ has m edges, and that for some $1 \le i \le s-1$ and $D > 0$ we have $\Delta_i(\mathcal{H}) \le D$. Given $t \ge s$, what is the maximum possible value of $k^t(\mathcal{H})$? In other words, what is

$$\max\{k^t(\mathcal{H}) \,:\, \mathcal{H} \text{ an } s\text{-graph with } m \text{ edges and } \Delta_i(\mathcal{H}) \le D\}?$$

Question 3. Suppose that an s-graph $\mathcal{H}$ has $k^u(\mathcal{H}) = p$ for some $u \ge s$, and that for some $1 \le i \le s-1$ and $D > 0$ we have $\Delta_i(\mathcal{H}) \le D$. Given $t \ge u$, what is the maximum possible value of $k^t(\mathcal{H})$? I.e., determine

$$\max\{k^t(\mathcal{H}) \,:\, \mathcal{H} \text{ an } s\text{-graph with } k^u(\mathcal{H}) = p \text{ and } \Delta_i(\mathcal{H}) \le D\}.$$

1.3 Related extremal problems

The area of extremal problems for hypergraphs is rich and deep. Among the most directly related problems are those involving the maximum number of cliques in an s-graph that contains no $(r+1)$-clique. The earliest such result is by Zykov [24]. He proved the following result for graphs.

Theorem 7 (Zykov)**.** *If $\mathcal{H}$ is a graph on n vertices containing no $(r+1)$-clique then $k^t(\mathcal{H}) \leq k^t(T_r(n))$. Here $T_r(n)$ is the* Turán graph*, that is to say it is the complete r-partite graph on n vertices whose parts are of sizes as equal as possible.*

The analogous result where we constrain G to have m edges is much more recent. The following result is due to Frohmader [8]. To describe the result we need to define the r-partite colex Turán graph. Let r be a positive integer. The *r-partite colex order* is the restriction of the colex order on $\binom{\mathbb{N}}{2}$ to $\{ij : i \not\equiv j \pmod r\}$. The *$r$-partite colex Turán graph with m edges*, $\mathrm{CT}_r(m)$, is the graph on vertex set $\mathbb{N}$ whose edge set consists of the first m edges in r-partite colex order. (Note that if $m = t_r(n)$, then the unique non-trivial component of $\mathrm{CT}_r(m)$ is isomorphic to $T_r(n)$.)

Theorem 8. *If G is a K_{r+1}-free graph with m edges and $2 \leq t \leq r$, then $k^t(G) \leq k^t\big(\mathrm{CT}_r(m)\big)$.*

In stark contrast, even the Turán problem for s-graphs with $s > 2$ is apparently intractable. For no $r > s \geq 3$ is the problem of determining

$$\max\{|\mathcal{H}| : \mathcal{H} \text{ is an } s\text{-graph on vertex set } [n] \text{ not containing an } (r+1)\text{-clique}\}$$

solved for all n, even asymptotically. (See Keevash's survey [14] for extensive discussion of this problem.) The hypergraph analogue of Theorem 8 seems no easier.

In a recent paper, Liu and Wang [19] determined the maximum number of t-cliques in an s-graph on n vertices containing at most k disjoint edges (for n sufficiently large).

In the context of hypergraphs with bounded degree, Jung [12] considered the question of minimizing the ratio $|\partial_{s-1}(\mathcal{H})|/|\mathcal{H}|$ for s-graphs $\mathcal{H}$ having bounded 1-degree. Jung's results have a similar spirit to ours, but are not directly comparable. In an opposite direction Füredi and Zhao [10] considered 3-graphs $\mathcal{H}$ with large minimum degree and gave asymptotically best possible lower bounds on the size of $\partial_2(\mathcal{H})$.

2 The Kruskal-Katona Theorem

The fundamental theorem given in Theorem 12 below was proved independently by Kruskal [18] and Katona [13]. It shows that for a given number of edges the s-graph with the most t-cliques and the smallest q-shadow is the *colex hypergraph*, whose edges form an initial segment in the *colexicographic* (or *colex*) *order*. Colex order is defined on finite subsets of $\mathbb{N}$ by $A < B$ iff $\max(A \triangle B) \in B$. The original version of the Kruskal-Katona theorem discussed only shadows, but the cliques version is a simple modification. We give the proof of the cliques bound for completeness. To do so it is useful to define another ordering.

Definition 9. The *retrolexicographic* (or *retlex*) order on finite subsets of $\mathbb{N}$ is defined by $A <_R B$ iff $\max(A \triangle B) \in A$. We write $\mathcal{R}_s(n,m)$ for the $<_R$-initial segment of size m in $\binom{[n]}{s}$.

In addition, given a ground set $[n]$ and an s-graph $\mathcal{A}$ on $[n]$, we define

$$\overline{\mathcal{A}} = \{[n] \setminus A : A \in \mathcal{A}\},$$

an $(n-s)$-graph on $[n]$ with the same size as $\mathcal{A}$.

Remark 10. The definition has the following symmetries with the colex order.

a) We have $A <_R B$ if and only if $A > B$, i.e., retlex is the reverse of colex order.

b) If both A and B are subsets of $[n]$ then, since $A \triangle B = ([n] \setminus A) \triangle ([n] \setminus B)$, we have $[n] \setminus A <_R [n] \setminus B$ if and only if $A < B$.

In particular for $0 \leq m \leq \binom{n}{s}$ we have

$$\mathcal{R}_s\Big(n, \binom{n}{s} - m\Big) = \binom{[n]}{s} \Big\backslash \, \mathcal{C}_s(m) \qquad \text{and} \qquad \overline{\mathcal{R}_s(n,m)} = \mathcal{C}_{n-s}(m).$$

Definition 11. For $\mathcal{A} \subseteq \binom{[n]}{s}$ an s-graph and $t > s$ we define the *upshadow* of $\mathcal{A}$ on level t by

$$U^t(\mathcal{A}) = \Big\{ T \in \binom{[n]}{t} \; : \; \exists\, A \in \mathcal{A} \text{ s.t. } A \subseteq T \Big\}.$$

Theorem 12 ([13, 18])**.** *For all $0 \leq q < s < t \leq n$, if $\mathcal{A}$ is an s-graph on vertex set V with $|V| = n$ then we have*

$$|\partial_q(\mathcal{A})| \geq |\partial_q(\mathcal{C}_s(m))|, \qquad k^t(\mathcal{A}) \leq k^t(\mathcal{C}_s(m)), \qquad \textit{and} \qquad |U^t(\mathcal{A})| \geq |U^t(\mathcal{R}_s(n,m))|$$

where $m = |\mathcal{A}|$.

Note that for cliques and shadows the optimal examples are independent of n (provided $m \leq \binom{n}{s}$), whereas the retlex initial segment depends in an essential way on n. Since the shadow bound is proven in [13, 18], we will prove only the clique and upshadow bounds.

Proof. We may assume without loss of generality that $V = [n]$. We'll start by proving the upshadow bound from the shadow bound. Given $\mathcal{E} \subseteq \binom{[n]}{s}$ and writing $\overline{\mathcal{E}} = \{[n] \setminus E \, : \, E \in \mathcal{E}\} \subseteq \binom{[n]}{n-s}$, we have

$$\begin{aligned}
\partial_{n-t}(\overline{\mathcal{E}}) &= \big\{ [n] \setminus T \, : \, |T| = t \text{ and } \exists\, ([n] \setminus E) \in \overline{\mathcal{E}} \text{ s.t. } ([n] \setminus T) \subseteq ([n] \setminus E) \big\} \\
&= \Big\{ [n] \setminus T \, : \, T \in \binom{[n]}{t} \text{ and } \exists\, E \in \mathcal{E} \text{ s.t. } E \subseteq T \Big\} = \overline{U^t(\mathcal{E})}.
\end{aligned}$$

Thus, by the shadow bound, to minimize $|U^t(\mathcal{E})| = |\overline{U^t(\mathcal{E})}|$ we can take $\overline{\mathcal{E}}$ to be a colex initial segment, i.e, by Remark 10 a), $\mathcal{E}$ to be a retlex initial segment. Now, for the clique bound, note that

$$K^t(\mathcal{A}) = \binom{[n]}{t} \Big\backslash \, U^t\Big(\binom{[n]}{s} \Big\backslash \, \mathcal{A}\Big).$$

Thus to maximize $|K^t(\mathcal{A})|$ we can take $\binom{[n]}{s} \setminus \mathcal{A}$ to be a retlex initial segment, i.e., by Remark 10 b), take $\mathcal{A}$ to be a colex initial segment. ∎

Remark 13. Using Remark 10 we can immediately read out of the proof of the previous theorem the value of $k^t(\mathcal{C}_s(m))$. We have

$$\begin{aligned}
K^t(\mathcal{C}_s(m)) &= \binom{[n]}{t} \Big\backslash \, U^t\Big(\binom{[n]}{s} \Big\backslash \, \mathcal{C}_s(m)\Big) \\
&= \binom{[n]}{t} \Big\backslash \, U^t\Big(\mathcal{R}_s\big(n, \tbinom{n}{s} - m\big)\Big) \\
&= \binom{[n]}{t} \Big\backslash \, \overline{\partial_{n-t}\Big(\overline{\mathcal{R}_s\big(n, \tbinom{n}{s} - m\big)}\Big)} \\
&= \binom{[n]}{t} \Big\backslash \, \overline{\partial_{n-t}\Big(\mathcal{C}_{n-s}\big(\tbinom{n}{s} - m\big)\Big)}
\end{aligned}$$

i.e.,

$$k_s^t(m) = k^t(\mathcal{C}_s(m)) = \binom{n}{t} - \partial_{n-t}^{n-s}\left(\tbinom{n}{s} - m\right).$$

2.1 Cascade notation

The standard way of describing initial segments of the colex order is the *cascade* notation, introduced by Kruskal in [18]. A good reference for the material in this subsection is Chapter 6 of the book [7] by Frankl and Tokushige.

Definition 14. We will say that an integer sequence $(n_s, n_{s-1}, \ldots, n_{s-\ell+1})$ is a *cascade* if it is strictly decreasing. We will define, for $s \geq 1$ and arbitrary cascades $(n_s, n_{s-1}, \ldots, n_{s-\ell+1})$ of length $\ell \geq 0$,

$$[n_s, n_{s-1}, \ldots, n_{s-\ell+1}]_s = \sum_{k=0}^{\ell-1} \binom{n_{s-k}}{s-k}.$$

We say that a cascade is a *strict s-cascade* if $n_{s-k} \geq s-k$ for all $0 \leq k \leq \ell-1$, and also $\ell \leq s$. In that case every term in (the sum defining) $[n_s, n_{s-1}, \ldots, n_{s-\ell+1}]_s$ is positive.

Remark 15. In checking that a cascade $(n_s, n_{s-1}, \ldots, n_{s-\ell+1})$ is strict it is sufficient to check that $n_{s-k} \geq s-k$ for $k = \ell-1$, because if so then for every $k < \ell-1$ we have

$$n_{s-k} \geq n_{s-\ell+1} + (\ell-1-k) \geq s-\ell+1+(\ell-1-k) = s-k.$$

Definition 16. If $\mathcal{B}$ is a family of sets, each disjoint from a fixed set A, we write $A+\mathcal{B}$ for the family

$$A + \mathcal{B} = \{A \cup B \,:\, B \in \mathcal{B}\}.$$

Lemma 17. *For all $m \geq 0$ and all $s \geq 1$ there exists a unique strict s-cascade such that $m = [n_s, n_{s-1}, \ldots, n_{s-\ell+1}]_s$. Indeed $(n_s, n_{s-1}, \ldots, n_{s-\ell+1})$ is the unique strictly decreasing sequence of length $\ell \geq 0$ satisfying*

$$\begin{gathered}
\binom{n_s}{s} < m < \binom{n_s+1}{s} \\
\binom{n_s}{s} + \binom{n_{s-1}}{s-1} < m < \binom{n_s}{s} + \binom{n_{s-1}+1}{s-1} \\
\vdots \\
\binom{n_s}{s} + \binom{n_{s-1}}{s-1} + \cdots + \binom{n_{s-\ell+2}}{s-\ell+2} < m < \binom{n_s}{s} + \binom{n_{s-1}}{s-1} + \cdots + \binom{n_{s-\ell+2}+1}{s-\ell+2} \\
\binom{n_s}{s} + \binom{n_{s-1}}{s-1} + \cdots + \binom{n_{s-\ell+1}}{s-\ell+1} = m.
\end{gathered}$$

If $(n_s, n_{s-1}, \ldots, n_{s-\ell+1})$ has length 1 then the first of these inequalities is satisfied with equality on the left. If $m = 0$ then we get the unique sequence of length 0 for all $s \geq 1$. Moreover, for all $m \geq 0$ and $s \geq 1$ the colex initial segment of $\binom{\mathbb{N}}{s}$ of length m is

$$\mathcal{C}_s(m) = \bigcup_{k=0}^{\ell-1}\left(\{n_{s-j}+1 \,:\, 0 \leq j < k\} + \binom{[n_{s-k}]}{s-k}\right)$$

where $(n_s, n_{s-1}, \ldots, n_{s-\ell+1})$ is the unique s-cascade such that $m = [n_s, n_{s-1}, \ldots, n_{s-\ell+1}]_s$.

Definition 18. For all $m \geq 0$ and all $s \geq 1$, we denote by $i_s(m)$ the unique s-cascade such that $m = [n_s, n_{s-1}, \ldots, n_{s-\ell+1}]_s$, guaranteed by Lemma 17.

Using cascade notation, we can exhibit lovely expressions for the number of cliques and the size of the shadow of a colex initial segment.

Lemma 19. *If* $(n_s, n_{s-1}, \ldots, n_{s-\ell+1})$ *is a strict* s*-cascade and* $m = [n_s, n_{s-1}, \ldots, n_{s-\ell+1}]_s$ *then*

$$k^t(\mathcal{C}_s(m)) = m'' = [n_s, n_{s-1}, \ldots, n_{s-\ell+1}]_t, \text{ and}$$

$$|\partial_q(\mathcal{C}_s(m))| = m' = [n_s, n_{s-1}, \ldots, n_{s-\ell+1}]_q.$$

Proof. Straightforward. See [7] for a proof of the shadow case when $q = s-1$. The general shadow result follows by induction, and the proof for cliques is similar. Note that neither the t-cascade nor the q-cascade need be strict. □

Definition 20. We denote the function that maps m to m'' by $k^t_s(m)$ and that sending m to m' by $\partial^s_q(m)$:

$$k^t_s(m) = k^t(\mathcal{C}_s(m)) \quad \text{and} \quad \partial^s_q(m) = |\partial_q(\mathcal{C}_s(m))|.$$

Corollary 21. *If* $\mathcal{H}$ *is an* s*-graph of size* m *and* $q < s < t$ *then*

$$k^t(\mathcal{H}) \leq k^t_s(m) \qquad \text{and} \qquad |\partial_q(\mathcal{H})| \geq \partial^s_q(m).$$

Proof. Immediate from Theorem 12. □

2.2 Uniqueness in Kruskal-Katona

It will be useful for us later to know when it is the case that $\mathcal{C}_s(m)$ is the unique extremal example for Corollary 21. We introduce two definitions from [9] by Füredi and Griggs.

Definition 22. Given $1 \leq q < s \leq n$ we say that m is a *jumping number* (or (s,q)-jumping number if we want to be more explicit) if $\partial^s_q(m+1) > \partial^s_q(m)$. We say that m is a *colex-unique number* if all s-graphs with m edges satisfying $|\partial_q(\mathcal{H})| = \partial^s_q(m)$ are isomorphic to $\mathcal{C}_s(m)$.

The following two theorems are proved in [9].

Theorem 23 (Füredi and Griggs [9])**.** *Suppose that* $1 \leq q < s \leq n$ *and that* $0 \leq m \leq \binom{n}{s}$ *is represented by the strict* s*-cascade* $m = [n_s, n_{s-1}, \ldots, n_{s-\ell+1}]_s$*. Then* m *is an* (s,q)*-jumping number if and only if* $\ell \leq q$*.*

Theorem 24 (Füredi and Griggs [9])**.** *Suppose that* $1 \leq q < s \leq n$ *and that* $0 \leq m \leq \binom{n}{s}$ *is represented by the strict* s*-cascade* $m = [n_s, n_{s-1}, \ldots, n_{s-\ell+1}]_s$*. Then* m *is a colex-unique number for all* $m \leq s+1$*. If* $m > s+1$ *then* m *is a colex-unique number if and only if one of the following is true:*

a) m *is a jumping number, i.e.* $\ell \leq q$*, or*

b) there exists $n' \leq n$ *such that* $m = \binom{n'}{s} - 1$*.*

For $m > s+1$ *conditions a) and b) are mutually exclusive.*

The next lemma and the subsequent corollary will help us in the process of tracing the criterion for uniqueness through the steps of the proof of Theorem 12.

Lemma 25. *Suppose that $u, v \geq 1$ and the cascade representations*

$$N = [n_u, n_{u-1}, \dots, n_{u-k+1}]_u \qquad \textit{and} \qquad M = [m_v, m_{v-1}, \dots, m_{v-\ell+1}]_v$$

satisfy $n_{u-k+1} = m_{v-\ell+1}$. Let $b = n_{u-k+1} = m_{v-\ell+1}$. Suppose moreover that

$$\{b, n_{u-k+2}, \dots, n_{u-1}, n_u\} \cup \{b, m_{v-\ell+2}, \dots, m_{v-1}, m_v\} = \{b, b+1, \dots, u+v-1\},$$

and

$$\{b, n_{u-k+2}, \dots, n_{u-1}, n_u\} \cap \{b, m_{v-\ell+2}, \dots, m_{v-1}, m_v\} = \{b\}.$$

Then $N + M = \binom{u+v}{u} = \binom{u+v}{v}$.

Proof. Consider first the case that $\min(u, v) = 1$. Without loss of generality we suppose that $u = 1$. Then $u + v - 1 = v$ so for some $1 \leq b \leq v$ we have

$$\begin{aligned} N + M &= [b]_1 + [v, v-1, v-2, \dots, b+1, b]_v \\ &= b + \sum_{i=b}^{v} \binom{i}{i} \\ &= b + (v - b + 1) = v + 1 = \binom{u+v}{u}. \end{aligned}$$

Now suppose that $u, v > 1$. By symmetry we may suppose that $n_u = u + v - 1$. If $k > 1$ then we let

$$N' = [n_{u-1}, n_{u-2}, \dots, b].$$

Note that the representations of N' and M satisfy the hypotheses of the lemma, with $u' = u - 1$ and $k' = k - 1$. By induction we get

$$N + M = \binom{u+v-1}{u} + N' + M = \binom{u+v-1}{u} + \binom{u+v-1}{u-1} = \binom{u+v}{u}.$$

On the other hand if $k = 1$ then we're forced to have $N = [u+v-1]_u$ and $M = [u+v-1]_v$, so

$$N + M = \binom{u+v-1}{u} + \binom{u+v-1}{v} = \binom{u+v-1}{u} + \binom{u+v-1}{u-1} = \binom{u+v}{u}. \qquad \square$$

Corollary 26. *Suppose that $1 \leq s < t \leq n$ and that $0 < m < \binom{n}{s}$. Let*

$$m = [n_s, n_{s-1}, \dots, n_{s-\ell+1}]_s$$

be the s-cascade representation of m. Then the $(n-s)$-cascade representation of $m' = \binom{n}{s} - m$ is

$$m' = [n'_{n-s}, n'_{n-s-1}, \dots, n'_{n-s-k+1}]_{n-s},$$

where $n_{s-\ell+1} = n'_{n-s-k+1}$ and, writing b for this value,

$$\begin{aligned} &\{b, n_{s-\ell+2}, \dots, n_{s-1}, n_s\} \cup \{b, n'_{n-s-k+2}, \dots, n'_{n-s-1}, n'_{n-s}\} = \{b, b+1, \dots, n-1\} \\ &\{b, n_{s-\ell+2}, \dots, n_{s-1}, n_s\} \cap \{b, n'_{n-s-k+2}, \dots, n'_{n-s-1}, n'_{n-s}\} = \{b\}. \end{aligned} \tag{$\dagger$}$$

In particular $k + \ell - 1 = n - b$, so $k = n - \ell - b + 1$.

Proof. With $n'_{n-s}, n'_{n-s-1}, \ldots, b$ defined to satisfy Eq. (†) it is easy to check that $n'_{n-s-k+1} = b \geq n-s-k+1$ and $k \leq n-s$. Using Remark 15 we deduce that $(n'_{n-s}, n'_{n-s-1}, \cdots, b)$ is a strict $(n-s)$-cascade. Then, by Lemma 25,

$$[n_s, n_{s-1}, \ldots, n_{s-\ell+1}]_s + [n'_{n-s}, n'_{n-s-1}, \ldots, n'_{n-s-k+1}]_{n-s} = \binom{s+(n-s)}{s} = \binom{n}{s}.$$

Thus $[n'_{n-s}, n'_{n-s-1}, \ldots, n'_{n-s-k+1}]_{n-s}$ is the $(n-s)$-cascade representation of $\binom{n}{s} - m$. $\square$

Theorem 27. *Suppose that $1 \leq s < t \leq n$ and that $0 < m < \binom{n}{s}$. Let*

$$m + 1 = [n_s, n_{s-1}, \ldots, n_{s-\ell+1}]_s$$

be the s-cascade representation of $m+1$, having length ℓ. Then m has $k_s^t(m+1) > k_s^t(m)$ if and only if $t \leq \ell + n_{s-\ell+1} - 1$. In this case we say that m is an (s,t)-clique-jumping number.

Proof. From Remark 13 we have

$$k^t(\mathcal{C}_s(m)) = \binom{n}{t} - \partial_{n-t}^{n-s}(\tbinom{n}{s} - m).$$

Thus $k^t(m+1) > k^t(m)$ exactly if we have

$$\partial_{n-t}^{n-s}(\tbinom{n}{s} - m) > \partial_{n-t}^{n-s}(\tbinom{n}{s} - m - 1)$$

i.e., $\binom{n}{s} - m - 1$ is an $(n-s, n-t)$-jumping number. By Corollary 26, the length of the $(n-s)$-cascade representation of $\binom{n}{s} - m - 1$ is $k = n - \ell - n_{s-\ell+1} + 1$, so by Theorem 23 we need $n - \ell - n_{s-\ell+1} + 1 \leq n - t$, i.e, $t \leq \ell + n_{s-\ell+1} - 1$. $\square$

Theorem 28. *Suppose that $1 \leq s < t \leq n$ and that $0 < m < \binom{n}{s}$. Let*

$$m = [n_s, n_{s-1}, \ldots, n_{s-\ell+1}]_s$$

be the s-cascade representation of m, having length ℓ. Then the colex s-graph $\mathcal{H} = \mathcal{C}_s(m)$ is unique up to isomorphism satisfying $|\mathcal{H}| = m$ and $k^t(\mathcal{H}) = k_s^t(m)$ if either $m \geq \binom{n}{s} - n + s - 1$ holds, or $m < \binom{n}{s} - n + s - 1$ and one of the following two (mutually exclusive) conditions holds:

a) $t \leq \ell + n_{s-\ell+1} - 1$ (equivalently $m-1$ is an (s,t)-clique-jumping number), or

b) for some $n - s + 2 \leq n' \leq n$ we have $m = \binom{n}{s} - \binom{n'}{n-s} + 1$.

Proof. By Remark 13, the colex s-graph $\mathcal{H} = \mathcal{C}_s(m)$ is unique up to isomorphism satisfying $|\mathcal{H}| = m$ and $k^t(\mathcal{H}) = k_s^t(m)$ if and only if all $(n-s)$-graphs with $\binom{n}{s} - m$ edges satisfying $|\partial_{n-t}(\mathcal{H})| = \partial_{n-t}^{n-s}(\binom{n}{s} - m)$ are isomorphic to $\mathcal{C}_{n-s}(\binom{n}{s} - m)$. Applying Theorem 24, and using Corollary 26 and Theorem 27 for condition a), yields the result. In condition b), note that $n' \leq n - s + 1$ and $m = \binom{n}{s} - \binom{n'}{n-s} + 1$ imply $m \geq \binom{n}{s} - n + s - 1$. $\square$

Corollary 29. *If $m = \binom{n'}{s}$ with $n' \geq t$ then $\mathcal{C}_s(m)$ is the unique s-graph $\mathcal{H}$, up to isomorphism, with m edges achieving $k^t(\mathcal{H}) = k_s^t(m)$.*

Proof. By Theorem 28, it suffices to show that either $\binom{n'}{s} \geq \binom{n}{s} - n + s - 1$, or condition a) is satisfied. For that condition note that $m = [n_s, n_{s-1}, \ldots, n_{s-\ell+1}]_s = [n']_s$ has length $\ell = 1$ and final entry $n_{s-\ell+1} = n'$, and we have $t \leq 1 + n' - 1$ by hypothesis. $\square$

2.3 Lovász Kruskal-Katona

Cascades have the merit of giving the precise values of $\partial_q^s(m)$ and $k_s^t(m)$, but are somewhat unwieldy to work with. There is a simpler form of the Kruskal-Katona theorem, due to Lovász [20], that is often strong enough. We work with the natural polynomial generalization of the binomial coefficient $\binom{n}{k}$ to real values of n.

Definition 30. For a real number x and natural number k, the generalized binomial coefficient is defined as $\binom{x}{k} = (x)(x-1)\cdots(x-k+1)/k!$.

Lemma 31 (Lovász [20])**.** *Let $\mathcal{H}$ be an r-graph. Write $|\mathcal{H}|$ in the form $\binom{u}{r}$, where $u \geq r-1$ is real. Then $|\partial_k(\mathcal{H})| \geq \binom{u}{k}$ for all $k \in [r]$.*

The clique version of this result is a straightforward consequence.

Theorem 32. *Let $s, t \in \mathbb{N}$ with $t \geq s$. Let $\mathcal{H}$ be an s-graph. Write $|\mathcal{H}|$ in the form $\binom{x}{s}$, where $x \geq s-1$ is real. Then $k^t(\mathcal{H}) \leq \binom{x}{t}$.*

Proof. Let $\mathcal{T} = K^t(\mathcal{H})$. Then $\mathcal{T}$ is a t-graph. Write $|\mathcal{T}|$ in the form $\binom{u}{t}$, where $u \geq t-1$ is real. By Lemma 31, the number of s-sets (edges of $\mathcal{H}$) contained in edges of $\mathcal{T}$ (t-cliques of $\mathcal{H}$) is at least $\binom{u}{s}$. The number of edges of $\mathcal{H}$ contained in t-cliques of $\mathcal{H}$ is at most the number of edges of $\mathcal{H}$, so we have $\binom{x}{s} = |\mathcal{H}| \geq \binom{u}{s}$. It is easy to check that $\binom{x}{s}$ is strictly increasing in x for $x \geq s-1$, and hence we must have $x \geq u$. Similarly we have $k^t(\mathcal{H}) = |\mathcal{T}| = \binom{u}{t} \leq \binom{x}{t}$. □

3 Steiner Shadows and Packing Shadows

In this section we define and discuss some important hypergraphs that turn out to be optimal examples in some cases of our problem.

Definition 33. A *Steiner system* with parameters i, r, n (abbreviated as an $S(i, r, n)$) is a collection of r-sets of some n-set V that covers each i-set of V exactly once. That is to say, it is an r-graph $\mathcal{A}$ on vertex set V such that for all $I \in \binom{V}{i}$ there exists a unique $A \in \mathcal{A}$ such that $I \subseteq A$.

It had been known for a long time (by straightforward counting arguments) that in order for a Steiner system with parameters i, r, n exist it must be the case that certain divisibility conditions are satisfied. In groundbreaking work Peter Keevash [15] showed (among other things) that for sufficiently large n these conditions are also sufficient.

Theorem 34 (Keevash 2014)**.** *For fixed i and r and for n sufficiently large, an $S(i, r, n)$ exists if and only if for all $0 \leq j < i$ we have that $(r-j)_{(i-j)}$ divides $(n-j)_{(i-j)}$.*

Corollary 35. *For fixed i and r, the set of n for which an $S(i, r, n)$ exists has positive lower density.*

Proof. The divisibility conditions are certainly satisfied if $n - i + 1$ is divisible by $r_{(i)}$, so the lower density of $\{n : \text{an } S(i, r, n) \text{ exists}\}$ is at least $1/r_{(i)}$. □

We can weaken the definition of a Steiner system to require only that each i-set is covered at most once (rather than exactly once), giving the following definition.

Definition 36. An *i-packing* of r-sets (abbreviated as a $P(i, r)$), also called a *partial Steiner system*, is a collection of r-sets of some set V that covers each i-set of V at most once. That is to say, it is an r-graph $\mathcal{A}$ on vertex set V such that for all $I \in \binom{V}{i}$ there exists at most one $A \in \mathcal{A}$ such that $I \subseteq A$. Equivalently, any distinct r-sets $A, B \in \mathcal{A}$ have $|A \cap B| < i$.

Existence of $P(i, r)$'s is guaranteed for all values of the parameters. For instance, a disjoint collection of r-sets is a $P(i, r)$ for all $i \geq 1$.

The hypergraphs that will be useful to us are not only Steiner systems and packings themselves, but their shadows on layers intermediate between i and r.

Definition 37. A *Steiner shadow* with parameters i, r, n, s, abbreviated $\partial_s S(i, r, n)$, is the s-shadow of an $S(i, r, n)$. A *packing shadow* with parameters i, r, s, abbreviated $\partial_s P(i, r)$, is the s-shadow of an i-packing of r-sets.

We will show later that Steiner shadows and packing shadows provide examples showing that the signpost results we prove are best possible (at least for some values of the parameters).

Lemma 38. *If $1 \leq i < s < r$ and $\mathcal{A}$ is a $P(i, r)$, then, if we write $\mathcal{H}$ for the s-graph $\partial_s(\mathcal{A})$, the following hold.*

a) For all $i \leq j \leq r$ we have $|\partial_j(\mathcal{A})| = \binom{r}{j}|\mathcal{A}|$. In particular, $\mathcal{H}$ has $\binom{r}{s}|\mathcal{A}|$ edges, and for all $s \leq t \leq r$ we have $k^t(\mathcal{H}) = |\partial_t(\mathcal{A})| = \binom{r}{t}|\mathcal{A}|$.

b) If $I \in \partial_i(\mathcal{H})$ then $\mathcal{H}(I) \cong K_{s-i}^{(r-i)}$, which implies that $d_{\mathcal{H}}(I) = \binom{r-i}{s-i}$ and $k^{t-i}(\mathcal{H}(I)) = \binom{r-i}{t-i}$. In particular $\Delta_i(\mathcal{H}) = \binom{r-i}{s-i}$.

In particular if $\mathcal{H}$ is a Steiner shadow $\partial_s S(i, r, n)$ then parts a) and b) hold with $|\mathcal{A}| = \binom{n}{i}/\binom{r}{i}$, and $\partial_i(\mathcal{H}) = \binom{[n]}{i}$.

Proof. Straightforward. $\square$

We show in the next two results that two conditions analogous to Item b) of Lemma 38 force a hypergraph to be a packing shadow or a Steiner shadow respectively.

Lemma 39. *Let $3 \leq i + 2 \leq s \leq t \leq r$ and suppose that $\mathcal{H}$ is an s-graph with $\Delta_i(\mathcal{H}) \leq \binom{r-i}{s-i}$. If we have $k^{t-i}(\mathcal{H}(I)) = \binom{r-i}{t-i}$ for every i-set I contained in an edge of $\mathcal{H}$, then $\mathcal{H}$ is a packing shadow $\partial_s P(i, r)$.*

Proof. We first note that Corollary 29 implies that for all $I \in \partial_i(\mathcal{H})$ we have $\mathcal{H}(I) \cong K_{r-i}^{(s-i)}$. For all such I we write A_I for the vertex set of $\mathcal{H}(I)$. Then $R_I = A_I \cup I$ has the property that for all s-sets $S \supseteq I$ we have $S \in \mathcal{H}$ if and only if $S \subseteq R_I$. We let $\mathcal{R} = \{R_I : I \in \partial_i(\mathcal{H})\}$. We'll show that $\mathcal{R}$ is a $P(i, r)$ and that $\mathcal{H} = \partial_s(\mathcal{R})$.

First let's show that if $I \in \partial_i(\mathcal{H})$ and $J \in \binom{R_I}{i}$ then also $J \in \partial_i(\mathcal{H})$ and $R_J = R_I$. We'll first prove the special case where $|J \cap I| = i - 1$. If $R_I \neq R_J$ then we can choose an s-set S in R_I containing $I \cup J$ and an element of $R_I \setminus R_J$, since $s \geq i + 2$. We have $I \subseteq S \subseteq R_I$, so $S \in \mathcal{H}$. Since $J \subseteq S$ we have $J \in \partial_i(\mathcal{H})$. Finally we have $J \subseteq S \not\subseteq R_J$, so $S \notin \mathcal{H}$. This contradiction implies that $R_I = R_J$. For any $J \in \binom{R_I}{i}$ there exists a sequence $I = J_0, J_1, \ldots, J_k = J$ of i-sets of R_I such that $|J_\ell \cap J_{\ell+1}| = i - 1$, and by the argument above we get that $R_{J_\ell} = R_I$ for all ℓ.

From this we can show that if $I \in \partial_i(\mathcal{H})$ then $\binom{R_I}{s} \subseteq \mathcal{H}$. To see this, consider $S \in \binom{R_I}{s}$ and pick $J \in \binom{S}{i}$. Since $J \subseteq S \subseteq R_I = R_J$ we have $S \in \mathcal{H}$.

Finally, set $\mathcal{R} = \{R_I : I \in \partial_i(\mathcal{H})\}$ as above. To show that $\mathcal{R}$ is a $P(i, r)$, suppose R_I and $R_{I'}$ are both in $\mathcal{R}$, and $J \subseteq R_I \cap R_{I'}$ is an i-set. Then by the result in the second paragraph

$R_I = R_J = R_{I'}$. The last thing we need to show is that $\mathcal{H} = \partial_s(\mathcal{R})$. If $S \in \mathcal{H}$ then for any i-set of S we have $I \subseteq S \subseteq R_I$, so $S \in \partial_s(\mathcal{R})$. On the other hand if $S \in \partial_s(\mathcal{R})$ then there exists $I \in \partial_i(\mathcal{H})$ with $S \subseteq R_I$ and hence $S \in \mathcal{H}$ by the result in the third paragraph. $\square$

The corresponding result for Steiner shadows is a corollary.

Corollary 40. *Let $3 \leq i+2 \leq s \leq t \leq r$. Let $\mathcal{H}$ be an s-graph with $\Delta_i(\mathcal{H}) \leq \binom{r-i}{s-i}$. If we have $k^{t-i}(\mathcal{H}(I)) = \binom{r-i}{t-i}$ for every i-set I of vertices of $\mathcal{H}$, then $\mathcal{H}$ is a Steiner shadow $\partial_s S(i,r,n)$.*

Proof. Let V be the vertex set of $\mathcal{H}$. Given $I \in \binom{V}{i}$ we have $k^{t-i}(\mathcal{H}(I)) = \binom{r-i}{t-i}$ and $t \leq r$, so $k^{t-i}(\mathcal{H}(I)) \geq 1$. Thus $\partial_i(\mathcal{H}) = \binom{V}{i}$. By Lemma 39, $\mathcal{H}$ is a packing shadow $P(i,r)$ with $\partial_i(\mathcal{H}) = \binom{V}{i}$, i.e a Steiner shadow $\partial_s S(i,r,n)$, where $n = |V|$. $\square$

4 Signpost Results for Hypergraphs

In this section we prove "signpost" versions of Theorems 3 and 4 for hypergraphs. We solve three related problems, fixing the numbers of vertices, edges, and cliques. For each problem we first prove an upper bound on the number of t-cliques, then, for degree bounds of a special form, characterize the extremal hypergraphs and prove the upper bound is asymptotically tight.

4.1 Hypergraphs with a fixed number of vertices

4.1.1 An upper bound on the number of t-cliques

We start with a bound on the number of t-cliques in an s-graph on n vertices with maximum degree at most Δ. The argument bounds the number of cliques that can contain a fixed i-set I, and deduces a bound on the total number of t-cliques.

Theorem 41. *Let $1 \leq i < s$ and suppose that $\mathcal{H}$ is an s-graph on n vertices such that $\Delta_i(\mathcal{H}) \leq \Delta$. Then*

$$k^t(\mathcal{H}) \leq \binom{n}{i} \frac{k^{t-i}_{s-i}(\Delta)}{\binom{t}{i}}.$$

If equality holds then for each $I \in \binom{[n]}{i}$ the neighborhood $\mathcal{H}(I)$ contains $k^{t-i}_{s-i}(\Delta)$ $(t-i)$-cliques.

Proof. We count pairs (I, K) where $I \in \binom{[n]}{i}$, $K \in K^t(\mathcal{H})$, and $I \subseteq K$. Counting by t-cliques in $\mathcal{H}$ we have a total of $\binom{t}{i} k^t(\mathcal{H})$. On the other hand consider $I \in \binom{[n]}{i}$. For cliques K that contain I all s-sets E such that $I \subseteq E \subseteq K$ must be in $\mathcal{H}$. Thus $|\{K : I \subseteq K \in K^t(\mathcal{H})\}| \leq k^{t-i}\big(\mathcal{H}(I)\big)$. Since by hypothesis $|\mathcal{H}(I)| = d_{\mathcal{H}}(I) \leq \Delta$ we have

$$|\{K : I \subseteq K \in K^t(\mathcal{H})\}| \leq k^{t-i}\big(\mathcal{H}(I)\big) \leq k^{t-i}_{s-i}(\Delta)$$

by Theorem 12. Thus, summarizing, we have

$$\binom{t}{i} k^t(\mathcal{H}) \leq \binom{n}{i} k^{t-i}_{s-i}(\Delta)$$

$$k^t(\mathcal{H}) \leq \binom{n}{i} \frac{k^{t-i}_{s-i}(\Delta)}{\binom{t}{i}}.$$

If we have equality then $k^{t-i}\big(\mathcal{H}(I)\big) = k^{t-i}_{s-i}(\Delta)$ for every $I \in \binom{[n]}{i}$. $\square$

From this result the following corollary is immediate from our known bounds on k_{s-i}^{t-i}.

Corollary 42. *Let $1 \le i < s$ and suppose that $\mathcal{H}$ is an s-graph on n vertices such that $\Delta_i(\mathcal{H}) \le \Delta$.*

a) If the cascade representation of Δ is given by $[n_{s-i}, n_{s-i-1}, \dots, n_{s-i-\ell+1}]_{s-i}$ then

$$k^t(\mathcal{H}) \le \binom{n}{i} \frac{[n_{s-i}, n_{s-i-1}, \dots, n_{s-i-\ell+1}]_{t-i}}{\binom{t}{i}}.$$

b) If $\Delta = \binom{x-i}{s-i}$ for some (not necessarily integral) $x \ge s$ then

$$k^t(\mathcal{H}) \le \binom{n}{i} \frac{\binom{x-i}{t-i}}{\binom{t}{i}} = \binom{n}{i} \frac{\binom{x}{t}}{\binom{x}{i}}.$$

Proof. The two parts follow from Theorem 41 together with Lemma 19 and Theorem 32 respectively. $\square$

4.1.2 Extremal hypergraphs and asymptotic tightness

For degree bounds of the form $\binom{r-i}{s-i}$, with r an integer, we show that Steiner shadows achieve the bound from Theorem 41, and that they are the only s-graphs that do when $i \le s-2$. We do not know whether other s-graphs achieve the bound when $i = s-1$.

Theorem 43. *Let $1 \le i < s \le t \le r$, where r is an integer, and suppose that $\mathcal{H}$ is an s-graph on n vertices.*

a) If $\mathcal{H}$ is a Steiner shadow $\partial_s S(i,r,n)$, then $\Delta_i(\mathcal{H}) = \binom{r-i}{s-i}$ and $k^t(\mathcal{H}) = \frac{\binom{n}{i}}{\binom{r}{i}}\binom{r}{t}$. I.e., $\mathcal{H}$ achieves the upper bound in Theorem 41.

b) If we further assume that $s \ne i+1$, then $\mathcal{H}$ satisfies both $\Delta_i(\mathcal{H}) \le \binom{r-i}{s-i}$ and $k^t(\mathcal{H}) = \frac{\binom{n}{i}}{\binom{r}{i}}\binom{r}{t}$ if and only if $\mathcal{H}$ is a Steiner shadow $\partial_s S(i,r,n)$.

Note that by Theorem 34 the set of n for which Steiner shadows $\partial_s S(i,r,n)$ exist has positive lower density.

Proof. For both parts, note that $k_{s-i}^{t-i}(\binom{r-i}{s-i}) = \binom{r-i}{t-i}$ and $\binom{r-i}{t-i}\binom{r}{i} = \binom{r}{t}\binom{t}{i}$ (as in Corollary 42), so $\frac{\binom{n}{i}}{\binom{r}{i}}\binom{r}{t} = \binom{n}{i}\frac{k_{s-i}^{t-i}(\Delta)}{\binom{t}{i}}$.

First, suppose $\mathcal{H} = \partial_s(\mathcal{A})$, where $\mathcal{A}$ is an $S(i,r,n)$. By Lemma 38, $\mathcal{H}$ has $\Delta_i(\mathcal{H}) = \binom{r-i}{s-i}$ and $k^t(\mathcal{H}) = \frac{\binom{n}{i}}{\binom{r}{i}}\binom{r}{t}$.

Now, suppose $s \ne i+1$ (so $3 \le i+2 \le s$) and $\mathcal{H}$ is an s-graph on n vertices such that $\Delta_i(\mathcal{H}) \le \binom{r-i}{s-i}$ and $k^t(\mathcal{H}) = \frac{\binom{n}{i}}{\binom{r}{i}}\binom{r}{t}$. By the condition for equality in Theorem 41, for each $I \in \binom{[n]}{i}$ the neighborhood $\mathcal{H}(I)$ contains $\binom{r-i}{t-i}$ $(t-i)$-cliques, and so by Corollary 40, $\mathcal{H}$ is a Steiner shadow $\partial_s S(i,r,n)$. $\square$

Now we show that the upper bounds given by Theorem 41 and Corollary 42 are asymptotically tight.

Theorem 44 (Rödl [21]). *Let $m(n,r,i)$ be the maximum number of edges in an i-packing of r-sets in V. Then $m(n,r,i) = (1-o_n(1))\frac{\binom{n}{i}}{\binom{r}{i}}$.*

Theorem 45. *For $1 \leq i < s \leq t \leq r \leq n$, let N be the maximum value of $k^t(\mathcal{H})$ over all s-graphs $\mathcal{H}$ on n vertices with $\Delta_i(\mathcal{H}) \leq \binom{r-i}{s-i}$. Then*

$$N = (1-o_n(1))\frac{\binom{n}{i}}{\binom{r}{i}}\binom{r}{t}.$$

Proof. By Theorem 44, let $\mathcal{A}$ be an i-packing of r-sets in V such that $|\mathcal{A}| = (1-o_n(1))\frac{\binom{n}{i}}{\binom{r}{i}}$. Then $\mathcal{H} = \partial_s(\mathcal{A})$ has $k^t(\mathcal{H}) = |\mathcal{A}|\binom{r}{t} = (1-o_n(1))\frac{\binom{n}{i}}{\binom{r}{i}}\binom{r}{t}$ by Lemma 38. For every $I \in \binom{V}{i}$, we have

$$d_{\mathcal{H}}(I) = \begin{cases} \binom{r-i}{s-i} & \text{if } I \in \partial_i(\mathcal{A}) \\ 0 & \text{otherwise,} \end{cases}$$

so $\Delta_i(\mathcal{H}) \leq \binom{r-i}{s-i}$. Together with Theorem 41 this implies $N = (1-o_n(1))\frac{\binom{n}{i}}{\binom{r}{i}}\binom{r}{t}$. $\square$

In the proof of Theorem 45, $\mathcal{A}$ covers $(1-o_n(1))\binom{n}{i}$ of the i-sets in V, i.e. almost all of them, so there exists $\mathcal{H}$ that is almost a Steiner shadow and almost attains the upper bound.

Remark 46. Theorem 34 gives an alternative proof of Theorem 45.

4.2 Hypergraphs with a fixed number of edges

We switch now to considering hypergraphs with a fixed number of edges.

4.2.1 An upper bound on the number of t-cliques

We write $K^t_{\mathcal{H}}(E)$ for the set of t-cliques in $\mathcal{H}$ containing the edge E and $k^t_{\mathcal{H}}(E)$ for $|K^t_{\mathcal{H}}(E)|$.

Lemma 47. *For any s-graph $\mathcal{H}$ and $t \geq s$,*

$$k^t(\mathcal{H})\binom{t}{s} = \sum_{E\in\mathcal{H}} k^t_{\mathcal{H}}(E).$$

Proof. Count the pairs (E,K), where $E \subseteq K \in K^t(\mathcal{H})$, in two ways. $\square$

Lemma 48. *Let $\mathcal{H}$ be an s-graph containing an edge $E \in \mathcal{H}$, and let $I \subsetneq E$ with $|I| = i$. Let K be a t-clique of $\mathcal{H}$ containing E. Then $K \setminus I$ is a $(t-i)$-clique in $\mathcal{H}(I)$, and $k^t_{\mathcal{H}}(E) \leq k^{t-i}_{\mathcal{H}(I)}(E\setminus I)$.*

Proof. We'll show that $K \mapsto K \setminus I$ is an injection from $K^t_{\mathcal{H}}(E)$ to $K^{t-i}_{\mathcal{H}(I)}(E \setminus I)$. It is immediate that if we can show that $K\setminus I \in K^{t-i}_{\mathcal{H}(I)}(E\setminus I)$ then the map is an injection. We have $|K\setminus I| = t-i$ since $I \subsetneq E \subseteq K$. Consider then an $(s-i)$-subset $F \subseteq K \setminus I$. We have $F \cup I \in \binom{K}{s} \subseteq \mathcal{H}$, hence $F = (F \cup I) \setminus I \in \mathcal{H}(I)$. Therefore $K \setminus I \in K^{t-i}(\mathcal{H}(I))$ and $K \setminus I \in K^{t-i}_{\mathcal{H}(I)}(E \setminus I)$. $\square$

Lemma 49. *Let $1 \le i < s \le t$ and suppose that $\mathcal{H}$ is an s-graph such that $\Delta_i(\mathcal{H}) \le \binom{x-i}{s-i}$ for some (not necessarily integral) $x \ge t-1$. If $I \subsetneq E \in \mathcal{H}$ and $\mathcal{J} = \mathcal{H}(I)$, then*

$$\frac{k^{t-i}(\mathcal{J})}{|\mathcal{J}|} \le \frac{(x-s)_{(t-s)}}{(t-i)_{(t-s)}},$$

where $k^{t-i}(\mathcal{J})$ is the number of $(t-i)$-cliques in the $(s-i)$-graph $\mathcal{J}$. If equality is achieved then $|\mathcal{J}| = \binom{x-i}{s-i}$ and $k^{t-i}(\mathcal{J}) = \binom{x-i}{t-i}$.

Proof. The number of edges in the neighborhood is $|\mathcal{J}| = d_{\mathcal{H}}(I) \le \Delta_i(\mathcal{H}) \le \binom{x-i}{s-i}$, so $|\mathcal{J}| = \binom{y}{s-i}$ for some $s-i-1 \le y \le x-i$. If $y \le t-i-1$, then $k^{t-i}(\mathcal{J}) = 0$, so the lemma holds. Otherwise, $y \ge t-i$. By Theorem 32, $k^{t-i}(\mathcal{J}) \le \binom{y}{t-i}$, so

$$\begin{aligned}\frac{k^{t-i}(\mathcal{J})}{|\mathcal{J}|} &\le \frac{\binom{y}{t-i}}{\binom{y}{s-i}} &(1)\\ &= \frac{y(y-1)\cdots(y-s+i+1)(y-s+i)\cdots(y-t+i+1)}{y(y-1)\cdots(y-s+i+1)}\frac{(s-i)!}{(t-i)!}\\ &= \frac{(y-s+i)_{(t-s)}}{(t-i)_{(t-s)}} \quad \text{using } s \le t\\ &\le \frac{(x-s)_{(t-s)}}{(t-i)_{(t-s)}}, &(2)\end{aligned}$$

since $(x-s)_{(t-s)}$ is a strictly increasing function of x for $x \ge t-1$, and we have $y+i > t-1$. If $\frac{k^{t-i}(\mathcal{J})}{|\mathcal{J}|} = \frac{(x-s)_{(t-s)}}{(t-i)_{(t-s)}}$, then equality holds in (2), so $y+i = x$, and $|\mathcal{J}| = \binom{x-i}{s-i}$. Then equality in (1) implies that $k^{t-i}(\mathcal{J}) = \binom{x-i}{t-i}$. $\square$

Remark 50. The expression $k^{t-i}_{s-i}(m)/m$ is not an increasing function of m, whereas $\frac{(x-s)_{(t-s)}}{(t-i)_{(t-s)}}$ is an increasing function of x. For values of m where

$$\frac{k^{t-i}_{s-i}(m)}{m} = \max_{m' \le m} \frac{k^{t-i}_{s-i}(m')}{m'} \tag{3}$$

we can improve Lemma 49 to say that if $\Delta_i(\mathcal{H}) \le m$ then

$$\frac{k^{t-i}(\mathcal{J})}{|\mathcal{J}|} \le \frac{k^{t-i}_{s-i}(m)}{m}.$$

For $m = \binom{x-i}{s-i}$ where x is an integer, it is easy to check that (3) holds. It is an interesting question to determine which values of m satisfy (3).

Theorem 51. *Let $1 \le i < s$ and suppose that $\mathcal{H}$ is an s-graph having m edges such that $\Delta_i(\mathcal{H}) \le \binom{x-i}{s-i}$ for some (not necessarily integral) $x \ge s$. Then, for all $t \ge s$,*

$$k^t(\mathcal{H}) \le m\frac{\binom{x}{t}}{\binom{x}{s}}.$$

If equality holds then for each $I \in \partial_i(\mathcal{H})$ we have $k^{t-i}(\mathcal{H}(I)) = \binom{x-i}{t-i}$.

Proof. If $t > x$ then $k^t(\mathcal{H}) = 0$ because any i-set I contained in a t-clique would have $d_{\mathcal{H}}(I) \geq \binom{t-i}{s-i} > \binom{x-i}{s-i}$. Therefore we may assume $t \leq x$. We will count

$$S = \{(I, E, K) : I \subsetneq E \subseteq K \in K^t(\mathcal{H}), |I| = i, |E| = s\}$$

in two ways. Counting by K, then E, then I, we obtain

$$|S| = k^t(\mathcal{H})\binom{t}{s}\binom{s}{i}.$$

Counting by I, then E, then K, and letting $\mathcal{J} = \mathcal{H}(I)$, we obtain

$$\begin{aligned}
|S| &= \sum_{I \in \partial_i(\mathcal{H})} \sum_{E \supseteq I} k^t_{\mathcal{H}}(E) \\
&\leq \sum_{I \in \partial_i(\mathcal{H})} \sum_{E \supseteq I} k^{t-i}_{\mathcal{J}}(E \setminus I) \quad \text{by Lemma 48} \\
&= \sum_{I \in \partial_i(\mathcal{H})} k^{t-i}(\mathcal{J})\binom{t-i}{s-i} \quad \text{by Lemma 47} \\
&\leq \binom{t-i}{s-i} \sum_{I \in \partial_i(\mathcal{H})} \frac{(x-s)_{(t-s)}}{(t-i)_{(t-s)}}|\mathcal{J}| \quad \text{by Lemma 49} \\
&= \binom{t-i}{s-i} \frac{(x-s)_{(t-s)}}{(t-i)_{(t-s)}} \sum_{I \in \partial_i(\mathcal{H})} d_{\mathcal{H}}(I) \\
&= \binom{t-i}{t-s} \frac{\binom{x-s}{t-s}}{\binom{t-i}{t-s}} \sum_{I \in \partial_i(\mathcal{H})} d_{\mathcal{H}}(I) \\
&= \binom{x-s}{t-s}\binom{s}{i}m.
\end{aligned}$$

Therefore, $k^t(\mathcal{H})\binom{t}{s}\binom{s}{i} = |S| \leq \binom{x-s}{t-s}\binom{s}{i}m$, and

$$k^t(\mathcal{H}) \leq \frac{\binom{x-s}{t-s}}{\binom{t}{s}}m = \frac{\binom{x}{t}}{\binom{x}{s}}m.$$

The last equation follows from the fact that $\binom{x}{t}\binom{t}{s} = \frac{(x)(x-1)\cdots(x-t+1)}{s!(t-s)!} = \binom{x}{s}\binom{x-s}{t-s}$.

If $k^t(\mathcal{H}) = m\frac{\binom{x}{t}}{\binom{x}{s}}$ then we have equality in the above application of Lemma 49 for every $I \in \partial_i(\mathcal{H})$. By Lemma 49, $k^{t-i}(\mathcal{H}(I)) = \binom{x-i}{t-i}$ for every $I \in \partial_i(\mathcal{H})$. □

4.2.2 Extremal hypergraphs and asymptotic tightness

For degree bounds of the form $\binom{r-i}{s-i}$, with r an integer, we show that packing shadows achieve the upper bound in Theorem 51, and that for $i \leq s-2$, they are the only s-graphs that achieve this bound. Again, we do not know whether only packing shadows achieve the bound when $i = s-1$.

Theorem 52. *Let $1 \leq i < s \leq t \leq r$, where r is an integer, and suppose that $\mathcal{H}$ is an s-graph having m edges.*

a) *If $\mathcal{H}$ is a packing shadow $\partial_s P(i,r)$, then $\Delta_i(\mathcal{H}) = \binom{r-i}{s-i}$ and $k^t(\mathcal{H}) = m\frac{\binom{r}{t}}{\binom{r}{s}}$. I.e., $\mathcal{H}$ achieves the upper bound in Theorem 51. In particular, if $\binom{r}{s} \mid m$, then $\mathcal{H} = \frac{m}{\binom{r}{s}} K_r^{(s)}$ achieves equality.*

b) *If we further assume that $s \neq i+1$, then $\mathcal{H}$ satisfies both $\Delta_i(\mathcal{H}) \leq \binom{r-i}{s-i}$ and $k^t(\mathcal{H}) = m\frac{\binom{r}{t}}{\binom{r}{s}}$ if and only if $\mathcal{H}$ is a packing shadow $\partial_s P(i,r)$.*

Proof. First, suppose $\mathcal{H} = \partial_s(\mathcal{A})$, where $\mathcal{A}$ is a $P(i,r)$. By Lemma 38, $\Delta_i(\mathcal{H}) = \binom{r-i}{s-i}$, and we have $m = |\mathcal{A}|\binom{r}{s}$ and $k^t(\mathcal{H}) = |\mathcal{A}|\binom{r}{t}$, so $k^t(H) = m\frac{\binom{r}{t}}{\binom{r}{s}}$.

If $\binom{r}{s} \mid m$, then $\frac{m}{\binom{r}{s}} K_r^{(r)}$ is a $P(i,r)$, and its s-shadow is $\frac{m}{\binom{r}{s}} K_r^{(s)}$. Note that

$$k^t\Big(\frac{m}{\binom{r}{s}} K_r^{(s)}\Big) = \frac{m}{\binom{r}{s}} k^t\big(K_r^{(s)}\big) = \frac{m}{\binom{r}{s}}\binom{r}{t}$$

and $\Delta_i\big(\frac{m}{\binom{r}{s}} K_r^{(s)}\big) = \binom{r-i}{s-i}$.

Now, suppose $s \neq i+1$ (so $3 \leq i+2 \leq s$) and $\mathcal{H}$ is an s-graph having m edges with $\Delta_i(\mathcal{H}) \leq \binom{r-i}{s-i}$ and $k^t(\mathcal{H}) = m\frac{\binom{r}{t}}{\binom{r}{s}}$. We have equality in the statement of Theorem 51. The last sentence of Theorem 51 states that $k^{t-i}(\mathcal{H}(I)) = \binom{r-i}{t-i}$ for every $I \in \partial_i(\mathcal{H})$. By Lemma 39, $\mathcal{H}$ is a packing shadow $\partial_s P(i,r)$. □

The bound given by Theorem 51 is asymptotically tight.

Theorem 53. *For $1 \leq i < s \leq t \leq r$ and $m \geq 1$, let M be the maximum value of $k^t(\mathcal{H})$ over all s-graphs $\mathcal{H}$ having m edges with $\Delta_i(\mathcal{H}) \leq \binom{r-i}{s-i}$. Then*

$$M = (1 - o_m(1))m\frac{\binom{r}{t}}{\binom{r}{s}}.$$

Proof. Given i, s, t, r, m, let $m = a\binom{r}{s} + b$, for $0 \leq b < \binom{r}{s}$. Then $M \geq k^t(aK_t^{(s)}) = a\binom{r}{t} = (1 - \frac{b}{m})m\frac{\binom{r}{t}}{\binom{r}{s}}$. Since $0 \leq b < \binom{r}{s}$, $\lim_{m\to\infty} \frac{b}{m} = 0$, so $M \geq (1 - o_m(1))m\frac{\binom{r}{t}}{\binom{r}{s}}$. Theorem 51 implies $M \leq m\frac{\binom{r}{t}}{\binom{r}{s}}$, completing the proof. □

4.3 Hypergraphs with a fixed number of cliques

In this section we consider s-graphs that have a fixed number of u-cliques, for some $u > s$. The numbers of vertices and edges are not specified. We will use the following lemma to connect this problem to our previous results.

Lemma 54. *Let $1 \leq i < s \leq u$ and suppose that $\mathcal{H}$ is an s-graph such that $\Delta_i(\mathcal{H}) \leq \binom{x-i}{s-i}$ for some (not necessarily integral) $x \geq s$. Then the u-graph $\mathcal{U} := K^u(\mathcal{H})$ satisfies $\Delta_i(\mathcal{U}) \leq \binom{x-i}{u-i}$.*

Proof. For any i-set I of vertices of $\mathcal{H}$, let $K = \mathcal{U}(I)$ and let $F = \mathcal{H}(I)$. We are given that $|F| = d_{\mathcal{H}}(I) \leq \binom{x-i}{s-i}$, so we have $|F| = \binom{y}{s-i}$ for some $y \leq x-i$. If $|F| = 0$ then $d_{\mathcal{U}}(I) = 0$, so we may assume that $y \geq s-i$. By Theorem 32, $k^{u-i}(F) \leq \binom{y}{u-i} \leq \binom{x-i}{u-i}$. Let E_I be an arbitrary $(u-i)$-edge of K. By definition it satisfies $E_I \cup I \in K^u(\mathcal{H})$, so every s-set in $E_I \cup I$ is an edge of $\mathcal{H}$, and every $(s-i)$-set in E_I is an edge of $\mathcal{H}(I)$. Therefore E_I is a $(u-i)$-clique in $\mathcal{H}(I) = F$. We have shown $K \subseteq K^{u-i}(F)$, so for every i-set I we have $d_{\mathcal{U}}(I) = |K| \leq k^{u-i}(F) \leq \binom{x-i}{u-i}$. □

4.3.1 An upper bound on the number of t-cliques

We generalize Theorem 51 as follows. The $s = u$ case is exactly Theorem 51.

Theorem 55. *Let $1 \le i < s \le u$ and suppose that $\mathcal{H}$ is an s-graph such that $k^u(\mathcal{H}) = p$ and $\Delta_i(\mathcal{H}) \le \binom{x-i}{s-i}$ for some (not necessarily integral) $x \ge s$. Then, for all $t \ge s$,*

$$k^t(\mathcal{H}) \le p\frac{\binom{x}{t}}{\binom{x}{u}}.$$

If equality holds then for each $I \in \partial_i(\mathcal{U})$ we have $k^{t-i}(\mathcal{U}) = \binom{x-i}{t-i}$, where $\mathcal{U} = K^u(\mathcal{H})$.

Proof. By Lemma 54, we can apply Theorem 51 to the u-graph $\mathcal{U} := K^u(\mathcal{H})$. Since $\mathcal{U}$ is a u-graph with p edges and $\Delta_i(\mathcal{U}) \le \binom{x-i}{u-i}$, Theorem 51 implies that for all $t \ge u$ we have $k^t(\mathcal{U}) \le p\frac{\binom{x}{t}}{\binom{x}{u}}$, with equality only if for each $I \in \partial_i(\mathcal{U})$ we have $k^{t-i}(\mathcal{U}(I)) = \binom{x-i}{t-i}$. Recall $s \le u \le t$. Given a t-clique T in the s-graph $\mathcal{H}$, every u-set in T is a u-clique of $\mathcal{H}$, so T is also a t-clique in the u-graph $\mathcal{U}$. Therefore $k^t(\mathcal{H}) \le k^t(\mathcal{U}) \le p\frac{\binom{x}{t}}{\binom{x}{u}}$. $\square$

4.3.2 Extremal hypergraphs and asymptotic tightness

When the degree bound is of the form $\binom{r-i}{s-i}$, with r an integer, we show that the upper bound given by Theorem 55 is achieved by the s-graphs in which the edges that contribute to the u-clique count form a packing shadow. By excluding the case $s = u$, which is addressed in Theorem 52, we find that these are the only s-graphs that achieve this bound. The case $s = i+1$ is included here. In particular, when $s = i+1$, all degree bounds $\Delta \ge t-i$ are of the form $\binom{r-i}{s-i}$ for some $r \ge t$, so are covered by Theorem 56.

Theorem 56. *Let $1 \le i < s < u \le t \le r$, where r is an integer, and suppose that $\mathcal{H}$ is an s-graph with $\Delta_i(\mathcal{H}) \le \binom{r-i}{s-i}$. Let $p = k^u(\mathcal{H})$. Then $k^t(\mathcal{H}) = p\frac{\binom{r}{t}}{\binom{r}{u}}$ if and only if the set of edges of $\mathcal{H}$ that are contained in a u-clique of $\mathcal{H}$ is a packing shadow $\partial_s P(i,r)$. In particular, if $\binom{r}{u} \mid p$, then $\mathcal{H} = \frac{p}{\binom{r}{u}}K_r^{(s)}$ achieves equality.*

Proof. First, let $\mathcal{E} = \{E \in \mathcal{H} : E \subset U \text{ for some } U \in K^u(\mathcal{H})\}$, and suppose $\mathcal{E} = \partial_s(\mathcal{A})$, where $\mathcal{A}$ is a $P(i,r)$. Note $k^u(\mathcal{E}) = k^u(\mathcal{H})$. Any edges in $\mathcal{H} \setminus \mathcal{E}$ are not contained in u-cliques of $\mathcal{H}$ so cannot be contained in t-cliques of $\mathcal{H}$. Therefore $k^t(\mathcal{E}) = k^t(\mathcal{H})$. By Lemma 38, we have $p = k^u(\mathcal{H}) = |\mathcal{A}|\binom{r}{u}$, and $k^t(\mathcal{H}) = |\mathcal{A}|\binom{r}{t}$, so $k^t(\mathcal{H}) = p\frac{\binom{r}{t}}{\binom{r}{u}}$.

If $\binom{r}{u} \mid p$, then $\frac{p}{\binom{r}{u}}K_r^{(r)}$ is a $P(i,r)$, and its s-shadow is $\frac{p}{\binom{r}{u}}K_r^{(s)}$. Note that

$$k^t\Big(\frac{p}{\binom{r}{u}}K_r^{(s)}\Big) = \frac{p}{\binom{r}{u}}k^t\big(K_r^{(s)}\big) = \frac{p}{\binom{r}{u}}\binom{r}{t}$$

and $\Delta_i\big(\frac{p}{\binom{r}{u}}K_r^{(s)}\big) = \binom{r-i}{s-i}$.

Now, suppose $k^t(\mathcal{H}) = p\frac{\binom{r}{t}}{\binom{r}{u}}$. By Lemma 54, the u-graph $\mathcal{U} := K^u(\mathcal{H})$ satisfies $\Delta_i(\mathcal{U}) \le \binom{r-i}{u-i}$. The last sentence of Theorem 55 states that for each $I \in \partial_i(\mathcal{U})$ we have $k^{t-i}(\mathcal{U}) = \binom{r-i}{t-i}$. By Lemma 39, $\mathcal{U}$ is a packing shadow $\partial_u P(i,r)$. Let $\mathcal{A}$ be a $P(i,r)$ such that $\mathcal{U} = \partial_u(\mathcal{A})$. Since $K^u(\mathcal{H}) = \mathcal{U}$, every edge S of $\mathcal{H}$ that is contained in a u-clique U of $\mathcal{H}$ is in $\partial_s(\mathcal{A})$, because there is some r-set $R \in \mathcal{A}$ such that $S \subseteq U \subseteq R$. $\square$

Theorem 55 is asymptotically tight, by a proof very similar to that of Theorem 53.

Theorem 57. *For $1 \le i < s \le u \le t \le r$ and $p \ge 1$, let P be the maximum value of $k^t(\mathcal{H})$ over all s-graphs $\mathcal{H}$ having $k^u(H) = p$ with $\Delta_i(\mathcal{H}) \le \binom{r-i}{s-i}$. Then*

$$P = (1 - o_p(1))p\frac{\binom{r}{t}}{\binom{r}{u}}.$$

4.3.3 A corollary on 2-graphs

We also obtain the following corollary giving the maximum number of t-cliques among 2-graphs with a fixed number of u-cliques and an arbitrary constant upper bound on the maximum degree.

Corollary 58. *Suppose $3 \le u \le t \le r$ and G is a graph such that $k^u(G) = p$ and $\Delta(G) \le r-1$. Then*

a) $k^t(G) \le p\binom{r}{t}/\binom{r}{u}$.

b) The maximum value of $k^t(G)$ over all such graphs is $(1 - o_p(1))p\binom{r}{t}/\binom{r}{u}$.

c) We have $k^t(G) = p\binom{r}{t}/\binom{r}{u}$ if and only if G (after removing any edge not contained in a u-clique) is a $(p/\binom{r}{u})K_r$ (possibly together with some isolated vertices). In particular, we have equality if and only if $\binom{r}{u} \mid p$.

Proof. Apply Theorem 55, Theorem 56, and Theorem 57 with $s = 2$ and $i = 1$. Note that a packing $P(1, r)$ is a set of disjoint r-sets, so its 2-shadow forms a set of disjoint r-cliques. □

Corollary 58 is a signpost answer to a question in the concluding remarks of [2].

5 Open Problems

Many interesting problems still remain. We list some of them here.

Problem 1. If $\Delta_i(\mathcal{H}) \le \binom{r-i}{s-i}$, where r is an integer, Theorem 43, Theorem 52, and Theorem 56 completely characterize the s-graphs that achieve the upper bounds given by Theorem 41 and Theorem 51 for $i \le s-2$, and Theorem 55 for $u \ne s$. In particular, these upper bounds cannot be achieved for some values of the problem parameters.

a) For values of i, r, and n for which Steiner systems $S(i, r, n)$ do not exist (either because they do not satisfy the necessary divisibility conditions or because n is too small—see Theorem 34), Theorem 43 shows that all s-graphs $\mathcal{H}$ on n vertices having $\Delta_i(\mathcal{H}) \le \binom{r-i}{s-i}$ have $k^t(\mathcal{H}) < \binom{n}{i}\binom{r}{t}/\binom{r}{i}$, although by Theorem 45, $\max\{k^t(\mathcal{H})\} = (1 - o_n(1))\binom{n}{i}\binom{r}{t}/\binom{r}{i}$. Which such s-graphs have the maximum number of t-cliques?

b) By Lemma 38, if $\mathcal{H} = \partial_s(\mathcal{A})$, then $|\mathcal{H}| = k^s(\mathcal{H}) = \binom{r}{s}|\mathcal{A}|$. Therefore, by Theorem 52, when $m \nmid \binom{r}{s}$, all s-graphs $\mathcal{H}$ having m edges and $\Delta_i(\mathcal{H}) \le \binom{r-i}{s-i}$ have $k^t(\mathcal{H}) < m\binom{r}{t}/\binom{r}{s}$, although by Theorem 53, $\max\{k^t(\mathcal{H})\} = (1 - o_m(1))m\binom{r}{t}/\binom{r}{s}$. Which such s-graphs have the maximum number of t-cliques?

c) Similarly, by Theorem 56, when $p \nmid \binom{r}{u}$, all s-graphs having $k^u(\mathcal{H}) = p$ and $\Delta_i(\mathcal{H}) \le \binom{r-i}{s-i}$ have $k^t(\mathcal{H}) < p\binom{r}{t}/\binom{r}{u}$, although by Theorem 57, $\max\{k^t(\mathcal{H})\} = (1 - o_p(1))p\binom{r}{t}/\binom{r}{u}$. Which such s-graphs have the maximum number of t-cliques?

Problem 2. Among s-graphs with $\Delta_{s-1}(\mathcal{H}) \le r-s+1$ (the $s=i+1$ case) we have determined the exact maximum number of t-cliques and found extremal s-graphs.

a) Are there s-graphs $\mathcal{H}$ on n vertices with $\Delta_{s-1}(\mathcal{H}) \le r-s+1$ that have $k^t(\mathcal{H}) = \frac{\binom{n}{s-1}}{\binom{r}{s-1}}\binom{r}{t}$ but are not Steiner shadows $\partial_s S(s-1,r,n)$?

b) Are there s-graphs $\mathcal{H}$ on m edges with $\Delta_{s-1}(\mathcal{H}) \le r-s+1$ that have $k^t(\mathcal{H}) = m\frac{\binom{r}{t}}{\binom{r}{s}}$ but are not packing shadows $\partial_s P(s-1,r)$?

Problem 3. We have characterized the extremal s-graphs and proved that our upper bounds are asymptotically tight only when the i-degree bound is $\binom{r-i}{s-i}$ for some integer r. Are the upper bounds given by Corollary 42, Theorem 51, and Theorem 55 tight when the i-degree bound does not have this form?

Problem 4. For which values of m does $\frac{k_s^t(m)}{m} = \max_{m' \le m} \frac{k_s^t(m')}{m'}$? (See Remark 50.)

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