Nexus Network Journal
https://doi.org/10.1007/s00004-023-00704-5

CONFERENCE PAPER

Heron's Legacy: An Example of Ancient Calculations Applied to Roman Imperial Architecture

Alicia Roca[1] · Francisco Juan-Vidal[1] · Luca Cipriani[2] · Filippo Fantini[3]

Accepted: 18 March 2023

Abstract
Heron of Alexandria is a well-known author in the field of mathematics and engineering, but his work is of great interest for understanding ancient construction problems related to architecture. The formulas translated and commented on by Heiberg (1914a, b) are analyzed through geometric diagrams and applied to famous Roman domed architecture.

Introduction

The subject of domes in the Roman world is complex. The scholar who intends to deal with it, in addition to direct acquaintance with the buildings, must contend with a wide and heterogeneous bibliography (De Angelis d'Ossat 1938:13–24, Sanpaolesi 1971: 3–64, Giuliani 2018:124–125, Lancaster 2005 And 2015, Taylor 2003) that, from time to time, either presents multidisciplinary features or pertains to individual areas of research (typological, constructive, logistical, geometric, and symbolic). Over the past few years, a strand of research has been gaining momentum that integrates knowledge of classical treatises with that of "technical manuals" such as the geometric calculations of Heron of Alexandria. Our goal is to comment on a selection of formulas from Johan Ludvig Heiberg's critical edition "*Heronis Alexandrini opera quae supersunt omnia*" (Heiberg 1914a, b), particularly books IV and V, and apply

✉ Filippo Fantini
filippo.fantini2@unibo.it

1 Institute of Multidisciplinary Mathematics (IMM), Universitat Politècnica de València / IRP, Camino de Vera SN, Valencia 46022, Spain

2 Alma Mater Studiorum – Universita' Di Bologna, via Tombesi dall'Ova 55, Ravenna 48121, Italy

3 Alma Mater Studiorum – Universita' Di Bologna, Viale del Risorgimento 2, Bologna 40136, Italy

them to real cases of buildings with domes to understand whether such geometric knowledge was being applied and how.

The Research

Beginning with a series of surveys and studies (Cipriani et al. 2020) conducted on the architecture of Villa Adriana, Baia and in Rome, it was possible to observe how the modular structures underlying the ancient design of centric spaces were based on recurring relationships, particularly aimed to facilitate the computational problems required for construction. Heron (I century AD) proposes a reinterpretation of Archimedes' earlier writings (particularly I, Corollary 34) from an applied perspective. We could almost define these writings as a kind of ancient computational tool that allows us to calculate many construction elements through easy-to-implement strategies. Frequently, the parameters from which the proposed exercises start are general and external to the structure to be calculated and do not always present a formalization similar to the contemporary way to solve analogous problems. Book IV contains *Definitiones* and *Geometrica*, both oriented to the study of two-dimensional shapes. Book V contains *Stereometrica* (*Stereometrica I* and *Stereometrica II*) and *De Mensuris*. Of particular interest to our subject is Book V. It consists of a series of formulas and procedures for calculating areas of flat shapes, surfaces and volumes of three-dimensional shapes (spheres, spherical caps, shells, quarter spheres), cones, pyramids, and trunks of pyramids. Some authors, albeit cautiously, propose the hypothesis that Heron wrote a manual on the design of domes "ta kamarika" (Conti and Martines 2010), salient elements of architectural production under Nero - a contemporary of the Alexandrian mathematician – (Fig. 1) and Hadrian (under whose principate the Pantheon and numerous domes between Tivoli and Baia were built) (Fig. 2). Our hypothesis is that among the calculations in "*Opera quae supersunt omnia*", collected, analyzed and thematically grouped by Johan Ludvig Heiberg, there may be hidden scraps of that lost text. Books IV and are plenty of examples referring explicitly to the calculation of areas and volumes of parts of buildings and in particular of roofing solutions. These can be very specific, using architectural vocabulary, or they have a more abstract and general focus and allude to building problems consistent with shapes typical in the architectural construction of the times (such as spheres or spherical caps).

Through a comprehensive screening of books IV and V, it could be conjectured that some of those examples were selected among those of the ancient volume related to the design of vaults and domes. In the numerical examples it appears systematic the use diameters of circumferences equal to 7 or multiples of 7. The reason is well known and already pointed out by other authors (Svenshon 2009; Fuchs 2022), since the resort to the approximation of π by 22/7, significantly simplifies the calculation of areas and volumes of curved spaces. Many of the domes we studied at Villa Adriana show that the diameters of hemispherical domes present 7 modules (Small Baths, Triclinium of Golden Court). The methods used by Heron involve different types of calculations, but also empirical systems for obtaining volumes and surfaces of building elements:

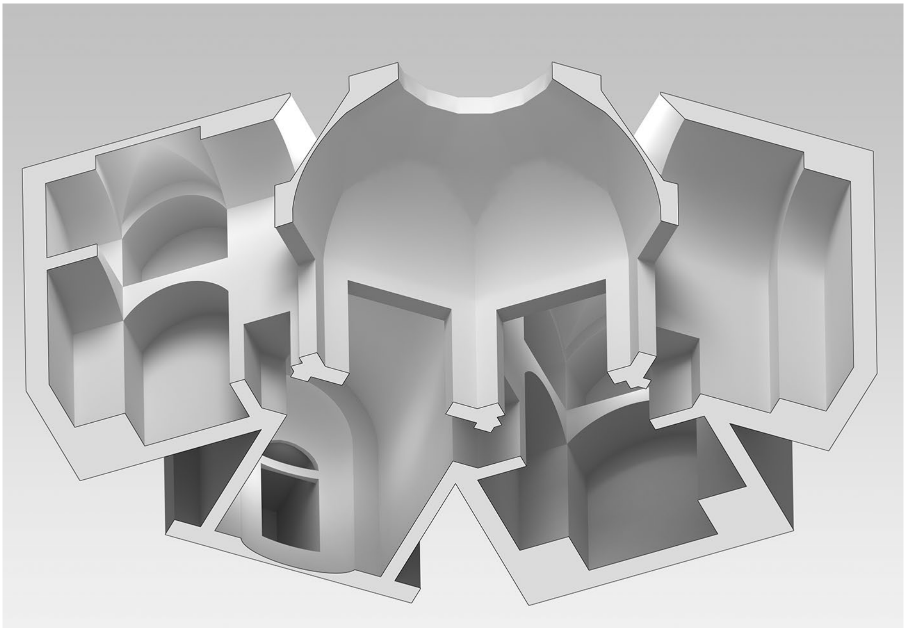

Fig. 1 Domus Aurea (Rome), Heron's practical formularies are part of the cultural context of the I century AD., which saw a high development of vaulted solutions in architecture, as is evident from the octagonal hall of the Domus Aurea

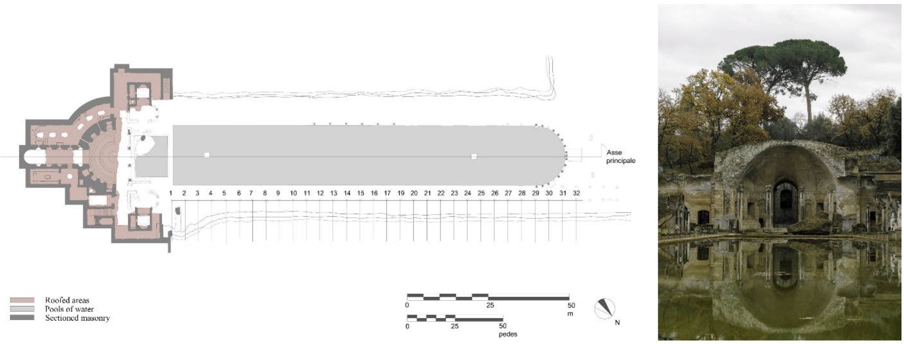

Fig. 2 One of the most famous examples of Hadrianic domes is the niche in the Serapeum of the Canopus at Hadrian's Villa (Tivoli, Rome), vivid evidence of the emperor's interest in developing complex and dramatic vaults

1) subtraction of volumes (extrados minus intrados);
2) use of "average" surfaces, halfway between the intrados and extrados of a dome to be multiplied by a constant thickness to obtain the volume. A similar methodology is also employed by Heron to calculate the number of spectators in theaters (Bianchini and Fantini 2015);
3) application of rectangular elements (patches such as cloth, animal skins, etc.) to be applied upon complex shapes and then wrapped onto a plane and then

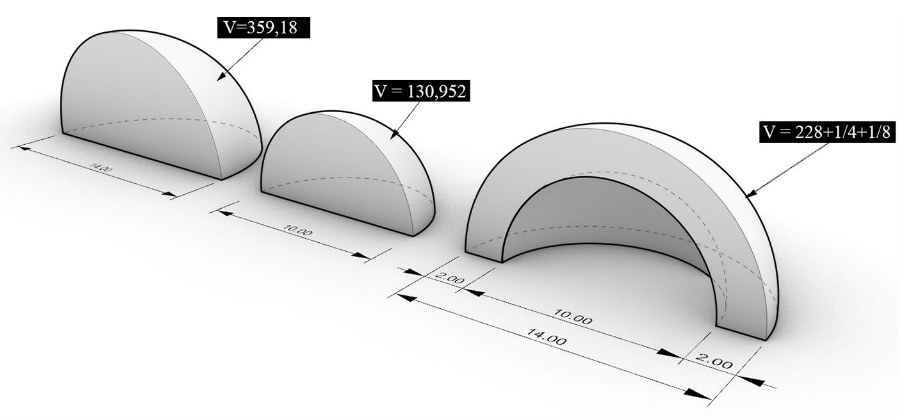

Fig. 3 Volume of a niche

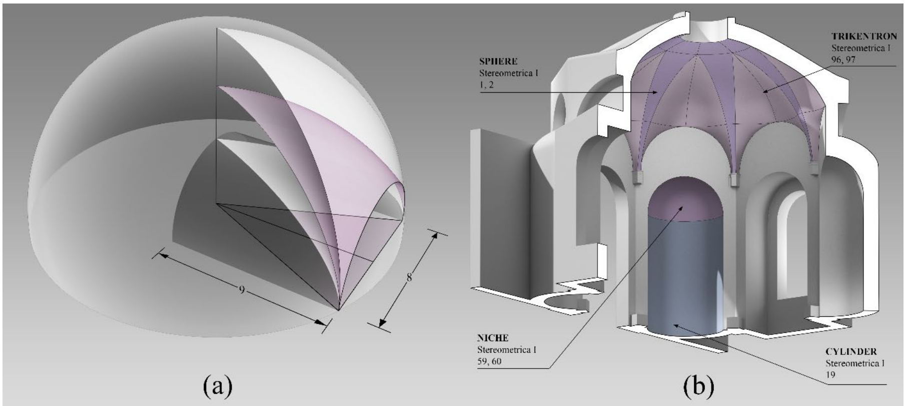

Fig. 4 (**a**) Surface of a *trikentron.* Calculations performed show how the area of the exercise is the product of the average between a spherical triangle and a portion of cross vault. (**b**) At Hadrian's Villa there are examples of the application of the *trikentron* from the Serapeum to the Vestibule of the Golden Court, but we have others in Rome (Horti Sallustiani) and Baia (Temple of Venus)

> measured. In this case, one might speculate that Heron is alluding to the use of scale models of particularly complex areas to quantify aspects important to the design process.

To the first category belong, for example, exercises 59 and 60 from *Stereometrica* II, "Volume of a niche (1/4 sphere)" (Fig. 3).

To the second category we emphasize the importance of what Heron calls the *trikentron* (*Stereometrica* I, 96), that apparently can be interpreted as a spherical triangle, which, as the word implies, is defined by "three centers," that is, by three arcs of a circle with centers located upon different planes (Fig. 4). The surface of the *trikentron* (S_T) of base (b) = 8 *pedes*, height (h) = 9 *pedes*, is calculated as a triangle

($S_{Triangle}$) to which a "supplement" is added to approximate the three-dimensionality of the surface:

$$S_{Triangle} = b \times h \times \frac{1}{2} = 8 \times 9 \times \frac{1}{2} = 36\ pedesconstrati$$

$$S_T = 36 + 36 \times \frac{1}{3} = 48\ pedesconstrati$$

By performing the 3D modelling through a CAD application, it is possible to verify how formula given by Heron provides a kind of average surface between that of a spherical triangle and a portion of a cylindrical cross vault.

It differs from the other exercises the example in 46 from *De Mensuris*, which is relevant because it examines the type of problems to be solved with empirical solutions:

> "*Regular surfaces are measured like this (as we have seen); irregular ones are divided into rectangles or "patches" (segments) as can be seen in the figure. When the surface is not flat, but irregular, such as that in the image of a statue, one must wrap it with a cloth or paper. After that it is unwrapped and measured.*"

Among the seemingly simple exercises that Heron proposes there is one type of particular importance that serves to calculate the spherical cap. This exercise, the 71 from *Stereometrica* I, while presenting itself as trivial, proposes measurements in close connection with the volume of the dome of the Pantheon that emerges from the cylindrical drum (Fig. 5).

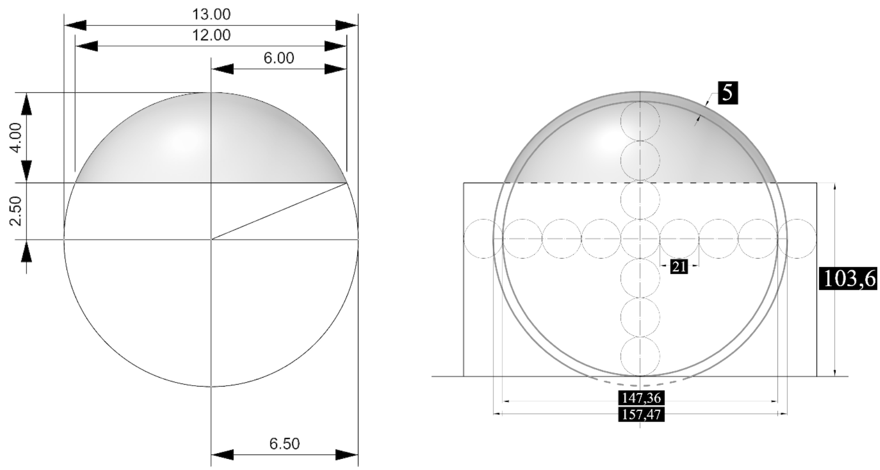

Fig. 5 On the left the exercise 71 to derive the volume of the spherical cap, and on the right the drawing of the relationship between cylinder and sphere in the Pantheon in Rome

Table 1 Summary table showing the measurements (in *pedes*) of the building elements examined in Book V of "*Heronis Alexandrini opera quae supersunt omnia*" in Johan Ludvig Heiberg's critical edition

Thicknesses of the walls	Diameters	Sides and heights
2	7	5
2+½	8	6
3	10	7
	12	9
	13	16
	14	20
	19	24
	24	30
		36

Volume of spherical cap (Vsc): radius (r) = 12 *pedes*, height h = 4 *pedes*

$$V_{sc} =? \left(\frac{1}{2} \times 12\right) \times \left(\frac{1}{2} \times 12\right) \times 3 + 4^2? \times 4 \times \frac{11}{21} = 259 + \frac{2}{3} + \frac{1}{7}$$

Conclusion

The screening on the exercises included in the collection of writings by Heiberg in the book IV e V (Table 1) can be considered an important complement to Vitruvius' fundamental De Architectura, which does not go systematically into the area of calculations necessary for planning and actual construction of buildings, with occasional exceptions.

The 2D and 3D drawings of the exercises provided by the Alexandrian mathematician and the update - according to the contemporary formalization - open several possibilities for deepening the interpretation of many buildings from the imperial age characterized by complex surfaces. An additional issue to be investigated is also the relationship between geometric patterns based on constructions made with ruler and compass, the use of regular modular grids and the set of formulas included in these texts. The measurements we find in the exercises also provide an additional scheme of interpretation to be applied to the study of individual buildings in the archaeological field through comparison with surveys. An aspect of great interest concerns the modular study related to the quantities provided by the Alexandrian mathematician; in fact, the measurements provided by Heron can alternatively be considered as true measurements in *pedes*, but also as the number of modules contained within certain quantities such as the diameters of the circles that define the intrados and extrados of domes and vaulted spaces (Fig. 6).

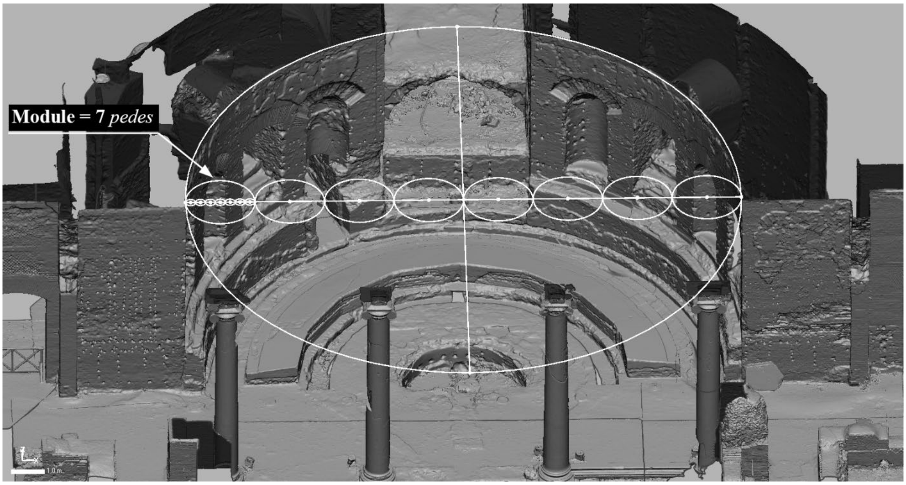

Fig. 6 Best-fitting circle of the Serapeum semi-dome. The modular grid of 7-*pedes* modules is present in the entire complex as well as in the large niche with a diameter of 56 *pedes* (8 modules of 7 *pedes*)

Funding Open access funding provided by Alma Mater Studiorum - Università di Bologna within the CRUI-CARE Agreement.

References

Bianchini, Carlo, and Filippo Fantini. 2015. Dimensioning of Ancient Buildings for Spectacles through *Stereometrica* and *De mensuris* by Heron of Alexandria. Nexus Netw J 17: 23–54.

Cinzia, Conti, and Giangiacomo Martines. 2010. Hero of Alexandria, Severus and Celer: Treatises and Vaulting at the Nero's Time. In *Mechanics and Architecture between epísteme and téchne*, ed. Anna Sinopoli, 79–96. Roma: Edizioni di storia e letteratura.

Cipriani, Luca, Filippo Fantini, and Silvia Bertacchi. 2020. Composition and Shape of Hadrianic Domes. Nexus Netw J 22: 1041–1061.

Fuchs, Wladyslaw. 2022. Studio geometrico della forma architettonica del tempio di Vesta ricostruito nel Foro Romano. In AEDES VESTAE. Archeologia, Architettura e Restauro, Atti della Giornata di Studio (Roma, Parco archeologico del Colosseo, 9 giugno 2021), 77–86. Rome: Gangemi Editore.

Giuliani, Cairoli Fulvio. 2018. *L'edilizia nell'antichità*. Roma: Carocci Editore.

De Angelis d'Ossat, Guglielmo. 1938. Sugli edifici ottagonali a cupola nell'antichità e nel Medio Evo. In *Atti del I Congresso Nazionale di Storia dell'Architettura* (29–31 Ottobre 1936-XV), 13–24. Firenze: G.C. Sansoni.

Heiberg, Johan Ludvig. 1914a. *Heronis Alexandrini opera quae supersunt omnia. Volumen IV: Heronis definitiones cum variis collectionibus. Heronis quae feruntur geometrica* Stuttgart: Teubner (rpt. 1976a).

Heiberg, Johan Ludvig. 1914b. *Heronis Alexandrini opera quae supersunt omnia: Volumen V: Heronis quae feruntur stereometrica et de mensuris*. Stuttgart: Teubner (rpt.1976b) 2.
Sanpaolesi, Piero. 1971. Strutture a cupola autoportanti. Palladio 3: 3–64.
Lancaster, Lynne C. 2005. *Concrete vaulted construction in Imperial Rome. Innovations in context*. Cambridge: Cambridge University Press.
Lancaster, Lynne C. 2015. Innovative Vaulting in Architecture of the Roman Empire, 1st to 4th Centuries CE. Cambridge: Cambridge University Press.
Svenshon, Helge. 2009. Heron of Alexandria and the Dome of Hagia Sophia in Istanbul. In *Proceedings of the Third International Congress on Construction History (Brandenburg University of Technology Cottbus, 20th – 24th May 2009)*, 1887 – 1394. Berlin: NEUNPLUS1.
Taylor, Rabun. 2003. Roman Builders: A Study in Architectural Process. Cambridge: Cambridge University Press.