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When does the onset of convection in an inclined porous layer become subcritical?

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ABSTRACT

We consider the onset of convective instability in an inclined porous layer heated from below. Linearised stability theory tells us that there always exists a band of wavenumbers within which small-amplitude disturbances will grow, but this is true only when the inclination of the layer is less than 31.49032°. At higher inclinations such disturbances always decay. However, it is also widely known that nonlinear convection may be computed for larger inclinations. This paper provides an initial explanation for how these two facts may be reconciled. It is generally assumed that the onset of convection in an inclined layer is supercritical, and, while this is certainly true when the layer is horizontal, there is no reason to assume that it remains so for other inclinations. The present paper, then, is a combined weakly-nonlinear and numerical investigation of the effect of inclination on the manner of onset. The weakly nonlinear analysis shows that the transition from a supercritical onset to a subcritical one takes place when the inclination is 24.247627°, and this is confirmed using a detailed and focussed set of nonlinear numerical simulations.

1. Introduction

The Darcy–Bénard problem has played a very substantial role within general stability theory and very many analytical and numerical tools have been devised to increase our understanding of it. At its simplest, the Darcy–Bénard problem consists of a fluid-saturated porous medium which is uniform in many different ways and where the horizontal plane boundaries that enclose the porous medium are held at constant but different temperatures. Instability is possible when the lower surface is hotter than the upper surface. The monograph by Nield and Bejan [1] devotes a very large amount of space discussing the many different aspects and extensions of this stability problem. However, for a layer of infinite horizontal extent, the critical Darcy–Rayleigh number is well-known to be $4\pi^2$ with the corresponding wavenumber, $k = \pi$.

Also important, but not studied anywhere near as extensively, is the corresponding inclined layer. Generally, fluid flows up the hot surface and down the cold one due to the direct action of buoyancy, and it is the additional presence of this nonstationary basic state which alters the stability characteristics from that of the horizontal layer. Caltagirone and Bories [2] found that streamwise vortices (longitudinal rolls) have the same critical wavenumber, but the critical Darcy–Rayleigh number is now $4\pi^2/\cos\alpha$, where α is the inclination. But when convection is confined to being two-dimensional (transverse rolls) the detailed stability characteristics become very much more complicated. Unlike longitudinal rolls, where instability is possible whenever $\cos\alpha > 0$, Caltagirone and Bories [2] have quoted $\alpha = 31.8°$ (strictly, $31° \ 48'$) as the maximum inclination for the onset of transverse rolls.

A much more comprehensive linear stability analysis of the inclined layer was undertaken by Rees and Bassom [3] who showed how the morphology of the neutral curves changes with inclination. They also found some neutral branches that correspond to unsteady modes, although the lowest value of the Darcy–Rayleigh number for any given inclination always corresponds to steady convection. Neutral curves eventually form closed loops as α gets close to $31.49032°$, and then they disappear at an isola point at that inclination. Thus, the layer is linearly stable at higher inclinations.

Without further knowledge, one might then be left wondering whether this maximum inclination is indeed the final chapter in the story for the onset of convection. One may also wonder if it is possible to obtain strongly nonlinear convection at higher inclinations simply because the layer is still being heated from below. And it could even be worth questioning whether the onset of convection is subcritical in some circumstances.

A definitive answer to these questions may be found in the very recent papers by Wen and Chini [4,5]. A Fourier-Chebyshev-tau pseudospectral solver was used by Wen and Chini [5] to compute strongly

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nonlinear two-dimensional convective flows, and they present an example of a nonlinear flow when the inclination angle is 35°, which is above the maximum inclination obtained from linear theory. In Wen and Chini [4] a nonlinear stability analysis is described and the main conclusion there is that the basic conduction state is not energy-stable for any inclination once the Darcy–Rayleigh number exceeds 91.6. Given that the onset of convection in a horizontal layer is supercritical, it is quite clear, then, that the general discrepancy between the linear theory and the nonlinear energy analysis means that the onset of convection must be subcritical for at least some range of inclinations. But the bifurcation must be supercritical for a complementary range of inclinations that includes a horizontal layer. At the very simplest, it must therefore be true that there will be an inclination where supercriticality undergoes a transition to subcriticality.

So the aim of this short paper is to use a weakly nonlinear analysis to derive a Landau equation for the amplitude of near-critical convection and, from this, to find the inclination at which this transition takes place. A finite difference code was then used to confirm the results of the weakly nonlinear analysis. Subject to restricting attention to the minimum of each neutral curve, we find that the onset of convection becomes subcritical once the inclination exceeds 24.247627°.

2. Governing equations

We are considering the onset and subsequent development of convection within a homogeneous and isotropic porous layer which is heated from below and inclined at an angle, α, to the horizontal. This configuration is illustrated in Fig. 1. We shall assume that the density of the fluid varies linearly with temperature, that the Boussinesq approximation holds, that Darcy's law is valid, and that the solid and fluid phases are in local thermal equilibrium. Given the restricted space that is available here, we begin by quoting the nondimensional form of the two-dimensional equations for buoyant flows in porous media,

$$\psi_{xx} + \psi_{zz} = \mathrm{Ra}\left[\theta_x \cos\alpha - \theta_z \sin\alpha\right], \tag{1}$$

$$\theta_t = \theta_{xx} + \theta_{zz} + \psi_z\theta_x - \psi_x\theta_z, \tag{2}$$

see Rees and Bassom [3] for details of the nondimensionalization. In these equations ψ and θ are the scaled streamfunction and temperature fields, respectively, while x is the Cartesian coordinate up the layer and z is the coordinate across the layer. The subscripts which appear in Eqs. (1) and (2) denote partial derivatives. The value, Ra, is the Darcy–Rayleigh number which is defined as,

$$\mathrm{Ra} = \frac{\rho g\beta(T_h - T_c)Kd}{\mu\kappa}. \tag{3}$$

Here, ρ is the reference density, g the acceleration due to gravity, β the coefficient of volumetric expansion, T_h (T_c) the fixed temperature of the lower (upper) boundary, K the permeability, d the thickness of the layer, μ the dynamic viscosity of the fluid and κ the thermal diffusivity of the porous medium. Apart from a slight notational change (i.e. y there has become z here) all of the above are precisely as described in Rees and Bassom [3].

Eqs. (1) and (2) are to be solved subject to the boundary conditions,

$$\psi = 0,\ \theta = 1 \quad \text{on} \quad z = 0 \quad \text{and} \quad \psi = 0,\ \theta = 0 \quad \text{on} \quad z = 1, \tag{4}$$

and we also assume that the resulting convection has period, A, in the x-direction where A will be defined below.

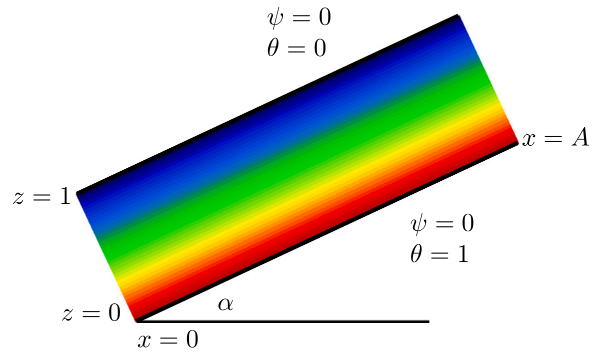

Fig. 1. Depicting an inclined porous layer heated from below. The domain is periodic with period, A, in the x-direction.

3. Weakly nonlinear analysis

The aim here is to describe very briefly the derivation of a Landau equation for the evolution of the amplitude of convection for values of Ra that are just above the threshold value, Ra_c. The coefficients of such an equation allow us to determine easily whether the onset of convection is supercritical (which it is when the layer is horizontal) or subcritical.

The weakly nonlinear analysis proceeds by expanding both ψ and θ as power series in ϵ where $|\epsilon| \ll 1$ in magnitude and where $\mathrm{Ra} = \mathrm{Ra}_0 + \epsilon^2\mathrm{Ra}_2$ defines the distance of the Darcy–Rayleigh number from the critical value, Ra_0. Therefore, we expand as follows,

$$\psi = \frac{(z^2 - z)}{2}\mathrm{Ra}\,\sin\alpha + \sum_{n=1}\epsilon^n\psi_n(x,z,t,\tau), \qquad \theta = (1 - z) + \sum_{n=1}\epsilon^2\theta_n(x,z,t,\tau), \tag{5}$$

where the $O(1)$ terms correspond to the basic state whose stability is being analysed. The slow time scale, τ, is defined using $t = \frac{1}{2}\epsilon^2\tau$, and it reflects the very slow evolution of convection with time when Ra is very close to neutrally stable conditions. The numerical coefficient, $\frac{1}{2}$, is present here purely for numerical convenience.

The $O(\epsilon)$ terms correspond to linear stability, and the equations for ψ_1 and θ_1 are,

$$\nabla^2\psi_1 = \mathrm{Ra}_0\Big[\theta_{1,x}\cos\alpha - \theta_{1,z}\sin\alpha\Big], \qquad \nabla^2\theta_1 = -\psi_{1,x} - (z - \tfrac{1}{2})\mathrm{Ra}_0\sin\alpha\,\theta_{1,x} + \theta_{1,t}. \tag{6}$$

These may be solved by first using the following substitution,

$$(\psi_1,\theta_1) = \tfrac{1}{2}B(\tau)\Big(i\,f_1(z), g_1(z)\Big)e^{ikx+i\sigma t} + \text{c.c.}, \tag{7}$$

where k is the wave number and σ is the oscillation frequency. Despite the fact that the Principle of the Exchange of Stabilities does not apply, Rees and Bassom [3] showed numerically that the base of the neutral curve always corresponds to $\sigma = 0$. The value, B, is the complex amplitude of the ensuing convection.

The equations for f_1 and g_1 are,

$$f_1'' - k^2 f_1 = \mathrm{Ra}_0\,[kg_1\cos\alpha - g_1'\sin\alpha], \qquad g_1'' - k^2 g_1 = kf_1 - (z - \tfrac{1}{2})\mathrm{Ra}_0 ik\sin\alpha\, g_1 + i\sigma g_1, \tag{8}$$

which need to be solved subject to $f_1 = g_1 = 0$ on $z = 0, 1$. Numerical solutions were obtained using a shooting method and the fourth-order Runge–Kutta scheme. In all cases 100 equally-spaced intervals were used and this yields at least six significant figures of accuracy. A curve-tracking variant with pseudo-arc-length continuation was used in order to be able to compute folds and to follow around the closed-loop neutral curves.

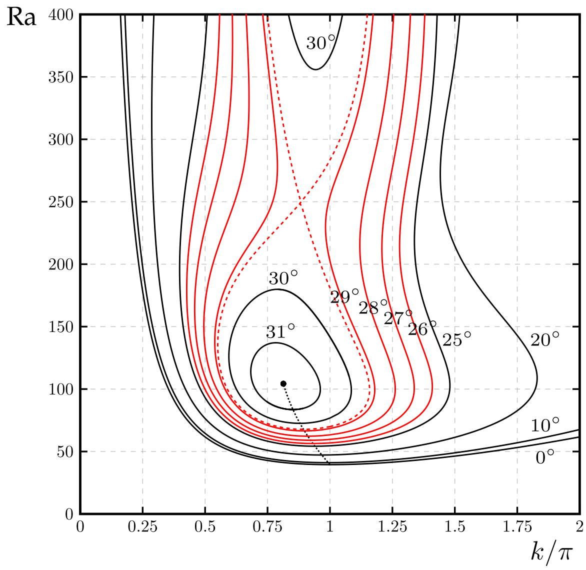

Fig. 2. Selected neutral curves for the given inclinations. The red dashed line corresponds to the saddle point at $\alpha = 29.234°$. The black disk shows the location of the isola point at $\alpha = 31.49032°$. The black dotted line follows the minima in the neutral curves from $\alpha = 0$ up to the isola point. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

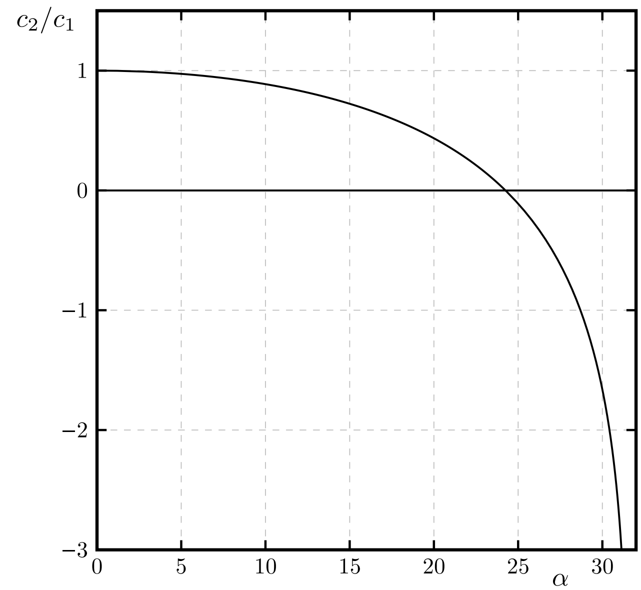

Fig. 3. The variation of c_2/c_1 with inclination. Negative values correspond to a subcritical instability. We note that c_1 always remains positive.

Although a detailed account of the variation in the shapes of the neutral curves with inclination has already been presented in Rees and Bassom [3], a subset of those neutral curves have been chosen to appear in Fig. 2 in order to improve their clarity for the present context.

We have also followed the extended-system approach given in Rees and Bassom [3] to obtain the locations of the minima in the neutral curves; this is shown as the dotted black line in Fig. 2. In the rest of the paper we shall confine ourselves solely to this locus.

We may now continue to $O(\epsilon^2)$ in the expansion. This yields,

$$\begin{aligned} \nabla^2\psi_2 &= \mathrm{Ra}_0\Big[\theta_{2,x}\cos\alpha - \theta_{2,z}\sin\alpha\Big], \\ \nabla^2\theta_2 &= -\psi_{2,x} - (z-\tfrac{1}{2})\mathrm{Ra}_0\sin\alpha\,\theta_{2,x} + \theta_{2,t} \\ &\quad + \psi_{1,x}\theta_{1,z} - \psi_{1,z}\theta_{1,x}. \end{aligned} \tag{9}$$

The nonlinear terms in Eq. (9) include those which are proportional to $B^2e^{2i(kx+\sigma t)}$, $\bar{B}^2e^{-2i(kx+\sigma t)}$ and $B\bar{B}$. Each provides a nonresonant forcing term so that Eqs. (9) may be solved numerically by using substitutions that are of similar form to Eq. (7). We omit the details of this for the sake of brevity.

Finally, the equations at $O(\epsilon^3)$ are now,

$$\begin{aligned} \nabla^2\psi_3 &= \mathrm{Ra}_0\Big[\theta_{3,x}\cos\alpha - \theta_{3,z}\sin\alpha\Big] \\ &\quad + \mathrm{Ra}_2\Big[\theta_{1,x}\cos\alpha - \theta_{1,z}\sin\alpha\Big], \\ \nabla^2\theta_3 &= -\psi_{3,x} - (z-\tfrac{1}{2})\mathrm{Ra}_0\sin\alpha\,\theta_{3,x} + \theta_{3,t} \\ &\quad - (z-\tfrac{1}{2})\mathrm{Ra}_2\sin\alpha\,\theta_{1,x} + \tfrac{1}{2}\theta_{1,\tau} \\ &\quad + \psi_{2,x}\theta_{1,z} + \psi_{1,x}\theta_{2,z} - \psi_{2,z}\theta_{1,x} - \psi_{1,z}\theta_{2,x}. \end{aligned} \tag{10}$$

We note that the nonlinear terms and those terms which involve Ra_2 and $\theta_{1,\tau}$ all contain components that are proportional to $e^{ikx+i\sigma t}$. Respectively, these components involve $B^2\bar{B}$, Ra_2B and B_τ as coefficients. It is common practice at this point in a weakly nonlinear analysis to apply a solvability condition which will then yield a Landau equation of the form,

$$B_\tau = c_1\mathrm{Ra}_2B - c_2B^2\bar{B}. \tag{11}$$

However, we have taken the unusual alternative approach of solving the appropriate equations numerically. Once more we shall suppress the details of how the shooting method was employed to find c_1 and c_2 but it became necessary to solve a 52^{nd} system to accomplish this.

It is important to note that, when the layer is horizontal and the normalisation boundary condition is set to $g_1'(0) = 1/\pi$, then the coefficients in (11) may be shown analytically to be $c_1 = c_2 = 1$, but the present numerical approach shows that c_1 and c_2 vary with inclination. We find that c_1 always remains positive as α increases from 0° to 31.49032°, the isola point. For relatively small inclinations the value of c_2 is positive, and this corresponds to a supercritical onset of convection where the steady-state weakly nonlinear solution is given by $B\bar{B} = (c_1/c_2)\mathrm{Ra}_2$. Fig. 3 shows the variation of c_2/c_1 with inclination, and it is seen easily that it takes negative values once α is above approximately 24° degrees. A more precise value for this transitional inclination was obtained by using a very slightly modified version of the same shooting method code: rather than computing c_2/c_1 for a fixed value of Ra, the modified code computed that value of Ra for which $c_2/c_1 = 0$. Thus, the transitional inclination was found to be,

$$\alpha = 24.247627°, \tag{12}$$

which is correct to six decimal places. Therefore, we conclude that the onset of convection is supercritical when α takes smaller values than this and is subcritical when it takes larger values. The critical Rayleigh number and wave number which correspond to Eq. (12) are,

$$\mathrm{Ra}_c = 52.851961, \qquad k_c = 2.969012. \tag{13}$$

4. Nonlinear simulations

The above weakly nonlinear analysis gives a precise criterion for the transition from a supercritical onset of convection to a subcritical onset, and the purpose of this section is to present a suitable set of fully nonlinear simulations of Eqs. (1) and (2) to confirm this conclusion.

The computations were undertaken using second-order accurate finite differences in space and the DuFort–Frankel scheme in time. The

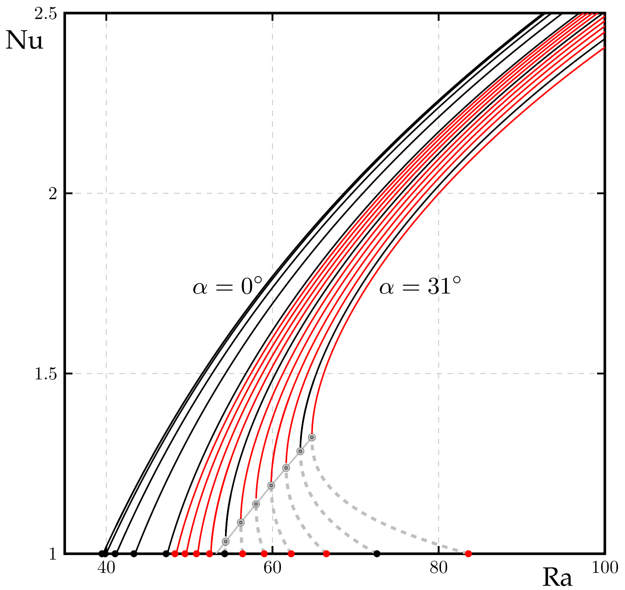

Fig. 4. The variation of Nu with the Darcy–Rayleigh number for a selection of inclinations. For further details, see the text. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

nonlinear terms in Eq. (2) were approximated using Arakawa's formulation [6], which is well-suited for problems involving instabilities due to its superior conservation properties [7]. The stream function equation, (1), was solved using a standard Correction Scheme multigrid method at each timestep.

As it was decided to restrict the present paper to cases which are equivalent to flow at the critical wavenumber, this means that the monotonic variation in wavenumber from $k_c = \pi$ when $\alpha = 0°$ to $k_c = 2.55532$ when $\alpha = 31.49032°$ at the isola point, is equivalent to the spatial period increasing monotonically from $A = 2$ to $A = 2.45886$. Given that A does not vary by a particularly large amount, it was decided to use a 64×32 uniform spatial mesh in all cases. Thus, the mesh aspect ratio varied between 1 and roughly 1.2194, which hardly affects iterative convergence. For each case, the computations proceeded until a steady flow was attained. Finally, the Nusselt number per unit distance in the x-direction was computed:

$$\text{Nu} = -\frac{1}{A}\int_0^A \frac{\partial\theta}{\partial z}\Big|_{z=0} dx. \tag{14}$$

Although a one-sided difference was used to compute $\partial\theta/\partial z$, it may be shown easily that this approximation does in fact have second-order accuracy. The integration in Eq. (14) was undertaken using the trapezium rule which also has second-order accuracy.

Fig. 4 shows how the Nusselt number varies with the Darcy–Rayleigh number for a suitable selection of inclinations. We note that Nu = 1 corresponds to a purely conductive state. The black curves correspond to inclinations which are multiples of 5° degrees, namely 0° (far left), 5°, 10°, 15°, 20°, 25° and 30°. The red curves correspond to the intermediate inclinations, 21° to 24° and 26° to 29° in steps of one degree, and also to 31°. Other inclinations are omitted to avoid cluttering. The horizontal axis also displays the critical values of Ra from linear theory; we note that there is a very small mismatch between the linear and nonlinear theories which is due to a small discretisation error. Indeed, a very small error is also incurred when attempting to extrapolate the Nusselt number curves back to Nu = 1. Nevertheless, the agreement between linearised theory and the predicted onset criterion from the nonlinear computations remains very good when $\alpha \lesssim 24°$ when the system exhibits a supercritical bifurcation.

When the inclination is larger than this, the Nusselt number curves appear to terminate well above Nu = 1 but this is the signature for the presence of a fold bifurcation. Thus, these curves should continue downwards while retreating back to larger values of Ra. Such curves are unstable in time, and are therefore inaccessible to a time-stepping code. However, the fold may be approached quite closely using the present unsteady solver by successively decreasing Ra by very small amounts and by using the previously-found steady solution as the initial condition for the next case.

Finally, the location of each fold is found by fitting a quadratic in Ra as a function of Nu to the last three data points and then by minimising that quadratic. These points are shown in Fig. 4 as grey disks and it is quite clear that these disks lie on an essentially straight line which can be extrapolated back to Nu = 1 to find the transitional values of Ra and α. This procedure yields $\alpha = 24.355$, a value which is only very slightly above the weakly nonlinear value given in Eq. (12), despite being the outcome of two different extrapolations.

In the range, $0° \le \alpha \le 24.247627°$, the positive slope of the Nusselt number curve at Nu = 1 indicates that onset is supercritical. Once α rises above its threshold value, the slope should be negative to reflect the subcritical nature of the onset. Therefore, there will be another branch of the Nusselt number curve which attaches to the point corresponding to the critical Rayleigh number for linear theory. For the purpose of illustration, these curves have been approximated by fitting a cubic which has exactly the same curvature at the fold and which recovers the linear theory when Nu = 1; these curves, which merely have the status of being reasonable guesses for the true curves, are shown as grey dotted lines. Because such curves are impossible to find using an unsteady solver, we plan to develop a direct steady solver to confirm the qualitative nature of these guessed branches.

5. Discussion and conclusion

In this short paper, we have described a two-pronged attack on the question of whether the onset of two-dimensional convection in a porous layer remains supercritical (as it is when the layer is horizontal) or whether it may become subcritical as the inclination increases. The works of Wen and Chini [4,5] give very strong reasons for expecting some sort of transition, but our task was to determine the precise conditions for the that transition. Both a weakly nonlinear theory and a two-dimensional unsteady finite difference scheme were developed and deployed for this purpose. It is clear that, despite the natural presence of very small differences caused by standard discretisation errors and by the use of some ad hoc extrapolations, the two studies validate one another and yield a definitive conclusion: the onset of convection is supercritical when $\alpha < 24.247627°$ but it is subcritical otherwise.

Whilst this conclusion is novel, it is perhaps not a surprise given that Wen and Chini [5] found that strongly nonlinear convection exists when $\alpha = 35°$. Clearly, the inclined layer is linearly stable at this inclination and therefore this solution must exist on an isolated solution branch that does not intersect with Nu = 1. It is now quite important to extend the present analysis by first approaching the isola point and then passing beyond it in terms of inclination. At present it is suspected that the closed-loop neutral curves play a central role in this for there will be two neutral locations, one at the bottom of the neutral loop (a subcritical bifurcation) and one at the top (possibly supercritical). We think that it is reasonable to expect the solution curves corresponding to the two neutral locations to join when α reaches the isola point, and then the resulting single curve will lift away from Nu = 1 as α increases further. However, this will need to be confirmed numerically and will require the use of a steady-state solver possibly based on a 2D spectral decomposition and Newton–Raphson iteration; such a scheme has been used in [8].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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