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ORIGINAL ARTICLE

An interpolation problem in the Denjoy–Carleman classes

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Abstract
Inspired by some iterative algorithms useful for proving the real analyticity (or the Gevrey regularity) of a solution of a linear partial differential equation with real-analytic coefficients, we consider the following question. Given a smooth function defined on $[a, b] \subset \mathbb{R}$ and given an increasing divergent sequence d_n of positive integers such that the derivative of order d_n of f has a growth of the type M_{d_n}, when can we deduce that f is a function in the Denjoy–Carleman class $C^M([a, b])$? We provide a positive result and show that a suitable condition on the gaps between the terms of the sequence d_n is needed.

KEYWORDS
Denjoy–Carleman classes, Gevrey functions, interpolation, quasi-analytic functions

MSC (2020)
41A17, 26E05, 26E10

1 | INTRODUCTION AND STATEMENT OF THE RESULTS

A way to prove the local regularity of the solutions of linear elliptic partial differential equations with real-analytic coefficients consists in the so called L^2–methods, that is, an iterative procedure based on the use of the elliptic estimate (see, e.g., [1, p. 207]). This approach extends to degenerate elliptic equations, such as sums of squares of vector fields, satisfying a subelliptic estimate (see, e.g., [2, 3] and [4] for recent applications of this method). Then, a natural question is which derivatives one should control in order to conclude that a function is in a given Gevrey class (or, in particular, is real-analytic). This is the main motivation for the present paper. In order to put in evidence the essential points in our proofs, we state our results for the Denjoy–Carleman classes, and, for the sake of simplicity, we limit our considerations to functions of one variable. To be definite: let $M_0 = 1, M_1, \ldots$ be a sequence of positive numbers and consider the Denjoy–Carleman class C^M

$$C^M([a, b]) = \{f \in C^\infty([a, b]) \mid \exists K > 0 \text{ such that } |f^{(n)}(x)| \le K^{n+1} M_n \text{ if } x \in [a, b],\ n = 0, 1, \ldots\}.$$

In what follows, we assume that the sequence M_n is log-convex. That is, for $j < \ell < k$, we have that

$$M_\ell \le M_j^{\frac{k-\ell}{k-j}} M_k^{\frac{\ell-j}{k-j}}. \tag{1.1}$$

We denote by $\mathbb{N}$ the set of all the non-negative integers while $\mathbb{N}_+$ stands for the set of positive integers. Condition (1.1) can be equivalently stated as

$$M_n^2 \leq M_{n-1}M_{n+1}, \qquad \forall n \in \mathbb{N}_+, \tag{1.2}$$

that is, M_{n+1}/M_n is an increasing sequence. In particular, if M_n is log-convex then

$$M_n^{\frac{1}{n}} \quad \text{is an increasing sequence for } n \in \mathbb{N}_+. \tag{1.3}$$

For the reader convenience, we provide the proofs of the assertions (1.1) $\iff$ (1.2) and (1.2) $\Rightarrow$ (1.3) in Appendix A.

We will assume the additional condition that there exists $m_0 \geq 0$ such that

$$M_j \leq M_k^{\frac{j}{k}} M_i^{\frac{j}{i}}, \text{ for } i, k > m_0 \text{ with } i < j \text{ and } j/i < k. \tag{1.4}$$

Example 1.1.

(i) Our first example of a sequence satisfying (1.1) and (1.4) is $M_n = n^{ns}$, $n \in \mathbb{N}_+$, for a suitable $s \geq 1$. This choice corresponds to the Gevrey class G^s (the set of all the real-analytic functions in the case $s = 1$). We point out that (1.4) holds with $m_0 = 1$.

(ii) Our second example is $M_0 = M_1 = 1$, $M_2 = \sqrt{M_3}$, $M_n = n^{ns_1}(\log n)^{ns_2}$ $(n = 3, 4, \ldots)$, for suitable $s_1, s_2 \geq 1$. This sequence is log-convex and satisfies (1.4) with $m_0 = e^2$. By the Denjoy–Carleman theorem, $C^M([a,b])$ is a quasi-analytic class for $s_1 = s_2 = 1$. (We recall that a class $C^M([a,b])$ is called quasi-analytic if for every $u \in C^M$ vanishing of infinite order at a point in $[a,b]$, it follows that u is identically zero on $[a,b]$.)

We observe that, for $s_1 > 1$ or $s_2 > 1$, $C^M([a,b])$ is a non-quasi-analytic class (other than Gevrey).

For the proof of the claims in (i) and (ii), we refer the interested reader to Appendix B.

Furthermore, we assume that the class C^M contains the real-analytic functions on $[a,b]$, that is, that there exists $c > 0$ such that

$$M_n \geq c^n n^n. \tag{1.5}$$

For a smooth function $f : [a,b] \longrightarrow \mathbb{R}$, we define

$$F_n = \max_{x \in [a,b]} |f^{(n)}(x)|, \qquad n \in \mathbb{N}. \tag{1.6}$$

Let $d_n > 0$ be an increasing divergent sequence of integers such that for a suitable $K > 0$,

$$F_{d_n} \leq K^{d_n+1} M_{d_n}, \qquad \text{for every } n \in \mathbb{N}. \tag{1.7}$$

We consider the following problem: given a function $f \in C^\infty([a,b])$ satisfying (1.7), under what condition on $\{d_n\}_n$ can we conclude that f belongs to the class C^M?

First, we observe that Condition (1.7) yields some additional properties on the function f when C^M is the class of real-analytic functions, that is, $M_k = k^k$, $k \in \mathbb{N}_+$. Precisely, we have the following:

Proposition 1.2. *Let $f \in C^\infty([a,b])$ and let $C^F([a,b])$ be the corresponding Denjoy–Carleman class defined by the sequence $\{F_n\}_n$ in (1.6). Let $d_n > 0$ be an increasing divergent sequence of integers. Assume that $F_{d_n} \leq K^{d_n+1} d_n^{d_n}$ for a suitable $K > 0$ and for every $n \in \mathbb{N}$. Then f is identically zero, whenever f vanishes of infinite order at a point of $[a,b]$.*

In other words, the class $C^F([a,b])$ is quasi-analytic (independently on the sequence d_n).

Remark 1.3. In general, we cannot deduce that the class $C^F([a,b])$, given by Proposition 1.2, is contained in the class of the real-analytic functions, this is a consequence of Theorem 1.6 with $M_n = n^n$.

If we make an additional assumption on the growth of $d_{n+1} - d_n$, we obtain the following

Theorem 1.4. *Let $f \in C^\infty([a,b])$, assume (1.1), (1.4), (1.5), (1.7) and that there exists $c_0 > 0$ such that*

$$d_{n+1}/d_n \le c_0, \quad \text{for} \quad n \in \mathbb{N}. \tag{1.8}$$

Then, $f \in C^M([a,b])$.

In particular, for $f \in C^\infty$ to be in the Gevrey class G^s, it is enough to control only a set of derivatives $f^{(d_n)}$ where the sequence $\{d_n\}_n$ satisfies (1.8).

Remark 1.5.

(i) We point out that the above result applies to both quasi-analytic and non-quasi-analytic Denjoy–Carleman classes.
(ii) Theorem 1.4 is related to the Carleman problem (see [7] and [9]).

Finally, we show that Theorem 1.4 may fail in the absence of a condition on the sequence $\{d_n\}_n$.

Theorem 1.6. *Let $\{M_n\}_n$ be an arbitrary sequence of positive numbers such that*

$$\limsup_{n\to+\infty} M_n^{\frac{1}{n}} = +\infty. \tag{1.9}$$

Then, we can find an increasing, divergent, sequence of positive integers $\{d_n\}_n$, and a function $f \in C^\infty([a,b])$, such that (1.7) holds true but $f \notin C^M([a,b])$.

2 | PROOFS

2.1 | Proof of Proposition 1.2

The proof of the Proposition 1.2 is an elementary computation based on the Taylor expansion (see, e.g., [8]). Indeed, let $x_0 \in [a,b]$, then by the Taylor formula, we find that

$$|f(x)| \le K^{d_n+1}\frac{{d_n}^{d_n}}{d_n!}|x-x_0|^{d_n} \le K(Ke|x-x_0|)^{d_n}, \qquad \forall x \in [a,b],$$

and taking the limit as $n \to \infty$ in the formula above, we deduce that $f(x) = 0$, for every $x \in [a,b]$ with $|x - x_0| < 1/(Ke)$. Then, the conclusion follows from finitely many interactions.

2.2 | Proof of Theorem 1.4

We want to show that $f \in C^M([a,b])$ if $f \in C^\infty([a,b])$ and $\{M_n\}_n$ is a log-convex sequence such that

(A) $M_j \le M_k^{\frac{j}{k}} M_i^{\frac{j}{i}}$, for $i,k > m_0$ with $i < j$ and $j/i < k$, for a suitable $m_0 \ge 0$;
(B) $M_n \ge c^n n^n$, for a suitable $c > 0$;
(C) $F_{d_n} \le K^{d_n+1} M_{d_n}$ (see (1.6)),

where $\{d_n\}_n$ is an increasing divergent sequence of natural numbers satisfying the gap condition,

$$\frac{d_{n+1}}{d_n} \le c_0, \qquad \forall n \in \mathbb{N}, \tag{2.1}$$

for a suitable positive integer c_0.

The idea of the proof consists in showing that the control of the derivatives of length d_n yields, by interpolation, an estimate for the intermediate derivatives (provided that the rescaled gap $(d_{n+1} - d_n)/d_n$ is bounded, uniformly w.r.t. n).

For this purpose, we recall an estimate due to Cartan and Gorny (see, e.g., [5, 6] and [9]).

Lemma 2.1. *Let g be a function m-times differentiable on the closed interval $[a,b]$ and set $G_\ell = \max_{x\in[a,b]} |g^{(\ell)}(x)|$, for every $\ell \in \mathbb{N}$. Then, for every $m \in \mathbb{N}$, with $m \ge 2$, and $k \in \{1, \dots, m-1\}$, one has*

$$G_k \le 2\left(\frac{e^2 m}{k}\right)^k G_0^{1-\frac{k}{m}} \left(\max\left\{m!\, G_0 \left(\frac{2}{b-a}\right)^m, G_m\right\}\right)^{\frac{k}{m}}. \tag{2.2}$$

Let $d_n < \ell < d_{n+1}$ and choose $g = f^{(d_n)}$, $m = d_{n+1} - d_n$ and $k = \ell - d_n$ in (2.2). It turns out that

$$\frac{1}{p} := 1 - \frac{k}{m} = \frac{d_{n+1} - \ell}{d_{n+1} - d_n}, \qquad \frac{1}{q} := \frac{k}{m} = \frac{\ell - d_n}{d_{n+1} - d_n},$$

and

$$\frac{d_n}{p} + \frac{d_{n+1}}{q} = \ell. \tag{2.3}$$

Whence (2.2) reads as

$$F_\ell \le 2\left(\frac{e^2(d_{n+1} - d_n)}{\ell - d_n}\right)^{\ell - d_n} \max\left\{\frac{(d_{n+1} - d_n)!^{\frac{1}{q}}}{((b-a)/2)^{\ell - d_n}} F_{d_n}, F_{d_n}^{\frac{1}{p}} F_{d_{n+1}}^{\frac{1}{q}}\right\}. \tag{2.4}$$

Now, in view of (2.1), we have that

$$\left(\frac{e^2(d_{n+1} - d_n)}{\ell - d_n}\right)^{\ell - d_n} = \left(e^2\left(1 + \frac{d_{n+1} - \ell}{\ell - d_n}\right)\right)^{\ell - d_n} \le e^{2(\ell - d_n) + d_{n+1} - \ell} \le e^{\ell + d_n\left(\frac{d_{n+1}}{d_n} - 2\right)} \le e^{\ell(c_0 - 1)}.$$

Furthermore,

$$\frac{(d_{n+1} - d_n)!^{\frac{1}{q}}}{((b-a)/2)^{\ell - d_n}} \le \frac{(d_{n+1} - d_n)^{\frac{d_{n+1} - d_n}{q}}}{((b-a)/2)^{\ell - d_n}} \le \left(\frac{2d_n(c_0 - 1)}{b-a}\right)^{\ell - d_n},$$

and, since $F_{d_n} \le K^{d_n+1} M_{d_n}$, we obtain that

$$F_\ell \le \max\left\{2e^{\ell(c_0-1)}\left(\frac{2d_n(c_0-1)}{b-a}\right)^{\ell - d_n} K^{d_n+1} M_{d_n}, 2e^{\ell(c_0-1)} K^{\frac{d_n+1}{p}} M_{d_n}^{\frac{1}{p}} K^{\frac{d_{n+1}+1}{q}} M_{d_{n+1}}^{\frac{1}{q}}\right\}. \tag{2.5}$$

We observe that by (2.3) and the fact that $d_n < \ell$, we have that

$$K^{\frac{d_n+1}{p}} K^{\frac{d_{n+1}+1}{q}} = K^{\ell+1}, \text{ and } K^{d_n+1} \le K^{\ell+1} \tag{2.6}$$

Moreover, by Assumption (A) above[1] and by the gap condition (2.1), we see that

$$M_{d_n}^{\frac{1}{p}} M_{d_{n+1}}^{\frac{1}{q}} \leq M_{d_n}^{\frac{1}{p}} M_{c_0}^{\frac{d_{n+1}}{c_0 q}} M_{d_n}^{\frac{d_{n+1}}{q d_n}}.$$

Whence, as a consequence of the log-convexity of $\{M_n\}_n$, (1.3), and (2.3), we find that

$$M_{d_n}^{\frac{1}{p}} M_{d_{n+1}}^{\frac{1}{q}} \leq M_{\ell}^{\frac{d_n}{\ell p}} M_{c_0}^{\frac{d_{n+1}}{c_0 q}} M_{\ell}^{\frac{d_{n+1}}{q\ell}} \leq M_\ell M_{c_0}^{\frac{\ell}{c_0}}. \tag{2.7}$$

Assumption (B) yields that

$$d_n^{\ell-d_n} \leq \left(\frac{M_{d_n}^{\frac{1}{d_n}}}{c} \right)^{\ell-d_n}. \tag{2.8}$$

Hence, taking together (2.5), (2.6), (2.7), (2.8) and using the log-convexity of $\{M_n\}_n$, (1.3), we deduce that

$$\begin{aligned} F_\ell &\leq \max\left\{ 2e^{\ell(c_0-1)} \left(\frac{2M_{d_n}^{\frac{1}{d_n}}(c_0-1)}{c(b-a)} \right)^{\ell-d_n} M_{d_n},\ 2e^{\ell(c_0-1)} M_\ell M_{c_0}^{\frac{\ell}{c_0}} \right\} K^{\ell+1} \\ &\leq \max\left\{ 2e^{\ell(c_0-1)} \left(\frac{2(c_0-1)}{c(b-a))} \right)^{\ell-d_n},\ 2e^{\ell(c_0-1)} M_{c_0}^{\frac{\ell}{c_0}} \right\} K^{\ell+1} M_\ell \leq C_1^\ell K^{\ell+1} M_\ell, \end{aligned}$$

where

$$C_1 = 2e^{c_0-1}\left(\frac{2(c_0-1)}{c(b-a)} + M_{c_0}^{\frac{1}{c_0}} \right).$$

Then, we conclude that there exists $K_1 > 0$ such that

$$F_n \leq K_1^{n+1} M_n, \qquad \text{for every } n \in \mathbb{N},$$

that is, $f \in C^M([a,b])$. This completes our proof of Theorem 1.4.

2.3 | Proof of Theorem 1.6

The proof is based on the following result, which ensures the existence, in every class C^N, of a function that attains the bounds N_j. Although different formulations are already present in the literature (see, e.g., [10] for the case of complex-valued functions), for the sake of completeness we provide its proof.

Lemma 2.2. *Let $\{N_j\}_j$ be a positive sequence satisfying (1.1). Then, there exists $f \in C^N([a,b])$ such that*

$$\left| f^{(j)}\left(\frac{a+b}{2} \right) \right| \geq N_j, \qquad \forall j \in \mathbb{N}. \tag{2.9}$$

Proof. Set $m_j = N_{j+1}/N_j$ and observe that, by (1.1), it is an increasing sequence. We note that

$$\left(\frac{1}{m_k} \right)^{k-j} \leq \frac{N_j}{N_k} \qquad \forall k, j \in \mathbb{N}. \tag{2.10}$$

Indeed, (2.10) trivially holds in the case of $j = k$. Moreover, if $j < k$, we have that

$$\left(\frac{1}{m_k}\right)^{k-j} \le \frac{1}{m_{k-1} \dots m_{k-(k-j)}} = \frac{N_j}{N_k}.$$

Finally, if $j > k$, we obtain

$$m_k^{j-k} \le m_k \dots m_{j-1} = \frac{N_j}{N_k}.$$

Summing up, (2.10) holds true.

Let us define

$$g(x) = \sum_{k=0}^{\infty} \frac{N_k}{(2m_k)^k}(\cos(2m_k x) + \sin(2m_k x)).$$

We notice that $g \in C^N(I)$ for every interval $I \subset \mathbb{R}$. Indeed, in light of (2.10), we get

$$|g^{(n)}(x)| \le \sum_{k=0}^{\infty} \frac{N_k}{(2m_k)^k}(2m_k)^n 2 \le N_n \sum_{k=0}^{\infty} \frac{2^{n+1}}{2^k} \le 2^{n+2} N_n.$$

Furthermore, we have that

$$|g^{(n)}(0)| = \left| \left(\sum_{k=0}^{\infty} \frac{N_k}{(2m_k)^{k-n}} \right) (\cos^{(n)}(0) + \sin^{(n)}(0)) \right|,$$

and, since

$$\begin{cases} \cos^{(n)}(0) = (-1)^{n/2} \text{ and } \sin^{(n)}(0) = 0, & \text{for } n \text{ even}, \\ \cos^{(n)}(0) = 0 \text{ and } \sin^{(n)}(0) = (-1)^{\frac{n-1}{2}}, & \text{for } n \text{ odd}, \end{cases}$$

we deduce that

$$|g^{(n)}(0)| = \sum_{k=0}^{\infty} \frac{N_k}{(2m_k)^{k-n}} \ge N_n$$

The conclusion follows by taking

$$f(x) := g\left(x - \frac{a+b}{2}\right), \qquad (x \in [a, b]).$$

Then, the proof of Theorem 1.6 reduces to show that given a positive sequence $\{M_n\}_n$ satisfying (1.9), there exist

- a sequence $\{N_n\}_n$ satisfying (1.1),
- two divergent sequences of positive integers $\{d_n\}_n$ and $\{i_n\}_n$,

so that

$$N_{d_n} = M_{d_n}, \qquad n = 0, 1, \dots$$

and

$$\frac{N_{i_n}}{M_{i_n}} = 2^{2^{i_n}}. \tag{2.11}$$

We point out that once the existence of such a sequence $\{N_n\}_n$ is established, Lemma 2.2 yields the existence of a function $f \in C^N([a,b])$ such that $N_{d_n} \leq M_{d_n}$. On the other hand, due to (2.11) , we have that $f \notin C^M([a,b])$. In other words, in general (if the gap $(d_{n+1} - d_n)/d_n$ is suitably large), the interpolation between d_n and d_{n+1} does not provide the bounds on the derivatives of f ensuring that $f \in C^M([a,b])$.

We construct inductively the sequences $\{N_n\}_n$, $\{d_n\}_n$, and $\{i_n\}_n$. For this purpose, it is useful to define

$$m_n = \frac{N_{n+1}}{N_n}.$$

We observe that N_n satisfies (1.1) if and only if m_n is an increasing sequence; furthermore, without loss of generality, we may assume that $M_0 = N_0 = 1$. Clearly, we have

$$m_0 \cdot \ldots \cdot m_{j-1} = \frac{N_j}{N_0} = N_j.$$

Let us fix an arbitrary positive integer i_0. We claim that there exists a positive integer $d_0 > i_0$ so that setting

$$\begin{cases} m_0 = \cdots = m_{i_0-1} = (2^{2^{i_0}} M_{i_0})^{\frac{1}{i_0}}, \\ m_{i_0} = \cdots = m_{d_0-1} = \left(\dfrac{M_{d_0}}{2^{2^{i_0}} M_{i_0}} \right)^{\frac{1}{d_0-i_0}} \end{cases} \tag{2.12}$$

we have

$$m_j \leq m_{j+1}, \qquad \text{for } j = 0, \ldots, d_0 - 2. \tag{2.13}$$

We observe that (2.13) is equivalent to the inequality

$$(2^{2^{i_0}} M_{i_0})^{\frac{1}{i_0}} \leq \left(\frac{M_{d_0}}{2^{2^{i_0}} M_{i_0}} \right)^{\frac{1}{d_0-i_0}} \iff (2^{2^{i_0}} M_{i_0})^{\frac{1}{i_0}} \leq M_{d_0}^{\frac{1}{d_0}}.$$

In view of (1.9), for every $c > 0$ there exists d_0 such that $M_{d_0}^{\frac{1}{d_0}} \geq c$. Hence, we find that

$$M_{d_0}^{\frac{1}{d_0}} \geq c \geq (2^{2^{i_0}} M_{i_0})^{\frac{1}{i_0}},$$

provided that $c > (2^{2^{i_0}} M_{i_0})^{\frac{1}{i_0}}$, that is, (2.13) holds true.

We point out that as a consequence of our construction, we have

$$\begin{cases} N_j = (2^{2^{i_0}} M_{i_0})^{\frac{j}{i_0}}, & \text{for } j = 0, \ldots, i_0, \\ N_j = 2^{2^{i_0}} M_{i_0} \left(\dfrac{M_{d_0}}{2^{2^{i_0}} M_{i_0}} \right)^{\frac{j-i_0}{d_0-i_0}}, & \text{for } j = i_0 + 1, \ldots d_0, \end{cases} \tag{2.14}$$

$$\frac{N_{i_0}}{M_{i_0}} = 2^{2^{i_0}}, \quad \text{and} \quad N_{d_0} = M_{d_0}. \tag{2.15}$$

Let us show that there exist two positive integers $d_1 > i_1 > d_0$ so that setting

$$\begin{cases} m_{d_0} = \cdots = m_{i_1-1} = \left(\dfrac{2^{2^{i_1}} M_{i_1}}{M_{d_0}}\right)^{\frac{1}{i_1-d_0}}, \\ m_{i_1} = \cdots = m_{d_1-1} = \left(\dfrac{M_{d_1}}{2^{2^{i_1}} M_{i_1}}\right)^{\frac{1}{d_1-i_1}}, \end{cases} \tag{2.16}$$

we have that

$$m_j \le m_{j+1}, \qquad \text{for } j = 0, \dots, d_1 - 2. \tag{2.17}$$

We observe that (2.17) is equivalent to

$$\begin{cases} \left(\dfrac{2^{2^{i_1}} M_{i_1}}{M_{d_0}}\right)^{\frac{1}{i_1-d_0}} \ge \left(\dfrac{M_{d_0}}{2^{2^{i_0}} M_{i_0}}\right)^{\frac{1}{d_0-i_0}}, \\ \left(\dfrac{M_{d_1}}{2^{2^{i_1}} M_{i_1}}\right)^{\frac{1}{d_1-i_1}} \ge \left(\dfrac{2^{2^{i_1}} M_{i_1}}{M_{d_0}}\right)^{\frac{1}{i_1-d_0}}. \end{cases} \tag{2.18}$$

As for the first inequality in (2.18), it is enough to show that

$$\left(\frac{2^{2^{i_1}} M_{i_1}}{M_{d_0}}\right)^{\frac{1}{i_1-d_0}} \ge \left(M_{d_0}\right)^{\frac{1}{d_0-i_0}},$$

which is equivalent to

$$2^{2^{i_1}} M_{i_1} \ge M_{d_0}^{\frac{i_1-i_0}{d_0-i_0}}.$$

Once more, by (1.9), this last inequality can be satisfied provided $i_1 > d_0$ is chosen suitably large. Furthermore, the second inequality in (2.18) can be rewritten as

$$M_{d_1} \ge \frac{\left(2^{2^{i_1}} M_{i_1}\right)^{\frac{d_1-d_0}{i_1-d_0}}}{M_{d_0}^{\frac{d_1-i_1}{i_1-d_0}}},$$

which can be fulfilled, due to (1.9), provided that $d_1 > i_1$ is large enough.

Summing up, we have constructed a sequence N_j, $j = 0, \dots d_1$, such that $m_j = \frac{N_{j+1}}{N_j}$ is an increasing sequence for $j = 0, \dots, d_1$, and

$$N_{d_0} = M_{d_0}, \quad N_{d_1} = M_{d_1}, \quad N_{i_0}/M_{i_0} = 2^{2^{i_0}} \quad \text{and} \quad N_{i_1}/M_{i_1} = 2^{2^{i_1}},$$

for suitable positive integers $0 < i_0 < d_0 < i_1 < d_1$.

Now, let us suppose that we have already defined $d_0 < i_0 < \dots < i_n < d_n$, and N_j, $j = 1, \dots d_n$, such that

$$\begin{cases} m_j = \dfrac{N_{j+1}}{N_j} \quad \text{are increasing for } j = 0, \dots, d_n - 1, \\ N_{d_k} = M_{d_k}, \qquad k = 0, \dots, n, \\ \dfrac{N_{i_k}}{M_{i_k}} = 2^{2^{i_k}}, \qquad k = 0, \dots, n. \end{cases}$$

In order to complete the proof of Theorem 1.6 it suffices to show that we can find i_{n+1}, d_{n+1} and N_j, $j = d_n + 1, \dots, d_{n+1}$ such that

1. $d_n < i_{n+1} < d_{n+1}$,
2. $m_j = \frac{N_{j+1}}{N_j}$ are increasing for $j = 0, \dots, d_{n+1}$,
3. $N_{d_{n+1}} = M_{d_{n+1}}$ and $\frac{N_{i_{n+1}}}{M_{i_{n+1}}} = 2^{2^{i_{n+1}}}$.

Set

$$\begin{cases} m_{d_n} = \cdots = m_{i_{n+1}-1} = \left(\dfrac{2^{2^{i_{n+1}}} M_{i_{n+1}}}{M_{d_n}} \right)^{\frac{1}{i_{n+1}-d_n}}, \\ m_{i_{n+1}} = \cdots = m_{d_{n+1}-1} = \left(\dfrac{M_{d_{n+1}}}{2^{2^{i_{n+1}}} M_{i_{n+1}}} \right)^{\frac{1}{d_{n+1}-i_{n+1}}}, \end{cases} \tag{2.19}$$

we have that

$$m_j \le m_{j+1}, \qquad \text{for } j = 0, \dots, d_{n+1} - 2. \tag{2.20}$$

We observe that (2.20) reduces to verifying

$$\begin{cases} \left(\dfrac{2^{2^{i_{n+1}}} M_{i_{n+1}}}{M_{d_n}} \right)^{\frac{1}{i_{n+1}-d_n}} \ge m_{d_n-1}, \\ \left(\dfrac{M_{d_{n+1}}}{2^{2^{i_{n+1}}} M_{i_{n+1}}} \right)^{\frac{1}{d_{n+1}-i_{n+1}}} \ge \left(\dfrac{2^{2^{i_{n+1}}} M_{i_{n+1}}}{M_{d_n}} \right)^{\frac{1}{i_{n+1}-d_n}}. \end{cases} \tag{2.21}$$

We observe that the first inequality in (2.21) is equivalent to

$$M_{i_{n+1}} \ge \frac{M_{d_n} m_{d_n-1}^{i_{n+1}-d_n}}{2^{2^{i_{n+1}}}}$$

which, by (1.9), can be satisfied provided that $i_{n+1} > d_n$ is large enough. Finally, the second inequality in (2.21), can be rewritten as

$$M_{d_{n+1}} \ge \frac{\left(2^{2^{i_{n+1}}} M_{i_{n+1}}\right)^{\frac{d_{n+1}-d_n}{i_{n+1}-d_n}}}{M_{d_n}^{\frac{d_{n+1}-i_{n+1}}{i_{n+1}-d_n}}}$$

which, once more by (1.9), can be satisfied provided that $d_{n+1} > i_{n+1}$ is large enough. Finally, as a consequence of (2.19) we have:

$$\frac{N_{i_{n+1}}}{N_{d_n}} = m_{d_n} \cdot m_{d_n+1} \cdot \dots \cdot m_{i_{n+1}-1} = \frac{2^{2^{i_{n+1}}} M_{i_{n+1}}}{M_{d_n}}$$
$$\frac{N_{d_{n+1}}}{N_{d_n}} = m_{d_n} \cdot m_{d_n+1} \cdot \dots \cdot m_{d_{n+1}-1} = \frac{M_{d_{n+1}}}{M_{d_n}}$$

Since $N_{d_n} = M_{d_n}$, this completes our proof of Theorem 1.6. □

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CONFLICT OF INTEREST

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ENDNOTE

[1] We observe that, without loss of generality, we can take c_0 and d_n larger than the number m_0 in (1.4).

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APPENDIX A

Let $M_0 = 1, M_1, M_2, \ldots$ be a sequence of positive numbers. In this section, we consider the following assertions

(A) For $j < \ell < k$, $M_\ell \leq M_j^{\frac{k-\ell}{k-j}} M_k^{\frac{\ell-j}{k-j}}$;
(B) $M_n^2 \leq M_{n-1}M_{n+1}$, for $n \in \mathbb{N}_+$;
(C) $M_n^{\frac{1}{n}}$ is an increasing sequence for $n \in \mathbb{N}_+$.

We show that $(A) \iff (B)$ and $(B) \Rightarrow (C)$.

We observe that (B) can be rephrased by requiring that M_n/M_{n-1} is an increasing sequence for $n \in \mathbb{N}_+$. Indeed, dividing both sides of the inequality in (B) by $M_{n-1}M_n$, we deduce that (B) holds if and only if M_n/M_{n-1} is an increasing sequence for $n \in \mathbb{N}_+$.

Now, let us show that $(B) \Rightarrow (C)$. (B) implies that

$$\frac{M_{n+1}}{M_n} \geq \frac{M_n}{M_{n-1}} \geq \frac{M_{n-1}}{M_{n-2}} \geq \ldots \geq \frac{M_1}{M_0} = M_1.$$

In particular, since M_{n+1}/M_n is greater or equal than each of the factors $M_n/M_{n-1}, M_{n-1}/M_{n-2}, \dots, M_1$, we find that it is greater or equal than the geometrical mean of these factors, that is,

$$\frac{M_{n+1}}{M_n} \geq \left(\frac{M_n}{M_{n-1}}\frac{M_{n-1}}{M_{n-2}} \dots M_1\right)^{\frac{1}{n}} = M_n^{\frac{1}{n}}.$$

Then, we find that $M_{n+1} \geq M_n^{\frac{n+1}{n}}$, that is, $M_n^{\frac{1}{n}}$ is an increasing sequence. This completes the proof of $(B) \Rightarrow (C)$.

Now, let us consider the implication $(A) \Rightarrow (B)$.

Choosing $j = n-1$, $\ell = n$ and $k = n+1$, the inequality in (A) can be rewritten as $M_n \leq M_{n-1}^{\frac{1}{2}} M_{n+1}^{\frac{1}{2}}$. Then, taking the square of both sides of the above inequality we deduce that $(A) \Rightarrow (B)$.

In order to show that $(B) \Rightarrow (A)$, we need the following

Lemma A.1. *Let $M_0 = 1, M_1, M_2, \dots$ be a sequence of positive numbers and assume that Condition (B) above holds. Then, for every $n \in \mathbb{N}$, the sequence*

$$\mathbb{N}_+ \ni h \mapsto \left(\frac{M_{n+h}}{M_n}\right)^{\frac{1}{h}} \text{ is increasing.}$$

Remark A.2. The geometrical content of the above result is that a function is convex if and only if its slope is increasing.

Proof. The proof of the lemma proceeds by induction on h. We observe that

$$\left(\frac{M_{n+h+1}}{M_n}\right)^{\frac{1}{h+1}} \geq \left(\frac{M_{n+h}}{M_n}\right)^{\frac{1}{h}}$$

is equivalent to the inequality

$$M_{n+h} \leq M_n^{\frac{1}{h+1}} M_{n+h+1}^{\frac{h}{h+1}}, \tag{A.1}$$

for every $h \in \mathbb{N}_+$.

For $h = 1$, (A.1) reduces to (B). Now, let us suppose that the inequality (A.1) is satisfied for a suitable natural number h and let us show that it is satisfied for $h+1$. Indeed, (B) implies that

$$M_{n+h+1} \leq M_{n+h}^{\frac{1}{2}} M_{n+h+2}^{\frac{1}{2}}. \tag{A.2}$$

Furthermore, by the inductive assumption, we have that

$$M_{n+h}^{\frac{1}{2}} \leq M_n^{\frac{1}{2(h+1)}} M_{n+h+1}^{\frac{h}{2(h+1)}}$$

and plugging the above inequality in (A.2), we find

$$M_{n+h+1} \leq M_n^{\frac{1}{2(h+1)}} M_{n+h+1}^{\frac{h}{2(h+1)}} M_{n+h+2}^{\frac{1}{2}}.$$

This last inequality yields that

$$M_{n+h+1}^{\frac{h+2}{2(h+1)}} \leq M_n^{\frac{1}{2(h+1)}} M_{n+h+2}^{\frac{1}{2}},$$

that is,

$$M_{n+h+1} \leq M_n^{\frac{1}{h+2}} M_{n+h+2}^{\frac{h+1}{h+2}}$$

and the proof of the lemma is completed. □

We observe that (A) can be rewritten as

(A′) for $j < \ell < k$, $\log M_\ell \le \frac{k-\ell}{k-j} \log M_j + \frac{\ell-j}{k-j} M_k$.

Now, Lemma A.1 implies that for every $n \in \mathbb{N}$ the sequence

$$\mathbb{N}_+ \ni h \mapsto \frac{\log(M_{n+h}) - \log(M_n)}{h} \quad \text{is increasing.}$$

Then for $j < \ell < k$, we find that

$$\frac{\log(M_\ell) - \log(M_j)}{\ell - j} \le \frac{\log(M_k) - \log(M_j)}{k - j},$$

that is, (A') holds. This completes the proof of the equivalence $(A) \iff (B)$.

APPENDIX B

In this section, we provide all the computations needed to justify the claims done in Example 1.

(i) Let $s \ge 1$ and $M_n = n^{ns}$, for $n \in \mathbb{N}_+$. In order to show that M_n is log-convex it suffices to show that the function $f(x) = xs \log x$ is convex for $x \ge 1$ (it is clear that, due to the limitation $s \ge 1$, the factor s here is immaterial). Since $f^{(2)}(x) = s/x > 0$, for $x > 0$, we deduce that M_n is log-convex.

Let us verify that M_n satisfies (1.4), that is, there exists $m_0 \ge 0$ such that

$$M_j \le M_k^{\frac{j}{k}} M_i^{\frac{j}{i}}, \text{ for } i, k > m_0 \text{ with } i < j \text{ and } j/i < k.$$

In this case, (1.4) can be written as

$$j^{js} \le k^{js} i^{js}, \text{ for } i, k > m_0 \text{ with } i < j \text{ and } j/i < k,$$

that is,

$$j \le ki, \text{ for } i, k > m_0 \text{ with } i < j \text{ and } j/i < k,$$

which trivially holds true with $m_0 = 1$.

(ii) Let $s_1, s_2 \ge 1$, $M_0 = M_1 = 1$, $M_2 = \sqrt{M_3}$, and $M_n = n^{ns_1}(\log n)^{ns_2}$ $(n = 3, 4, \ldots)$. We claim that M_n is log-convex. Recalling that the log-convexity of M_n is equivalent to the fact that the sequence M_{n+1}/M_n is increasing, in order to prove that M_n is log-convex it suffices to verify that $M'_0 = M'_1 = 1$, $M'_2 = 3^{3s_1/2}$, $M'_n = n^{ns_1}$ $(n = 3, 4, \ldots)$ and $M''_0 = M''_1 = 1$, $M''_2 = (\log 3)^{3s_2/2}$, $M''_n = (\log n)^{ns_2}$ $(n = 3, 4, \ldots)$ are separately log-convex. In light of (i) above, M'_n is log-convex and the proof reduces to show that M''_n has the same regularity.

We have that

$$\frac{M''_1}{M''_0} \le \frac{M''_2}{M''_1} \iff 1 \le (\log 3)^{3s_2},$$

$$\frac{M''_2}{M''_1} \le \frac{M''_3}{M''_2} \iff M''_3 \le M''_3,$$

$$\frac{M''_3}{M''_2} \le \frac{M''_4}{M''_3} \iff M''^3_3 = (\log 3)^9 \le M''^2_4 = (\log 4)^8.$$

In order to prove this last inequality it suffices to show that

$$\log 3 \le \log 4 / \log 3. \tag{B.1}$$

Using the concavity of the logarithm, we have that

$$\log 3 \leq \log e + \frac{1}{e}(3-e) = \frac{3}{e} \leq \frac{3}{2+1/2} = \frac{6}{5}, \tag{B.2}$$

and

$$\log 3 \leq \log 4 - \frac{1}{4} \iff \frac{\log 4}{\log 3} \geq 1 + \frac{1}{4\log 3}.$$

The inequality above and (B.2) yield

$$\frac{\log 4}{\log 3} \geq 1 + \frac{5}{24}\left(> 1 + \frac{1}{5} \geq \log 3\right),$$

and (B.1) follows. It remains to show that $(\log n)^{ns_2}$ is log-convex for $n \geq 3$. Once more, this fact can be reduced to a one-variable problem: it suffices to verify that the function $f(x) = xs_2 \log(\log(x))$ is convex for $x \geq 3$. Now, $f^{(2)}(x) = s_2(\log x - 1)/[(\log x)^2 x] > 0$, for $x > e$, and the log-convexity of M''_n (and M_n) follows.

It remains to show that M_n satisfies (1.4). Clearly, it suffices to verify that n^{ns_1} and $(\log n)^{ns_2}$ satisfy separately (1.4). The factor n^{ns_1} was already treated in (i) above. Then the proof reduces to show that

$$(\log j)^{js_2} \leq (\log k)^{js_2}(\log i)^{js_2}, \text{ for } i,k > m_0 \text{ with } i < j \text{ and } j/i < k,$$

that is,

$$\log j \leq \log k \log i, \text{ for } i,k > m_0 \text{ with } i < j \text{ and } j/i < k. \tag{B.3}$$

Now, the inequality $j/i < k$ and the monotonicity of the logarithm imply that

$$\log j < \log i + \log k.$$

Now, taking $m_0 = e^2$, we have that

$$\log i + \log k \leq \log i \log k, \text{ for } i,k > m_0,$$

then (B.3), and this completes the proof that M_n satisfies (1.4).

Finally, the fact that the class $C^M([a,b])$ is quasi-analytic for $s_1 = s_2 = 1$ while it is non-quasi-analytic for $s_1 > 1$ or $s_2 > 1$, follows by the Denjoy–Carleman theorem (see, e.g., [8]). Indeed, since $M_n^{\frac{1}{n}}$ is increasing (in view of the log-convexity of M_n), the Denjoy–Carleman theorem can be stated as follows:

the class $C^M([a,b])$ is quasi-analytic if and only if

$$\sum_{n=1}^{\infty} \frac{1}{M_n^{\frac{1}{n}}} = +\infty. \tag{B.4}$$

Hence, since $\sum_{n=1}^{\infty} \frac{1}{M_n^{\frac{1}{n}}}$ behaves as

$$\sum_{n=3}^{\infty} \frac{1}{n^{s_1}(\log n)^{s_2}}, \qquad (s_1, s_2 \geq 1),$$

using the generalized integral associated with the series above, we conclude that the series in (B.4) diverges if and only if $s_1 = s_2 = 1$.

We point out that, by Lemma 2.2, the class $C^M([a,b])$ is a proper sub-class of $G^{s_1+\varepsilon}$, for every $\varepsilon > 0$.