On the Cauchy-Dirichlet problem for fully nonlinear equations with fractional time derivative

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Abstract

We consider quite general fully nonlinear mixed Cauchy-Dirichlet problems with a Caputo derivative $\mathbb{D}^\alpha$ with respect to the time variable and α in $(0, 2)$. Under natural conditions, we show the existence of a local solution u such that $\mathbb{D}^\alpha u$ and the second order space derivatives $D_{x_i x_j} u$ belong to the class $C^{\frac{\alpha\theta}{2},\theta}([0, T] \times \overline{\Omega})$, for some T positive, with $\theta \in (0, 1)$. Moreover, we show the uniqueness of global solutions in the same class of functions.

1 Introduction

se1

The aim of this paper is the study of fully nonlinear Cauchy-Dirichlet systems in the form

$$\begin{cases} \mathbb{D}^\alpha u(t,x) = F(t,x,(D^\rho_x u(t,x))_{|\rho|\le 2}), \quad t \in [0,T], x \in \Omega, \\ u(t,x') = g(t,x'), \quad (t,x') \in [0,T] \times \partial\Omega, \\ D^k_t u(0,x) = u_k(x), \quad x \in \overline{\Omega}, k \in \mathbb{N}_0, k < \alpha, \end{cases} \tag{1.1}$$

eq2.1

Here F is a real valued function with domain $[0, T_0] \times \overline{\Omega} \times \mathbb{R}^{N(n)}$, with $T_0 \in \mathbb{R}^+$, Ω is an open bounded subset of $\mathbb{R}^n$ with appropriately regular boundary $\partial\Omega$, $N(n) = n^2 + n + 1$. The main feature of F is that, if we indicate with $(t, x, p) = (t, x, (p_\rho)_{|\rho|\le 2})$ ($\rho \in \mathbb{N}^n_0$) the generic element of $[0, T_0] \times \overline{\Omega} \times \mathbb{R}^{N(n)}$,

$$\sum_{|\rho|=2} D_{p_\rho} F(t, \cdot, p)\xi^\rho \ge \nu(t, x, p)|\xi|^2 \quad \forall \xi \in \mathbb{R}^n$$

with ν continuous and positive (for precise assumptions see Section 3). $\mathbb{D}^\alpha$ indicates the Caputo time derivative of order $\alpha \in \mathbb{R}^+$, which is described in Definition 1.1.

Some variations of this situation are briefly discussed in Remarks 3.4 and 3.5.

Problems with fractional time derivatives arise as mathematical models of complex systems which exhibit anomalous diffusion, appearing, for example, in strongly porous materials and percolation clusters. A clear discussion of anomalous diffusion taking to fractional order derivatives is given, for example, in [8]. See also [10] for further applications and motivations.

Before illustrating more in detail the content of the present paper, we cite some literature dealing with nonlinear problems with fractional time derivatives.

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A general problem in the form (1.1) is studied in [9], looking for viscosity solutions in the case $\alpha \in (0,1)$, both with Dirichlet and Neumann boundary conditions. The assumptions on F, which is assumed to be merely continuous, allow a degenerate ellipticity. Assuming that a subsolution and a supersolution exist, a theorem of existence and uniqueness of a solution is proved. Such solution is (a priori) just continuous.

Abstract quasilinear evolution equations in the form

$$\begin{cases} \mathbb{D}^\alpha u(t) + A(u)u = f(u) + h(t), \\ \\ u(0) = x \end{cases} \tag{1.2}$$

eq0.3

are discussed in [3]. The authors work in continuous interpolation spaces allowing a singularity in $t = 0$, but such that, for u in the class of solutions, $u(0)$ is defined. Operators $A(u)$ are supposed to be positive (in the sense that their resolvent set contains $(-\infty, 0]$, which maximal decrease of $(\lambda - A(u))^{-1}$) and results of maximal regularity are proved for corresponding linear problems. These results are employed to prove the existence of local solutions of systems in the form (1.2) and are applied to quasilinear Cauchy-Dirichlet problems in the form

$$\begin{cases} \mathbb{D}^\alpha u(t) - (\sigma(u_x))_x = h(t,x), \quad x \in (0,1), \quad t \geq 0 \\ \\ u(0,x) = u_0(x), \quad x \in (0,1), \\ \\ u(t,0) = u(t,1) = 0 \end{cases} \tag{1.3}$$

eq1.3A

in case $h(\cdot, 0) = h(\cdot, 1) = 0$. Their abstract linear theory is applicable because the operator

$$A(u) = \sigma'(u_x) D_x^2$$

is positive in the class of Hölder continuous functions vanishing in $\{0,1\}$. Unfortunately, this does not happen in the larger class of Hölder continuous functions. So, if, for example, $h(\cdot, 0)$ or $h(\cdot, 1)$ do not vanish identically, the application of their abstract theory becomes problematic.

Zacher studies in [11] the quasilinear Cauchy-Dirichlet problem

$$\begin{cases} \mathbb{D}^\alpha u = \sum_{i=1}^n \sum_{j=1}^n D_{x_i}(a_{ij}(u) D_{x_j} u) = f, \quad t \in [0,T], x \in \Omega, \\ \\ u(t,x') = g(t,x'), \quad (t,x') \in [0,T] \times \partial\Omega, \\ \\ u(0,x) = u_0(x), \quad x \in \overline{\Omega}, k \in \mathbb{N}_0, k < \alpha, \end{cases}$$

with $\alpha \in (0,1)$, in an L^p setting. Notably, he proves a result of uniqueness and global existence of a solution.

A result of global existence for a problem similar to (1.3), with the Riemann-Liouville D_t^α replacing $\mathbb{D}^\alpha$ is proved in [2], under the condition $0 < \sigma_0 \leq \sigma'(y) \leq \sigma_1 < \infty$, and $\frac{\sigma_1}{\sigma_0}$ sufficiently small. The same thesis [2] contains a result of maximal regularity for the abstract equation (with a suitable initial condition)

$$D_t^\alpha u(t) + Au(t) \ni f(t),$$

with A $m-$accretive in the Hilbert space H.

The majority of papers dedicated to nonlinear evolution equations with fractional time derivative concerns semilinear systems. For them, we refer to the preprint [4], which contains a large bibliography. We mention also [1], where the authors study a problem in the form

$$\begin{cases} \mathbb{D}^\alpha u(t) = Au(t) + f(t, u(t)), t > 0, \quad t \geq 0, \\ \\ u(0) = u_0, \end{cases}$$

with A linear sectorial operator in the Banach space X. They prove the existence and uniqueness of a maximal local mild solution. It is considered the case that, for some interval $[\gamma_0, \gamma_1]$, f is Lipschitz continuous in the variable u from $X^{1+\epsilon} = D(A^{1+\epsilon})$ to $X^{\gamma(\epsilon)}$ with $\gamma(\epsilon) \geq \rho\epsilon$, $\rho > 1$. If X is ordered, for different functions f, called f_1 and f_2, with $f_1 \leq f_2$, a comparison between the corresponding solutions u_j is proved.

The aim of this paper is to show a result of local existence and global uniqueness of a solution to (1.1), extending the classical linear $C^{1+\frac{\theta}{2},2+\theta}$ theory in case $\alpha = 1$ to the case $\alpha \in (0,2)$ and fully nonlinear problems. In the linear case with $\alpha = 1$ necessary and sufficient conditions on f, g,u_0, g are known in order that there exists a unique solution u in the class $C^{1+\frac{\theta}{2},2+\theta}$ (see [7], Theorem 5.1.16). The fully nonlinear case was studied in [7], chapter 8.5.3, with first order boundary conditions, by the method of linearization. This method can be easily adapted to the case of Dirichlet boundary conditions.

Concerning the case $\alpha \neq 1$, the problem has to be slightly reformulated; the belonging of u to $C^{1+\frac{\theta}{2},2+\theta}([0,T] \times \overline{\Omega})$ is equivalent to the belonging of $D_t u$ and $D_{x_i x_j} u$ to $C^{\frac{\theta}{2},\theta}([0,T] \times \overline{\Omega})$ $(1 \leq i,j \leq n$, the space dimension of $\Omega)$, but its natural generalization for $\alpha \in (0,2)$ that the belonging of u to $C^{\alpha+\frac{\alpha\theta}{2},2+\theta}([0,T] \times \overline{\Omega})$ is equivalent to the belonging of $\mathbb{D}^\alpha u$ and $D_{x_i x_j} u$ to $C^{\frac{\alpha\theta}{2},\theta}([0,T] \times \overline{\Omega})$ is false (see Remark 2.3) and, in fact, no regularity in time better than C^α can be, in general, expected. So we shall look for a solution u such that $\mathbb{D}^\alpha u$ and $D_{x_i x_j} u$ belong to $C^{\frac{\alpha\theta}{2},\theta}([0,T] \times \overline{\Omega})$. It turns out that these conditions imply that $u \in C^{\frac{\alpha\theta}{2}}([0,T]; C^2(\overline{\Omega})) \cap B([0,T]; C^{2+\theta}(\overline{\Omega}))$ (B stands for "bounded", see the following for the notation). So we shall employ the linearization method, looking for a fixed point to which the contraction mapping theorem can be applied not in the class $C^{\alpha+\frac{\alpha\theta}{2},2+\theta}([0,T] \times \overline{\Omega})$, but in the larger class $C^{\frac{\alpha\theta}{2}}([0,T]; C^2(\overline{\Omega})) \cap B([0,T]; C^{2+\theta}(\overline{\Omega}))$. Such fixed point u has $\mathbb{D}^\alpha u$ in $C^{\frac{\alpha\theta}{2},\theta}([0,T] \times \overline{\Omega})$ and so our result is a generalization of the result described in the case $\alpha = 1$. The main tool of this project is Theorem 2.1, which is an extension to the case $\alpha \in (0,2)$ of the classical maximal regularity result prescribing necessary and sufficient conditions for solutions in the class $C^{1+\frac{\theta}{2},2+\theta}([0,T] \times \overline{\Omega})$ in the linear case, with $\alpha = 1$.

The content of the paper is the following: in this Section 1 we introduce some notations and basic results which we shall employ.

Section 2 contains the linear theory. We begin by stating the crucial Theorem 2.1 and Proposition 2.2, and extend them, with Theorem 2.8, to the nonautonomous case. In Section 3, we consider the nonlinear problem (1.1), showing a result of existence of a local solution (Theorem 3.2) and a result of uniqueness of a global solution (Theorem 3.3), with possible extensions to more general domains of F (Remark 3.4) and to certain quasilinear problems (Remark 3.5).

Now we introduce some notations which we are going to use in the paper.

C will indicate a positive real constant we are not interested to precise (the meaning of which may be different from time to time). In a sequence of inequalities, we shall write $C_0, C_1, C_2, \ldots$. If C depends on α, we shall write $C(\alpha)$ or $C_0(\alpha), C_1(\alpha), C_2(\alpha), \ldots$. Analogously, if δ is a parameter depending on $\alpha, \beta, \ldots$, we shall write $\delta(\alpha, \beta, \ldots)$.

If $\alpha \in \mathbb{R}$, $[\alpha]$ will indicate the maximum integer less or equal than α. $\mathbb{R}^+$ will indicate the set of (strictly) positive real numbers.

If f is a function defined in the cartesian product $[0,T] \times A$, we shall not distinguish between f and the function $t \to f(t,\cdot)$, which has domain $[0,T]$ and may be with values in some space of functions with domain A.

If X is a complex Banach space with norm $\|\cdot\|$, $B([0,T];X)$ will indicate the class of bounded functions with values in X with domain $[0,T]$; if A is a (generally unbounded) linear operator from $D(A) \subseteq X$ to X, $\rho(A)$ will indicate the resolvent set of A. If A is a closed operator in X, $A : D(A)(\subseteq X) \to X$, $D(A)$, equipped with the norm

$$\|x\|_{D(A)} := \|x\| + \|Ax\|$$

is a Banach space.

If X and Y are Banach spaces, we shall indicare with $\mathcal{L}(X,Y)$ the Banach space of linear, bounded operators from X to Y, equipped with its natural norm. In case $X = Y$, we shall simply write $\mathcal{L}(X)$.

Given a function f with domain $\overline{\Omega}$, with Ω subset of $\mathbb{R}^n$, γf will indicate the trace of f on the boundary $\partial\Omega$ of Ω.

If $\beta \in \mathbb{N}_0$ and Ω is an open, bounded subset of $\mathbb{R}^n$, we shall indicate with $C^\beta(\overline{\Omega})$ the class of complex valued functions which are continuous in $\overline{\Omega}$, together with their derivatives (extensible by continuity to $\overline{\Omega}$) of order not exceeding β. If $\beta \in \mathbb{R}^+ \setminus \mathbb{N}$, $C^\beta(\overline{\Omega})$ will indicate the class of functions in $C^{[\beta]}(\overline{\Omega})$ whose derivatives of order $[\beta]$ are Hölder continuous of order $\beta - [\beta]$ in $\overline{\Omega}$. We shall equip $C^\beta(\overline{\Omega})$ with the norm

$$\|f\|_{C^\beta(\overline{\Omega})} := \max\{\max_{\|\rho| \le [\beta]} \|D^\rho f\|_{C(\overline{\Omega})}, \max_{|\rho|=[\beta]} \sup_{x,y\in\overline{\Omega}, x\neq y} \frac{|D^\rho f(x) - D^\rho f(y)|}{|x-y|^{\beta-[\beta]}}\}. \quad (1.4)$$ eq2.1A

These definitions admit natural extensions to function with values in a Banach space X. In this case, we shall use the notation $C^\beta(\overline{\Omega}; X)$ (in particular $C^\beta([a,b]; X)$ in case $\Omega = (a,b) \subseteq \mathbb{R}$). All these classes will be assumed to be equipped with natural norms, obtained from (1.4) replacing the absolute value with the norm in X.

By local charts, if $\partial\Omega$ is sufficiently regular, we can consider also the spaces $C^\beta(\partial\Omega; X)$.

If $\alpha, \beta \in [0, \infty)$, $T \in \mathbb{R}^+$ and Ω is an open bounded subset of $\mathbb{R}^n$, we set

$$C^{\alpha,\beta}([0,T] \times \overline{\Omega}) := C^\alpha([0,T]; C(\overline{\Omega})) \cap B([0,T]; C^\beta(\overline{\Omega})).$$

This space will be equipped with the norm

$$\|f\|_{C^{\alpha,\beta}([0,T]\times\overline{\Omega})} := \max\{\|f\|_{C^\alpha([0,T];C(\overline{\Omega}))}, \|f\|_{B([0,T];C^\beta(\overline{\Omega}))}\}.$$

An analogous meaning will have $C^{\alpha,\beta}([0,T] \times \partial\Omega)$. If X is a Banach space, $Lip([0,T]; X)$ will indicate the class of Lipschitz continuous functions from $[0,T]$, equipped with the natural norm

$$\|f\|_{Lip([0,T];X)} := \max\{\|f\|_{C([0,T];X)}, \sup_{s,t\in[0,T], s\neq t} \frac{\|f(t) - f(s)\|}{|t-s|}\}.$$

We pass to define the Caputo time derivative of order $\alpha \in \mathbb{R}^+$. Such derivative can be introduced in different ways (see, for example, [10] 2.4). We adopt the following definition: if $f \in C([0,T]; X)$ (with X Banach space), we introduce the operator J_α defined as

$$J_\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds.$$

It turns out that J_α is injective and, if $\alpha \ge m$, with $m \in \mathbb{N}_0$, $J_\alpha f \in C^m([0,T]; X)$, and $(J_m f)^{(m)} = f$. This justifies the following

de2.2 **Definition 1.1.** *Let $\alpha \in \mathbb{R}^+$, $m = [\alpha]$ (the integer part of α), $u \in C^m([0,T]; X)$, with $T \in \mathbb{R}^+$ and X Banach space. We write that the Caputo derivative $\mathbb{D}^\alpha u$ is defined (in $[0,T]$) if there exists $f \in C([0,T]; X)$ such that*

$$u(t) - \sum_{j\in\mathbb{N}_0, j<\alpha} \frac{t^j}{j!} u^{(j)}(0) = J_\alpha f(t) \quad \forall t \in [0,T].$$

In this case we define

$$\mathbb{D}^\alpha u := f.$$

Of course, in case $\alpha \in \mathbb{N}$, $\mathbb{D}^\alpha u$ is defined if and only if $u \in C^\alpha([0,T]; X)$ and $\mathbb{D}^\alpha u$ coincides with the classical derivative $D^\alpha u$. It is also easily seen that, if $0 < T < T_1$ and $\mathbb{D}^\alpha u$ is defined in $[0, T_1]$, it is defined also in $[0,T]$ and the derivative in $[0,T]$ is the restriction of the derivative in $[0, T_1]$.

We shall employ also the following

pr2.3 **Proposition 1.2.** *Let Ω be an open, bounded subset in $\mathbb{R}^n$, locally lying on one side of its boundary $\partial\Omega$, which is a submanifold of $\mathbb{R}^n$ of dimension $n-1$ and class C^r, with $r \ge 1$. Let $0 \le \beta_0 < \beta_1 \le r$. Then, $\forall \xi \in (0,1)$ $C^{(1-\xi)\beta_0+\xi\beta_1}(\overline{\Omega}) \in J_\xi(C^{\beta_0}(\overline{\Omega}), C^{\beta_1}(\overline{\Omega}))$, that is, there exists C positive such that, if $f \in C^{\beta_1}(\overline{\Omega})$,*

$$\|f\|_{C^{(1-\xi)\beta_0+\xi\beta_1}(\overline{\Omega})} \le C \|f\|^{1-\xi}_{C^{\beta_0}(\overline{\Omega})} \|f\|^{\xi}_{C^{\beta_1}(\overline{\Omega})};$$

the same holds if we replace $\overline{\Omega}$ with $\partial\Omega$.

Proof. See [5], Proposition 1.1.

□

We shall employ also the following result of extension:

pr2.5 **Proposition 1.3.** *Let Ω fulfill the conditions of Propositin 1.2. Then there exists an element R of $\mathcal{L}(C(\partial\Omega), C(\overline{\Omega}))$ such that $\gamma(Rg) = g \ \forall g \in C(\partial\Omega)$. Moreover, $\forall \xi \in [0, r]$ the restriction of R to $C^{\xi}(\partial\Omega)$ belongs to $\mathcal{L}(C^{\xi}(\partial\Omega), C^{\xi}(\overline{\Omega}))$.*

For a proof, see [6], Lemma 2.1(IV).

2 Linear equations

se2A

We begin by considering the linear problem

$$\begin{cases} \mathbb{D}^{\alpha} u(t,x) = A(x, D_x)u(t,x) + f(t,x), \quad t \in [0,T], x \in \Omega, \\ u(t,x') = g(t,x'), \quad (t,x') \in [0,T] \times \partial\Omega, \\ D_t^k u(0,x) = u_k(x), \quad x \in \overline{\Omega}, k \in \mathbb{N}_0, k < \alpha. \end{cases} \tag{2.1}$$ eq1.3

(A1) Ω is an open, bounded subset in $\mathbb{R}^n$ lying on one side of its boundary $\partial\Omega$, which is a $n-1-$submanifold of $\mathbb{R}^n$ of class $C^{2+\theta}$, with $\theta \in (0,2) \setminus \{1\}$.

(A2) $\alpha \in (0,2)$, $A(x, D_x) = \sum_{|\rho| \leq 2} a_{\rho}(x) D_x^{\rho}$, with $a_{\rho} \in C^{\theta}(\overline{\Omega})$, a_{ρ} complex valued; $A(x, D_x)$ is assumed to be elliptic, in the sense that $\sum_{|\rho|=2} a_{\rho}(x)\xi^{\rho} \neq 0 \ \forall \xi \in \mathbb{R}^n \setminus \{0\}$; moreover,

$$|Arg(\sum_{|\rho|=2} a_{\rho}(x)\xi^{\alpha})| < (1 - \frac{\alpha}{2})\pi, \quad \forall x \in \overline{\Omega}, \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

(A3) $\alpha\theta < 2$, $\theta \neq \frac{2}{\alpha} - 1$.

The following result is proved in [6], Theorem 1.2:

th1.2 **Theorem 2.1.** *Suppose that (A1)-(A3) hold. Then the following conditions are necessary and sufficient in order that (2.1) has a unique solution u in $C([0,T]; C^2(\overline{\Omega})) \cap B([0,T]; C^{2+\theta}(\overline{\Omega}))$, with $\mathbb{D}^{\alpha} u, A(\cdot, D_x)u$ belonging to $C^{\frac{\alpha\theta}{2},\theta}([0,T] \times \overline{\Omega})$:*
(I) $f \in C^{\frac{\alpha\theta}{2},\theta}([0,T] \times \overline{\Omega})$;
(II) if $k \in \mathbb{N}_0$, $k < \alpha$, $u_k \in C^{\theta + 2(1 - \frac{k}{\alpha})}(\overline{\Omega})$.
(III) $g \in C([0,T]; C^2(\partial\Omega)) \cap B([0,T]; C^{2+\theta}(\partial\Omega))$, $\mathbb{D}^{\alpha} g$ exists and belongs to $C^{\frac{\alpha\theta}{2},\theta}([0,T] \times \partial\Omega)$;
(IV) if $k \in \mathbb{N}_0$, $k < \alpha$, $\gamma u_k = D_t^k g(0)$;
(V) $\gamma[A(\cdot, D_x)u_0 + f(0)] = \mathbb{D}^{\alpha} g(0)$.

The following regularity result will be employed:

pr1.5A **Proposition 2.2.** *Suppose that (A1) holds and let $u \in C([0,T]; C^2(\overline{\Omega})) \cap B([0,T]; C^{2+\theta}(\overline{\Omega}))$, $\alpha \in (0,2)$, $\mathbb{D}^{\alpha} u \in C([0,T]; C(\overline{\Omega})) \cap B([0,T]; C^{\theta}(\overline{\Omega}))$. Then $u \in C^{\frac{\alpha\theta}{2}}([0,T]; C^2(\overline{\Omega}))$. A similar result holds if $\overline{\Omega}$ is replaced by $\partial\Omega$.*

Proof. See [6], Lemma 3.1.

□

re1.6 **Remark 2.3.** As already observed, if the assumptions of Theorem 2.1 are satisfied and $\alpha = 1$, the solution u belongs to $C^{1+\frac{\theta}{2}}([0,T]; C(\overline{\Omega}))$, so that $u \in C^{1+\frac{\theta}{2}, 2+\theta}([0,T] \times \overline{\Omega})$. If $\alpha \neq 1$, in general u does not belong to $C^{\alpha + \frac{\alpha\theta}{2}}([0,T]; C(\overline{\Omega}))$. See for this Remark 4.2 in [6].

We shall need also some information concerning the dependence of the solution u of (2.1) on the data and T. Moreover, we want to study a nonautonomous version of (2.1). So we begin with the following proposition, concerning the dependence of some norms of the solution on T.

pr1.5 **Proposition 2.4.** *Consider problem (2.1), with the assumptions (A1)-(A3). Let* $T_0 \in \mathbb{R}^+$*. Then there exists* $C(T_0)$ *in* $\mathbb{R}^+$ *such that,* $\forall T \in (0, T_0]$*, the solution* u *as in Theorem 2.1 satisfies the estimate*

$$\begin{aligned}&\|\mathbb{D}^\alpha u\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\overline{\Omega})} + \|u\|_{B([0,T];C^{2+\theta}(\overline{\Omega}))} + \|u\|_{C^{\frac{\alpha\theta}{2}}([0,T];C^2(\overline{\Omega}))} \\ &\le C(T_0)(\|f\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\overline{\Omega})} + \textstyle\sum_{k<\alpha} \|u_k\|_{C^{\theta+2(1-\frac{k}{\alpha})}(\overline{\Omega})} \\ &+\|\mathbb{D}^\alpha g\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\partial\Omega)} + \|g\|_{B([0,T];C^{2+\theta}(\partial\Omega))} + \|g\|_{C^{\frac{\alpha\theta}{2}}([0,T];C^2(\partial\Omega))}).\end{aligned} \tag{2.2}$$

eq1.4A

Proof. Let R be the operator described in Proposition 1.3. We set

$$v(t, x) := R[g(t, \cdot)](x).$$

Then $v \in C^{\frac{\alpha\theta}{2}}([0, T]; C^2(\overline{\Omega})) \cap B([0, T]; C^{2+\theta}(\overline{\Omega}))$, $\mathbb{D}^\alpha v$ is defined and belongs to $C^{\frac{\alpha\theta}{2},\theta}([0, T] \times \overline{\Omega})$. In fact, for example,

$$g(t, \cdot) - \sum_{k<\alpha} \frac{t^k}{k!} D_t^k g(0) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \mathbb{D}^\alpha g(s, \cdot) ds,$$

so that

$$v(t, \cdot) - \sum_{k<\alpha} \frac{t^k}{k!} D_t^k v(0) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} R(\mathbb{D}^\alpha g(s, \cdot)) ds$$

and

$$\mathbb{D}^\alpha v = R(\mathbb{D}^\alpha g).$$

So we have

$$\begin{aligned}&\|v\|_{B([0,T];C^{2+\theta}(\overline{\Omega}))} + \|v\|_{C^{\frac{\alpha\theta}{2}}([0,T];C^2(\overline{\Omega}))} + \|\mathbb{D}^\alpha v\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\overline{\Omega})} \\ &\le C_0(\|g\|_{B([0,T];C^{2+\theta}(\partial\Omega))} + \|g\|_{C^{\frac{\alpha\theta}{2}}([0,T];C^2(\partial\Omega))} + \|\mathbb{D}^\alpha g\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\partial\Omega)}),\end{aligned} \tag{2.3}$$

eq2.2A

with C_0 independent of T. We deduce

$$\begin{aligned}\|A(\cdot, D_x)v\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\overline{\Omega})} &\le C_1(\|v\|_{C^{\frac{\alpha\theta}{2}}([0,T];C^2(\overline{\Omega}))} + \|v\|_{B([0,T];C^{2+\theta}(\overline{\Omega}))}) \\ &\le C_2(\|g\|_{C^{\frac{\alpha\theta}{2}}([0,T];C^2(\partial\Omega))} + \|g\|_{B([0,T];C^{2+\theta}(\partial\Omega))}),\end{aligned} \tag{2.4}$$

eq2.4

with C_2 independent of T. By difference we have:

$$\begin{cases} \mathbb{D}^\alpha(u - v)(t, x) = A(x, D_x)(u - v)(t, x) + \phi(t, x), & t \in [0, T], x \in \Omega, \\ (u - v)(t, x') = 0, \quad (t, x') \in [0, T] \times \partial\Omega, \\ D_t^k(u - v)(0, x) = u_k(x) - R\gamma u_k(x), & x \in \overline{\Omega}, k \in \mathbb{N}_0, k < \alpha, \end{cases}$$

with

$$\phi(t, x) = f(t, x) - (\mathbb{D}^\alpha v(t, x) - A(x, D_x)v(t, x)).$$

We set

$$\tilde{\phi}(t, \cdot) = \begin{cases} \phi(t, \cdot) & \text{if} \quad t \in [0, T], \\ \phi(T, \cdot) & \text{if} \quad t \in [T, T_0]. \end{cases}$$

Then $\tilde{\phi} \in C^{\frac{\alpha\theta}{2},\theta}([0, T_0] \times \overline{\Omega}))$. We consider the problem

$$\begin{cases} \mathbb{D}^\alpha z(t, x) = A(x, D_x)z(t, x) + \tilde{\phi}(t, x), & t \in [0, T_0], x \in \Omega, \\ z(t, x') = 0, \quad (t, x') \in [0, T] \times \partial\Omega, \\ D_t^k z(0, x) = u_k(x) - R\gamma u_k(x), & x \in \overline{\Omega}, k \in \mathbb{N}_0, k < \alpha, \end{cases} \tag{2.5}$$

th2.4

By Theorem 2.1, (2.5) has a unique solution z in $C([0,T_0];C^2(\overline{\Omega}))\cap B([0,T_0];C^{2+\theta}(\overline{\Omega}))$ with $\mathbb{D}^\alpha z$ belonging to $C^{\frac{\alpha\theta}{2},\theta}([0,T_0]\times\overline{\Omega})$. Moreover, as

$$\|\tilde{\phi}\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T_0]\times\overline{\Omega}))} = \|\phi\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\overline{\Omega}))},$$

$$\|\mathbb{D}^\alpha z\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T_0]\times\overline{\Omega}))} + \|z\|_{B([0,T_0];C^{2+\theta}(\overline{\Omega}))} + \|z\|_{C^{\frac{\alpha\theta}{2}}([0,T_0];C^2(\overline{\Omega}))}$$

$$\leq C_0(T_0)(\|\tilde{\phi}\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T_0]\times\overline{\Omega}))} + \textstyle\sum_{k<\alpha}\|u_k - R\gamma u_k\|_{C^{\theta+2(1-\frac{k}{\alpha})}(\overline{\Omega})})$$

$$\leq C_1(T_0)(\|\phi\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\overline{\Omega}))} + \textstyle\sum_{k<\alpha}\|u_k\|_{C^{\theta+2(1-\frac{k}{\alpha})}(\overline{\Omega})})$$

$$\leq C_2(T_0)(\|f\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\overline{\Omega}))} + \|g\|_{C^{\frac{\alpha\theta}{2}}([0,T];C^2(\partial\Omega))} + \|g\|_{B([0,T];C^{2+\theta}(\partial\Omega))} + \|\mathbb{D}^\alpha g\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T_0]\times\partial\Omega))}$$

$$+ \textstyle\sum_{k<\alpha}\|u_k\|_{C^{\theta+2(1-\frac{k}{\alpha})}(\overline{\Omega})}). \tag{2.6}$$

eq2.5A

So, from (2.3) and (2.6) we deduce

$$\|\mathbb{D}^\alpha u\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\overline{\Omega}))} + \|u\|_{B([0,T];C^{2+\theta}(\overline{\Omega}))} + \|u\|_{C^{\frac{\alpha\theta}{2}}([0,T];C^2(\overline{\Omega}))}$$

$$\leq \|\mathbb{D}^\alpha v\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\overline{\Omega}))} + \|v\|_{B([0,T];C^{2+\theta}(\overline{\Omega}))} + \|v\|_{C^{\frac{\alpha\theta}{2}}([0,T];C^2(\overline{\Omega}))}$$

$$+\|\mathbb{D}^\alpha z\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T_0]\times\overline{\Omega}))} + \|z\|_{B([0,T_0];C^{2+\theta}(\overline{\Omega}))} + \|z\|_{C^{\frac{\alpha\theta}{2}}([0,T_0];C^2(\overline{\Omega}))}$$

$$\leq C(T_0)(||g||_{B([0,T];C^{2+\theta}(\partial\Omega))} + |||g||_{C^{\frac{\alpha\theta}{2}}([0,T];C^2(\partial\Omega))} + \|\mathbb{D}^\alpha g\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\partial\Omega)}$$

$$+\|f\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\overline{\Omega}))} + \textstyle\sum_{k<\alpha}\|u_k\|_{C^{\theta+2(1-\frac{k}{\alpha})}(\overline{\Omega})})$$

on account of (2.3) and (2.4).

□

We examine the variation of the constant $C(T_0)$ in (2.2) as we modify $A(x, D_x)$.

le2.5 **Lemma 2.5.** *Suppose that the conditions (A1)-(A3) are fulfilled. Let $T_0 \in \mathbb{R}^+$, $0 < T \leq T_0$ and let f, u_k $(k < \alpha), g$ satisfy the conditions (I)-(IV) in the statement of Theorem 2.1. Let $B(x, D_x) = \sum_{|\rho|\leq 2} b_\rho(x) D_x^\rho$, with $b_\rho \in C^\theta(\overline{\Omega})$ and*

$$\max_{|\rho|\leq 2}\|b_\rho - a_\rho\|_{C(\overline{\Omega})} \leq \delta, \quad \max_{|\rho|\leq 2}\|b_\rho - a_\rho\|_{C^\theta(\overline{\Omega})} \leq R,$$

with δ, R positive. We consider the system

$$\begin{cases} \mathbb{D}^\alpha u(t,x) = B(x,D_x)u(t,x) + f(t,x), \quad t \in [0,\tau], x \in \Omega, \\ u(t,x') = g(t,x'), \quad (t,x') \in [0,T]\times\partial\Omega, \\ D_t^k u(0,x) = u_k(x), \quad x \in \overline{\Omega}, k \in \mathbb{N}_0, k < \alpha, \end{cases} \tag{2.7}$$

eq2.5

with $\gamma[B(\cdot, D_x)u_0 + f(0)] = \mathbb{D}^\alpha g(0)$.

Then there exists δ_0 positive, depending on $(a_\rho)_{\rho\leq 2}$, R and T_0, such that, if $\delta \leq \delta_0$, (2.7) has a unique solution u in $C([0,T];C^2(\overline{\Omega})) \cap B([0,T];C^{2+\theta}(\overline{\Omega}))$, with $\mathbb{D}^\alpha u, B(\cdot, D_x)u$ belonging to $C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\overline{\Omega})$; moreover,

$$\|\mathbb{D}^\alpha u\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\overline{\Omega}))} + \|u\|_{B([0,T];C^{2+\theta}(\overline{\Omega}))} + \|u\|_{C^{\frac{\alpha\theta}{2}}([0,T];C^2(\overline{\Omega}))}$$

$$\leq C(T_0, R, (a_\rho)_{|\rho|\leq 2})(\|f\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\overline{\Omega}))} + \textstyle\sum_{k<\alpha}\|u_k\|_{C^{\theta+2(1-\frac{k}{\alpha})}(\overline{\Omega})}$$

$$+\|\mathbb{D}^\alpha g\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\partial\Omega))} + \|g\|_{B([0,T];C^{2+\theta}(\partial\Omega))} + \|g\|_{C^{\frac{\alpha\theta}{2}}([0,T];C^2(\partial\Omega))}).$$

Proof. We write (2.7) in the form

$$\begin{cases} \mathbb{D}^\alpha u(t,x) = A(x,D_x)u(t,x) + [B(x,D_x) - A(x,D_x)]u(t,x) + f(t,x), \quad t \in [0,\tau], x \in \Omega, \\ u(t,x') = g(t,x'), \quad (t,x') \in [0,T] \times \partial\Omega, \\ D_t^k u(0,x) = u_k(x), \quad x \in \overline{\Omega}, k \in \mathbb{N}_0, k < \alpha. \end{cases}$$

It is easily seen that, if δ is sufficiently small, $B(x,D_x)$ satisfies the condition (A2). So, by Theorem 2.1, a solution u with the declared regularity exists. From (2.2) and Proposition 2.4

$$\|\mathbb{D}^\alpha u\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\overline{\Omega}))} + \|u\|_{B([0,T];C^{2+\theta}(\overline{\Omega}))} + \|u\|_{C^{\frac{\alpha\theta}{2}}([0,T];C^2(\overline{\Omega}))}$$

$$\leq C(\theta,T_0)(\|f\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\overline{\Omega}))} + \textstyle\sum_{k<\alpha} \|u_k\|_{C^{\theta+2(1-\frac{k}{\alpha})}(\overline{\Omega})}$$

$$+\|\mathbb{D}^\alpha g\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\partial\Omega))} + \|g\|_{B([0,T];C^{2+\theta}(\partial\Omega))} + \|g\|_{C^{\frac{\alpha\theta}{2}}([0,T];C^2(\partial\Omega))}$$

$$+\|[B(\cdot,D_x) - A(\cdot,D_x)]u\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\overline{\Omega}))}).$$

and, employing Proposition 1.2,

$$\|[B(\cdot,D_x) - A(\cdot,D_x)]u\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\overline{\Omega}))}$$

$$\leq \textstyle\sum_{|\rho|\leq 2}(\|a_\rho - b_\rho\|_{C(\overline{\Omega})}\|D_x^\rho u\|_{C^{\frac{\alpha\theta}{2}}([0,T];C(\overline{\Omega}))} + \sum_{j=0}^{[\theta]} \|a_\rho - b_\rho\|_{C^j(\overline{\Omega})}\|D^\rho u\|_{B([0,T];C^{\theta-j}(\overline{\Omega}))}$$

$$+ \textstyle\sum_{j=0}^{[\theta]} \|a_\rho - b_\rho\|_{C^{j+\theta-[\theta]}(\overline{\Omega})}\|D^\rho u\|_{B([0,T];C^{[\theta]-j}(\overline{\Omega}))})$$

$$\leq C(n,\Omega)(\delta\|u\|_{C^{\frac{\alpha\theta}{2}}([0,T];C^2(\overline{\Omega}))} + \textstyle\sum_{j=0}^{[\theta]} \delta^{1-\frac{j}{\theta}} R^{\frac{j}{\theta}} \|u\|_{B([0,T];C^{\theta-j+2}(\overline{\Omega}))}$$

$$+ \textstyle\sum_{j=0}^{[\theta]} \delta^{\frac{[\theta]-j}{\theta}} R^{1-\frac{[\theta]-j}{\theta}} \|u\|_{B([0,T];C^{2+[\theta]-j)}(\overline{\Omega}))})$$

$$\leq C(n,\Omega)[\omega_\theta(\delta,R)(\|u\|_{C^{\frac{\alpha\theta}{2}}([0,T];C^2(\overline{\Omega}))} + \|u\|_{B([0,T];C^{2+\theta}(\overline{\Omega}))}) + R\|u\|_{B([0,T];C^2(\overline{\Omega}))}],$$

with $\lim_{\delta\to 0} \omega_\theta(R,\delta) = 0$. Taking δ so small that $C(\theta,T_0)C(n,\Omega)\omega_\theta(\delta,R) \leq \frac{1}{2}$, we deduce

$$\|\mathbb{D}^\alpha u\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\overline{\Omega}))} + \|u\|_{B([0,T];C^{2+\theta}(\overline{\Omega}))} + \|u\|_{C^{\frac{\alpha\theta}{2}}([0,T];C^2(\overline{\Omega}))}$$

$$\leq 2C(\theta,T_0)(\|f\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\overline{\Omega}))} + \textstyle\sum_{k<\alpha} \|u_k\|_{C^{\theta+2(1-\frac{k}{\alpha})}(\overline{\Omega})}$$

$$+\|\mathbb{D}^\alpha g\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\partial\Omega))} + \|g\|_{B([0,T];C^{2+\theta}(\partial\Omega))} + \|g\|_{C^{\frac{\alpha\theta}{2}}([0,T];C^2(\partial\Omega))}$$

$$+C(n,\Omega)R\|u\|_{B([0,T];C^2(\overline{\Omega}))}).$$

Now we fix θ' in $(0,\theta) \setminus \{1\}$. Then, there exists $C(\theta',T_0)$ positive such that

$$\|\mathbb{D}^\alpha u\|_{C^{\frac{\alpha\theta'}{2},\theta'}([0,T]\times\overline{\Omega}))} + \|u\|_{B([0,T];C^{2+\theta'}(\overline{\Omega}))} + \|u\|_{C^{\frac{\alpha\theta'}{2}}([0,T];C^2(\overline{\Omega}))}$$

$$\leq C(\theta',T_0)(\|f\|_{C^{\frac{\alpha\theta'}{2},\theta'}([0,T]\times\overline{\Omega}))} + \textstyle\sum_{k<\alpha} \|u_k\|_{C^{\theta'+2(1-\frac{k}{\alpha})}(\overline{\Omega})}$$

$$+\|\mathbb{D}^\alpha g\|_{C^{\frac{\alpha\theta'}{2},\theta'}([0,T]\times\partial\Omega))} + \|g\|_{B([0,T];C^{2+\theta'}(\partial\Omega))} + \|g\|_{C^{\frac{\alpha\theta'}{2}}([0,T];C^2(\partial\Omega))}$$

$$+\|[B(\cdot,D_x) - A(\cdot,D_x)]u\|_{C^{\frac{\alpha\theta'}{2},\theta'}([0,T]\times\overline{\Omega}))})$$

and

$$\|[B(\cdot,D_x)-A(\cdot,D_x)]u\|_{C^{\frac{\alpha\theta'}{2},\theta'}([0,T]\times\overline{\Omega}))}$$

$$\leq \sum_{|\rho|\leq 2}(\|a_\rho-b_\rho\|_{C(\overline{\Omega})}\|D_x^\rho u\|_{C^{\frac{\alpha\theta'}{2}}([0,T];C(\overline{\Omega}))}+\sum_{j=0}^{[\theta]}\|a_\rho-b_\rho\|_{C^j(\overline{\Omega})}\|D^\rho u\|_{B([0,T];C^{\theta'-j}(\overline{\Omega}))}$$

$$+\sum_{j=0}^{[\theta]}\|a_\rho-b_\rho\|_{C^{j+\theta'-[\theta]}(\overline{\Omega})}\|D^\rho u\|_{B([0,T];C^{[\theta]-j}(\overline{\Omega}))})$$

$$\leq C(n,\Omega)\omega_{\theta'}(\delta,R)(\|u\|_{C^{\frac{\alpha\theta'}{2}}([0,T];C^2(\overline{\Omega}))}+\|u\|_{B([0,T];C^{2+\theta'}(\overline{\Omega}))}),$$

with $\lim\limits_{\delta\to 0}\omega_{\theta'}(R,\delta)=0$. Taking δ so small that $C(\theta)\omega_{\theta'}(R,\delta)\leq\frac{1}{2}$, we deduce

$$\|u\|_{B([0,T];C^2(\overline{\Omega}))}\leq\|u\|_{B([0,T];C^{2+\theta'}(\overline{\Omega}))}$$

$$\leq 2C(\theta',T_0)(\|f\|_{C^{\frac{\alpha\theta'}{2},\theta'}([0,T]\times\overline{\Omega}))}+\sum_{k<\alpha}\|u_k\|_{C^{\theta'+2(1-\frac{k}{\alpha})}(\overline{\Omega})}$$

$$+\|\mathbb{D}^\alpha g\|_{C^{\frac{\alpha\theta'}{2},\theta'}([0,T]\times\partial\Omega))}+\|g\|_{B([0,T];C^{2+\theta'}(\partial\Omega))})$$

$$\leq C(\theta',\Omega,T_0)(\|f\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\overline{\Omega}))}+\sum_{k<\alpha}\|u_k\|_{C^{\theta+2(1-\frac{k}{\alpha})}(\overline{\Omega})}$$

$$+\|\mathbb{D}^\alpha g\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\partial\Omega))}+\|g\|_{B([0,T];C^{2+\theta}(\partial\Omega))})$$

which completes the proof.

□

co2.6 **Corollary 2.6.** *Suppose that Ω is as in (A1). Consider, for R,ν,ϵ positive the class of coefficients*

$$\mathcal{C}(R,\nu,\epsilon):=\{(a_\rho)_{|\rho|\leq 2}:a_\rho\in C^\theta(\overline{\Omega}),\|a_\rho\|_{C^\theta(\overline{\Omega})}\leq R,|\sum_{|\rho|=2}a_\rho(x)\xi^\rho|\geq\nu|\xi|^2\quad\forall x\in\overline{\Omega},\xi\in\mathbb{R}^n,$$

$$|Arg(\sum_{|\rho|=2}a_\rho(x)\xi^\alpha)|\leq(1-\tfrac{\alpha}{2})\pi-\epsilon,\forall x\in\overline{\Omega},\xi\in\mathbb{R}^n\setminus\{0\}\}.$$

Given $(a_\rho)_{|\rho|\leq 2}$ in $\mathcal{C}(R,\nu,\epsilon)$, consider problem (2.1). Then, for any T_0 positive, the constant $C(T_0)$ in the estimate (2.2) can be chosen independently of $(a_\rho)_{|\rho|\leq 2}$ and $T\in(0,T_0]$.

Proof. Let $T_0\in\mathbb{R}^+$. By Lemma 2.5, for any $(a_\rho)_{|\rho|\leq 2}\in\mathcal{C}(R,\nu,\epsilon)$ there exists $\delta((a_\rho)_{|\rho|\leq 2})$ positive such that the conclusion holds for

$$\{(b_\rho)_{|\rho|\leq 2}\in\mathcal{C}(R,\nu,\epsilon):\max_{|\rho|\leq 2}\|b_\rho-a_\rho\|_{C(\overline{\Omega})}<\delta((a_\rho)_{|\rho|\leq 2})\}.$$

The conclusion follows from the fact that $\mathcal{C}(R,\nu,\epsilon)$ is compact in $C(\overline{\Omega})^{n^2+n+1}$, by the theorem of Ascoli-Arzelà.

□

Now we consider the nonautonomous system

$$\begin{cases}\mathbb{D}^\alpha u(t,x)=A(t,x,D_x)u(t,x)+f(t,x),\quad t\in[0,T],x\in\Omega,\\ u(t,x')=g(t,x'),\quad (t,x')\in[0,T]\times\partial\Omega,\\ D_t^k u(0,x)=u_k(x),\quad x\in\overline{\Omega},k\in\mathbb{N}_0,k<\alpha,\end{cases}\tag{2.8}$$ eq1.8

with the following conditions:

(B1) $\theta \in \mathbb{R}^+ \setminus \mathbb{N}$, $0 < \theta < 2$; Ω *is an open, bounded subset in* $\mathbb{R}^n$ *lying on one side of its boundary* $\partial\Omega$, *which is a* $n-1-$*submanifold of* $\mathbb{R}^n$ *of class* $C^{2+\theta}$.

(B2) $\alpha \in (0,2)$, $A(t,x,D_x) = \sum_{|\rho|\le 2} a_\rho(t,x)D_x^\rho$, *with* $a_\rho \in C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\overline{\Omega}))$, *complex valued;* $\forall t \in [0,T]$ $A(t,x,D_x)$ *is assumed to be elliptic, in the sense that* $\sum_{|\rho|=2} a_\rho(t,x)\xi^\rho \neq 0$ $\forall \xi \in \mathbb{R}^n \setminus \{0\}$; *we suppose, moreover, that*

$$|Arg(\sum_{|\rho|=2} a_\rho(t,x)\xi^\alpha)| < (1-\frac{\alpha}{2})\pi, \quad \forall x \in \overline{\Omega}, \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

le1.14 **Lemma 2.7.** *Suppose that (B1)-(B2) and the conditions (I)-(IV) of Theorem 2.1 hold; moreover,*

$$\gamma[A(0,\cdot,D_x)u_0 + f(0,\cdot)] = \mathbb{D}^\alpha g(0).$$

Suppose also that, for some $\tau \in [0,T)$, *(2.8) has a solution* $\tilde{u}$ *in* $[0,\tau]$, *with* $\mathbb{D}^\alpha \tilde{u} \in C^{\frac{\alpha\theta}{2}}([0,\tau]\times\overline{\Omega}))$, $\tilde{u} \in C([0,\tau]; C^2(\overline{\Omega})) \cap B([0,\tau]; C^{2+\theta}(\overline{\Omega}))$.

Then there exists $\delta \in]0,T]$, *independent of* τ, *such that (2.8) has a unique solution* u *in* $[0, \min\{\tau+\delta, T\}]$ *with* $\mathbb{D}^\alpha u \in C^{\frac{\alpha\theta}{2},\theta}([0,\tau+\delta]\times\overline{\Omega})$, $u \in C([0,\tau+\delta]; C^2(\overline{\Omega})) \cap B([0,\tau+\delta]; C^{2+\theta}(\overline{\Omega}))$ *and* $u_{|[0,\tau]} = \tilde{u}$.

Proof. Of course, if $\tau = 0$, no function $\tilde{u}$ is given. In this case, we show only the existence and uniqueness of a solution with the declared properties in $[0,\delta]$.

Let $\delta \in (0, T-\tau]$. We consider the class

$$X_{\tau,\delta} := \{U \in C^{\frac{\alpha\theta}{2}}([0,\tau+\delta]); C^2(\overline{\Omega})) \cap B([0,\tau+\delta]); C^{2+\theta}(\overline{\Omega})) : U_{|[0,\tau]} = \tilde{u}\}.$$

In case $\tau = 0$, we simply set

$$X_{0,\delta} := \{U \in C^{\frac{\alpha\theta}{2}}([0,\delta]); C^2(\overline{\Omega})) \cap B([0,\delta]); C^{2+\theta}(\overline{\Omega})) : U(0) = u_0\}.$$

$X_{\tau,\delta}$ is a complete metric space with the distance

$$d(U_1,U_2) := \max\{\|U_1-U_2\|_{C^{\frac{\alpha\theta}{2}}([0,\tau+\delta];C^2(\overline{\Omega}))}, \|U_1-U_2\|_{B([0,\tau+\delta];C^{2+\theta}(\overline{\Omega}))}\}$$

$$= \max\{\|U_1-U_2\|_{C^{\frac{\alpha\theta}{2}}([\tau,\tau+\delta];C^2(\overline{\Omega}))}, \|U_1-U_2\|_{B([\tau,\tau+\delta];C^{2+\theta}(\overline{\Omega}))}\}.$$

An element of $X_{\tau,\delta}$ is, for example,

$$U(t,x) = \begin{cases} \tilde{u}(t,x) & if \quad t \in [0,\tau], \\ \tilde{u}(\tau,x) & if \quad t \in [\tau,\tau+\delta]. \end{cases}$$

We recall that, by Proposition 2.4, if a solution with the declared properties exists, it belongs to $X_{\tau,\delta}$. For each U in $X_{\tau,\delta}$ we consider the system

$$\begin{cases} \mathbb{D}^\alpha u(t,x) = A(\tau,x,D_x)u(t,x) + [A(t,x,D_x) - A(\tau,x,D_x)]U(t,x) + f(t,x), \quad t \in [0,(\tau+\delta)\wedge T], x \in \overline{\Omega}, \\ u(t,x') = g(t,x'), \quad (t,x') \in [0,T]\times\partial\Omega, \\ D_t^k u(0,x) = u_k(x), \quad x \in \overline{\Omega}, k \in \mathbb{N}_0, k < \alpha. \end{cases} \tag{2.9}$$
eq1.9

By Theorem 2.1, (2.9) has a unique solution $u = S(U)$, with $\mathbb{D}^\alpha u \in C^{\frac{\alpha\theta}{2},\theta}([0,\tau+\delta]\times\overline{\Omega})$, $u \in C([0,\tau+\delta]; C^2(\overline{\Omega})) \cap B([0,\tau+\delta]; C^{2+\theta}(\overline{\Omega}))$. On account of the uniqueness of the solution in $[0,\tau]$ in the autonomous case, we deduce that $S(U)_{|[0,\tau]} = \tilde{u}$ and so $S(U) \in X_{\tau,\delta}$. Therefore, if $U_1, U_2 \in X_{\tau,\delta}$ and $v = S(U_1) - S(U_2)$,

$$\begin{cases} \mathbb{D}^\alpha v(t,x) = A(\tau,x,D_x)v(t,x) + [A(t,x,D_x) - A(\tau,x,D_x)](U_1-U_2)(t,x), \quad t \in [0,(\tau+\delta)\wedge T], x \in \overline{\Omega}, \\ v(t,x') = 0, \quad (t,x') \in [0,T]\times\partial\Omega, \\ D_t^k v(0,x) = 0, \quad x \in \overline{\Omega}, k \in \mathbb{N}_0, k < \alpha. \end{cases}$$

By Corollary 2.6, there exists $C(\theta)$ positive, independent of τ, such that

$$d(S(U_1), S(U_2)) \le C(\theta)\|[A(t,x,D_x) - A(\tau,x,D_x)](U_1 - U_2)\|_{C^{\frac{\alpha\theta}{2},\theta}([\tau,\tau+\delta]\times\overline{\Omega})}.$$

Let $R \in \mathbb{R}^+$, with $\max_{|\rho|\le 2}\|a_\rho(t,\cdot)\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\overline{\Omega})} \le R$. Then

$$\|a_\rho(t,\cdot) - a_\rho(\tau,\cdot)\|_{C(\overline{\Omega})} \le R|t-\tau|^{\frac{\alpha\theta}{2}}.$$

If $|\rho| \le 2$,we deduce, for $0 \le \sigma \le \theta$,

$$\|a_\rho(t,\cdot) - a_\rho(\tau,\cdot)\|_{C^\sigma(\overline{\Omega})} \le C(\sigma)|t-\tau|^{\frac{\alpha(\theta-\sigma)}{2}} R.$$

Arguing as in the proof of Lemma 2.5, we deduce

$$d(S(U_1), S(U_2)) \le \omega_1(R,\delta) d(U_1,U_2) + C_1 R\|U_1 - U_2\|_{C([\tau,\tau+\delta];C^2(\overline{\Omega}))},$$

with $\lim_{\delta\to 0}\omega_1(R,\delta) = 0$ and C_1 independent of τ. We have also, for $\sigma \in [0, \frac{\alpha\theta}{2}]$,

$$\|a_\rho\|_{C^\sigma([\tau,\tau+\delta];C(\overline{\Omega}))} \le R\delta^{\frac{\alpha\theta}{2}-\sigma}.$$

So, again arguing as in the proof of Lemma 2.5, taking $\theta' \in (0,\theta)\setminus\{1\}$, we deduce

$$\|S(U_1) - S(U_2)\|_{C([\tau,\tau+\delta];C^2(\overline{\Omega}))} \le \|U_1 - U_2\|_{B([\tau,\tau+\delta];C^{2+\theta'}(\overline{\Omega}))}$$

$$\le \omega_2(R,\delta) d(U_1,U_2),$$

again with $\omega_2(R,\delta)$ independent of τ and $\lim_{\delta\to 0}\omega_2(R,\delta) = 0$. We deduce that

$$d(S^2(U_1), S^2(U_2))$$

$$\le \omega_1(R,\delta) d(S(U_1), S(U_2)) + C_1 R\|S(U_1) - S(U_2)\|_{C([\tau,\tau+\delta];C^2(\overline{\Omega}))}$$

$$\le \omega_1(R,\delta)(\omega_1(R,\delta) d(U_1,U_2) + C_1 R\|U_1 - U_2\|_{C([\tau,\tau+\delta];C^2(\overline{\Omega}))}) + C_1 R\,\omega_2(R,\delta) d(U_1,U_2).$$

We conclude that, for some δ positive independent of τ, S^2 is a contraction in $X_{\tau,\delta}$ and the conclusion follows.

□

From Lemma 2.7 we immediately deduce the following

th1.15 **Theorem 2.8.** *Suppose that (B1)-(B2) hold. Consider system (2.8). Then the following conditions are necessary and sufficient in order that there exist a unique solution u such that $\mathbb{D}^\alpha u$ belongs to $C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\overline{\Omega})$ and $u \in C^{\frac{\alpha\theta}{2}}([0,T];C^2(\overline{\Omega})) \cap B([0,T];C^{2+\theta}(\overline{\Omega}))$:*
(I) $f \in C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\overline{\Omega})$;
(II) if $k \in \mathbb{N}_0$, $k < \alpha$, $u_k \in C^{\theta+2(1-\frac{k}{\alpha})}(\overline{\Omega})$;
(III $g \in C[0,T];C^2(\partial\Omega)) \cap B([0,T];C^{2+\theta}(\partial\Omega))$, $\mathbb{D}^\alpha g \in C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\overline{\Omega})$;
(IV) $\gamma u_k = D_t^k g(0)$ $(k < \alpha)$;
(V) $\gamma[A(0,\cdot,D_x)u_0 + f(0)] = \mathbb{D}^\alpha g(0)$.

Proof. The necessity of conditions (I)-(V) can be proved with the same arguments in the proof of Theorem 2.1 (see [6]).

Concerning the sufficiency, we can construct a solution with the declared regularity, employing Lemma 2.7 several times.

To show that the solution is unique, it suffices to consider the case that all data vanish. In this case, let $\tilde{u} \ne 0$ be a solution with the declared regularity. We set

$$\tau := inf\{t \in [0,T] : \tilde{u}(t,\cdot) \ne 0\}.$$

Then, $\tau \in [0,T)$ and $\tilde{u}_{|[0,\tau]} = 0$. By Lemma 2.7, the restriction $\tilde{u}_{|[0,\tau]}$ can be extended in a unique way to a solution u with the prescribed regularity in $[0, \tau+\delta]$, for some $\delta \in (0, T-\tau]$, necessarily coinciding with $\tilde{u}$ in $[0, \tau+\delta]$. But, owing to the uniqueness, $u(t) = 0$ $\forall t \in [0, \tau+\delta]$, implying $\tilde{u}(t) = 0$ $\forall t \in [0, \tau+\delta]$, which is in contradiction with the definition of τ.

□

We conclude with the following nonautonomous analog of Proposition 2.4:

pr1.16 **Proposition 2.9.** *Suppose that (B1)-(B2) hold, with T replaced by T_0. Let $0 < T \le T_0$ and consider problem (2.8), with the conditions (I)-(V) in Theorem 2.8 satisfied. Then there exists $C(T_0)$ positive, independent of T, such that*

$$\|\mathbb{D}^\alpha u\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\overline{\Omega})} + \|u\|_{C^{\frac{\alpha\theta}{2}}([0,T];C^2(\overline{\Omega}))} + \|u\|_{B([0,T];C^{2+\theta}(\overline{\Omega}))}$$

$$\le C(T_0)(\|f\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\overline{\Omega})} + \textstyle\sum_{k<\alpha} \|u_k\|_{C^{\theta+2(1-\frac{k}{\alpha})}(\overline{\Omega})}$$

$$+\|\mathbb{D}^\alpha g\|_{C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\partial\Omega)} + \|g\|_{C^{\frac{\alpha\theta}{2}}([0,T];C^2(\partial\Omega))} + |g\|_{B([0,T];C^{2+\theta}(\partial\Omega))}).$$

Proof. It is analogous to the proof of Proposition 2.4. □

3 Fully nonlinear problems

se2

Now we consider a system in the form (1.1), with the following assumptions:

(D1) $T_0 \in \mathbb{R}^+$, Ω is an open, bounded subset in $\mathbb{R}^n$ lying on one side of its boundary $\partial\Omega$, which is a $n-1-$submanifold of $\mathbb{R}^n$ of class $C^{2+\theta}$, with $\theta \in (0,1)$, $F : [0,T_0]\times\overline{\Omega}\times\mathbb{R}^{N(n)} \to \mathbb{R}$, with $N(n) = n^2+n+1$;

(D2) $\alpha \in (0,2)$, $\theta \neq \frac{2}{\alpha} - 1$;

(D3) $\forall (t,x) \in [0,T_0] \times \overline{\Omega}$ $F(t,x,\cdot) \in C^1(\mathbb{R}^{N(n)})$ and $\nabla_p F(t,x,\cdot)$ is locally Lipschitz continuous in $\mathbb{R}^{N(n)}$, uniformly in bounded subsets of $[0,T_0] \times \overline{\Omega} \times \mathbb{R}^{N(n)}$;

(D4) $\forall (x,p) \in \overline{\Omega} \times \mathbb{R}^{N(n)}$

$$\|F(\cdot,x,p)\|_{C^{\frac{\alpha\theta}{2}}([0,T_0])} + \|\nabla_v F(\cdot,x,p)\|_{C^{\frac{\alpha\theta}{2}}([0,T_0])} \le C(|p|);$$

with $C(|p|)$ positive, nondecreasing in $\mathbb{R}^+$;

(D5) $\forall (t,p) \in [0,T_0] \times \mathbb{R}^{N(n)}$

$$\|F(t,\cdot,p)\|_{C^\theta(\overline{\Omega})} + \|\nabla_v F(t,\cdot,p)\|_{C^\theta(\overline{\Omega})} \le C(|p|);$$

(D6) $\forall (t,x,p) \in [0,T_0]\times\overline{\Omega}\times\mathbb{R}^{N(n)}$, $\forall q \in \mathbb{R}^n$, $\sum_{|\rho|=2} D_{p_\rho} F(t,x,p) q^\rho \ge \nu(t,x,p)|q|^2$, with ν continuous and positive.

In order to solve (1.1), we apply Taylor's formula: we have

$$F(t,x,p+q) = F(t,x,p) + \sum_{|\rho|\le 2} D_{p_\rho} F(t,x,p) q_\rho + r(t,x,p,q), \qquad (3.1)$$ eq3.1

with $r(t,x,p,0) = 0$, $\nabla_q r(t,x,p,0) = 0$.

From (D1)-(D6), we can write (1.1) in the form

$$
\begin{cases}
\mathbb{D}^\alpha u(t,x) = F(t,x,(D_x^\rho u_0(x))_{|\rho|\le 2}) + \sum_{|\rho|\le 2} D_{p_\rho} F(t,x,(D_x^\sigma u_0(x))_{|\sigma|\le 2})(D_x^\rho u(t,x) - D_x^\rho u_0(x)) \\
+r(t,x,(D_x^\rho u_0(x))_{|\rho|\le 2},(D_x^\rho u(t,x) - D_x^\rho u_0(x))_{|\rho|\le 2}), \quad t\in[0,T], x\in\overline{\Omega}, \\
u(t,x') = g(t,x'), \quad (t,x')\in[0,T]\times\partial\Omega, \\
D_t^k u(0,x) = u_k(x), \quad x\in\overline{\Omega}, k\in\mathbb{N}_0, k<\alpha.
\end{cases}
\tag{3.2}
$$
eq2.2

We begin with the following elementary lemma:

le3.1 **Lemma 3.1.** *Suppose that (D1)-(D6) hold and let r be as in (3.1). Then:*
(I) let $M\in\mathbb{R}^+$; then $\forall\epsilon>0$ there exists $\delta(M,\epsilon)>0$ such that, if $(t,x,p,q)\in[0,T_0]\times\overline{\Omega}\times\mathbb{R}^{N(n)}\times\mathbb{R}^{N(n)}$ and $|p|\le M$, $|q|\le\delta(M,\epsilon)$, $|\nabla_q r(t,x,p,q)|\le\epsilon$;
(II) $\forall M\in\mathbb{R}^+$ there exists $L(M)$ positive such that, if $t,t'\in[0,T_0]$, $x,x'\in\overline{\Omega}$, $p,p',q,q'\in\mathbb{R}^{N(n)}$ and $\max\{|p|,|q|,|p'|,|q'|\}\le M$,

$$
|\nabla_q r(t,x,p,q) - \nabla_q r(t',x',p',q')| \le L(M)(|t-t'|^{\frac{\alpha\theta}{2}} + |x-x'|^\theta + |p-p'| + |q-q'|).
$$

Proof. It follows immediately from

$$
\nabla_q r(t,x,p,q) = \nabla_p F(t,x,p+q) - \nabla_p F(t,x,p).
$$

□

Let $R\in\mathbb{R}^+$ and $T\in(0,T_0]$. We set

$$
\begin{aligned}
X(R,T) := \{U \in C^{\frac{\alpha\theta}{2}}([0,T];C^2(\overline{\Omega})) \cap B([0,T];C^{2+\theta}(\overline{\Omega})) :\\
U(0)=u_0, \max\{\|U-u_0\|_{C^{\frac{\alpha\theta}{2}}([0,T];C^2(\overline{\Omega}))}, \|U-u_0\|_{B([0,T];C^{2+\theta}(\overline{\Omega}))}\} \le R\}.
\end{aligned}
\tag{3.3}
$$
eq3.3A

$X(R,T)$ is a complete metric space with the metric

$$
d(U_1,U_2) := \max\{\|U_1-U_2\|_{C^{\frac{\alpha\theta}{2}}([0,T];C^2(\overline{\Omega}))}, \|U_1-U_2\|_{B([0,T];C^{2+\theta}(\overline{\Omega}))}\}.
$$

We want to prove the following

th3.2 **Theorem 3.2.** *Suppose that (D1)-(D6) hold. Suppose, moreover, that:*
(I) if $k\in\mathbb{N}_0$, $k<\alpha$, $u_k\in C^{\theta+2(1-\frac{k}{\alpha})}(\overline{\Omega})$,
(II) $g\in C([0,T_0];C^2(\partial\Omega))\cap B([0,T_0];C^{2+\theta}(\partial\Omega))$, $\mathbb{D}^\alpha g$ exists and belongs to $C^{\frac{\alpha\theta}{2}\theta}([0,T];\partial\Omega)$;
(III) if $k\in\mathbb{N}_0$, $k<\alpha$, $\gamma u_k = D_t^k g(0)$;
(IV) $\gamma[F(0,\cdot,(D_x^\rho u_0)_{|\rho|\le 2})] = \mathbb{D}^\alpha g(0)$.
Then, for some $R_0>0$, if $R\ge R_0$, there exists $T(R)\in(0,T_0]$ such that, if $0<T\le T(R)$, (1.1) has a unique solution in $X(R,T)$.

Proof. Let $U\in X(R,T)$. We consider the problem

$$
\begin{cases}
\mathbb{D}^\alpha u(t,x) = F(t,x,(D_x^\rho u_0(x))_{|\rho|\le 2}) + \sum_{|\rho|\le 2} D_{p_\rho} F(t,x,(D_x^\sigma u_0(x))_{|\sigma|\le 2})(D_x^\rho u(t,x) - D_x^\rho u_0(x)) \\
+r(t,x,(D_x^\rho u_0(x))_{|\rho|\le 2},(D_x^\rho U(t,x) - D_x^\rho u_0(x))_{|\rho|\le 2}), \quad t\in[0,T], x\in\overline{\Omega}, \\
u(t,x') = g(t,x'), \quad (t,x')\in[0,T]\times\partial\Omega, \\
D_t^k u(0,x) = u_k(x), \quad x\in\overline{\Omega}, k\in\mathbb{N}_0, k<\alpha.
\end{cases}
\tag{3.4}
$$
eq3.3

Then, by Theorem 2.8, (3.4) has a unique solution u in $C([0,T];C^2(\overline{\Omega}))\cap B([0,T];C^{2+\theta}(\overline{\Omega}))$, with $\mathbb{D}^\alpha u \in C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\overline{\Omega})$ and, by Proposition 2.2, $u \in C^{\frac{\alpha\theta}{2}}([0,T];C^2(\overline{\Omega}))$. It is clear that U solves (1.1) if and only if it is a fixed point of the mapping $S(U) := u$. If $u = S(U)$ and $v = S(V)$, we have

$$\begin{cases} \mathbb{D}^\alpha(u-v)(t,x) = \sum_{|\rho|\le 2} D_{p_\rho}F(t,x,(D_x^\sigma u_0(x))_{|\sigma|\le 2})(D_x^\rho u(t,x) - D_x^\rho v(t,x)) \\ +r(t,x,(D_x^\rho u_0(x))_{|\rho|\le 2},(D_x^\rho U(t,x) - D_x^\rho u_0(x))_{|\rho|\le 2}) \\ -r(t,x,(D_x^\rho u_0(x))_{|\rho|\le 2},(D_x^\rho V(t,x) - D_x^\rho u_0(x))_{|\rho|\le 2}), \quad t\in[0,T], x\in\overline{\Omega}, \\ (u-v)(t,x') = 0, \quad (t,x')\in[0,T]\times\partial\Omega, \\ D_t^k(u-v)(0,x) = 0, \quad x\in\overline{\Omega}, k\in\mathbb{N}_0, k<\alpha. \end{cases}$$

We set

$$R(U)(t,x) := r(t,x,(D_x^\rho u_0(x))_{|\rho|\le 2},(D_x^\rho U(t,x) - D_x^\rho u_0(x))_{|\rho|\le 2}).$$

If $U \in X(R,T)$,

$$\|U - u_0\|_{C([0,T];C^2(\overline{\Omega}))} \le RT^{\frac{\alpha\theta}{2}}.$$

We set

$$M_0 := \max_{x\in\overline{\Omega}} |(D^\sigma u_0(x))_{|\sigma|\le 2}|.$$

Let $\eta \in \mathbb{R}^+$. Referring to Lemma 3.1, we take T such that $RT^{\frac{\alpha\theta}{2}} \le \delta(M_0,\eta)$. Then if, $U,V \in X(R,T)$,

$$\|R(U) - R(V)\|_{C([0,T]\times\overline{\Omega})} \le \eta\|U-V\|_{C([0,T];C^2(\overline{\Omega}))}.$$

Let $t,s\in[0,T], x\in\overline{\Omega}$. Then

$$\begin{aligned} &|R(U)(t,x) - R(V)(t,x) - (R(U)(s,x) - R(V)(s,x))| \\ &= |\int_0^1 \nabla_q R(t,x,(D_x^\rho u_0(x))_{|\rho|\le 2},(D_x^\rho V(t,x) - D_x^\rho u_0(x) + \tau(D_x^\rho U(t,x) - D_x^\rho V(t,x))_{|\rho|\le 2}) \\ &\cdot(D_x^\rho U(t,x) - D_x^\rho V(t,x))_{|\rho|\le 2})d\tau \\ &-(\int_0^1 \nabla_q R(s,x,(D_x^\rho u_0(x))_{|\rho|\le 2},(D_x^\rho V(s,x) - D_x^\rho u_0(x) + \tau(D_x^\rho U(s,x) - D_x^\rho V(s,x))_{|\rho|\le 2}) \\ &\cdot(D_x^\rho U(s,x) - D_x^\rho V(s,x))_{|\rho|\le 2})d\tau)| \\ &\le |\int_0^1(\nabla_q R(t,x,(D_x^\rho u_0(x))_{|\rho|\le 2},(D_x^\rho V(t,x) - D_x^\rho u_0(x) + \tau(D_x^\rho U(t,x) - D_x^\rho V(t,x))_{|\rho|\le 2})d\tau \\ &- \int_0^1(\nabla_q R(s,x,(D_x^\rho u_0(x))_{|\rho|\le 2},(D_x^\rho V(s,x) - D_x^\rho u_0(x) + \tau(D_x^\rho U(s,x) - D_x^\rho V(s,x))_{|\rho|\le 2})d\tau| \\ &|(D_x^\rho U(t,x) - D_x^\rho V(t,x))_{|\rho|\le 2})| \\ &+ \int_0^1 |\nabla_q R(s,x,(D_x^\rho u_0(x))_{|\rho|\le 2},(D_x^\rho V(s,x) - D_x^\rho u_0(x) + \tau(D_x^\rho U(s,x) - D_x^\rho V(s,x))_{|\rho|\le 2})|d\tau \\ &|(D_x^\rho U(t,x) - D_x^\rho V(t,x))_{|\rho|\le 2}) - (D_x^\rho U(s,x) - D_x^\rho V(s,x))_{|\rho|\le 2})|. \end{aligned}$$

By Lemma 3.1 (II)

$$\begin{aligned} &|\int_0^1(\nabla_q R(t,x,(D_x^\rho u_0(x))_{|\rho|\le 2},(D_x^\rho V(t,x) - D_x^\rho u_0(x) + \tau(D_x^\rho U(t,x) - D_x^\rho V(t,x))_{|\rho|\le 2}) \\ &- \int_0^1(\nabla_q R(s,x,(D_x^\rho u_0(x))_{|\rho|\le 2},(D_x^\rho V(s,x) - D_x^\rho u_0(x) + \tau(D_x^\rho U(s,x) - D_x^\rho V(s,x))_{|\rho|\le 2})d\tau| \\ &\le C(R)|t-s|^{\frac{\alpha\theta}{2}}. \end{aligned}$$

Moreover,as $D_x^\rho U(0,x) = D_x^\rho V(0,x)$,

$$|(D_x^\rho U(t,x) - D_x^\rho V(t,x))_{|\rho|\le 2})| \le CT^{\frac{\alpha\theta}{2}} \|U - V\|_{C^{\frac{\alpha\theta}{2}}([0,T];C^2(\overline{\Omega}))}.$$

Finally,

$$\int_0^1 |\nabla_q R(s,x,(D_x^\rho u_0(x))_{|\rho|\le 2}, (D_x^\rho V(s,x) - D_x^\rho u_0(x) + \tau(D_x^\rho U(s,x) - D_x^\rho V(s,x))_{|\rho|\le 2})| d\tau \le \eta$$

if $RT^{\frac{\alpha\theta}{2}} \le \delta(M_0,\eta)$ and

$$|(D_x^\rho U(t,x) - D_x^\rho V(t,x))_{|\rho|\le 2}) - (D_x^\rho U(s,x) - D_x^\rho V(s,x))_{|\rho|\le 2})|$$

$$\le C_0|t-s|^{\frac{\alpha\theta}{2}} \|U - V\|_{C^{\frac{\alpha\theta}{2}}([0,T];C^2(\overline{\Omega}))}$$

So

$$\|R(U) - R(V)\|_{C^{\frac{\alpha\theta}{2}}([0,T];C(\overline{\Omega})} \le (C(R)T^{\frac{\alpha\theta}{2}} + C_0\eta)\|U - V\|_{C^{\frac{\alpha\theta}{2}}([0,T];C^2(\overline{\Omega})} \quad (3.5) \quad \boxed{\texttt{eq3.4}}$$

if $T \le T_0(R,\eta) \le T_0$.

Let $t \in [0,T]$ and $x, y \in \overline{\Omega}$. Then, if $U, V \in X(R,T)$,

$$|R(U)(t,x) - R(V)(t,x) - (R(U)(t,y) - R(V)(t,y))|$$

$$= |\int_0^1 \nabla_q R(t,x,(D_x^\rho u_0(x))_{|\rho|\le 2}, (D_x^\rho V(t,x) - D_x^\rho u_0(x) + \tau(D_x^\rho U(t,x) - D_x^\rho V(t,x))_{|\rho|\le 2})$$

$$\cdot(D_x^\rho U(t,x) - D_x^\rho V(t,x))_{|\rho|\le 2})d\tau$$

$$-(\int_0^1 \nabla_q R(t,y,(D_x^\rho u_0(y))_{|\rho|\le 2}, (D_x^\rho V(t,y) - D_x^\rho u_0(y) + \tau(D_x^\rho U(t,y) - D_x^\rho V(t,y))_{|\rho|\le 2})$$

$$\cdot(D_x^\rho U(t,y) - D_x^\rho V(t,y))_{|\rho|\le 2})d\tau)|$$

$$\le |\int_0^1 (\nabla_q R(t,x,(D_x^\rho u_0(x))_{|\rho|\le 2}, (D_x^\rho V(t,x) - D_x^\rho u_0(x) + \tau(D_x^\rho U(t,x) - D_x^\rho V(t,x))_{|\rho|\le 2})d\tau$$

$$-\int_0^1 (\nabla_q R(t,y,(D_x^\rho u_0(y))_{|\rho|\le 2}, (D_x^\rho V(t,y) - D_x^\rho u_0(y) + \tau(D_x^\rho U(t,y) - D_x^\rho V(t,y))_{|\rho|\le 2})d\tau|$$

$$|(D_x^\rho U(t,x) - D_x^\rho V(t,x))_{|\rho|\le 2})|d\tau$$

$$+\int_0^1 |\nabla_q R(t,y,(D_x^\rho u_0(y))_{|\rho|\le 2}, (D_x^\rho V(t,y) - D_x^\rho u_0(y) + \tau(D_x^\rho U(t,y) - D_x^\rho V(t,y))_{|\rho|\le 2})|d\tau$$

$$|(D_x^\rho U(t,x) - D_x^\rho V(t,x))_{|\rho|\le 2}) - (D_x^\rho U(t,y) - D_x^\rho V(t,y))_{|\rho|\le 2})|,$$

implying

$$\|R(U) - R(V)\|_{B([0,T];C^\theta(\overline{\Omega}))} \le \eta(\|U - V\|_{C^{\frac{\alpha\theta}{2}}([0,T];C^2(\overline{\Omega})} + \|U - V\|_{B([0,T];C^{2+\theta}(\overline{\Omega})}), \quad (3.6) \quad \boxed{\texttt{eq3.5}}$$

if $0 < T \le T_1(R,\eta) \le T_0$. By (3.5), (3.6) and Proposition 2.9, for any η and R positive there exists $T_2(R,\eta)$ in $(0,T_0]$ such that, if $0 < T \le T_2(R,\eta)$, and $U_1, U_2 \in X(R,T)$,

$$d(u_1,u_2) \le \frac{1}{2} d(U_1,U_2),$$

with u_j solution of (3.4), taking $U = U_j$ $(j \in \{1,2\})$.

Let U_0 be the solution of (3.4) with $U = u_0$ (constant function). Then, if $U \in X(R,T)$, we have

$$d(u,u_0) \le d(u,U_0) + d(U_0,u_0) \le \frac{1}{2} d(U,u_0) + d(U_0,u_0) \le \frac{R}{2} + d(U_0,u_0) \le R$$

if

$$R \geq 2\,max\{\|U_0 - u_0\|_{C^{\frac{\alpha\theta}{2}}([0,T_0];C^2(\overline{\Omega}))}, \|U_0 - u_0\|_{B([0,T_0];C^{2+\theta}(\overline{\Omega}))}\}.$$

With such choice of R, by the contraction mapping theorem, S is a contraction in $X(R,T)$ and the conclusion follows.

□

We show a result of global uniqueness:

th3.3 **Theorem 3.3.** *Suppose that (D1)-(D6) and (I)-(IV) in the statement of Theorem 3.2 hold. Let, for $j \in \{1,2\}$, $v_j \in C([0,T];C^2(\overline{\Omega})) \cap B([0,T];C^{2+\theta}(\overline{\Omega}))$, with $\mathbb{D}^\alpha v_j \in C^{\frac{\alpha\theta}{2},\theta}([0,T]\times\overline{\Omega})$, be solutions of (3.2). Then $v_1 \equiv v_2$.*

Proof. We argue by contradiction, assuming that $u \neq v$. We set

$$\tau = \inf\{t \in [0,T] : v_1(t) \neq v_2(t)\}.$$

Then $0 \leq \tau < T$ and $v_1(\tau) = v_2(\tau)$. So we have, for $j = 1,2$,

$$\begin{cases} \mathbb{D}^\alpha v_j(t,x) = F(t,x,(D_x^\rho v_1(\tau,x))_{|\rho|\leq 2}) + \sum_{|\rho|\leq 2} D_{p_\rho}F(t,x,(D_x^\rho v_1(\tau,x))_{|\rho|\leq 2})(D_x^\rho v_j(t,x) - D_x^\rho v_1(\tau,x)) \\ +r(t,x,(D_x^\rho v_1(\tau,x))_{|\rho|\leq 2},(D_x^\rho v_j(t,x) - D_x^\rho v_1(\tau,x))_{|\rho|\leq 2}), \quad t \in [0,T], x \in \overline{\Omega}, \\ v_j(t,x') = g(t,x'), \quad (t,x') \in [0,T]\times\partial\Omega, \\ D_t^k v_j(0,x) = u_k(x), \quad x \in \overline{\Omega}, k \in \mathbb{N}_0, k < \alpha, \end{cases}$$

implying

$$\begin{cases} \mathbb{D}^\alpha (v_1 - v_2)(t,x) = \sum_{|\rho|\leq 2} D_{p_\rho}F(t,x,(D_x^\rho v_1(\tau,x))_{|\rho|\leq 2})(D_x^\rho v_1(t,x) - D_x^\rho v_2(t,x)) \\ +r(t,x,(D_x^\rho v_1(\tau,x))_{|\rho|\leq 2},(D_x^\rho v_1(t,x) - D_x^\rho V(\tau,x))_{|\rho|\leq 2}) \\ -r(t,x,(D_x^\rho v_1(\tau,x))_{|\rho|\leq 2},(D_x^\rho v_2(t,x) - D_x^\rho v_1(\tau,x))_{|\rho|\leq 2}), \quad t \in [0,T], x \in \overline{\Omega}, \\ v_1(t,x') - v_2(t,x') = 0, \quad (t,x') \in [0,T]\times\partial\Omega, \\ D_t^k (v_1 - v_2)(0,x) = 0, \quad x \in \overline{\Omega}, k \in \mathbb{N}_0, k < \alpha. \end{cases}$$

Let $0 < \delta \leq T - \tau$. By Proposition 2.9, there exists C_0 positive independent of δ such that

$$\|v_1 - v_2\|_{C^{\frac{\alpha\theta}{2}}([0,\tau+\delta];C^2(\overline{\Omega}))} + \|v_1 - v_2\|_{B([0,\tau+\delta];C^{2+\theta}(\overline{\Omega}))}$$

$$\leq C_0\|r(\cdot,\cdot,(D_x^\rho v_1(\tau,\cdot))_{|\rho|\leq 2},(D_x^\rho v_1 - D_x^\rho v_1(\tau,\cdot))_{|\rho|\leq 2})$$

$$-r(\cdot,\cdot,(D_x^\rho v_1(\tau,\cdot))_{|\rho|\leq 2},(D_x^\rho v_2 - D_x^\rho v_1(\tau,\cdot))_{|\rho|\leq 2})\|_{C^{\frac{\alpha\theta}{2},\theta}([0,\tau+\delta]\times\overline{\Omega})}$$

$$= C_0\|r(\cdot,\cdot,(D_x^\rho v_1(\tau,\cdot))_{|\rho|\leq 2},(D_x^\rho v_1 - D_x^\rho v_1(\tau,\cdot))_{|\rho|\leq 2})$$

$$-r(\cdot,\cdot,(D_x^\rho v_1(\tau,\cdot))_{|\rho|\leq 2},(D_x^\rho v_2 - D_x^\rho v_1(\tau,\cdot))_{|\rho|\leq 2})\|_{C^{\frac{\alpha\theta}{2},\theta}([\tau,\tau+\delta]\times\overline{\Omega})}.$$

Let $R \in \mathbb{R}^+$ be such that, for $j = 1,2$, $\max\{\|v_j\|_{C^{\frac{\alpha\theta}{2}}([\tau,T];C^2(\overline{\Omega}))}, \|v_j\|_{B([\tau,T];C^{2+\theta}(\overline{\Omega}))}\} \leq R$. Then, arguing as in the proof of Theorem 3.2, we can see that, $\forall\epsilon > 0$ there exists $\delta(R,\epsilon) \in (0, T-\tau]$, such that, if

$0 < \delta \leq \delta(R, \epsilon)$,

$$\|r(\cdot, \cdot, (D_x^\rho v_1(\tau, \cdot))_{|\rho| \leq 2}, (D_x^\rho v_1 - D_x^\rho v_1(\tau, \cdot))_{|\rho| \leq 2})$$

$$-r(\cdot, \cdot, (D_x^\rho v_1(\tau, \cdot))_{|\rho| \leq 2}, (D_x^\rho v_2 - D_x^\rho v_1(\tau, \cdot))_{|\rho| \leq 2})\|_{C^{\frac{\alpha\theta}{2}, \theta}([\tau, \tau+\delta] \times \overline{\Omega})}$$

$$\leq \epsilon(\|v_1 - v_2\|_{C^{\frac{\alpha\theta}{2}}([\tau, \tau+\delta]; C^2(\overline{\Omega}))} + \|v_1 - v_2\|_{B([\tau, \tau+\delta]; C^{2+\theta}(\overline{\Omega}))})$$

$$= \epsilon(\|v_1 - v_2\|_{C^{\frac{\alpha\theta}{2}}([0, \tau+\delta]; C^2(\overline{\Omega}))} + \|v_1 - v_2\|_{B([0, \tau+\delta]; C^{2+\theta}(\overline{\Omega}))})$$

We conclude that

$$\|v_1 - v_2\|_{C^{\frac{\alpha\theta}{2}}([0, \tau+\delta]; C^2(\overline{\Omega}))} + \|v_1 - v_2\|_{B([0, \tau+\delta]; C^{2+\theta}(\overline{\Omega}))}$$

$$\leq C_0 \epsilon(\|v_1 - v_2\|_{C^{\frac{\alpha\theta}{2}}([0, \tau+\delta]; C^2(\overline{\Omega}))} + \|v_1 - v_2\|_{B([0, \tau+\delta]; C^{2+\theta}(\overline{\Omega}))}),$$

implying (if $C_0 \epsilon < 1$) $v_{1|[0, \tau+\delta]} = v_{2|[0, \tau+\delta]}$, in contradiction with the definition of τ.

□

re3.4A **Remark 3.4.** Theorems 3.2 and 3.3 hold if (D1)-(D6) are relaxed in the following way: the domain of F is $[0, T_0] \times \overline{\Omega} \times O$ with O open subset in $\mathbb{R}^{N(n)}$ containing $(D_x^\rho u_0(x))_{|\rho| \leq 2}$; moreover, the estimates of F and its derivatives in (D3)-(D5) are uniform in any compact subset of $[0, T_0] \times \overline{\Omega} \times O$.

Observe that, if $U \in X(R, T)$ (see (3.3)) and $|\rho| \leq 2$,

$$|D_x^\rho U(t, x) - D_x^\rho u_0(x)| \leq R T^{\frac{\alpha\theta}{2}}, \quad \forall (t, x) \in [0, T] \times \overline{\Omega}.$$

re3.4 **Remark 3.5.** Theorems 3.2 and 3.3 can be extended to the case that

$$F(t, x, (p_\sigma)_{|\sigma| \leq 2}) = \sum_{|\sigma|=2} a_\sigma(t, x, (p_\rho)_{|\rho| \leq 1}) p_\sigma + b(t, x, (p_\rho)_{|\rho| \leq 1}),$$

with

(a) if $|\sigma| = 2$, $a_\sigma, b : [0, T_0] \times \overline{\Omega} \times O \to \mathbb{C}$, with O open subset of $\mathbb{C}^{n+1}$ containing $(D^\rho u_0(x))_{|\rho| \leq 1}$ $\forall x \in \overline{\Omega}$;

(b) the derivatives of F appearing in (D3)-(D5) are intended by identifing $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2(n+1)}$.

(c) $\sum_{|\sigma|=2} a_\sigma(t, x, (p_\rho)_{|\rho| \leq 1}) q^\sigma \neq 0$ $\forall (t, x, (p_\rho)_{|\rho| \leq 1}) \in [0, T_0] \times \overline{\Omega} \times O$, $\forall q \in \mathbb{R}^n \setminus \{0\}$;

(d) $|Arg(\sum_{|\sigma|=2} a_\sigma(t, x, (p_\rho)_{|\rho| \leq 1}) q^\sigma)| < (1 - \frac{\alpha}{2})\pi$ $\forall (t, x, (p_\rho)_{|\rho| \leq 1}) \in [0, T_0] \times \overline{\Omega} \times O$, $\forall q \in \mathbb{R}^n \setminus \{0\}$.

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